New Functional Techniques and Methods of Path Integration
by SCOTT B ANDERSON

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Physics
Montana State University

Abstract:
We develop three new functional techniques. The first is the method of delta functionals whereby a path integral having a Hamiltonian linear in position is reduced to quadratures through the evaluation of an equation of evolution. We augment this technique through the introduction of three canonical transformations which may be used to simplify the path integral. We also construct a new path integral from coherent states for a time—dependent harmonic oscillator. Finally we solve a new quantum mechanical problem and introduce the concept of a functional anti-derivative.

NEW FUNCTIONAL TECHNIQUES AND METHODS
OF PATH INTEGRATION

by

Scott Buckingham Anderson

A thesis submitted in partial fulfillment
of the requirements for the degree

of

Doctor of Philosophy

in

Physics

MONTANA STATE UNIVERSITY
Bozeman, Montana

June 1984

APPROVAL

of a thesis submitted by

Scott Buckingham Anderson

This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

May 25, 1984 — Date

George F. Tuthill — Chairperson, Graduate Committee

Approved for the Major Department

May 25, 1984 — Date

Robert J. Swenson — Head, Major Department

Approved for the College of Graduate Studies

June 7, 1984 — Date

Henry L. Parsons — Graduate Dean

STATEMENT OF PERMISSION TO USE

In presenting this thesis in partial fulfillment of the requirements for a doctoral degree at Montana State University, I agree that the Library shall make it available to borrowers under rules of the Library. I further agree that copying of this thesis is allowable only for scholarly purposes, consistent with ''fair use'' as prescribed in the U.S. Copyright Law. Requests for extensive copying or reproduction of this thesis should be referred to University Microfilms International, 300 North Zeeb Road, Ann Arbor, Michigan 48106, to whom I have granted ''the exclusive right to reproduce and distribute copies of the dissertation in and from microfilm and the right to reproduce and distribute by abstract in any format.''

Signature Scott B. Anderson

Date May 25, 1984

TABLE OF CONTENTS

ABSTRACT

We develop three new functional techniques. The first is the method of delta functionals whereby a path integral having a Hamiltonian linear in position is reduced to quadratures through the evaluation of an equation of evolution. We augment this technique through the introduction of three canonical transformations which may be used to simplify the path integral. We also construct a new path integral from coherent states for a time-dependent harmonic oscillator. Finally we solve a new quantum mechanical problem and introduce the concept of a functional anti-derivative.

CHAPTER 1

BASIC CONCEPTS

This thesis is devoted to the development of new functional methods. The primary emphasis is upon the explicit calculation of path integrals which are functional integral representations of the Green's functions for the Schroedinger equation. The author first provides a review of the basic concepts which are standard and are concisely reviewed by Marinov (1980). The history of path integration is also discussed towards the end of the chapter. To involve the reader in a participatory manner, the thesis is written in the first person plural.

A word about conventions and notation. We use natural units where h = c = 1. In many calculations we will also take the mass and frequency to be unity. Vectors in three dimensions will be boldface and 4-vectors will simply be in lower case. Scalar products will be denoted by a period, eg: **a.b**. The product symbol, usually denoted by an upper-case π, will be taken to be II. Our metric convention for the Minkowski tensor is (+,-,-,-) and Greek letters will be used for tensor indices in spacetime. We write a function f with argument x as usual: f(x). We write a functional g with argument F(t) as g[F]. Remember that a functional is a number that depends upon a function for its value (eg: the area under a curve, $A[F] = \int F(x)\, dx$).

We start with the Schroedinger equation for the wave function Ψ in one dimension

$$H\Psi = i\partial_t \Psi \qquad (1.1)$$

with initial condition $\Psi(0,x)$. Since it is a linear partial differential equation we may solve it in terms of the initial wave function $\Psi(0,x)$ if we can construct a Green's function K that satisfies

$$(\partial_t + iH(p,x))K(x,t|a,r) = \delta(x-a)\delta(t-r). \qquad (1.2)$$

If this can be done we may then write $\Psi(x,t)$ as

$$\Psi(x,t) = \int da\, K(x,t|a,0)\,\Psi(a,0) \qquad (1.3)$$

which requires the limit for K

$$\text{limit } t \to r,\ K(x,t|a,r) \to \delta(x-a). \qquad (1.4)$$

We will also take $K(x,t|a,r) = 0$ for $t < r$ which may be enforced by the introduction of the factor $\Theta(t-r)$, the unit step function. We also note that K itself satisfies Eq. (1.3)

$$K(c,t|a,r) = \int db\, K(c,t|b,s)K(b,s|a,r). \qquad (1.5)$$

which is the defining property of a semigroup. Because K has this quality of propagating the wave function and itself forward into the future it is often called the propagator and that nomenclature will be used in this thesis.

Before we show how K may be represented by a functional integral we discuss some of its properties which make it a useful quantity to know. First we note that K has a standard construction in terms of the energy eigenfunctions $\Psi_E(x)$ with $T = s - r$,

$$K(b,s|a,r) = \sum_E e^{-iET}\Psi_E(b)\Psi^*_E(a). \tag{1.6}$$

A useful function constructed from the propagator is the spectral function Y(T) which is defined as

$$Y(T) = \int da\ K(a,T|a,0) \tag{1.7}$$

and by using Eq. (1.6) we easily see that

$$Y(T) = \sum_E e^{-iET}. \tag{1.8}$$

Information on the ground state energy may be found by taking the limit $T \rightarrow -i\infty$ of Y(T) since only the $\exp(-iE_0T)$ term will survive to contribute to the sum.

The Fourier transform of K(T) is also of interest. Writing (we suppress the spatial dependence)

$$K(\omega) = \int dT\ K(T)\ e^{i\omega T} \tag{1.9}$$

and once again using Eq. (1.6) we find that

$$K(\omega) = i \sum_E \Psi_E(b)\Psi^*_E(a)\ (\omega - E)^{-1} \tag{1.10}$$

which reveals that $K(\omega)$ has poles at bound state energies and a cut along the continuum.

Another quantity of interest that may be calculated from the propagator is the generating functional Z[F]. It is found from the propagator K[F](b|a) where a driving term F(t)x has been added to the Lagrangian. Specifically, if $\Psi_0(x)$ is the ground state wave function for the undriven system then

$$Z[F] = N \int dbda\ \Psi^*_0(b)\Psi_0(a)\ K[F](b,s|a,r). \tag{1.11}$$

We require that the source $F(t)$ be turned off at the endpoints $F(s) = F(r) = 0$ and determine the normalization constant N through the boundary condition $Z[0] = 1$.

Z[F] is of enormous importance as it is (in quantum field theory) the generating functional for the N-point Green's functions. They may be found by taking N functional derivatives of Z[F] and then setting the source F(t) to zero. A functional derivative is somewhat like a partial derivative. We give some examples:

$$\frac{\delta}{\delta F(t)} F(s) = \delta(t-s),$$

$$\frac{\delta}{\delta F(t)} \int F(s)G(s)\, ds = G(t),$$

$$\frac{\delta}{\delta F(t)} \exp\{\iint dxdy\, F(x)G(x,y)F(y)\} = 2\int dx\, G(t,x)F(x).$$

We now wish to discuss the representation of K by a functional integral. This was first done in the context of quantum mechanics by Feynman (1948) in the early Forties based upon an observation by Dirac that the form for the infinitesimal propagator $K(\Delta t)$ is approximately the exponential of the classical action. Feynman took this form as axiomatic and succeeded in constructing all of quantum mechanics from his functional integral representation of the propagator K. For a very interesting anecdotal recounting of those halcyon days we recommend reading Feynman's (1972) Nobel lecture. We will take a different route and derive K as a functional integral from standard quantum theory. The derivation that we follow is the route taken by Abers and Lee (1973).

To begin we write K in the Schroedinger representation as

$$K(b,s|a,r) = \langle b|e^{-iHT}|a\rangle. \tag{1.12}$$

Then we insert N-1 resolutions of the identity $\Sigma |x\rangle\langle x| = 1$ to write K as

$$K = \Sigma \langle b|e^{-iH\Delta t}|x_{N-1}\rangle\langle x_{N-1}|e^{-iH\Delta t}|x_{N-2}\rangle \times \ldots$$
$$\times \ldots \times \langle x_2|e^{-iH\Delta t}|x_1\rangle\langle x_1|e^{-iH\Delta t}|a\rangle \tag{1.13}$$

where we have defined $\Delta t = t_n - t_{n-1}$ and $N\Delta t = T = s - r$. For convenience we will also define $x_N = b$ and $x_o = a$.

Now we examine the infinitesimal propagator $K(\Delta t) = \langle x_n|e^{-iH\Delta t}|x_{n-1}\rangle$. We may insert a complete set of momentum eigenstates $|p\rangle$ to write $K(\Delta t)$ as

$$K(\Delta t) = \Sigma \langle x_n|p_{n-1/2}\rangle\langle p_{n-1/2}|x_{n-1}\rangle$$
$$\times \exp\{-i\Delta t H(p_{n-1/2}, x_n, x_{n-1})\}. \tag{1.14}$$

We will discuss the explicit form of the Hamiltonian in a moment in connection with the factor-ordering problem. Using the explicit form for $\langle x|p\rangle$ allows us to rewrite $K(\Delta t)$ as

$$K(\Delta t) = \int dp_{n-1/2}\ (2\pi)^{-1} \exp(ip_{n-1/2}(x_n - x_{n-1})) \tag{1.15}$$
$$\times \exp\{-i\Delta t(H(p_{n-1/2}, x_n) + H(p_{n-1/2}, x_{n-1}))/2\}.$$

The choice for the Hamiltonian in the second term on the right is somewhat ambiguous and is related to the factor-ordering problem of quantum mechanics. The usual form taken is $H(p_{n-1/2}, (x_n+x_{n-1})/2)$ but we eschew the conventional wisdom and write H in the Euler approximation form as it will allow us to deduce the precise ordering of operators in the Schroedinger equation that K satisfies.

Because all the $K(\Delta t)$'s are exponential in form we may concatenate them to write K(T) as

$$K(T) = \Pi_1^{N-1} \int dx_n \; \Pi_1^N \int dp_{n-1/2} \,/2\pi \qquad (1.16)$$
$$\times \exp\{i \Sigma_1^N p_{n-1/2}(x_n - x_{n-1})\}$$
$$\times \exp\{-i \Sigma_1^N (H(p_{n-1/2},x_n) + H(p_{n-1/2},x_{n-1}))\Delta t/2\}$$

or symbolically as

$$K = \int [dx\; dp/2\pi] \exp(i\int p\dot{x} - H(p,x)\; dt). \qquad (1.17)$$

The above integral is to be interpreted as a functional integral where one integrates over functions rather than points as in a conventional Riemann integral. In particular we are instructed in the integral for K to integrate over all momentum functions p(t) and over all position functions x(t) that start at x(0) = a and end at x(T) = b. In a certain sense we have 'integrated' the Schroedinger equation. The question that comes to mind of course is how on earth would one ever evaluate an integral over functions and that is one of the main topics of this thesis.

We would now like to make some general remarks about the functional integral for K. We notice that the exponent in the integral is just the action S of classical mechanics in Hamiltonian form. This suggests an interpretation for K. A particle may travel from point a to point b via many paths. Classically only one path is selected, the one that has the minimal amount of action. However in quantum mechanics (QM) anything that can happen will happen with a certain amplitude. The prescription in QM is that the total amplitude for an event is the sum of the amplitudes for all possible ways the event can occur. The amplitude for a particle going from a to b along a certain path is $\exp\{iS[\text{path}]\}$ and $K(b,s|a,r)$, the total amplitude for the particle to go from a to b, is just the sum (or integral) of the amplitudes for all possible paths. Hence the terminology of the path integral.

We now turn to the question of what is the equation solved by K? A good discussion of this question for our time-slicing procedure is given in the paper by Mayes and Dowker (1973). In the first appendix we show that the functional integral represents a solution to the specific Schroedinger equation

$$i\frac{\partial}{\partial t} K(q,t|a,r) = H_S(p,q)\ K(q,t|a,r) \tag{1.18}$$

where the quantum operator p is represented as $p = -i\frac{\partial}{\partial q}$. The subscript S on the Hamiltonian stands for a very specific ordering of the operators p and q. We represent this ordering by writing $H_S(p,q) = \frac{1}{2}(H(p|q) + H(q|p))$ where $H(p|q)$ is a right ordered function of q which means that in a power series expansion of $H(p,q)$

$$H(p,q) = \Sigma\, H_{mn} \frac{p^m}{m!} \frac{q^n}{n!} \tag{1.19}$$

all of the quantum q's are placed to the right of the quantum p's, ie:

$$H(p|q) = \Sigma H_{mn} \frac{p^m}{m!} \frac{q^n}{n!} \tag{1.20}$$

and similarly $H(q|p)$ is left ordered

$$H(q|p) = \Sigma H_{mn} \frac{q^n}{n!} \frac{p^m}{m!} . \tag{1.21}$$

A Green's function solution to this Schroedinger equation may be constructed from the path integral for K by writing $G(b,s|a,r) = \theta(s-r)K(b,s|a,r)$ with $\theta(t)$ the unit step function. G satisfies Eq. (1.2)

$$(\frac{\partial}{\partial t} + iH_S(p,q))G = \delta(q-a)\delta(t-r). \tag{1.22}$$

We will also be concerned with a generalization of K namely

$$K(b,a) = C(b)C(a)\frac{D(b)}{D(a)}\int[dp/2\pi][dq]$$

$$x \exp(i \int_r^s p\dot{q}-H(p,q)\ dt) \tag{1.23}$$

where $C(q)$ and $D(q)$ are arbitrary functions of position. This propagator possesses the semigroup property

$$K(c|a) = \int db\ C(b)^{-2}\ K(c|b)K(b|a) \tag{1.24}$$

and has the limit as $s \to r$ that $K(b|a) \to C(b)^2\ \delta(b-a)$. It also satisfies the Schroedinger equation

$$i\frac{\partial}{\partial t}\ K(q,t|a,r) = H_S(p,q)\ K(q,t|a,r) \tag{1.25}$$

where the operator $p = -iE(q)\frac{\partial}{\partial q}E(q)^{-1} = -i\frac{\partial}{\partial q} + i\frac{E'(q)}{E(q)}$ with $E = CD$.

Following Faddeev and Slavnov (1980, p.26) we may derive an interesting form for K when

$$H = \frac{p^2}{2} + V(x).$$

Then K is

$$K = \int [dx\ dp/2\pi]\ \exp\{i\int p\dot{x} - p^2/2 - V(x)\ dt\}. \quad (1.26)$$

We perform a canonical transformation $P = p + \dot{x}$ with $dP = dp$.

$$K = \int [dx\ dP/2\pi]\ \exp\{i\int (P + \dot{x})\dot{x} - (P + \dot{x})^2/2 - V\ dt\}.$$

The cross terms $P\dot{x}$ cancel giving (1.27)

$$K = (\int [dP/2\pi]\ \exp\{-i\int P^2/2\ dt\})$$
$$\times (\int [dx]\ \exp\{i\int \dot{x}^2/2 - V(x)\ dt\}). \quad (1.28)$$

The momentum integrations may be considered a normalization factor that may be absorbed into the quasi-measure [dx] allowing us to write

$$K = \int [dx]\ \exp\{i\int \dot{x}^2/2 - V(x)\ dt\}. \quad (1.29)$$

We see that the exponent is just the action S written in Lagrangian form. This was Feynman's original form for his version of quantum mechanics. We note that in general one must start with Eq. (1.15) as the fundamental form for K. We will see in chapter 6 that even this form must be modified should the mechanics take place on other manifolds than Euclidean space.

The subject of this thesis is the investigation and development of new methods for dealing with functionals, in particular we provide a new technique called the method of delta functionals (MDF) for the explicit calculation for a certain class of path integrals. We extend our technique through the introduction of three canonical transformations. Also discussed are the generalizations of MDF to relativity and group manifolds. In Chapter 7 we construct a new path integral for time-dependent coherent states. Finally we report the results of our investigation of the generating functional for a new exactly solvable quantum mechanical system which is the one-dimensional analogue of general relativity. In the penultimate chapter we introduce a new kind of functional integration, functional anti-differentiation, in connection with the perturbation series for this new solvable system.

Historically, the first explicit path integration was that of the quadratic Lagrangian (harmonic oscillator) reported in Feynman's (1948) seminal paper. The next new path integral was that of the inverse quadratic potential calculated in the paper of Peak and Inomata (1969) some twenty years later. Basically the result was obtained from the observation that a free particle in polar coordinates has a r^{-2} centrifugal potential already. The propagator for the Coulomb potential was evaluated by Duru and Kleinert (1979) by transforming it into an harmonic oscillator in 4 dimensions. Recently the transformation introduced by Duru and Kleinert was used by Duru (1983) to obtain the Morse potential propagator. This transformation may be used to obtain other propagators which have not yet appeared in print. One sees that the calculation of new propagators has depended upon the creation of suitable transformations. Relativistic path integrals were first constructed by Feynman in his 1948 paper and in Feynman (1951). Group space propagators were first discussed by Schulman (1968) and definitively treated by Marinov and Terentyev (1979). Coherent states were first used to construct a path integral by Klauder (1960).

CHAPTER 2

METHODS OF PATH INTEGRATION

In the previous chapter we derived a representation of the propagator as a functional integral. Now we wish to examine the principal techniques for evaluating functional integrals in order to be able to compare and contrast them with our technique which will be presented in the next chapter. The example used in all cases will be the harmonic oscillator (HO) with Lagrangian $L = (\dot{x}^2 - \omega^2 x^2)/2$ and Hamiltonian $H = (p^2 + \omega^2 x^2)/2$.

The first method we illustrate hinges upon the fact that Gaussian or quadratic integrals concatenate, scilicet the integral of a product of two Gaussians is again a Gaussian. This was noticed by Feynman in his first paper on path integrals. We follow the derivation given by Khandekar and Lawande (1975). We first write the Lagrangian (L) path integral for K in a time-sliced or discretized form (2.1)

$$K = A^{-N} \Pi \int dx_n \exp\{\frac{i}{2} \Sigma (x_n - x_{n-1})^2/\Delta t - \omega^2 x_n{}^2 \Delta t\}.$$

A is the normalization constant induced by the momentum integrations.

Suppose we do the x_1, x_2,...., x_{n-1} integrals. We would be left with an integral

$$\int dx_n \exp\{\frac{i}{2\Delta t}((x_{n+1} - x_n)^2 - (\omega\Delta t x_{n+1})^2\}$$

$$\times \exp\{\frac{i}{2\Delta t}(\Omega_n a^2 + \mu_n x_n{}^2 - 2\beta_n x_n a)\} \qquad (2.2)$$

which upon integration would produce a factor multiplying the exponential of

$$\exp\{\frac{i}{2\Delta t}(\Omega_{n+1} a^2 + \mu_{n+1} x_{n+1}{}^2 - 2\beta_{n+1} x_{n+1} a)\}. \qquad (2.3)$$

By using Eq. (B.9) we may obtain the following recursion relations for Ω, μ, and β which will allow us to evaluate the concatenation of Gaussian integrals. We define $\alpha = \mu + 1$.

$$\alpha_{n+1} = 2 - \omega^2\Delta t^2 - 1/\alpha_n. \qquad (2.4)$$

$$\Omega_{n+1} = \Omega_n - \beta_n{}^2/\alpha_n. \qquad (2.5)$$

$$\beta_{n+1} = \beta_n/\alpha_n. \qquad (2.6)$$

To solve the key equation Eq. (2.4) we define $Q_{n+1}/Q_n = \alpha_n$. Then Eq. (2.4) becomes

$$Q_{n+2} - 2Q_{n+1} + Q_n = -\omega^2\Delta t^2 Q_{n+1}. \qquad (2.7)$$

In the limit as $\Delta t \to 0$, Eq. (2.7) reduces to the differential equation for the harmonic oscillator

$$d^2Q/dt^2 + \omega^2 Q = 0. \qquad (2.8)$$

The solution with the correct boundary condition $Q_o = 0$ is $Q(t) = \dot{Q}_o \sin(\omega t)/\omega$. The other functions may be found from

$$\mu(t)/\Delta t = \dot{Q}(t)/Q(t) = \omega\cot(\omega t), \qquad (2.9)$$

$$\beta(t)/\Delta t = \dot{Q}_o/Q(t) = \omega\csc(\omega t), \text{ and} \qquad (2.10)$$

$$\dot{\Omega}(t)/\Delta t = -\dot{Q}_o{}^2/Q(t)^2 \qquad (2.11)$$

with solution $\Omega(t)/\Delta t = \omega\cot(\omega t)$. (2.12)

Hence the exponential term resulting from the path integral is

$$\exp\{\frac{i\omega}{2\sin(\omega T)}((b^2 + a^2)\cos(\omega T) - 2ba)\}. \qquad (2.13)$$

The pre-exponential factors concatenate to

$$A^{-N}(2\pi i)^{(N-1)/2}\Delta t^{N/2}(Q(T)/Q_0)^{-1/2}. \qquad (2.14)$$

For a convergent functional integral we must have $A = 2\pi i\Delta t$ which gives as our result for the HO propagator $(T = s - r)$

$$K(b,s|a,r) = (2\pi i\sin(\omega T)/\omega)^{-1/2}$$
$$\times \exp\{\frac{i\omega}{2\sin(\omega T)}((b^2 + a^2)\cos(\omega T) - 2ba)\}. \qquad (2.15)$$

The success of this method is a consequence of the fact that Gaussian integrals concatenate. To generalize this method requires the discovery or creation of other concatenatory integrals involving Gaussians. This has only been done in one other instance where the inverse quadratic propagator was found using Eq. (B.11). Hence we see that this method has a severe limitation; a suitable concatenatory integral must be found antecedent to any calculation and this is a very difficult task.

A second method that is used to evaluate quadratic path integrals utilizes an expansion about the classical path. This was also developed by Feynman but the evaluation of the Fredholm determinant seems to first appear in Montroll (1952). Examine the HO path integral in Lagrangian form

$$K = \int[dx]\exp\{\frac{i}{2}\int \dot{x}^2 - \omega^2x^2 \, dt\} \qquad (2.16)$$

with boundary conditions $x(T) = b$ and $x(0) = a$. Now we expand $x(t)$ about the classical path $q(t)$ which satisfies the equation of motion $d^2q/dt^2 + \omega^2 q = 0$. Specifically we write $x(t) = q(t) + y(t)$ and note that $dx = dy$ and $y(0) = y(T) = 0$. For quadratic Lagrangians (and only for quadratic L) we have $S[q + y] = S[q] + S[y]$. Hence we may write K as

$$K = \exp\{iS[q]\} \int [dy] \exp\{iS[y]\}. \tag{2.17}$$

We remark that all of the position dependence upon b and a is contained in the first factor which is simply the exponential of the classical action S_c (the action evaluated along the classical path). This is the familiar

$$S_c(b,a) = \frac{\omega}{2\sin(\omega T)}((b^2 + a^2)\cos(\omega T) - 2ba). \tag{2.18}$$

The second, path integral factor is a function of T alone (due to the fact that y(t) vanishes at the endpoints). We now turn to its evaluation. Because of the periodic boundary conditions we may write the integral as

$$\int [dy] \exp\{-i\int y(D^2 + \omega^2)y/2\} \tag{2.19}$$

where we have performed an integration by parts and defined $D^2 = d^2/dt^2$. This is a continuum version of Eq. (B.5) and we use it to write Eq. (2.19) formally as

$$C(\mathrm{Det}[D^2 + \omega^2])^{-1/2}. \tag{2.20}$$

C is a formally infinite normalization constant. We have used the standard notation of capital Det for determinants of continuous matrices (Fredholm determinants). The problem is now to evaluate the this determinant.

The time-sliced version of this determinant is

$$A^{-N}(2\pi i\Delta t)^{(N-1)/2}(\det M)^{-1/2} \tag{2.21}$$

where the NxN matrix M is (2.22)

$$M = \begin{vmatrix} 2-\omega^2\Delta t^2 & -1 & 0 & 0 & \cdots\cdots \\ -1 & 2-\omega^2\Delta t^2 & -1 & 0 & \cdots\cdots \\ 0 & -1 & 2-\omega^2\Delta t^2 & -1 & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

Using the fact that A must equal $2\pi i\Delta t$ allows us to write the factor Eq. (2.21) as

$$(2\pi i)^{-1/2}\ (\Delta t \det M)^{-1/2}. \tag{2.23}$$

By inspection of the form for M given above we see that the quantity $f = \Delta t \det M$ satisfies the recursion relation

$$f_{n+1} - 2f_n + f_{n-1} = -\omega^2\Delta t^2 f_n. \tag{2.24}$$

In the limit as Δt goes to zero this equation reduces to the differential equation for the HO

$$D^2 f + \omega^2 f = 0 \tag{2.25}$$

with initial conditions $f(0) = 0$ and $\dot{f}(0) = 1$. The solution is $f(t) = \sin(\omega t)/\omega$ and consequently upon substitution of this result into Eq.(2.23) and Eq. (2.17) we arrive at the HO propagator given by Eq. (2.15).

This method is the principal technique used in the literature to evaluate path integrals as it is easily generalized to higher dimensions and field theory. However as should be obvious, this method may only be employed to calculate Gaussian path integrals and has often led to the claim that only Gaussian path integrals may be evaluated. This claim is not true in quantum mechanics but it is certainly a fact that at present the only path integrals that have been evaluated in quantum field theory are indeed Gaussian.

The third method we review exploits the recently developed formalism of the star product. A seeming advantage of path integrals is that they are classical objects. One need never work with operators in Hilbert space. The star product extends this advantage to such things as the commutator and the Heisenberg equations of motion. The use of the star product allows one to map operations in Hilbert space to (perhaps simpler) operations involving ordinary functions. The concept of the star product is developed by Bayen et al. (1978).

Specifically the star product of two phase space functions f and g is defined as

$$(f*g)(x,p) = f(x,p)\exp\{-(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)\}g(x,p). \quad (2.26)$$

The quantum mechanical commutator goes over to

$$[f,g] \dashrightarrow -i(f*g - g*f) \quad (2.27)$$

and in an interesting (but unremarked upon) paper Sharan (1979) has shown that if we define a star exponential as

$$(\exp*f)(x,p) = 1 + f + f*f/2! + \ldots \quad (2.28)$$

then the phase space path integral may be written as

$$\int[dxdp/2\pi]\exp\{i\int p\dot{x} - H(p,x)\ dt\}$$

$$= \int dp/2\pi\ e^{ip(b-a)}\ (\exp*-iHT)(p,\tfrac{b+a}{2}) \ . \quad (2.29)$$

The obvious difficulty here is the evaluation of the star exponential. In the case of a quadratic Hamiltonian $H = (p^2 + x^2)/2$ it may be done as follows. Let G(t,H) =

$\exp^* - iHt$. Next we use the fact that for any function f

$$H^*f(H) = Hf - f'/4 - Hf''/4 \tag{2.30}$$

to write a partial differential equation for G

$$i\frac{\partial G}{\partial t} = HG - \frac{1}{4}\frac{\partial G}{\partial H} - \frac{H}{4}\frac{\partial^2 G}{\partial H^2} . \tag{2.31}$$

Fourier transforming in both variables to a new function

$$G(s,\omega) = \int\int dH\, dt/2\pi\; e^{i\omega t}\, e^{isH}\, G(t,H) \tag{2.32}$$

allows us to write a differential equation

$$i(\omega + is/4)G(s,\omega) = (1 + s^2/4)dG/ds \tag{2.33}$$

which has solution

$$G(\omega,s) = (1 + s^2/4)^{-1/2} \exp\{2i\omega \tan^{-1}(s/2)\}. \tag{2.34}$$

Inverting this result gives

$$G(t,H) = \sec(t/2)\exp\{-2iH\tan(t/2)\} \tag{2.35}$$

and inserting this expression into Sharan's form for the path integral leads to

$$K = \int dp/2\pi\; e^{ip(b-a)} \sec(T/2)$$
$$\times \exp\{-i\tan(T/2)(p^2 + (b+a)^2/2\} \tag{2.36}$$

which upon integration gives again Eq. (2.15) with $\omega = 1$.

One advantage of this method is that it is not limited to Gaussians. However the mathematical technology for evaluating the star exponential is extremely rudimentary and at present only star exponentials with general quadratic arguments of p and x have been calculated, cf. Bayen and Maillard (1982). Also in evaluating the star exponential we had to solve a partial differential equation which is somewhat contrary to the spirit of the path integral.

There are two other techniques proposed for evaluating path integrals. One is Lee's (1976) continuum calculus. It is deficient in that the endpoint dependence must be extracted somehow and it still requires the evaluation of a continuous determinant.

The other method which has recieved a lot of attention is the Fourier transform technique of Dewitt et al. (1979) and Mizrahi (1976). At the present time this method may only be used to calculate Gaussian path integrals. Its virtue is that it supplies a somewhat rigorous definition of the path integral without a time-slicing procedure.

CHAPTER 3

THE METHOD OF DELTA FUNCTIONALS

At this point we have surveyed the present techniques of path integration. We now wish to present a new technique called the method of delta functionals and use it to solve the general Schroedinger equation (Eq. (1.2)) where H has the specific form (3.1)

$$i\partial_t K(q,t|a,r) = [F(p,t) - \frac{1}{2}(qG(p,t)+G(p,t)q)]K(q,t|a,r)$$

This seems to be a somewhat simple partial differential equation but we will show how when specialized to second order in p the solutions to this equation can accommodate nearly all of the known propagators including the following potentials.

$V(q) = \frac{q^2}{2} - qF(t)$ The forced harmonic oscillator.

$V(q) = \frac{q^2}{2} + \frac{1}{q^2}$ The inverse quadratic potential.

$V(q) = \frac{1}{q^2} + \frac{e^2}{q}$ The Coulomb potential.

$V(q) = e^q$ The Morse potential.

Hence as a byproduct of our investigation we will bring to light the underlying equivalence of all of the propagator solutions to the above potentials. Specifically; in the subsequent sections we concern ourselves with finding the general propagator solution to Eq. (3.1). To achieve this end we will write the solution as a path integral. To evaluate the path integral we develop a new technique, the method of delta functionals (MDF), and apply it to our problem.

In the next chapter with the solution in hand we will construct three canonical transformations that allow us generate further solutions to more complex Hamiltonians than that of Eq. (3.1). With the presentation of each transformation we work a physical example (where H is $O(p^2)$) illustrating the use of the formalism.

The idea of MDF was conceived by us after examination of a calculation by Katz (1965). The basic observation there was that phase space path integrals with Hamiltonians that only depended upon momentum ($H = H(p)$) were easy to integrate because the functional integration over position gave rise to an infinite product of delta functions. In MDF we extend this idea to Hamiltonians that depend linearly upon position and we present our reasoning without the use of time-slicing.

To begin we write the phase space path integral representation for the propagator solution to Eq. (3.1)

$$K(b,s|a,r) = \int [dq\,dp/2\pi] \exp(i\int_r^s \dot{p}q - F(p) + qG(p)\, dt). \qquad (3.2)$$

Notice that F and G may be arbitrary functions of time but we shall rarely make this explicit.

If we perform an integration by parts in the exponent we may write K as

$$K = \int [dq dp/2\pi] \exp(ipq| \exp(i\int q(G - \dot{p}) - F\, dt) \quad (3.3)$$

where we use the notation $pq| = p(s)b - p(t)a$.

The functional integration over position is a representation of a delta functional. Recognizing this allows us to rewrite K as

$$K = \int [dp]/2\pi \exp(ipq| \delta[\dot{p}-G(p)] \exp(-i\int F\, dt) \quad (3.4)$$

To proceed any further requires an interpretation of the delta functional. Heuristically its presence forces the contributing paths in momentum space to obey the equation of evolution

$$\frac{dp}{dt} = G(p,t). \quad (3.5)$$

The solutions to this first order differential equation are trajectories that may be labelled by the initial condition p_o. In the spirit of a sum over paths we may expect to be able to replace the momentum functional integration by an equivalent integration over p_o which labels all of the contributing paths. Schematically we have

$$\int [dp]\, \delta[\dot{p}-G(p)]\, f[p] \dashrightarrow \int dp_o\, f[p(p_o)] \quad (3.6)$$

where f is some arbitrary functional and $p(p_o,t)$ is the solution to $\dot{p} = G(p,t)$ with initial condition p_o. However there is a correction to the naive replacement scheme of Eq. (3.6). The propagator must satisfy the semigroup property Eq. (1.5). Inserting our replacements for Eq. (3.4) into Eq. (1.5) gives

$$\int dq_o/2\pi \; db \; dp_o/2\pi \; \exp(i(q(t)c-q(s)b)) \; \exp(i(p(s)b-p(r)a))$$
$$\times \; \exp(-i\int F(q(q_o)) \; dt) \; \exp(-i\int F(p(p_o)) \; dt). \quad (3.7)$$

The b integral is a representation for a delta function so we may write Eq. (3.7) as

$$\int dq_o/2\pi \; dp_o \; \exp(i(q(t)c-p(r)a) \; \exp(-i\int dt \; F)\exp(-i\int dt \; F)$$
$$\times \; \delta(q(s) - p(s)). \quad (3.8)$$

Writing the delta function in terms of q_o

$$\delta(q(s) - p(s)) = \delta(q_o - p_o) \; (dq(q_o,s)/dq_o)^{-1} \quad (3.9)$$

and integrating over q_o gives

$$\int dp_o/2\pi \; (dp(p_o)/dp_o)^{-1} \; \exp(ipq| \; \exp(-i\int F(p) \; dt). \quad (3.10)$$

Clearly to preserve the semigroup property we must include a factor $(dp(p_o,s)/dp_o \; dp(p_o,r)/dp_o)^{-1/2}$ in our replacement scheme to cancel the factor arising from the delta function. Hence the correct replacement is

$$\int[dp] \; \delta[\dot{p}-G(p)] \; f[p] \; \text{-->} \; \int dp_o \; \left(\frac{dp(p_o,s)}{dp_o} \; \frac{dp(p_o,r)}{dp_o}\right)^{1/2}$$
$$\times \; f[p(p_o,t)]. \quad (3.11)$$

Using this result in Eq. (3.4) we may write the general solution to Eq. (3.1) as

$$K(b,s|a,r) = \int dp_o/2\pi \; \left(\frac{dp(p_o,s)}{dp_o} \; \frac{dp(p_o,r)}{dp_o}\right)^{1/2} \quad (3.12)$$
$$\times \; \exp(i(p(p_o,s)b-p(p_o,r)a) \; \exp(-i\int_r^s F(p(p_o,t) \; dt)$$

where $p(p_o,t)$ is the solution to the equation of evolution

$$\frac{dp}{dt} = G(p,t) \quad (3.13)$$

with initial condition $p_o = p(t = 0)$.

One might feel a lack of confidence in our solution due to the heuristic reasoning involved. However it is easily verified that this is the correct solution by simply checking to see that it does indeed satisfy Eq. (3.1). In Appendix A we present a more rigorous derivation of the replacement scheme Eq. (3.11) using a time-slicing definition of the path integral.

To illustrate the method we present the example of the free particle. The propagator is written as a path integral

$$K(b|a) = \int[dq dp/2\pi] \exp(i\int_0^T p\dot{q} - p^2/2 \, dt). \tag{3.14}$$

Integrating by parts and recognizing the delta functional allows us to write

$$K = \int[dp]/2\pi \exp(ipq| \exp(-i\int p^2/2 \, dt) \, \delta[\dot{p}]. \tag{3.15}$$

The solution to $dp/dt = 0$ is $p = \text{constant} = p_o$ and the semigroup factor $dp(p_o)/dp_o = 1$. Hence we may write K as

$$K = \int dp_o/2\pi \exp(i(p_o(b-a) - \int_0^T p_o{}^2/2 \, dt). \tag{3.16}$$

Performing the time integration in the exponent gives

$$K = \int dp_o/2\pi \exp(i(p_o(b-a) - p_o{}^2T/2) \tag{3.17}$$

and integrating over p_o leads to the result

$$K(b|a) = (2\pi iT)^{-1/2} \exp(\frac{i}{2T}(b-a)^2). \tag{3.18}$$

We now make some remarks about the technique. It is easily generalized to higher dimensions. In doing so, one will encounter matrix equations of evolution with solutions of the form

$$p(t) = A(t).p_o. \tag{3.19}$$

The semigroup factors in this case are easily found to be

$[\det A(s) \det A(r)]^{1/2}$.

Also as we shall see one often deals with delta functions of complex quantities. The prescription is to simply ignore this fact. The integration is always over real p_o.

As noted, this technique was suggested by an observation by Katz. In our notation his observation (based upon time-slicing) was

$$\int [dp]\ \delta[\dot{p}]\ F[p] = \int dp_o\ F[p_o]. \qquad (3.20)$$

Recently we uncovered one other paper that contains formulae of this sort. This is the article by Cambell et al. (1976). Using time-slicing they derive the formula (in our notation)

$$\int [dp]\ \delta[\dot{p}-G(p)]\ e^{ipx}\Big| \qquad (3.21)$$

$$= \int dp\ \exp\{i(pb-p(0)a)\}\exp\{-\frac{1}{2}\int_0^T ds\ \mathrm{Re}\ \frac{\partial G}{\partial p}(p(s),s)\}$$

where $p(t)$ is a solution to $dp/dt = G(p)$. If one compares this equation with Eq. (A.36), one sees that this is equivalent to our formula (restricted to $F = 0$) except for the requirement that the real part of $G'(p)$ be used. This is incorrect. We also feel it is simpler to write the semigroup factor as $(dp/dp_o)^{1/2}$.

CHAPTER 4

CANONICAL TRANSFORMATIONS

Now that we have a new method for calculating path integrals we wish to extend our technique so that we may move beyond the restriction of Hamiltonians linear in position. To this end we will introduce in this chapter three canonical transformations and illustrate each with a worked example. In all cases our strategy is to transform the Schroedinger equation and then write down the path integral that corresponds to it. This may be done because our time-slicing definition of the path integral enables us to determine the Schroedinger equation to which the path integral is a solution. We refer the reader to the discussion concerning Eq. (1.18). Consequently we suffer from no factor-ordering ambiguity in our derivations. We shall see that in general when one transforms the path integral, correction terms must be added to the naively transformed Hamiltonian. In most cases this correction may be cast into the form of an effective potential.

LINEAR MOMENTUM TRANSFORMATION

The first canonical transformation that may be applied to our phase-space path integral in order to bring it into a more convenient form is a linear transformation in p(t).

$$p = P + f(q),\ q = Q,\ dp = dP,\ dq = dQ. \tag{4.1}$$

$$K(b|a) = \int [dq\ dp/2\pi]\ \exp(i \int p\dot{q} - H(p,q)\ dt) \tag{4.2}$$

$$= \int [dQ\ dP/2\pi]\ \exp(i \int (P+f(Q))\dot{Q} - H(P+f(Q),Q)\ dt).$$

In general for Hamiltonians of higher order than p^2 a correction term U must be added to H to preserve the correct ordering of the quantum operators. We do not calculate U for this transformation as all physical examples are $O(p^2)$. Correction terms must be calculated for the transformations we discuss later in the chapter and it will become apparent from their evaluation how the correction for this transformation would be calculated.

As shown in Appendix A we have the useful results:

$$\int_r^s f(q)\dot{q}\ dt = \int_a^b f(q)\ dq = g| \tag{4.3}$$

where $dg/dq = g'(q) = f(q)$ and $g|$ is defined to be $g| = g(b)-g(a)$. Consequently such a term will be a constant as far as position integration is concerned. We also have an integration by parts identity

$$\int_r^s p\dot{q}\ dt = pq| - \int_r^s \dot{p}q\ dt \tag{4.4}$$

and note that $[dq][dp/2\pi] = [dq/2\pi][dp]/2\pi$.

To illustrate the utility of this transformation we work out the propagator for the standard one-dimensional harmonic oscillator. The action is

$$S = \int p\dot{q} - \frac{p^2}{2} - \frac{\omega^2 q^2}{2}\ dt. \tag{4.5}$$

By performing a linear transformation in p (basically completing the square)

$$p = P - i\omega Q, \quad q = Q, \tag{4.6}$$

we may write the action in the more convenient form for use in the path integral

$$S = \int P\dot{Q} - i\omega Q\dot{Q} - P^2/2 + i\omega PQ \, dt. \tag{4.7}$$

Inserting this into the path integral and integrating out the total derivative gives (4.8)

$$K = \exp(\frac{\omega Q^2}{2}|)\int[dPdQ/2\pi] \exp(i\int P\dot{Q} - P^2/2 + i\omega PQ \, dt)$$

At this point we note that this path integral is still a solution to the appropriate Schroedinger equation. From Eq. (1.25) K is a solution to

$$i\frac{\partial}{\partial t}K = [(-i\frac{\partial}{\partial Q}+i\omega Q)^2/2-(Qi\omega(-i\frac{\partial}{\partial Q}+i\omega Q)+(-i\frac{\partial}{\partial Q}+i\omega Q)i\omega Q)/2]K$$

$$= [-\frac{1}{2}\frac{\partial^2}{\partial Q^2} + \frac{\omega^2 Q^2}{2}]K. \tag{4.9}$$

Using MDF, Eq. (4.8) may be immediately rewritten as

$$K = \exp(\frac{\omega Q^2}{2}|)\int dp_o/2\pi \; e^{iPQ|} \; (\frac{dP(s)}{dP_o}\frac{dP(r)}{dP_o})^{1/2}$$

$$x \exp(-i\int_r^s P^2/2 \, dt) \tag{4.10}$$

where $dP/dt = i\omega P$. The solution to this differential equation is $P(P_o,t) = P_o e^{i\omega(t-t_o)}$. Choosing $t_o = (s+r)/2$ causes the semigroup factors to equal unity (4.11)

$$(\frac{dP(s)}{dP_o}\frac{dP(r)}{dP_o})^{1/2} = (e^{i\omega(s-r)/2}e^{i\omega(r-s)/2})^{1/2} = 1.$$

The time integral in the exponent is easily done giving

$$-i\int_r^s P^2/2 \, dt = -\frac{iP_o^2}{2\omega}\sin(\omega T) \tag{4.12}$$

with $T = s - r$. Hence (4.13)

$$K(b,s|a,r) = \exp(\frac{\omega}{2}(b^2-a^2)) \int dP_o/2\pi$$

$$x \exp(iP_o(be^{i\omega T/2}-ae^{-i\omega T/2})) \exp(\frac{-iP_o}{2\omega}\sin(\omega T))$$

$$= (\frac{\omega}{2\pi i\sin(\omega T)})^{1/2} \exp(\frac{i\omega}{2\sin(\omega T)}((b^2+a^2)\cos(\omega T)-2ab))$$

which is the usual expression for the harmonic oscillator propagator (see Eq. (2.15). To use this transformation, one must be clever in completing the square. This is done for various propagators of physical interest in Appendix C.

POINT CANONICAL RESCALING

The next canonical transformation we examine is a point rescaling of the position. Let (4.14)

$$p = P/f'(Q),\ q = f(Q),\ dp = dP/f'(Q),\ dq = f'(Q).$$

For convenience we also define $g(Q) = f'(Q)^2$. We interpret $g(Q)$ as a kind of 'metric' on a one dimensional space. Then

$$K(b|a) = \int[dq\ dp/2\pi]\ \exp(i \int p\dot{q} - H(p,q)\ dt) \quad (4.15)$$

is transformed to ($b = f(B)$ and $a = f(A)$)

$$K(B|A) = (g(B)g(A))^{-1/4} \int[dQ\ dP/2\pi]\ \exp(i \int P\dot{Q}\ dt)$$
$$\times \exp(-i \int H(g(Q)^{-1/2}P, f(Q)) + U(P,Q)\ dt). \quad (4.16)$$

$U(P,Q)$ is an correction term that must be included for the following reason. The rescaled K is a solution to the rescaled Schroedinger equation where the Hamiltonian (H) is $H_S(g^{-1/4}Pg^{-1/4}, f(Q))$. However our path integral is a solution where the Hamiltonian has its operators ordered according to S. We define a symmetrizing operation Sym( ) which takes the operators in a function and permutes them until the symmetrized form is reached, ie: there is some $h_S(p,q)$ such that $h_S(p,q) = \mathrm{Sym}(f(p,q))$. Hence the Hamiltonian used in our path integral must be

$$\mathrm{Sym}(H_S(g^{-1/4}Pg^{-1/4}, f(Q))) \tag{4.17}$$

and this is in general not equal to $\bar{H}_S(P,Q)$ where

$$\bar{H}(P,Q) = H(g^{-1/2}P,Q). \tag{4.18}$$

However, one may write

$$\bar{H}_S(P,Q) + U(P,Q) = \mathrm{Sym}(H_S(g^{-1/4}Pg^{-1/4}, f(Q))) \tag{4.19}$$

and interpret U as a necessary quantum correction needed to maintain a specific operator ordering.

As an example we work out the case for $H = p^2/2m + V(q)$. Then

$$H_S(g^{-1/4}Pg^{-1/4}, f) = (2m)^{-1}(g^{-1/4}Pg^{-1/2}Pg^{-1/4})+V. \tag{4.20}$$

Using the commutation relation $[P,Q] = -i$ one finds

$$\mathrm{Sym}(H_S(g^{-}Pg^{-})) = (2m)^{-1}(g^{-1}P^2+P^2g^{-1})/2 - \frac{g''}{8mg^2} + \frac{9g'^2}{32mg^3} + V. \tag{4.21}$$

Now $\bar{H}(P,Q) = P^2/2mg + V$ so $\bar{H}_S(P,Q) = (2m)^{-1}(g^{-1}P^2+P^2g^{-1})/2 + V$. It is now easily seen that the necessary U is

$$U(P,Q) = \frac{1}{8mg^2}\left(\frac{9g'^2}{4g} - g''\right). \tag{4.22}$$

The example we work to illustrate this transformation has an inverse quadratic potential. Examine the action (4.23)

$$S = \int \frac{m}{2}\dot{y}^2 - \frac{m}{2}\Omega^2 y^2 - g/y^2\, dt = \int p\dot{y} - \frac{p^2}{2m} - \frac{m}{2}\Omega^2 y^2 - g/y^2\, dt$$

where Ω is a function of time $\Omega(t)$. To place this Hamiltonian into an appropriate form for the use of MDF we perform a rescaling point canonical transformation. Let $p \to (1+\alpha q)^{1/2}p$, $y \to 2(1+\alpha q)^{1/2}/\alpha$. Then the rescaled action is $(m = 1)$

$$S = \int p\dot{q} - (1+\alpha q)p^2/2 - \omega^2 q^2/2(1+\alpha q) + qF - \omega^2/\alpha^2 + F/\alpha + \alpha^2/32(1+\alpha q)\ dt \qquad (4.24)$$

where $\varepsilon = 2\Omega(t)$, $\varepsilon^2 = \omega^2 - 2\alpha F(t)$, and $g = 2\omega^2/\alpha^4 - 1/8$. Note that ω and g are constants. Expressed this way the particle appears to be a forced harmonic oscillator with a position-dependent mass.

The subtraction of the effective potential term $U = \alpha^2/32(1+\alpha q)$ neatly cancels the last term of the action.

As it stands we must still modify the action if we are to apply MDF, so we perform a linear transformation letting $p \to p + i\omega q/(1+\alpha q)$ which allows us to write S as

$$S = i\omega(b-a)/\alpha - (i\omega/\alpha^2)\ln\{(1+\alpha b)/(1+\alpha a)\} - \omega^2 T/\alpha^2 + \int_r^s p\dot{q} - (1+\alpha q)p^2/2 - i\omega pq + qF + F/\alpha\ dt \qquad (4.25)$$

and the path integral we wish to evaluate as

$$K(b|a) = \left(\frac{1+\alpha b}{1+\alpha a}\right)^{\omega/\alpha^2} (1+\alpha b)^{1/4}(1+\alpha a)^{1/4} \qquad (4.26)$$
$$\times \exp(-\omega(b-a)/\alpha - i\omega^2 T/\alpha^2)\ \exp(i\int F(t)/\alpha\ dt)$$
$$\times \int\left[\frac{dq\,dp}{2\pi}\right] \exp\{i\int dt\ p\dot{q} - (1+\alpha q)p^2/2 - i\omega pq + Fq\}.$$

Notice the introduction of the fourth root factors as per Eq. (4.16). The pre-integration factors will be called P1. Integrating by parts in the exponent and noting the delta functional gives (4.27)

$$K = (P1/2\pi)\int[dp]\ \delta[\dot{p}+\beta p^2+i\omega p-F]\ e^{ipq|}\ e^{-i\int p^2/2\ dt}$$

where $\beta = \alpha/2$.

The solution to $dp/dt = F(t) - i\omega p - \beta p^2$ may be found through a dependent variable substitution. Define a function f(t) by

$$p(t) = \dot{f}(t)/\beta f(t) - i\omega/2\beta. \tag{4.28}$$

Substitution of this form for p(t) into the differential equation for p leads to an equation for f ($D^2 = d^2/dt^2$)

$$D^2 f(t) + (\omega^2/4 - \beta F(t))f(t) = 0. \tag{4.29}$$

This is the equation for a harmonic oscillator with a time dependent frequency $\varepsilon/2 = \Omega$. A discussion of this system is contained in Appendix D. To solve this equation (formally) we construct a Green's function which satisfies

$$(D^2 + \Omega^2)G(t,s) = \delta(t-s), \tag{4.30}$$

then the solution f(t) may be expressed in terms of f at an earlier time s by

$$f(t) = G(t,s)\overleftrightarrow{\partial}s f(s) \tag{4.31}$$

where $A\overleftrightarrow{\partial}tB = AdB/dt - (dA/dt)B$. We are dealing with the retarded Green's function $G(t,s) = 0$ if $t < s$. Using G we may write the solution $p(p_o,t)$ as ($Gts = G(t,s)$)

$$p(p_o,t) = \frac{1}{\beta}\left(\frac{\dot{G}ts(\beta po + i\omega/2) - \dot{G}t\dot{s}}{Gts(\beta p_o + i\omega/2) - Gt\dot{s}}\right) - \frac{i\omega}{2\beta} \tag{4.32}$$

with the 'dot' signifying time differentiation.

The semigroup factors are found by differentiating the above equation for p with respect to the initial condition p_o. The expression may be shown to be

$$dp(t)/dp_o = (f_o/f(t))^2$$

$$= (Gto(\beta p_o + i\omega/2) - Gt\dot{o})^{-2} \tag{4.33}$$

where we have used the constancy of the regular Wronskian to set (see Eq. (D.12))

$$\dot{G}t\dot{s}Gts - \dot{G}tsGt\dot{s} = 1. \tag{4.34}$$

Using MDF and taking the arbitrary initial time $t_o = r$ allows us to rewrite the path integral as $(p = p_o)$

$$K = (P1/2\pi) \int dp\ (f_r/f_s)\ e^{ipq|}\ \exp\{-i\int p^2/2\ dt\}. \quad (4.35)$$

The time integral in the exponent may be shown to be

$$-i\beta\ F(t)/\alpha\ dt + i(p(s)-p)/\alpha - (2\omega/\alpha^2)\ln(f_s/f_r) + i\omega^2 T/\alpha^2. \quad (4.36)$$

By defining a new variable of integration $u = f_s/f_r$ and using the above result for the time integral we may after some algebra put Eq. (4.35) into the form

$$K = (2\pi\beta Gsr)^{-1}\ ((1+\alpha b)(1+\alpha a))^{1/4}\ (\frac{1+\alpha b}{1+\alpha a})^{\omega/\alpha^2}$$
$$\times \exp\{(2i/\alpha^2 Gsr)((1+\alpha b)G\dot{s}r - (1+\alpha a)Gs\dot{r})\}$$
$$\times \int du\ u^{-(\mu+1)}\ \exp\{(-i/\beta Gsr)(Au + B/u)\} \quad (4.37)$$

with $B = b + 1/\alpha$ etc. The remaining integral is a representation of a modified Bessel function of order $\mu = 2\omega/\alpha^2$ and recognizing this allows us to write K as

$$K = (2/i\alpha Gsr)((1+\alpha b)(1+\alpha a))^{1/4}$$
$$\times \exp\{(2i/\alpha^2 Gsr)((1+\alpha b)G\dot{s}r - (1+\alpha a)Gs\dot{r})\}$$
$$\times I_\mu\{(-4i/\alpha^2 Gsr)((1+\alpha b)(1+\alpha a))^{1/2}\}. \quad (4.38)$$

We may rescale this back to the original position variables of interest $1+\alpha b \to \alpha^2 b^2/4$ to write finally the propagator for the inverse quadratic potential with time dependent frequency as $(\mu = \frac{1}{2}(1+8g)^{1/2})$

$$K(b,s|a,r) = (ba)^{1/2}\ (iGsr)^{-1}$$
$$\times \exp\{\frac{i}{2Gsr}(b^2 G\dot{s}r - a^2 Gs\dot{r})\}\ I_\mu\{-\frac{iba}{Gsr}\} \quad (4.39)$$

which agrees perfectly with the result given by Khandekar and Lawande (1975).

For constant $\omega = \Omega$, we may use the explicit expression for $G_{sr} = \sin(\omega T)/\omega$ to rewrite K as

$$K(b,a) = (ba)^{1/2} \; (\omega/i\sin(\omega T)) \; \exp\{i\frac{\omega}{2}\cot(\omega T)(b^2+a^2)\}$$
$$\times \; I_\mu\{-i \frac{\omega ba}{\sin(\omega T)}\}. \qquad (4.40)$$

There is an interesting limit to this propagator that we would now like to discuss. We wish to see what would happen in the limit g --> 0. Naively one might have assumed that this would reproduce the harmonic oscillator result. However we shall see that this is not the case. The limit g --> 0 is equivalent to μ --> 1/2. For $\mu = 1/2$ we may introduce a simple expression for I_μ

$$I_{1/2}(-iz) = e^{-i3\pi/4} \, (2\pi z)^{-1/2} \; (e^{iz} - e^{-iz}) \qquad (4.41)$$

and substituting this form into Eq. (4.40) gives

$$K = (\omega/2\pi i\sin(\omega T))^{1/2}\{\exp(\frac{i\omega}{2\sin(\omega T)}((b^2+a^2)\cos(\omega T)-2ba))$$
$$- \exp(\frac{i\omega}{2\sin(\omega T)}((b^2+a^2)\cos(\omega T)+2ba))\} \qquad (4.42)$$

which may be recognized as

$$K(\mu = 1/2) = K_{HO}(b|a) - K_{HO}(-b|a). \qquad (4.43)$$

K_{HO} is the propagator for the harmonic oscillator given by Eq. (4.13). From a discussion presented in Chapter 6 (see Eq. (6.20)) we note that the form given by Eq. (4.43) is that of the propagator for a harmonic oscillator with a hard reflecting wall at the origin. This might have been anticipated from the fact that for non-zero g there is always a hard core at the origin preventing the particle from travelling from the positive side of the axis to the negative side and vice-versa. Hence since for any non-zero value of g there is a hard core, we must have a hard core in the limit as g goes to zero. This fact has been thoroughly discussed by Klauder (1979) but in a path integral context has given rise to an erroneous statement by Schulman (1981, p. 345; that the limit is the HO) and a misguided paper by Langguth and Inomata (1979). These latter authors try to obtain the naive HO limit by essentially modifying the Bessel function. This is an incorrect approach. There is no need to modify the Bessel function as the naive limit gives in fact the wrong result. A hard core must persist.

TIME RESCALING

We now turn to a third transformation that may be used to evaluate path integrals. This is a rescaling of time first introduced in connection with path integrals by Duru and Kleinert (1979) in their calculation of the propagator for the Coulomb potential. Our derivation of this rescaling however, differs substantially from theirs.

Examine the path integral form for the propagator:

$$K = \int [dxdp/2\pi] \exp\{i\int p\dot{x} - H\, dt\}. \quad (4.44)$$

This propagator is a solution to ($p = -i\partial x$)

$$H_S(p,x)K = i\partial tK. \quad (4.45)$$

Let us now insert the unit constraint functional

$$F[x(t)] = 1 = \int_{-\infty}^{\infty} ds_b\, \delta(s_b - s_a - \int f(x(t))\, dt) \quad (4.46)$$

and we remark that ($S = s_b - s_a$)

$$\delta(S - \int f\, dt) = \delta(T - \int_{s_a}^{s_b} (f)^{-1}\, ds)/f(b). \quad (4.47)$$

This allows us to rewrite Eq. (4.44) as

$$K = \int ds_b\ f(b)^{-1}\ [dxdp/2\pi]\ \delta(T - \int ds/f)$$
$$\times \exp\{i\int p\dot{x} - H\ dt\}. \quad (4.48)$$

The presence of the constraint enables us to use a new time variable s with ds = f dt. We write

$$K = \int_{s_a}^{\infty} ds_b\ f(b)^{-1}\ [dxdp/2\pi]\ \delta(T - \int ds/f)$$
$$\times \exp\{i\int p\dot{x} - H/f\ ds\} \quad (4.49)$$

where we have used the $\theta(T) = \theta(S)$ to rewrite the lower limit of the ds_b integration. Now we insert the integral representation of the delta function to write

$$K = \int_0^{\infty} dS\ f(b)^{-1} \int_{-\infty}^{\infty} dE/2\pi\ e^{iET}\ [dxdp/2\pi]$$
$$\times \exp\{i\int_0^S p\dot{x} - H/f - E/f\ ds\}. \quad (4.50)$$

We have changed variables from s_b to S in the above.

The asymmetry implied by the f(b) term poses the question whether K as written still satisfies the Schroedinger equation Eq. (4.45). The answer is no. Duru and Kleinert get around this by using an ad hoc averaging procedure to eliminate the asymmetry. Pak and Sokmen (1983) also use an average to write a symmetric integral. The correct way to eliminate the asymmetry is to make sure K that continues to satisfy Eq. (4.45) which implies the addition of correction terms to the Hamiltonian just as we did had to do with the point canonical rescaling transformation. The correction term U can be found from requiring the quantum U_S to satisfy

$$U_S = H(f)^{-1} - \mathrm{Sym}(H(f)^{-1}) \quad (4.51)$$

which ensures that Eq. (4.45) is satisfied. For H(p,x) of the form

$$H = p^2/2g(x) + V(x) \qquad (4.52)$$

we find (remember U_S is a quantum operator and $' = d/dx$)

$$U_S = \frac{1}{4}(pf'(gf^2)^{-1} + (gf^2)^{-1}f'p) + \frac{1}{4}g'f'(gf)^{-2} \qquad (4.53)$$

with corresponding classical correction U to the path integral Hamiltonian

$$U = \frac{ipf'}{2gf^2} + \frac{1}{4}\,\frac{g'f'}{g^2f^2}\ . \qquad (4.54)$$

It is of interest in the case where H is of the form Eq. (4.52) (and we will restrict ourselves to this case from now on) to eliminate the cross terms arising from the correction U. This process will also result in the explicit elimination of the apparent asymmetry (which has been dealt with through the addition of U but is not manifest). To cancel the cross terms we make a linear transformation and let $p \rightarrow p - if'/2f$. Integrating out the total time derivatives gives ($C = (g(b)g(a))^{-1/4}$)

$$K = C \int dS\ (f(b)f(a))^{-1/2}\ dE/2\pi\ e^{iET}\ [dxdp/2\pi] \qquad (4.55)$$

$$\times \exp\{i\int p\dot{x} - \frac{p^2}{2gf} - \frac{f'^2}{8gf^3} - \frac{g'f'}{4g^2f^2} - \frac{V(x)}{f} - \frac{E}{f}\ ds\}.$$

Notice that the correction terms now take the form of an effective potential and that the symmetry in the endpoints b and a is now manifest.

Suppose the f was chosen to cancel the g in the p term. This would lead to a propagator K

$$K = C^{-1} \int dS\ dE/2\pi\ e^{iET}\ [dxdp/2\pi]$$

$$\times \exp\{i\int p\dot{x} - p^2/2 + g'^2/8g^2 - gV - gE\ ds\}. \qquad (4.56)$$

The effective potential is now simply $g'^2/8g^2$. Also note that the endpoint factor is now C^{-1}.

At this point it would be instructive to work an example. Let $V(x) = e/x$. The path integral we wish to evaluate is

$$K = \int [dxdp/2\pi] \exp\{i\int p\dot{x} - p^2/2 + e/x \, dt\}. \qquad (4.57)$$

Let $ds = dt/x$. The correction $U = -ip/2$. Substituting this into the above and rewriting in terms of s and S gives

$$K = \int\int dS \; b \; dE/2\pi \; e^{-iET} [dxdp/2\pi]$$
$$\times \exp\{i\int p\dot{x} - xp^2/2 + ip/2 + e + Ex \, ds\}. \qquad (4.58)$$

We turn to the evaluation of the interior propagator G

$$G = \int [dxdp/2\pi] \exp\{i\int p\dot{x} - Hx - U + xE \, ds\}. \qquad (4.59)$$

Using MDF we may rewrite G as (neglecting factor e^{ieS})

$$G = \int [dp]/2\pi \; e^{ipx|} \; \delta[\dot{p} + p^2/2 - E] \; e^{-\int p/2 \, ds}. \qquad (4.60)$$

This path integral is very similar to the one worked in the evaluation of the inverse quadratic propagator (see for example Eq. (4.27)) and we omit presenting a detailed calculation. Using the methods illustrated by the previous example we may solve the differential equation in the delta functional, find the semigroup factors, and perform the time integration in the exponent with result (including e^{ieS})

$$G = e^{ieS} \exp\{i\alpha \coth(\omega S)(b+a)\} \; (2\pi\alpha/i\sinh(\omega S)) \; (a/b)^{1/2}$$
$$\times I_1\{-\frac{2i\alpha(ba)^{1/2}}{\sinh(\omega S)}\}. \qquad (4.61)$$

We have defined $\alpha = (2E)^{1/2}$ and $\omega = (E/2)^{1/2}$.

Substituting this result into Eq. (4.58) gives the expression

$$K = (ba)^{1/2} \int_0^\infty dS \int_{-\infty}^\infty dE\ e^{-iET}\ e^{ieS}\ (i\sinh(\omega S))^{-1}$$
$$\times \exp\{i\alpha\coth(\omega S)(b+a)\}\ I_1\{-2i\alpha\frac{(ba)^{1/2}}{\sinh(\omega S)}\} \quad (4.62)$$

for the V = e/x propagator.

We point out that this time transformation enables us to write the propagator of interest as the double Fourier transform of another propagator which may be easier to evaluate. We see that we might profitably combine the point rescaling transformation with the time transformation to put the previous statement into practice. Duru (1983) has used this idea to evaluate the propagator for the Morse potential and this concept is the subject of the work of Pak and Sokmen. Examine the propagator

$$K(b|a) = \int[dxdp/2\pi]\ \exp\{i\int p\dot{x} - p^2/2 - V(x)\ dt\} \quad (4.63)$$

when rescaled by $x = f(q)$ ($g(q) = f'^2$). With $b = f(B)$ and $a = f(A)$ we may reexpress K in terms of the rescaled quantities as

$$K(B|A) = (g(B)g(A))^{-1/4} \int[dqdp/2\pi]$$
$$\times \exp\{i\int p\dot{q} - p^2/2g - V(f(q)) - V_e\ dt\} \quad (4.64)$$

where the effective potential V_e is

$$V_e = 9g'^2/32g^3 - g''/8g^2. \quad (4.65)$$

Now we perform a time rescaling to eliminate the g from the kinetic term in the Hamiltonian. Let $ds = dt/g$, then (4.66)

$$K(B|A) = (g(B)g(A))^{1/4} \iint dS\ dE/2\pi\ e^{iET}\ [dqdp/2\pi]$$
$$\times \exp\{i\int p\dot{q} - p^2/2 - V(f(q))g(q) - Eg(q) - V_e\}$$

with a new effective potential V_e given by

$$V_e = 5g'^2/32g^2 - g''/8g. \qquad (4.67)$$

This result has also been recently obtained by Pak and Sokmen through a different method. They find the correction term by examining variable transformations in the time-sliced path integral. This is a notoriously subtle calculation (see for example the three differing values of the correction term given by McLaughlin and Schulman (1971), Gervais and Jevicki (1976), and Gerry (1983) for the same time-sliced path integral) and is only applicable to Hamiltonians quadratic in momenta. Pak and Sokmen use the result of Gervais and Jevicki (which appears to be the correct one) to obtain precisely the correction given by Eq. (4.66). Our method gives an independent check of their work while providing the physical motivation for the need of such a correction term. It is also more general in that it is again not restricted to quadratic Hamiltonians. Also it is an important check of the consistency of this transformation since we use a quite different time-slicing definition of the path integral than they do. There can be no factor-ordering ambiguity in the expression for the effective potential so it is significant that both methods lead to the same result.

CHAPTER 5

RELATIVISTIC PROPAGATORS

We now extend the formalism of MDF to relativistic propagators. There are two approaches one may take. We briefly describe one technique and then turn to the second.

The first method merely uses the relativistic action in Hamiltonian form. However this is not straightforward. The relativistic action S in Lagrangian form is

$$S = -\int m(\dot{x}^2)^{1/2}\, ds. \tag{5.1}$$

The variable s is any path parameter. Sometimes the proper time is used which requires s such that $\dot{x}^2 = 1$. The canonical momentum is $p = m\dot{x}(\dot{x}^2)^{1/2}$ with consequence $p^2 = m^2$. The Hamiltonian is $p.\dot{x} - L = L - L = 0$! This occurs because the Lagrangian is invariant under arbitrary time rescalings s --> t(s) which requires an identically vanishing H. Hence the action in Hamiltonian form is naively

$$S = -\int p.\dot{x}\, ds, \tag{5.2}$$

but this cannot be right because the condition $p^2 = m^2$ is not enforced and does not arise from the equations of motion. This may be corrected through the introduction of a Lagrange multiplier α and writing S as

$$S = -\int p.\dot{x} - \alpha(p^2 - m^2)\, ds. \tag{5.3}$$

The multiplier α is to be treated as a new independent variable. The path integral corresponding to this action for a free relativistic particle is

$$K = \int [dxdp/2\pi][d\alpha] \exp\{-i\int p.\dot{x} - \alpha(p^2 - m^2) ds\}. \quad (5.4)$$

The multiplier is functionally integrated over giving rise to a delta functional $\delta[p^2 - m^2]$ ensuring that only paths that satisfy this relation contribute to the path integral.

While this method is probably the correct way to define relativistic path integrals there is another, more convenient technique that uses the fifth parameter formalism of Schwinger (1951) and Feynman (1951). That is the technique we now describe and use in our example. The demonstration of the equivalence of these two techniques using the Faddeev-Popov determinant is shown in the papers of Bardakci and Samuel (1978) and Krausz (1981).

To begin we examine the Klein-Gordon equation that a relativistic propagator must satisfy,

$$(\partial^2 + m^2)K(x|a) = \delta(x-a). \quad (5.5)$$

In operator notation with $G = \partial^2 + m^2$ this takes the form

$$GK = 1. \quad (5.6)$$

Hence

$$K = G^{-1} = i\int_0^\infty ds\ e^{-isG}\ . \quad (5.7)$$

Taking matrix elements of both sides using $\langle b|$ and $|a\rangle$ gives

$$\langle b|K|a\rangle = K(b|a) = i\int ds\ \langle b|e^{-isG}|a\rangle \quad (5.8)$$

and recognizing the term on the right-hand side as simply the Schroedinger representation of a propagator with

Hamiltonian G allows us to write K in the form (5.9)

$$K(b|a) = i\int_0^\infty ds \int [dxdp/2\pi] \exp\{-i\int_0^s p.\dot{x} - p^2 + m^2\, dt\}.$$

We may apply the method of MDF to either of these forms (Eq. (5.4) and Eq. (5.9)) for the relativistic propagator but the second equation usually turns out to be the more convenient.

As an example we work out the propagator for a particle in a constant electromagnetic (EM) field where the Maxwell field tensor F = constant. The action S is (notice that t is not time, merely a path parameter)

$$S = -\int p.\dot{x} - (p - eA)^2 + m^2\, dt. \qquad (5.10)$$

For a constant F, A = −F.x/2. To place this expression into a form in which MDF may be used we must cancel out the quadratic expression in A^2. To do this we remark upon several identities for F = constant. First we define the dual of F, $F^* = \varepsilon^{\alpha\beta\rho\sigma}F_{\rho\sigma}/2$ and the complex vector **n** in terms of the constant **E** and **B** fields, $\mathbf{n} = \mathbf{E} + i\mathbf{B} = n\hat{n}$. Then

$$F^T = -F \text{ and } F^{*T} = -F^*, \qquad (5.11)$$

$$FF^* = F^*F = \mathbf{E}.\mathbf{B}, \qquad (5.12)$$

$$F^2 - F^{*2} = \mathbf{E}^2 - \mathbf{B}^2, \qquad (5.13)$$

$$(F + iF^*)^2 = (\mathbf{E} + i\mathbf{B})^2 = \mathbf{n}^2 = n^2, \qquad (5.14)$$

$$\exp\{a(F+iF^*)\} = \cosh(an) + (F+iF^*)\sinh(an)/n. \qquad (5.15)$$

We perform a linear transformation letting

$$p \dashrightarrow p - \frac{e}{2n}F(F+iF^*).x \qquad (5.16)$$

which transforms the action into

$$S = -\int dt\ p.\dot{x} - \frac{e}{2n}\dot{x}.F(F+iF^*).x - p^2 + \frac{e^2}{4}x.F^2.x$$
$$- \frac{e^2}{4n^2}x.F(F+iF^*)^2F.x - ep.F.x + \frac{e}{n}p.F(F+iF^*).x$$
$$- \frac{e^2}{4n}x.F(F+iF^*)F.x + m^2 \qquad (5.17)$$
$$= \frac{e}{4n}x.F(F+iF^*).x| - m^2s - \int p.\dot{x} - p^2 - ep.N.x\ dt.$$

where we have defined the matrix N as

$$N = F(1 - F(F+iF^*)/n). \qquad (5.18)$$

Using MDF we may reduce the interior path integral in Eq. (5.9) to quadratures

$$\exp\{(ie/4n)(x.F(F+iF^*).x|\}\ e^{-im^2s}\ (2\pi)^{-4} \int d^4p_0\ e^{-ip.x|}$$
$$\times\ \exp\{i\int p^2\ dt\} \qquad (5.19)$$

with p(t) satisfying $\dot{p} = -\ ep.N$.

The succeeding steps are the standard ones of solving the differential equation in terms of p_0, calculating the semigroup factors (which equal 1 if the symmetric time limits s/2 and -s/2 are used for the time integral in the exponent), evaluating the time integral in the exponent (Eq. (5.15) is useful), and then performing the integral over the initial momenta using Eq. (B.5). The result for K is

$$K = i\int_0^\infty ds\ e^{-im^2s}\ (4\pi)^{-2}\ \det[ieF/\sinh(esF)]^{1/2}$$
$$\times\ \exp\{(-ie/4)(b-a).F\coth(esF).(b-a)\}. \qquad (5.20)$$

CHAPTER 6

GROUP MANIFOLDS

In the previous chapters we have discussed the method of delta functionals and canonical transformations for non-relativistic and relativistic path integrals. In all cases the underlying manifold was the real number line. In this chapter we discuss what happens when the manifold is changed to a circle or the half line. We do this to illustrate how MDF may be generalized to cases when the momentum is discrete or doesn't exist. Our examination will also provide a deeper understanding of what is involved in the construction of a path integral. We first examine the case of the circle.

The manifold of the circle is the interval $(0,2\pi)$ with the endpoints identified. This creates a compact manifold which as a consequence requires the momentum to take on discrete values. Marinov and Terentyev (1979) suggest a way to deal with this complication. Their prescription is to extend the range of position integration from $(0,2\pi)$ to $(-\infty,\infty)$, to work out the corresponding propagator, and then to take that propagator and sum over all homotopically equivalent classes. For example in the case of the circle we have

$$K = \int_0^{2\pi} [d\theta] \exp\{iS\}. \tag{6.1}$$

Extending the limits of integration creates a new propagator

$$G = \int_{-\infty}^{\infty} [d\theta] \exp\{iS\}. \tag{6.2}$$

To find K in terms of $G(\Delta\theta, T)$ where $\Delta\theta$ is in $(-\infty,\infty)$ we define a new variable $\Delta\theta = \Delta\alpha + 2\pi n$ where $\Delta\alpha$ is in $(0,2\pi)$ and $n = \pm 1, \pm 2$, etc. The homotopically equivalent classes are labelled by n (the winding number) so we have

$$K(\Delta\alpha, T) = \sum_{-\infty}^{\infty} G(\Delta\alpha + 2\pi n, T). \tag{6.3}$$

A word about 'homotopically equivalent classes'. We might as well have said 'quantum mechanically equivalent classes'. An observer is blind to any revolutions the particle might have made around the circle. One only 'sees' the initial and final endpoints. Hence a trip from α to μ without a revolution and one from α to μ with n revolutions are quantum mechanically equivalent. The dictum of quantum mechanics is to add up the amplitudes for all possible ways an event may occur, in this case to sum over all revolutions or 'homotopically equivalent classes'.

We may formalize this prescription. Let H be a Hamiltonian which takes its values on the manifold of some group R (usually the real number line). Suppose H is invariant under the action of a group G with elements g and let M be the manifold (coset space) of the factor group R/G. Then we may write the propagator of the manifold M in terms of the propagator of the manifold R by

$$K_M(b|a) = \sum_g K_R(gb|a). \qquad (6.4)$$

In the case of the circle, R is the real number line which is the group manifold for the non-compact translation group T with elements e^a. Translations by $2\pi n$ form a subgroup Z and the factor group T/Z is the group U(1) with elements $e^{i\theta}$. The group manifold for U(1) is just the circle $(0,2\pi)$. The action of a group element g of Z upon b is just $gb = b + 2\pi n$ and so

$$K_M(b|a) = \sum_g K_R(gb|a) = \sum_{-\infty}^{\infty} K_R(b + 2\pi n|a). \qquad (6.5)$$

We may generalize the construction of Marinov by adding in a phase factor c so that Eq. (6.4) becomes

$$K_M(b|a) = \sum_g c(g)K_R(gb|a). \qquad (6.6)$$

The c(g) are essentially determined by the boundary conditions that K_M must satisfy.

We now wish to construct the path integral on the circle and see that it does reduce to Eq. (6.5). In doing so we will generalize MDF. To begin we must redo the derivation of the path integral illustrated by Eq. (1.13). The propagator may be written as

$$K(\mu|\alpha) = (\Pi \int_0^{2\pi} d\theta_n)\langle\mu|\theta_{N-1}\rangle \ldots \langle\theta_1|\alpha\rangle. \qquad (6.7)$$

We need to evaluate the infinitesmal propagator $\langle\theta_{n+1}|\theta_n\rangle$ which in the Schroedinger representation is

$$\langle\theta_{n+1}|e^{-iH\Delta t}|\theta_n\rangle \qquad (6.8)$$

We insert a complete set of momentum eigenstates,

$$\langle\theta|p\rangle = (2\pi)^{-1/2}\, e^{in\theta} \qquad (6.9)$$

where we have defined n = p which is some integer, into

Eq. (6.8) to give

$$\langle\theta_{n+1}|\theta_n\rangle = \Sigma \exp\{ip\Delta\theta - iH(p,\theta)\Delta t\}. \quad (6.10)$$

This allows us to write K symbolically as (6.11)

$$K(\mu|\alpha) = \int_0^{2\pi} \Sigma [d\theta][p/2\pi] \exp\{i\int p\dot{\theta} - H(p,\theta)\, dt\}.$$

We illustrate the use of this symbolism by working out the propagator for the free particle confined to a circle. For a free particle we have

$$K = \Sigma [p/2\pi] \int [d\theta] \exp\{i\int p\dot{\theta} - p^2/2I\, dt\} \quad (6.12)$$

where I is the moment of inertia. Using the idea of MDF we integrate by parts and recognize the position integral as a delta functional to write K as

$$K = \Sigma [p]/2\pi\, e^{ip\theta|}\, \delta[\Delta p] \exp\{-i\int p^2/2I\, dt\}. \quad (6.13)$$

The solution to the discrete equation $\Delta p = 0$ is $p(t) = p$ where p is a constant (remember that p at any time is an integer). The semigroup factors are unity. Hence the propagator is (writing p as m to emphasize the discrete nature of p)

$$K = \Sigma (2\pi)^{-1} \exp\{im(\mu-\alpha) - im^2T/2I\}. \quad (6.14)$$

By using the theta function (not the step function) defined by

$$\theta(\alpha,T) = \Sigma_{-\infty}^{\infty} \exp\{im\alpha - m^2T/2\} \quad (6.15)$$

we may write K as

$$K(\mu|\alpha) = (2\pi)^{-1}\, \theta(\mu-\alpha, T/I). \quad (6.16)$$

The Marinov prescription would have us evaluate the infinite limit propagator G which is

$$G = (I/2\pi iT)^{1/2} \exp\{iI(\mu-\alpha)^2/2T\} \quad (6.17)$$

and write K as

$$K = (I/2\pi iT)^{1/2} \quad \Sigma \exp\{iI(\mu+2\pi m-\alpha)^2/2T\}. \tag{6.18}$$

The result is actually equal to the expression given by Eq. (6.14) since the theta function satisfies the identity

$$\Theta(\alpha,T) = (2\pi/iT)^{1/2} \quad \Sigma \exp\{i(\alpha+2\pi n)^2/2T\}. \tag{6.19}$$

We now turn to a more involved case, that of a particle confined to the positive real axis. The real number line R (excluding zero) is the group manifold for multiplication of the reals. A subgroup is Z_2 the group consisting of the two elements 1 and −1. Z_2 is essentially the parity group. The factor group $M = R/Z_2$ is the group of multiplication for the positive reals and the manifold is just the positive real axis. The generalized Marinov construction gives for the propagator

$$K_M(b|a) = K_R(b|a) - K_R(-b|a) \tag{6.20}$$

where we have chosen the phase factors c(g) to equal g. This was decided by the requirement that K vanish at the wall at x = 0.

We now wish to derive this result from a construction of the propagator. As before, the construction hinges upon the evaluation of the infinitesmal propagator

$$\langle x_+|e^{-iH\Delta t}|x\rangle. \tag{6.21}$$

However in this case we cannot insert momentum eigenstates since they do not exist. This is related to the fact that there must always be a wave reflected from the wall which prevents the eigenfunctions from satisfying the boundary condition at the wall where they must vanish. We can, however construct eigenfunctions $|q\rangle$ of the kinetic energy. These are

$$\langle x|q\rangle = (2/\pi)^{1/2} \sin(qx). \tag{6.22}$$

They are orthonormal and complete on the positive real axis. We may insert them into Eq. (6.21) to write the infinitesmal propagator as

$$\int_0^\infty dq\ (2/\pi)\ \sin(qb)\sin(qa)\ e^{-iH\Delta t}. \tag{6.23}$$

Unfortunately, this expression may not be concatenated. To avoid this we create a special notation and define

$$\mathrm{Sin}[p\dot{x}] = \Pi\ \sin(px_+)\sin(px). \tag{6.24}$$

This functional admits an integration by parts identity

$$\mathrm{Sin}[p\dot{x}] = \sin(p(s)b)\ \mathrm{Sin}[\dot{p}x]\ \sin(p(r)a) \tag{6.25}$$

and provides a representation of a delta functional

$$\int_0^\infty [2dp/\pi]\ \mathrm{Sin}[p\dot{x}] = \delta[\dot{x}] \quad (\text{for } x > 0). \tag{6.26}$$

Using this notation we may write the propagator on the half line as

$$K(b|a) = \int_0^\infty [2dpdx/\pi]\mathrm{Sin}[p\dot{x}]\exp\{-i\int H(p,x)\ dt\}. \tag{6.27}$$

We now derive the Marinov expression. Examine the double integral

$$\int_0^\infty dx\ 2dq/\pi\ \sin(px)\sin(qx)\sin(qa)\ e^{-iH\Delta t} \tag{6.28}$$

which occurs in the concatenation of the infinitesmal propagator. We may extend the integration limits of x by defining $H(x)$ on the extended interval as $H(|x|)$. This allows us to write Eq. (6.28) as

$$\int_0^\infty 2dq/\pi \int_{-\infty}^\infty dx/2 \; \sin(px)\cos(q(x-a)) \; e^{-iH\Delta t} \qquad (6.29)$$

since $\sin(qx)\sin(qa) = \cos(q(x-a)) - \cos(qx)\cos(qa)$ and the $\cos^2$ term must vanish as it gives an odd integrand. We may further extend the limits of the 'momentum' integration to $(-\infty,\infty)$ by extending $H(q,|x|)$ to $H(|q|,|x|)$ and writing

$$\int dxdq/2\pi \; \sin(px) \; \exp\{iq(x-a) - iH\Delta t\} \qquad (6.30)$$

for Eq. (6.29). We can do this for every pair of momentum and position integrations except the last two in the concatenation. Therefore we may write the propagator as

$$K(b|a) = \int_{-\infty}^\infty dc \int_0^\infty 2dp/\pi \; e^{-iH(p,c)\Delta t} \; \sin(pb)\sin(pc)$$
$$\times \int [dxdp/2\pi] \; \exp\{i\int p\dot{x} - H \, dt\}. \qquad (6.31)$$

The interior path integral is over all paths from a to c. In the limit as $\Delta t \to 0$ we may perform the p integration to give a pair of delta functions

$$K = \int dc \; (\delta(b-c) - \delta(b+c)) \; K_R(c|a) \qquad (6.32)$$

which gives back Eq. (6.20).

The importance of this derivation is not only to show how the method of delta functionals may be generalized but also to show that one must always construct the propagator from scratch on a new manifold. This fact has not always been appreciated. For example Shulman (1981, p. 40) states that the propagator on the half line would obtained by simply restricting the position functional integrations to the positive axis. As we have seen, this is incorrect.

CHAPTER 7

TIME-DEPENDENT COHERENT STATE PATH INTEGRALS

Recently, coherent states (eigenstates of the annihilation operator) for a time-dependent harmonic oscillator have been discussed with the hope that these generalized coherent states might be used to describe interactions between molecules in lasers, cf. Lewis and Riesenfeld (1968). In quantum field theory, coherent states are used to construct path integrals which figure prominently in a functional formalism, cf. Itzykson and Zuber (1980). In particular, a path integral built from time-dependent coherent states might be of value in discussing quantum field theories in curved spacetime, cf. Berger (1982) and Ichinose (1982). Our interest in time-dependent coherent states is the fact that they allow an easy evaluation of the generating functional. We calculate the generating functional for the time -dependent harmonic oscillator at the end of the chapter.

We present the derivation of a time-dependent coherent state path integral in this chapter. We further show that these states are eigenstates of an annihilation operator that diagonalizes a new Hamiltonian obtained through a time-dependent canonical transformation. Hence this chapter is an extension of the philosophy of Chapter 4 where we solved path integrals through canonical transformations.

TIME-DEPENDENT COHERENT STATES

We first briefly discuss the properties of time-dependent coherent states (TDCS) needed for the construction of the path integral. The derivation of the path integral is presented in the latter part of the chapter.

To begin, examine the Lagrangian

$$L = \frac{\dot{q}^2}{2} - \frac{\omega^2(t)q^2}{2} . \tag{7.1}$$

and the associated Hamiltonian

$$H = \frac{p^2}{2} + \frac{\omega^2(t)q^2}{2} . \tag{7.2}$$

One would normally diagonalize H by introducing operators a, a^+ and write q(t) and p(t) as

$$q(t) = (a(t)+a^+(t))/(2\omega(t))^{1/2} \tag{7.3}$$

$$p(t) = i(a^+(t)-a(t))(\omega(t)/2)^{1/2}, \tag{7.4}$$

but the Heisenberg equations of motion

$$H = \omega(t)(a^+a+1/2) \tag{7.5}$$

$$\frac{da^+}{dt} = i[H,a^+] + \frac{\partial a^+}{\partial t} = i\omega(t)a^+ + \frac{\dot{\omega}(t)}{2\omega(t)}a \tag{7.6}$$

$$\frac{da}{dt} = i[H,a] + \frac{\partial a^+}{\partial t} = -i\omega a + \frac{\dot{\omega}}{2\omega}a^+ \tag{7.7}$$

lead to coupled equations of motion for a and a^+. This indicates a mixing of the positive and negative frequencies or modes, spoiling the interpretation of a and a^+ as creation and annihilation operators.

More useful would be a description of q and p in terms of operators that satisfy uncoupled equations of motion. To this end, one can write q and p in terms of s and γ, functions to be determined. Let

$$q = (a+a^+)s(t) \quad , \quad p = i\dot{\gamma}(a^+-a)s + (a+a^+)\dot{s} \qquad (7.8)$$

where $\dot{\gamma} = 1/2s^2$.

One finds that the equations of motion for a, a^+ can be uncoupled if s(t) satisfies the differential equation

$$\frac{d^2s}{dt} + \omega^2 s - \frac{1}{4s^3} = 0 \ . \qquad (7.9)$$

As pointed out by Lewis and Riesenfeld (1968), s(t) may be any particular solution of this differential equation. The only condition we will make is that for ω = constant, s(t) --> $(2\omega)^{-1/2}$ and $\dot{\gamma}$ --> ω. This enables us to compare our results in the time-independent limit. Notice that γ as defined here is the negative of that γ discussed in Appendix D. Now a and a^+ satisfy

$$\dot{a}^+ = i\dot{\gamma}(t)a^+, \ \dot{a} = -i\dot{\gamma}(t)a \ . \qquad (7.10)$$

The equation for s(t) can be cast in a more suggestive form by writing $f(t) = s(t)e^{i\gamma(t)}$. Then f(t) satisfies

$$d^2f(t)/dt + \omega^2(t)f(t) = 0, \qquad (7.11)$$

the differential equation for a harmonic oscillator with time-dependent frequency (TDHO).

The object of interest as regards path integrals is the classical phase space action S, where

$$S = \int (p\dot{q}-\dot{p}q)/2 - H \ dt. \qquad (7.12)$$

Expressing H in terms of a, a^+ (7.13)

$$H = \frac{(a+a^+)^2}{2}(\dot{s}^2-s\ddot{s}) + \frac{i\dot{s}}{s}(a^{+2}-a^2) + \frac{\dot{\gamma}}{2}(a^+a+aa^+)$$

and noting (7.14)

$$\frac{\dot{p}q-p\dot{q}}{2} = \frac{i}{2}(\dot{a}a^{+}-a\dot{a}^{+}) + \frac{i\dot{s}}{s}(a^{+2}-a^{2}) + \frac{1}{2}(a+a^{+})^{2}(\dot{s}^{2}-s\ddot{s})$$

one finds S to be

$$S = \int (\dot{a}^{+}a-\dot{a}a^{+})/2i - (\dot{\gamma}/2)(a^{+}a+aa^{+})\ dt. \tag{7.15}$$

We define $K = (\dot{\gamma}/2)(a^{+}a+aa^{+})$, $I = K/\dot{\gamma}$, and $N = a^{+}a$. Classically and quantum mechanically, K is seen to be a new Hamiltonian, arrived at through a linear time-dependent canonical transformation. I is the much-discussed Lewis-Riesenfeld invariant and N is a quantum mechanical number operator.

One expects the transition amplitude from an initial coherent eigenstate $|z_i\rangle$ to a final one $\langle z_f^*|$ to have a path integral representation

$$\langle z_f^*|z_i\rangle = \int [dz^*dz/2\pi i]\ \exp\{iS[z^*,z]\} \tag{7.16}$$

where the precise meaning of this symbolic notation will be elaborated in the next section. It will be shown that this is indeed the case with one important caveat, to be discussed.

TDCS can be constructed from the eigenstates of K. Let these eigenstates be given by $|n\rangle$, n = 0, 1, 2, 3,... such that

$$K|n\rangle = \dot{\gamma}(n+1/2)|n\rangle. \tag{7.17}$$

The eigenvalue n is time-independent. These eigenstates do not evolve in time with the original Hamiltonian as the generator of time displacements. As shown by Lewis and Riesenfeld, a state vector that does evolve via H can be constructed by multiplying the eigenstates by a time-dependent phase factor. Specifically they find

$$|n\rangle_s = e^{i\alpha_n(t)}|n\rangle \tag{7.18}$$

where α_n satisfies the differential equation

$$\frac{d\alpha_n(t)}{dt} = \langle n|i\frac{\partial}{\partial t} - H(t)|n\rangle \ . \tag{7.19}$$

In the case of the TDHO we have introduced operators a, a^+ which have the properties (7.20)

$$K = \dot{\gamma}(a^+a+1/2), \quad a^+|n\rangle = (n+1)^{1/2}\ |n+1\rangle,$$

$$a|n\rangle = n^{1/2}\ |n-1\rangle.$$

If one fixes the phase of $|0\rangle$ by requiring

$$\langle 0|\frac{\partial}{\partial t}|0\rangle = \frac{i}{2}(s\ddot{s}-\dot{s}^2) \tag{7.21}$$

so as to agree in the limit $s \dashrightarrow (2\omega)^{-1/2}$ one finds

$$\dot{\alpha}_n(t) = -(n+1/2)\dot{\gamma}. \tag{7.22}$$

Hence the Schroedinger state vector may be written as

$$|n(t)\rangle_s = e^{-i\gamma(n+1/2)}|n,t\rangle = e^{-i\int_0^t dt\ K}\ |n,t\rangle \tag{7.23}$$

where the time-dependence of the K eigenstates has been made explicit. We may now regard the $|n,t\rangle$ as 'rotating' interaction state vectors $|n\rangle$ which evolve in time via the new Hamiltonian K as observed from the 'rotating' frame of reference. Therefore we once again suppress the time-dependence and write the state vectors as $|n\rangle$. Thus we have implicitly defined a rotating reference frame in which K generates time translations such that Eq. (7.23) transforms back to the standard Schroedinger frame in which H generates time translations. As these vectors form a complete orthonormal set, any vector built from them will also evolve according to K, in particular the unnormalized eigenstates of a, $|z)$ where $a|z) = z|z)$. These are our coherent states.

The states $|z)$ can be formed by $|z) = e^{za}\ |0\rangle$, the eigenvalues z are time-independent and the states evolve (as seen from the rotating frame) according to

$$|z(t)) = e^{-i\int_0^t dt\ K}\ |z)\ . \tag{7.24}$$

It is these TDCS $|z)$ that will be used to construct the path integral which is the subject of the next section. Notice that since the eigenvalues are time-independent one has the important resolution of the identity

$$1 = \int dz^* dz/2\pi i\ |z)(z^*|\ e^{-z^* z} \tag{7.25}$$

where $dz^* dz/2\pi i = d(\mathrm{Re}z)d(\mathrm{Im}z)/\pi$.

CONSTRUCTION OF THE PATH INTEGRAL

Coherent states were first used to construct a path integral by Klauder (1960) and have now become part of the standard texts. We now follow the treatment of Faddev and Slavnov (1980) to present a brief derivation of the TDCS path integral.

Any operator A has a kernel associated with it by mapping it from Hilbert space into the complex numbers. We may write

$$\begin{aligned}(z^*|A|z) &= \Sigma\ (z^*|n\rangle\langle n|A|m\rangle\langle m|z) \\ &= \Sigma\ \frac{(z^*)^n}{n!}\ A_{nm}\ \frac{z^m}{m!} = A(z^*, z),\end{aligned} \tag{7.26}$$

and any two kernels A, B are convolved by

$$(z^*|BA|z) = \int \frac{dw^* dw}{2\pi i}\ e^{-w^* w}\ B(z^*, w)A(w^*, z)\ . \tag{7.27}$$

A normal symbol for A is a function $A^N(z^*,z)$ found by writing A as a function of a, a^+ in normal order and then letting the arguments a, a^+ become complex c-numbers z and z^* respectively.

The relationship between kernel A and A^N can be shown to be

$$A(z^*,z) = e^{z^*z} A^N(z^*,z) \quad . \tag{7.28}$$

Let the new Hamiltonian K be given in normal order and examine the kernel of the evolution operator U for short times Δt.

$$U^N(t+\Delta t,t) = e^{-iK(z^*,z,t)\Delta t} \tag{7.29}$$

and the kernel is

$$U(t+\Delta t,t) = e^{z^*z-iK(z^*,z,t)\Delta t} . \tag{7.30}$$

Convolving N such kernels together such that $N\Delta t = t-r$ gives

$$U(z_f^*,t|z_i,r) = \int \Pi \frac{dz_n^* dz_n}{2\pi i} (e^{z_N^* z_{N-1}} x\ldots xe^{z_1^* z_0})$$

$$x(e^{-z_{N-1}^* z_{N-1}} x\ldots xe^{-z_1^* z_1})(e^{-iK(z_N^*,z_{N-1})\Delta t}$$

$$xe^{-iK(z_{N-1}^*,z_{N-2})\Delta t} x\ldots xe^{-iK(z_1^*,z_0)\Delta t}) . \tag{7.31}$$

Passing to the limit $\Delta t \longrightarrow 0$, $N \longrightarrow \infty$, and symmetrizing in z_f^* and z_i one may write symbolically

$$(z_f^*,t|z_i,r) = (z_f^*|e^{-i\int dt\ K}|z_i) = U(t,r) \tag{7.32}$$

$$= \int [\frac{dz^*dz}{2\pi i}] \exp\{i\int dt\ (\frac{\dot{z}^*z-z^*\dot{z}}{2i})-K(z^*,z)\} \exp \frac{1}{2}(z_f^*z_f+z_i^*z_i)$$

which was the conjectured form. Once again, it is emphasized that this gives the propagator for the 'rotating' states. This is the form for the propagator as seen from the rotating frame of reference implicitly defined by Eq. (7.23).

As a check, one can use this expression to find the propagator for the position $\langle b,t|a,r\rangle$. Inserting complete sets of TDCS one may write

$$\langle b,t|a,r\rangle = \int \frac{dz^*_f dz_f}{2\pi i} \frac{dz^*_i dz_i}{2\pi i} e^{-z^*_f z_f} e^{-z^*_i z_i}$$

$$\times\langle b|z_f(t))(z^*_f|e^{-i\int dt\, K}|z_i)(z^*_i(r)|a\rangle \qquad (7.33)$$

where $\langle b|$ and $|a\rangle$ are position eigenstates.

Since $K(z^*,z) = \dot{\gamma}(t)(z^*z+1/2)$ one has to evaluate a Gaussian path integral which as we have seen simply has the exponent evaluated along the classical path. This result follows from the functional analogue of the method of steepest descent or by concatenating the infinitesmal propagators. The classical equations of motion for z^* and z are

$$\dot{z}^*(t) = i\dot{\gamma}(t)z^*(t) \ , \qquad \dot{z}(t) = -i\dot{\gamma}(t)z(t) \qquad (7.34)$$

with solutions (7.35)

$$z^*(t) = z^*_f e^{i\int_s^t dt\, \dot{\gamma}(t)} \ , \quad z(t) = z_i e^{-i\int_r^t dt\, \dot{\gamma}(t)} \ .$$

The exponent becomes

$$z^*_f z_i e^{-i\sigma(t,r)} - i\sigma(t,r)/2 \qquad (7.36)$$

where $\sigma(t,r) = \gamma(t) - \gamma(r)$ hence the TDCS propagator is given by

$$(z^*_f,t|z_i,r) = \exp\{z^*_f z_i e^{-i\sigma} - i\sigma/2\} \ . \qquad (7.37)$$

The wavefunctions $\langle x|z)$ can be found by writing a in terms of operator expressions for x and p in the position representation and solving the resulting eigenvalue differential equation. Specifically (7.38)

$$\langle b|z_f(t)) = \left(\frac{\pi}{\dot{\gamma}(t)}\right)^{-1/4} \exp\left\{-\frac{z_f^2}{2} + \frac{z_f b}{s(t)} - \left(\dot{\gamma}(t) - i\frac{\dot{s}(t)}{s(t)}\right)\frac{b^2}{2}\right\}$$

$$(z_i^*(r)|a\rangle = \left(\frac{\pi}{\dot{\gamma}(r)}\right)^{-1/4} \exp\left\{-\frac{z_i^{*2}}{2} + \frac{z_i^* a}{s(r)} - \left(\dot{\gamma}(r) + i\frac{\dot{s}(r)}{s(r)}\right)\frac{b^2}{2}\right\}$$

so the propagator should be given by

$$K(b|a) = (\pi)^{-1/2}(\dot{\gamma}(t)\dot{\gamma}(r))^{1/4} e^{-i\sigma(t,r)/2} \tag{7.39}$$

$$\times \exp\left\{-\left(\dot{\gamma}(t) - i\frac{\dot{s}(t)}{s(t)}\right)\frac{b^2}{2} - \left(\dot{\gamma}(r) + i\frac{\dot{s}(r)}{s(r)}\right)\frac{a^2}{2}\right\} \times \int \frac{dw^* dw}{2\pi i}\frac{dz^* dz}{2\pi i}$$

$$\times \exp\left\{-\frac{w^2}{2} + \frac{wb}{s(t)} - w^*w + w^* z e^{-i\sigma} - \frac{z^{*2}}{2} + \frac{z^* a}{s(r)} - z^* z\right\}.$$

Using the formula Eq. (B.7) one finds (7.40)

$$K(b|a) = \left[\frac{(\dot{\gamma}(t)\dot{\gamma}(r))^{1/2}}{2\pi i \sin\sigma(t,r)}\right]^{1/2} \exp\left\{\frac{i}{2}\left(\frac{\dot{s}(t)}{s(t)}b^2 - \frac{\dot{s}(r)}{s(r)}a^2\right)\right\}$$

$$\times \exp\left\{\frac{i}{2\sin\sigma(t,r)}\left((b^2\dot{\gamma}(t) + a^2\dot{\gamma}(r)) \times \cos\sigma(t,r) - 2(\dot{\gamma}(t)\dot{\gamma}(r))^{1/2} ab\right)\right\}$$

which is precisely the result given by Khandekar and Lawande (1975).

The generating functional Z is found from the coherent state path integral by letting $z_f^* = z_i = 0$ and is

$$Z = \exp\{-i\sigma/2\}. \tag{7.41}$$

In terms of an ancillary function Ω which is a particular solution to

$$i\dot{\Omega} + \Omega^2 = \omega^2 \tag{7.42}$$

with limit $\Omega = \omega$ when ω is a constant, Z may be written as

$$Z = \exp\left\{-\frac{i}{2}\int \Omega(s)\, ds\right\}. \tag{7.43}$$

CHAPTER 8

NEW SOLVABLE QUANTUM SYSTEM

In Chapter 4 we used the path integral solution to the Hamiltonian

$$H = \frac{p^2}{2}(1+\mu x) + \frac{\omega^2 x^2}{2(1+\mu x)} - \frac{\mu^2}{32(1+\mu x)} \qquad (8.1)$$

to find the propagator for the inverse quadratic potential. In this chapter we wish to present the complete solution to the quantum mechanical problem posed by this Hamiltonian.

This solution is of interest because it is a non-trivial generalization of the standard harmonic oscillator which provides a new testing ground for the development of functional methods which might be applied to more sophisticated problems.

Also of interest is that the problem is the 1-dimensional theory of non-linear scalar gravity and hence its solution may provide some insight into questions concerning the more complex 4-dimensional theory.

As we have seen this Hamiltonian is related to the inverse quadratic potential, however no complete solution to the inverse quadratic potential can be given as a generating functional cannot be constructed thus preventing the development of a quantum field-theoretical perturbation theory.

The solution to the classical problem is illuminating because there are several analogous features in the quantum solution. Varying the action leads to the Euler-Lagrange equations of motion which are

$$(D^2 + \omega^2)x = \mu(\dot{x}^2 + \omega^2 x^2)/2(1+\mu x). \quad (8.2)$$

Since the Lagrangian is invariant under time translations the energy is conserved and takes the form

$$E = (\dot{x}^2 + \omega^2 x^2)/2(1+\mu x). \quad (8.3)$$

Therefore the equations of motion may be written as

$$(D^2 + \omega^2)x = \mu E$$

with immediate solution

$$x(t) = ae^{-i\omega t} + a^+ e^{i\omega t} + \mu E\omega^{-2}. \quad (8.4)$$

The particle oscillates about an equilibrium position $x_0 = \mu E\omega^{-2}$ with amplitude $A = (\mu^2 E^2 \omega^{-4} + 2E\omega^{-2})^{1/2}$ and frequency ω.

The quantum mechanical Hamiltonian associated with this problem (with the correct ordering of operators) is

$$H = (1+\mu x)^{1/4} p (1+\mu x)^{1/2} p (1+\mu x)^{1/4}/2 \quad (8.5)$$
$$+ \omega^2 x^2/2(1+\mu x) - \mu^2/32(1+\mu x).$$

This may be put into the more convenient form

$$H = p(1+\mu x)p/2 + \omega^2 x^2/2(1+\mu x). \quad (8.6)$$

The quantum-mechanical Heisenberg equations of motion are found to be

$$\dot{p} = -\mu p^2/2 - (2+\mu x)\omega^2 x(1+\mu x)^{-2}/2 \quad (8.7)$$

and

$$\dot{x} = ((1+\mu x)p + p(1+\mu x))/2 \quad (8.8)$$

which may be combined with result

$$(D^2 + \omega^2)x = \mu H, \qquad (8.9)$$

the exact analogue of the classical equations. We may take the vacuum expectation value (VEV) of this equation to find

$$(D^2 + \omega^2)\langle x\rangle = \omega^2\langle x\rangle = \mu\langle H\rangle = \mu E_o. \qquad (8.10)$$

Therefore the VEV of the 'field' x(t) is

$$\langle x\rangle = \mu E_o\omega^{-2}. \qquad (8.11)$$

This result will be used to check our construction of the generating functional in the next chapter.

The solution to Eq. (8.9) is

$$x(t) = Be^{-i\omega t}(2\omega)^{-1/2} + B^+e^{i\omega t}(2\omega)^{-1/2} + \mu H\omega^{-2}. \qquad (8.12)$$

The operators B and B^+ are non-canonical creation and annihilation operators and with the operator $C = [B,B^+] = 1 + \mu^2 H\omega^{-2}$ are easily found to form the Lie algebra of SO(2,1).

At this point it is also convenient to rescale the operators. Let

$$P \rightarrow (2\omega)^{1/2}p,\ x \rightarrow x/(2\omega)^{1/2},\ \mu \rightarrow (2\omega)^{1/2}\mu. \qquad (8.13)$$

This canonical transformation allows us to write the rescaled Hamiltonian as

$$H = \omega(p(1+\mu x)p + x^2/4(1+\mu x)). \qquad (8.14)$$

This Hamiltonian may be diagonalized through the introduction of canonical creation and annihilation operators. These are defined via three successive sets of transformations.

$$q = x/2(1+\mu x) + ip \quad (8.15)$$
$$q^+ = x/2(1+\mu x) - ip$$
$$x = (q+q^+)/(1-\mu(q+q^+))$$
$$p = i(q^+-q)/2$$

$$a = q/(1-\mu q) \quad (8.16)$$
$$a^+ = q^+/(1-\mu q^+)$$
$$q = a/(1+\mu a)$$
$$q^+ = a^+/(1+\mu a^+)$$

$$A = a(1-\mu^2 a^+ a)^{-1/2} \quad (8.17)$$
$$A^+ = a^+(1-\mu^2 a a^+)^{-1/2}$$
$$a = A(1+\mu^2 A^+ A)^{-1/2}$$
$$a^+ = A^+(1+\mu^2 AA^+)^{-1/2}$$

The a's are non-canonical creation and annihilation operators with $[a,a^+] \neq 1$ and $[H,a] = -\omega a$, $[H,a^+] = \omega a^+$. The canonical operators satisfy the standard commutation relations $[A,A^+] = 1$ and $[H,A] = -\omega A$, $[H,A^+] = \omega A$. The operators B and B^+ in terms of A and A^+ are

$$B = A(1+A^+A/\gamma)^{1/2} \quad (8.19)$$
$$B^+ = A^+(1+AA^+/\gamma)^{1/2} \quad (8.20)$$

where $\gamma = 2\omega\mu^{-2}$. Using this result we find the commutator

$$[x,\mathring{x}] = i(1+\mu x) \quad (8.21)$$

and the time-dependent commutator (8.22)

$$i[x(s),x(r)] = \sin(\omega T)(1+\mu^2 H\omega^{-2})/\omega - \mu(\mathring{x}(s)-\mathring{x}(r))\omega^{-2}.$$

The Hamiltonian may be written in terms of the non-canonical operators as

$$H = \frac{\omega}{2}\left(\frac{aa^+}{1-\mu^2 aa^+} + \frac{a^+a}{1-\mu^2 a^+a}\right) \tag{8.23}$$

and in terms of the canonical operators as

$$H = \omega(AA^+ + A^+A)/2 = \omega(A^+A + 1/2). \tag{8.24}$$

The spectrum may be determined with creation and annihilation operators in the same manner as for the harmonic oscillator and is found to be the same, namely $E_n = \omega(n+1/2)$.

We may construct eigenstates $|n\rangle$ of a number operator $N = A^+A$ with the properties

$$N|n\rangle = n|n\rangle \tag{8.25}$$

$$H|n\rangle = \omega(n+1/2)|n\rangle \tag{8.26}$$

$$A|n\rangle = (n)^{1/2}|n-1\rangle \tag{8.27}$$

$$A^+|n\rangle = (n+1)^{1/2}|n+1\rangle \tag{8.28}$$

Coherent states (eigenstates of A) may be constructed out of the ground state as follows

Normalized: $|z\rangle = \exp(zA^+ - z^*A)|0\rangle$, $A|z\rangle = z|z\rangle$ (8.29)

Unnormalized: $|z) = \exp(zA^+)|0\rangle$, $A|z) = z|z)$ (8.30)

with $|z\rangle = \exp(-z^*z/2)|z)$ and

$$\int d^2z/\pi\ |z\rangle\langle z| = \int d^2z/\pi\ e^{-z^*z}\ |z)(z| = 1 \tag{8.31}$$

where $d^2z/\pi = dz^*dz/2\pi i = d\mathrm{Re}z\,d\mathrm{Im}z/\pi$. The energy eigenfunctions $\langle n|z) = z^n/(n!)^{1/2}$.

The spatial energy eigenfunctions $\langle x|n\rangle$ are most easily found by examining the propagator from Chapter 4 which is (in terms of the unrescaled variables)

$$K = ((1+\mu b)(1+\mu a))^{1/4}\ e^{i\omega^2 T/\mu^2}\ (\omega/i\mu\sin(\omega T/2)) \tag{8.32}$$

$$\times\ \exp\{(i\omega\cot(\omega T/2)/\mu^2)((1+\mu b) + (1+\mu a))\}$$

$$\times\ I\gamma\{-\frac{2i\omega((1+\mu b)(1+\mu a))^{1/2}}{\mu^2\sin(\omega T/2)}\}.$$

Using the Hille-Hardy formula Eq. (B.13) we may rewrite the propagator as

$$K = \sum e^{-i\omega T(n+1/2)}\ (2\omega/\mu)\ ((1+\mu b)(1+\mu a))^{1/4}$$

$$\times\ (n!/\Gamma(n+\gamma+1))\ (e^{-u/2}\ u^{\gamma/2}\ L_n^{\gamma}(u))$$

$$\times\ (e^{-v/2}\ v^{\gamma/2}\ L_n^{\gamma}(v)) \tag{8.33}$$

where L_n^{γ} is the Laguerre polynomial and Γ the Gamma function. A glance at Eq. (1.6) allows us to pick out the spectrum as $\omega(n+1/2)$ and the eigenfunctions as (8.34)

$$\langle u|n\rangle = (n!/\Gamma(n+\gamma+1))^{1/2}\ (2\omega u)^{1/4}\ e^{-u/2}\ u^{\gamma/2}\ L_n^{\gamma}(u)$$

and in terms of the rescaled variables $u = (1+\mu x)/\mu^2$ and $\gamma = 1/\mu^2$.

We are now finished with the presentation of the exact solution to the problem. We have found the spectrum of the Hamiltonian, diagonalized it, and constructed the various wavefunctions and propagator.

The problem of calculating VEVs hinges upon the construction of a generating functional $Z[F] = \langle 0|T\exp(i\int F(t)x(t))|0\rangle$ where T is the time-ordering symbol placing all operators in chronological order. For example the VEV $\langle x(t)x(s)\rangle$ (time-ordering will be implicitly understood from now on) may be found by taking two functional derivatives and two ordinary derivatives of Z[F]. For example

$$\langle x(t)x(s)\rangle = \frac{\delta}{i\delta F(t)} \frac{\delta}{i\delta F(s)} Z[F]|_{F=0} \quad (8.35)$$

Now Z[F] in the presence of a potential or interaction is hard to find but fortunately it may be written in terms of Zo[F] the generating functional for a free field in the presence of a source F(t). The expression is

$$Z[F] = \exp(-i \int V(\delta/i\delta F(t))\, dt) Zo[F]. \quad (8.36)$$

Physically Z[F] represents the amplitude for the vacuum to remain undisturbed in the presence of the force or source F(t). In quantum field theory that amplitude is taken to be unity in the absence of sources requiring the normalization Z[0] = 1. In Feynman diagram language this amounts to neglecting all disconnected vacuum graphs or bubbles.

In this case we are actually able to solve for the functional Z[F]. Using its definition as $\langle 0|0\rangle$ in the presence of sources we may write

$$Z[F] = \int\int db\, da\, \langle 0|b\rangle\langle b,s|a,r\rangle\langle a|0\rangle \quad (8.37)$$

in the limit as s and r go to plus and minus infinity. Now $\langle b,s|a,r\rangle$ is the propagator for the system driven by some arbitrary time-dependent force F(t). This was calculated in Chapter 4 and was found to be (unrescaled)

$$K = (2/i\mu D(s,r))\ ((1+\mu b)(1+\mu a))^{1/4}\ e^{-i\int F/\mu\, dt}\ e^{i\omega^2 T/\mu^2}$$
$$\times \exp\{(2i/\mu^2 D(s,r))((1+\mu b)D(\dot{s},r) - (1+\mu a)D(s,\dot{r}))\}$$
$$\times I\gamma\{(-4i/\mu^2 D(s,r))((1+\mu b)(1+\mu a))^{1/2}\} \quad (8.38)$$

where $(D^2 + \varepsilon^2/4)D(s,r) = \delta(s,r)$ with

$$\varepsilon^2(s) = \omega^2 - 2\mu F(s) \quad \text{(unrescaled)}$$
$$= \omega^2 - 4\mu\omega F(s) \text{ (rescaled).}$$

Using the previously given ground state wave functions from Eq. (8.34) and employing Eq. (B.12) from Appendix B enables us to perform the double integration giving the result (rescaled)

$$Z[F] = e^{-i\int F/\mu \, dt} \, e^{i\omega T(\gamma+1)/2} \times \left(\frac{i}{\omega} e^{-i\omega T/2} \frac{d^2}{ds\,dr} e^{i\omega T/2} D(s,r)\right)^{-(\gamma+1)}. \quad (8.39)$$

This result may be simplified by constructing an explicit solution for D(s,r). Define a new function Ω by

$$i\dot{\Omega} + \Omega^2 = \varepsilon^2/4 \quad (8.40)$$

with limit $\Omega = \varepsilon/2$ as $\varepsilon \to$ const. Then by using the identity Eq. (D.13) and Eq. (D.30) we may put this all together to write as our final expression for the exact generating functional (rescaled)

$$Z[F] = \exp(-i\int F/\mu) \exp[-(\gamma+1)\mathrm{Tr}\ \ln(\delta - \mu\omega F D c)]. \quad (8.41)$$

CHAPTER 9

STRUCTURE OF THE GENERATING FUNCTIONAL

In this chapter we will compare the results obtained in the last chapter with the corresponding results for the harmonic oscillator (HO). In particular, we wish to examine the limit $\mu \to 0$ closely to see if the limit is the naive free HO or some pseudo-free HO (with perhaps a hard core). We will also derive the exact equations of motion formally satisfied by the generating functional and see that paradoxically they are not satisfied by Z. This is a very curious result. Finally we discuss the perturbation theory for the exact generating functional Z and introduce the last new functional technique, the concept of a functional anti-derivative.

First we examine the limit of the propagator K to see if it reduces to the HO result Eq. (2.15). The exact K is

$$K = -i((1+\mu a)(1+\mu b))^{1/4}\ e^{i\omega\gamma T/2}\ (\omega\gamma/2)^{1/2}$$
$$\times\ (\sin(\omega T/2))^{-1}\ \exp\{\frac{i\gamma}{2}\cot(\frac{\omega T}{2})((1+\mu b)+(1+\mu a))\}$$
$$\times\ I_\gamma\{-i\gamma((1+\mu b)(1+\mu a))^{1/2}(\sin(\omega T/2)^{-1}\}. \qquad (9.1)$$

The limit $\mu \to 0$ is the same as $\gamma = 2\omega\mu^{-2} \to \infty$. Noting that $I_\gamma(-iz) = e^{-i\pi\gamma/2}\ J_\gamma(z)$ and using the asymptotic limit for J for large argument and order

$$J_\gamma(\gamma/\cos\beta) \sim (2/\pi\gamma\tan\beta)^{1/2} \cos(\gamma\tan\beta-\gamma\beta-\pi/4) \qquad (9.2)$$

we find that

$$\beta = \cos^{-1}\{\sin(\omega T/2)(1+\mu(a+b)+ab\mu^2)^{-1/2}\} \qquad (9.3)$$

and

$$\tan\beta = (\cos(\omega T/2)^2+\mu(a+b)+ab\mu^2)^{1/2}\sin(\omega T/2)^{-1}. \qquad (9.4)$$

Substituting these expressions into Eq. (9.1) allows us to write the limit for K in the form of a product of a pre-exponential factor and a sum of exponential factors.

The limit as $\mu \to 0$ of the pre-exponential factors is easily found to be

$$e^{-i\pi/4} (\omega/2\pi i\sin(\omega T))^{1/2}. \qquad (9.5)$$

The exponential factors are found by expanding $\tan\beta - \beta$ to order μ^2. After a great deal of algebra one finds

$$\tan\beta-\beta \sim (\omega T-\pi)/2 + \cot(\omega T/2)(1+\mu(a+b)/2)$$
$$- \mu^2((b^2+a^2)\cos(\omega T) - 2ab)/4\sin(\omega T). \qquad (9.6)$$

Substituting this into Eq. (9.1) for K gives as the limit

$$K = K_{HO} + K_{HO}^{*}\exp\{i\gamma(\omega T - \pi + 2\cot(\omega T/2))\}. \qquad (9.7)$$

In the limit $\gamma \to \infty$ the second term on the right will oscillate so rapidly relative to the first term that it will fade out of existence relative to the first term. Notice that the term multiplying $i\gamma$ is strictly real. Hence the limit as $\mu \to 0$ of the propagator (and consequently the wavefunctions) is the HO. The wall at $x = -1/\mu$ moves off to $-\infty$ and the reflected (complex conjugate) term fades away.

Now we wish to examine the limit for the generating functional Z as given by Eq. (8.41). To do this it is best to examine the structure of the generating functional. We introduce the concept of the connected generating functional W which is just the logarithm of Z,

$$W = \ln(Z). \qquad (9.8)$$

In the case we are considering,

$$W = (-i/\mu)\int F - (1+\gamma)\mathrm{Tr}\ \ln[\delta - \mu FD/2]. \qquad (9.9)$$

In the above and from here on out the Green's functions D (and sometimes G) will be considered causal. We also introduce the functional M which is defined by

$$Mst...u[F] = Z^{-1} \frac{\delta}{i\delta Fs} \frac{\delta}{i\delta Ft} \cdots \frac{\delta}{i\delta Fu} Z[F]. \qquad (9.10)$$

When the source F is set to zero the functional M becomes M^0, an N-point Green's function $\langle x(s)x(t)...x(u)\rangle$ which is a VEV of a product of fields at different times. One of the primary objectives in quantum field theory is to calculate these N-point Green's functions. It is for this reason that the generating functional Z is introduced in the first place. To proceed further we also introduce one last functional P

$$Pst...u[F] = \frac{\delta}{i\delta Fs} \frac{\delta}{i\delta Ft} \cdots \frac{\delta}{i\delta Fu} W[F]. \qquad (9.11)$$

We illustrate the relationship between the M's and Ps with a few examples:

$$Ms = Ps, \qquad (9.12)$$

$$Mst = (PsPt + Pst), \qquad (9.13)$$

$$Mstu = (PsPtPu + PtPsu + PuPst + PsPtu + Pstu). \qquad (9.14)$$

A knowledge of the Ps enables one to construct the M's and consequently the N-point Green's functions. We now present the general formula for P

$$Ps = -(1/\mu + i\mu(1+\gamma)Dss/2), \tag{9.15}$$

$$Pt_1t_2\ldots t_n = (\mu/2i)^n(1+\gamma) \quad Dt_1t_2Dt_2t_3\ldots Dt_nt_1. \tag{9.16}$$

Notice in the above formula for $n > 1$ that the sum is over all $(n-1)!$ permutations of the set $\{t_2, t_3, \ldots, t_n\}$. To illustrate the use of this formalism and to check our construction we calculate the VEV of the field x(t) which was found in Chapter 8 to be (Eq. (8.11) with $E_o = \omega/2$)

$$\langle x(t)\rangle = \mu/2\omega. \tag{9.17}$$

Using our construction $\langle x\rangle$ is

$$\langle x(t)\rangle = M^o t = P^o t = -(1/\mu + i\mu(1+\gamma)D^o ss/2) \tag{9.18}$$

$$= -(1/\mu + i\mu(1+2\omega\mu^{-2})(i/\omega)/2) = \mu/2\omega. \tag{9.19}$$

The answers check.

We have now presented a complete analysis of the structure of the generating functional Z and have shown how to calculate any quantity of interest. We now return to the discussion of the limit as $\mu \to 0$.

To show that the limit for Z is the HO we may show that the limit for W is the HO result of

$$W_{HO} = \frac{i}{2} \iint FtG^o tsFs. \tag{9.20}$$

The Green's function $G^o ts$ is the solution to the time-independent harmonic oscillator differential equation

$$(D^2 + \omega^2)G^o ts = \delta ts. \tag{9.21}$$

Expanding the exact W (Eq. (9.9)) out to order μ gives

$$W \sim -\frac{i}{\mu}\int Fs(1+i\omega D^o ss) + \frac{\omega}{4}\iint Fs(D^o st)^2 Ft + O(\mu). \tag{9.22}$$

Remember that a superscript ^{o} indicates the $F = 0$ in the functional. Now we have the following relationships:

$$D^{o}st = \frac{i}{\omega} e^{-i\omega|T|/2}, \quad (9.23)$$

$$D^{o}ss = i/\omega, \quad (9.24)$$

$$G^{o}st = \frac{i}{2\omega} e^{-i\omega|T|} = \frac{\omega}{2i}(D^{o}st)^2. \quad (9.25)$$

Substituting the facts into the expression for W given by Eq. (9.19) leads to the result

$$\text{limit } \mu \to 0,\ W \to W_{HO}. \quad (9.26)$$

Now we discuss an interesting paradox. We have found the correct generating functional Z and have examined its limits. We now wish to check whether it satisfies its equation of motion. An equation of motion for Z may be derived from the functional integral for Z using the fact that the integral of a derivative vanishes, therefore

$$0 = \int [dx]\ (\delta/\delta x)\ \exp\{iS\}. \quad (9.27)$$

We may pull the functional derivative out of integral using the fact that $x = \delta/i\delta F$ giving

$$0 = i\delta S[x]/\delta x\ Z[F] \text{ with } x = \delta/i\delta F. \quad (9.28)$$

Working out the derivative of the action leads to the equation of motion for Z

$$0 = [-(1+\mu x)^{-1}(D^2+\omega^2)x + (\mu/2)(1+\mu x)^{-2}(\dot{x}^2+\omega^2 x^2-\mu^2/16)$$
$$+ F + i\mu/2\Delta t(1+\mu x)]\ Z[F]. \quad (9.29)$$

The last term in the above sum arises from a correction that must be added to the Lagrangian which is $i\ln(1+\mu x)/2\Delta t$. This creates an effective action which is used in the equation of motion. We may multiply the above equation by $(1+\mu x)^2$ to write a new equation of motion

$$0 = [-(1+\mu x)(D^2+\omega^2)x + \frac{\mu}{2}(\dot{x}^2 + \omega^2 x^2 - \frac{\mu^2}{16}) + F(1+\mu x)^2 - \frac{3i\mu}{2\Delta t}(1+\mu x)]\ Z[F]. \qquad (9.30)$$

Notice that $[x,F] = -i/\Delta t$. Working out the right-hand side of the equation involves the addition of some 30 terms which we omit from presenting. When the dust clears, one finds that the right-hand side has the value

$$-\frac{3\mu}{8}(\omega + \frac{\mu^2}{4}). \qquad (9.31)$$

This is of course non-zero. Consequently we are left with a contradiction. Either we have calculated the generating functional incorrectly or the equation of motion is not satisfied. We feel that the latter is the proper explanation and present a simple argument for it.

By shifting the variables to $\mu^2 z/4 = (1+\mu x)$ one may put the equation into the form

$$0 = [-z(D^2 + 2\Omega^2)z/4 + z^2/8 - 3iz/2\Delta t + g]\ Z'[F] \qquad (9.32)$$

where Z' is just $\mathrm{Det}^{-(1+\gamma)}$ and g is the constant appearing in the Lagrangian for the inverse quadratic potential $g = \gamma^2/2 - 1/8$. Examine the above equation when $Z' = \mathrm{Det}^{\beta}$. The first three terms arise from the rescaled TDHO. Consequently those terms operating upon Z' must give a term like $C\beta(1+2\beta)$ since the roots for β are 0 and $-1/2$. The constant C cannot be a function of μ. Therefore Eq. (9.32) when evaluated must be of the form

$$2\beta^2 + \beta + Cg(\mu) = 0. \qquad (9.33)$$

When evaluated the constant $C = -1/4$. The point is that the

β found from this method cannot satisfy the simple relation

$$\langle x \rangle = -\beta\mu/2\omega - 1/\mu = \mu/2\omega = \mu E_0 \omega^{-2} \tag{9.34}$$

if we demand as we must that C not be a function of μ. So we have a paradox. The formal equations of motion for Z cannot be satisfied.

We now turn to an examination of the perturbation theory for Z in terms of the harmonic oscillator. Basically the question we ask is if we did not know the exact Z what information about it would we obtain through perturbation theory. In our ignorance of the exact solution we could construct a perturbation expansion based upon the form (see Eq. (8.36) in this regard)

$$Z = \exp\{i\int \frac{\mu x}{2}(\frac{\omega^2 x^2}{1+\mu x} - p^2) + \frac{\mu^2}{32(1+\mu x)}\} Z^o[F,J] \tag{9.35}$$

where $x = \delta/i\delta F$ and $p = \delta/i\delta J$ with

$$Z^o = \exp\{\frac{i}{2}\int\int \omega^2 J G^o J + F G^o F - 2 F G^o \dot{J}\} \tag{9.36}$$

and the Green's function G^o defined as in Eq. (9.21) and Eq. (9.25). An examination of the above leads to some curious conclusions. First it does not look much like the exact solution. In fact from our knowledge of the exact solution we know for example that the P^ostu goes as $\mu^3 + \mu$ which implies that the perturbation series for P^ostu in powers of μ must somehow cut off at order μ^3 with all higher order terms vanishing through some mechanism. This seems very improbable. Second, we know that the exact P^ostu involves the Green's function D^o which is in a real sense the square root of G^o. It is difficult to imagine how products of G^o

can possibly create D^0. So just from a superficial examination of the perturbative form for Z we expect little if any information about the exact Z. In certain sense perturbation theory fails for Z. To circumvent this we create a new perturbation series for Z.

If we rescale L back to the inverse quadratic form of Eq. (4.23) we may write Z as

$$Z = e^{-i\int F/\mu} \exp\{-ig\int (-4i\delta/\mu\delta F)^{-1}\} \,\mathrm{Det}[F]^{-1/2}. \quad (9.37)$$

This seems to be on the right track. The first factor is easily recognized to be present in the exact Z and the determinant is also present. However the power of the determinant is incorrect. It was arrived at by pulling the perturbative term out of the functional integral. But we know from the discussion in Chapter 4 that the limit g --> 0 is not the HO. It is rather the HO with a reflecting wall. Hence the determinant term which is the generating functional for the TDHO must be replaced by the generating functional for a TDHO with a reflecting wall. This may be worked out using the discussion in Chapter 6 and we rewrite Eq. (9.37) correctly as

$$Z = \exp\{-2i\mu^{-2}\int F\} \exp\{\tfrac{g}{2}\int (\delta/\delta F)^{-1}\} \,\mathrm{Det}[F]^{-3/2} \quad (9.38)$$

where for convenience in what is to follow we have rescaled F --> $2F/\mu$. As we have remarked this perturbative form for Z contains many features present in the exact Z. The problem now is to interpret the term $(\delta/\delta F)^{-1}$.

In the theory of ordinary functions through the use of Laplace transforms a term $(d/dx)^{-1}$ may be shown to be the anti-derivative. We appeal to this result to interpret the similar term $(\delta/\delta F)^{-1}$ to be a functional anti-derivative. Once this has been done we must now create a method for calculating the anti-derivatives of $Det^{-3/2}$. This may be done by recognizing that the Fredholm determinant is a linear functional of Ft, ie: $\delta^2/\delta Ft^2 \, Det[F] = 0$. By introducing the projection operator $(1 - Fs\delta/\delta Fs)\, Det = Det^{(s)}$ and the functional derivative notation $\delta/\delta Fs \, Det = Det_{(s)}$ we may write the determinant symbolically as $Det = Det^{(s)} + FsDet_{(s)}$. The anti-derivative $(\delta/\delta Fs)^{-1} \, Det^{\beta}$ is easily found by writing

$$(\delta/\delta Fs)^{-1} \, Det^{\beta} = \int^{Fs} dS \, (Det^{(s)} + SDet_{(s)})^{\beta}$$
$$= Det^{\beta+1}/(\beta+1)Det_{(s)}. \qquad (9.39)$$

Using the fact that $Det_{(s)} = -Dss \, Det$ we may write the above as

$$(\delta/\delta Fs)^{-1} \, Det^{\beta} = (-(\beta+1)Dss)^{-1} \, Det^{\beta}. \qquad (9.40)$$

Proceding along the lines discussed we find that the second functional anti-derivative is (for $\beta = -3/2$)

$$(\delta/\delta Fs)^{-1}(\delta/\delta Ft)^{-1} \, Det^{-3/2} = -4Det^{-3/2} \, (Dst)^{-1} \qquad (9.41)$$
$$\times \, (DssDtt - Dst^2)^{-1/2} \arctan\{(DssDttDst^{-2} - 1)^{1/2}\}.$$

Beyond this we have not been able to explicitly compute the anti-derivatives. An important fact to note about the new perturbation expansion is that it is an expansion in powers of g which means an expansion in inverse powers of μ!

CHAPTER 10

SUMMARY AND SPECULATION

In the preceeding chapters we have introduced three new functional techniques. The principle one is the method of delta functionals. By functionally integrating over the position we showed how linear Hamiltonians gave rise to a delta functional which constrained the momentum space paths to obey an equation of evolution. We were therefore able to reduce the momentum functional integral to quadratures.

To augment our technique we developed three canonical transformations which may be utilized in order to place the Schroedinger equation into a form where MDF might be applied. To illustrate each transformation, we worked a physical example.

In the succeeding chapters we showed how one could extend MDF to the relativistic case and to the situation where the dynamics took place on a group manifold. There the momentum might take on discrete values or not even exist.

In Chapter 7 we left our discussion of MDF and turned to a new functional method, the construction of a path integral from time-dependent coherent states. We then used our new path integral to evaluate the generating functional for the time-dependent harmonic oscillator. We also checked our path integral by deriving the known expression for the TDHO propagator.

We presented another topic in Chapters 8 and 9. There we solved the one-dimensional analogue of general relativity. All the techniques discussed earlier in this thesis were brought to bear upon the construction of the generating functional for this problem. Then with the solution in hand we discussed some of its peculiarities. In particular we showed that the generating functional appears not to satisfy its formal equation of motion. This is a very curious result and could have some bearing upon other formal functional equations.

We also discussed the formal perturbation theory for our generating functional and argued that the theory seems to produce nonsensical results. A new perturbation theory was constructed which utilized the concept of a functional anti-derivative. This concept is the last new functional technique we developed. We illustrated how to calculate functional anti-derivatives and worked one example. We also presented the next formula in order of complexity.

Of the three methods presented in this thesis the development of MDF is the most complete and consequently the most exhausted. There would appear to be little more MDF could be applied to. We have used it to solve the most general problem it can be used to solve. The creation of

MDF was a response to the tedious methods for explicitly evaluating then available. But its usefulness is limited. One interesting result from its development is the bringing to light of the common denominator that all (at present) known calculable path integrals possess. They may all be placed into a form where the Hamiltonian is at most linear in position.

The second technique, that of the TDCS path integral has more possible applications. We have used it in particular to find the generating functional for the TDHO. Other possible uses are sketched in the opening paragraph of Chapter 7. An interesting result from our construction is the recognition of the rotating frame of reference and its role in simplifying one's understanding of the quantum dynamics of the TDHO and TDCS.

The new method that is the most interesting to us, perhaps because it is the least developed of the three and has yet to lose its mystery, is the concept of the functional anti-derivative introduced in the preceeding chapter. The concept of the antiderivative is important but also important is the method introduced for calculating it. In a sense it allows one to sum a certain set of Feynman diagrams non-perturbatively. The same method may be used to evaluate a term like

$$\exp\{\delta/\delta F\}\ \mathrm{Det}[F]. \qquad (10.1)$$

This would be of use in investigating Liouville field theory where the Lagrangian density takes the form

$$L = (\partial x)^2 - e^x. \tag{10.2}$$

Liouville field theory is the higher dimensional analogue of the Morse potential and has been the subject of recent investigation in conjunction with string models for quarks.

Also the observation that the new perturbation theory becomes an expansion in inverse powers of the coupling constant is of potential importance. An obstacle in the present theory of quantum chromodynamics is the large value of the coupling constant rendering perturbation theory in powers of the coupling constant suspect. With the new perturbation theory one might now have a way around this difficulty.

REFERENCES CITED

REFERENCES CITED

Abers, E.S. and Lee, B.W., Gauge theories, Phys. Rep. C **9**, 1 (1973).

Bardacki, K. and Samuel, S., Local field theory for solitons, Phys. Rev. D **18**, 2859 (1978).

Bayen, F., Flato, M., Fronsdal, C., Lichnerowitz, A., and Sternheimer, D., Deformation theory and quantization. II. physical applications, Ann. Phys. **110**, 111 (1978).

Bayen, F. and Maillard, J.M., Star exponentials of the elements of the inhomogeneous symplectic Lie algebra, Lett. Math. Phys. **6**, 491 (1982).

Berger, B.K., Comments on the coherent-state representation of a scalar field in the early universe, Phys. Rev. D **25**, 2208 (1982).

Cambell, W.B., Finkler, P., Jones, C.E., and Misheloff, M.N., Path integrals with arbitrary generators and the eigenfunction problem, Ann. Phys. **96**, 293 (1976).

Dewitt-Morette, C., Maheshwari, A., and Nelson, B., Path integration in non-relativistic quantum mechanics, Phys. Rep. C **50**, 255 (1979).

Duru, I.H., Morse-potential Green's function with path integrals, Phys. Rev. D **28**, 2689 (1983).

Duru, I.H. and Kleinert, H., Solution of the path integral for the H-atom, Phys. Lett. **84B**, 185 (1979).

Faddeev, L.D. and Slavnov, A.A., Gauge Fields: Introduction to Quantum Theory, (Benjamin/Cummings, New York, 1980).

Feynman, R.P., Space-time approach to non-relativistic quantum mechanics, Rev. Mod. Phys. **20**, 367 (1948).

Feynman, R.P., An operator calculus having applications in quantum electrodynamics, Phys. Rev. **84**, 108 (1951).

Feynman, R.P., The development of the space-time view of quantum electrodynamics, Nobel Lectures: Physics 1963-1970, (Elsevier, Amsterdam, 1972) 155.

Gerry, C.G., Remarks on canonical transformations in phase-space path integrals, J. Math. Phys. **24**, 874 (1983).

Gervais, J.L. and Jevicki, A., Point canonical transformations in the path integral, Nucl. Phys. B **110**, 93 (1976).

Ichinose, I., Quantum field theory in a time-dependent gravitational field, Phys. Rev. D **25**, 365 (1982).

Itzykson, C. and Zuber, J-B., Quantum Field Theory, (McGraw-Hill, New York, 1980).

Katz, A., Classical Mechanics, Quantum Mechanics, Field Theory, (Academic Press, New York, 1965).

Khandekar, D.C. and Lawande, S.V., Exact propagator for a time-dependent harmonic oscillator with and without a singular perturbation, J. Math. Phys. **16**, 384 (1975).

Klauder, J., Quantization of spinor fields, Ann. Phys. **11**, 123 (1960).

Klauder, J., New measures for nonrenormalizable quantum field theory, Ann. Phys. **117**, 19 (1979).

Krausz, F.G., Path integral forms for the Klein-Gordon wavefunctions, J. Phys. A **14**, 2911 (1981).

Langguth, W. and Inomata, A., Remarks on the Hamiltonian path integral in polar coordinates, J. Math. Phys. **20**, 499 (1979).

Lee, L.L., Continuum calculus and Feynman's path integrals, J. Math. Phys. **17** 1988 (1976).

Lewis Jr., H.R. and Riesenfeld, W.B., An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field, J. Math. Phys. **10**, 1458 (1969).

Marinov, M.S., Path integrals in quantum theory: an outlook of basic concepts, Phys. Rep. C **60**, 1 (1980).

Marinov, M.S. and Terentyev, M.V., Dynamics on the group manifold and path integral, Fortschr. Phys. **27**, 511 (1979).

Mayes, I.W. and Dowker, J.S., Hamiltonian orderings and functional integrals, J. Math. Phys. **14**, 434 (1973).

McLaughlin, D.W. and Schulman, L.S., Path integrals in curved spaces, J. Math. Phys. **12**, 2520 (1971).

Mizrahi, M.M., Phase-space path integrals, without limiting procedure, J. Math. Phys. **19**, 298 (1978).

Montroll, E.W., Markoff chains, Weiner integrals, and quantum theory, Commun. Pure Appl. Math. **5**, 415 (1952).

Pak, N.K. and Sokmen, I., General new time formalism in the path integral, ICTP preprint IC/83/147 (1983).

Peak, D. and Inomata, A., Summation over Feynman histories in polar coordinates, J. Math. Phys. **10**, 1422 (1969).

Richtmeyer, R.D., Principles of Advanced Mathematical Physics, Vol. 1, (Springer-Verlag, New York, 1978).

Schulman, L.S., A path integral for spin, Phys. Rev. **176** (1968).

Schulman, L.S., Techniques and Applications of Path Integration, (Wiley, New York, 1981).

Schwinger, J., On gauge invariance and vacuum polarization, Phys. Rev. **82**, 664 (1951).

Sharan, P., Star-product representation of path integrals, Phys. Rev. D **20**, 414 (1979).

APPENDICES

APPENDIX A

TIME-SLICING DERIVATIONS

In this appendix we derive equations (1.4), (1.5), (1.18), (3.11), (4.3) and (4.4) from our definition of the time-sliced phase-space path integral of Eq. (1.16) which we write as (A.1)

$$K(b|a) = (\Pi \int dp_{n-}/2\pi)(\Pi \int dq_n)\ e^{iS_{ea}(p_{n-},q_n,\Delta t)}$$

in the limit $N \to \infty$, $\Delta t \to 0$ and where $n\pm = n\pm 1/2$. We do not discuss whether the limit as so defined actually exists but simply manipulate the formal time-sliced quantities in the following. The lattice action S_{ea} is defined as

$$S_{ea} = (\textstyle\int_r^s p\dot{q} - H(p,q)\ dt)_{ea} \quad (A.2)$$

$$= \Sigma\ p_{n-}(q_n - q_{n-1}) - 1/2(H(p_{n-},q_n) + H(p_{n-},q_{n-1}))\Delta t.$$

The Hamiltonian term is defined in such a way that variation of the lattice action leads to the Euler approximation (ea) to Hamilton's equations of motion $p = -\frac{\partial H}{\partial q}(p,q)$, $q = \frac{\partial H}{\partial p}(p,q)$

$$(p_{n+} - p_{n-}) = -\frac{\Delta t}{2}\left(\frac{\partial}{\partial q_n}H(p_{n+},q_n) + \frac{\partial}{\partial q_n}H(p_{n-},q_n)\right), \quad (A.3)$$

$$(q_n - q_{n-1}) = \frac{\Delta t}{2}\left(\frac{\partial}{\partial p_{n-}}H(p_{n-},q_n) + \frac{\partial}{\partial p_{n-}}H(p_{n-},q_{n-1})\right). \quad (A.4)$$

It is obvious that Eq. (1.5) is satisfied by inspection of Equations (A.1) and (A.2). The infinitesimal propagator K_i is (A.5)

$$K_i(b,c) = \int dp/2\pi\ \exp(ip(b-c) - i\Delta t(H(p,b)+H(p,c))/2)$$

and has the limit as $\Delta t \to 0$ (Eq. (1.4))

$$K_i = \int dp/2\pi\ e^{ip(b-c)} = \delta(b-c). \quad (A.6)$$

To establish Eq. (1.18), Schroedinger's equation, we first find the representation for the momentum p. Examine

$$K(b,T+\Delta t|a,0) = \int dc\ K_i(b,c)K(c,T|a,0) \quad (A.7)$$

and the limit as $\Delta t \to 0$ (A.8)

$$\lim \frac{\partial K}{\partial b}(b,T+\Delta t) = \int dpdc/2\pi \left(ip - i\frac{\Delta t}{2}\frac{\partial K}{\partial b}(p,b)\right) K_i \; K(c|a)$$

$$= \int dpdc/2\pi \; (ip) \; e^{ip(b-c)} \; K(c|a) = \frac{\partial}{\partial b}K(b|a). \tag{A.9}$$

This allows us to identify the operator representation of p as $p = \partial/i\partial b$. Eq. (A.7) may be written as

$$K = \int dcdp/2\pi \; e^{ip(b-c)}$$

$$\times \exp(-i\Delta t(H(p,b)+H(p,c))/2) \; K(c|a). \tag{A.10}$$

We may pull the exp(H) terms out from inside the integral by using the representation for p. To pull H(p,b) out we must order all the b's to the left of the p's. Then

$$K = \exp(-iH(b|p)\Delta t/2)$$

$$\times \int dc \; \exp(-iH(p,c)\Delta t/2) \; \delta(b-c) \; K(c|a) \tag{A.11}$$

$$= \exp(-iH(b|p)\Delta t/2) \; \Sigma \; H_{mn} \frac{p^m}{m!} \int dc \; \frac{c^n}{n!} \; \delta(b-c) \; K(c|a)$$

where we have expanded H in a Taylor series. We may now integrate with impunity with result

$$K(T+\Delta t) = \exp(-i(H(b|p)+H(p|b))\Delta t/2) \; K(T). \tag{A.12}$$

Using $\frac{\partial K}{\partial t} = \lim \; (K(T+\Delta t)-K(T))/\Delta t$ we may write

$$i\frac{\partial}{\partial t}K(b,t|a,r) = H_S(p,q) \; K(b,t|a,r) \tag{A.13}$$

which was to be shown (Eq. (1.18)).

Next we demonstrate Eq.s (4.3) and (4.4). Notice that

$$\int_r^s p\dot{q} \; dt_{ea} =$$

$$\Sigma \; p_{n-}(q_n-q_{n-1}) = (p_{N-}b-p_1a) - \Sigma \; q_n(p_{n+}-p_{n-})$$

$$= pq| - \int_r^s p\dot{q} \; dt_{ea} \tag{A.14}$$

which establishes Eq. (4.4). Examine

$$\int_r^s h(q)\dot{q} \; dt_{ea} = \Sigma \; (h(q_n)+h(q_{n-1}))(q_n-q_{n-1})/2 \tag{A.15}$$

and let $dg/dq = h(q)$. In the Euler approximation we may write

$$(g_n - g_{n-1}) = (h(q_n) + h(q_{n-1}))(q_n - q_{n-1})/2 \qquad (A.16)$$

and substituting this into Eq. (A.15) gives

$$\int_r^s h(q)\dot{q}\ dt_{ea} = \Sigma\ (g_n - g_{n-1}) = g| = g(b) - g(a) \qquad (A.17)$$

which is Eq. (4.3).

Finally we derive Eq. (3.11). Examine the functional integral

$$K = \int [dp/2\pi][dq]\ \exp(i\int_r^s p\dot{q} + qG(p)\ dt)\ H[p]. \qquad (A.18)$$

The time-slicing representation is given by

$$K = \int\ (\Pi\ dp_{n-}/2\pi)(\Pi\ dq_n)\ H[p_n]\ e^{i\ \Sigma p_{n-}(q_n - q_{n-1})}$$
$$\times\ e^{i\ \Sigma g(p_{n-})(q_n + q_{n-1})\Delta t/2} \qquad (A.19)$$

Here $N\Delta t = s-r$, $q_0 = q(r) = a$, and $q_N = q(s) = b$.

Performing an integration by parts in the exponent lets us write (A.20)

$$K = \int [dq/2\pi][dp]/2\pi\ e^{ipq|}\ \exp(i\int\ q(\dot{p} - G(p))\ dt)\ H[p].$$

Noting that the position functional integral is a representation of a delta functional, we write

$$K = \int [dp]/2\pi\ e^{ipq|}\ \delta[\dot{p} - G(p)]\ H[p]. \qquad (A.21)$$

Using time-slicing Eq.s (A.20) and (A.21) are written as

$$K = \int\ (\Pi\ dp_{n-}/2\pi)(\Pi\ dq_n)\ H[p_n]\ e^{ib(p_{N-} + g(p_{N-})\Delta t/2)}$$
$$\times\ e^{-ia(p_{1-} - g(p_{1-})\Delta t/2)} \qquad (A.22)$$
$$\times\ \exp\{-i\ \Sigma q_n((p_{n+} - p_{n-}) - (g(p_{n+}) + g(p_{n-}))\Delta t/2)\}$$

and

$$K = (2\pi)^{-1}\int(\Pi\ dp_n\ H[p_n])$$
$$\times\ e^{ib(p_N+g(p_N)\Delta t/2)}\ e^{-ia(p_1-g(p_1)\Delta t/2)}$$
$$\times\ \Pi\ \delta((p_{n+1}-p_n)-(g(p_{n+1})+g(p_n))\Delta t/2) \qquad (A.23)$$

where the $\pm$ subscript has been dispensed with. Notice that the delta functions constrain p(t) to obey the numerical Euler approximation to the differential equation $dp/dt = g(p)$, viz: $p_{n+1}-p_n = (g(p_{n+1})+g(p_n))\Delta t/2$.

It is desirable to collapse the concatenation of delta functions by integrating 'down from the top' (performing the p_N integration, then the p_{N-1} integration, etc.) and 'up from the bottom' to leave a single p_o integration to be performed. Here to avoid confusion, read as 'p oh' rather than 'p zero'. To perform the integrations it is necessary to express the argument of the delta function in terms of the integrated variable. There are two cases:

a. $\delta((p_{n+1}-p_n)-(g(p_{n+1})+g(p_n))\Delta t/2)$

$$= \delta(p_{n+1}-p_n-\tilde{g}(p_n)\Delta t)\ \times\ (d(p_{n+1}-g(p_{n+1})\Delta t/2)/dp_{n+1})^{-1}$$
$$= \delta(p_{n+1}-p_n-\tilde{g}(p_n)\Delta t)/(1-g'(p_{n+1})\Delta t/2) \qquad (A.25)$$

where the root of $p_{n+1}-g(p_{n+1})\Delta t/2 = p_n+g(p_n)\Delta t/2$ is written as $p_{n+1} = p_n+\tilde{g}(p_n)\Delta t$.

Similarly

b. $\delta(\ldots) = \delta(p_{n+1}-\tilde{g}(p_{n+1})\Delta t-p_n)/(1+g'(p_n)\Delta t/2)$ (A.26)

and to order Δt, these can be expressed as

a. $\delta(p_{n+1}-p_n-\tilde{g}(p_n)\Delta t)(1+g'(p_{n+1})\Delta t/2)$ (A.27)

b. $\delta(p_{n+1}-p_n-\tilde{g}(p_{n+1})\Delta t)(1-g'(p_n)\Delta t/2)$. (A.28)

Integrating down from the top gives a product

$$\Pi\ (1+g'(p_n)\Delta t/2) = \Pi\ e^{g'(p_n)\Delta t/2} \tag{A.29}$$

and up from the bottom, another $\Pi\ e^{-g'(p_n)\Delta t/2}$ with errors of Δt^2 and every integration will force the momenta at each time to become a recursive function of p_o. Schematically: p_N --> $p_N(p_{N-1})$ --> ... --> $p_N(p_o)$. We replace this by the exact $p(p_o,T)$ (similarly for all p_n) with an error of Δt^2.

Now the propagator is written as (A.30)

$$K = \int dp_o/2\pi\ H[p_n(p_o)]\ \Pi\ e^{g'(p_m)\Delta t/2} \times \Pi\ e^{-g'(p_m)\Delta t/2}$$
$$\times\ e^{ib(p_N(p_o)+g(p_N)\Delta t/2)} \times e^{-ia(p_1(p_o)-g(p_1)\Delta t/2)}.$$

In the limit as Δt-->0, N-->∞, with $N\Delta t$ = s-r, displaying the sums as Riemann integrals

$$K = \int_{-\infty}^{\infty} dp_o/2\pi\ H[p(p_o,t)]\ e^{ibp(p_o,s)} \times e^{-iap(p_o,r)}$$
$$\times\ e^{\int_o^s g'(p)/2\ dt} \times e^{-\int_r^o g'(p)/2\ dt}. \tag{A.31}$$

Noting that

$$\int_r^s g'(p)\ dt = \int dg(p)/dp\ dt \tag{A.32}$$
$$= \int dg\ (dt/dp) = \int dg/g = \ln[g(s)/g(r)]$$

the last two factors can be expressed as

$\left(\frac{g(s)}{g(o)}\frac{g(r)}{g(o)}\right)^{1/2}$ and recognizing that $\frac{dp(t)}{dp_o} = \frac{g(p)dt}{g(p_o)dt} = \frac{g(t)}{g(o)}$

the integral may be written as (A.33)

$$K = \int_{-\infty}^{\infty} dp_o/2\pi\ H[p(p_o,t)]e^{ibp(s)-iap(r)}\left(\frac{dp(s)}{dp_o}\frac{dp(r)}{dp_o}\right)^{1/2}$$

where p(t) satisfies dp/dt = g(p), which was to be shown (Eq. (3.11)).

A few words are in order at this point. First, only terms up to order Δt have been kept which is all that is required for the concatenation to give the correct result.

Secondly, in most instances the function g(p) will be complex. For a discussion of delta functions with complex arguments see Richtmeyer (1978, p.38). The integration is always over a real-valued p_o.

Thirdly, for matrix equations of the form $\mathbf{p}(t) = A(t).\mathbf{p}_o$ delta functions of the type

$$\delta(A(t).\mathbf{p}_o) = \delta(\mathbf{p}_o)/\det A \qquad (A.34)$$

will be encountered so the requisite semigroup factors are $(\det A(s)\det A(r))^{1/2}$ with '$d\mathbf{p}/d\mathbf{p}_o$' = detA, the Jacobian.

APPENDIX B

FORMULAE AND INTEGRALS

Table of Integrals and Formulae

$$\int_{-\infty}^{\infty} dx\, x^n\, e^{-ax^2+bx} = d^n/db^n\, (\pi/a)^{1/2}\, e^{b^2/4a} \tag{B.1}$$

$$\int_{-\infty}^{\infty} ds\, dt\, \exp\{-2rst-(s-ib)^2 - (t-ia)^2\}$$
$$= \pi(1-r^2)^{-1/2}\, \exp\{\frac{2abr-r^2(a^2+b^2)}{1-r^2}\} \tag{B.2}$$

$$\int_{-\infty}^{\infty} ds\, dt\, \exp(-2rst-as^2-bt^2) = \pi(ab-r^2)^{-1/2} \tag{B.3}$$

$$\int_{-\infty}^{\infty} ds\, dt\, \exp(-2rst-as^2+2bs-ct^2+2dt)$$
$$= \pi(ac-r^2)^{-1/2}\, \exp\{\frac{b^2c+d^2a-2rbd}{ac-r^2}\} \tag{B.4}$$

The equations above are all special cases of the following integral.

$$\int_{-\infty}^{\infty} d^N x\, \exp(-x.A.x + \mathbf{b}.x)$$
$$= (\pi)^{N/2}\, (\det A)^{-1/2}\, \exp\{\mathbf{b}.A^{-1}.\mathbf{b}/4\}. \tag{B.5}$$

This integral admits a continuum or path integral generalization

Next we list some formulae for complex Gaussian integrals where the measure $dz^*dz/2\pi i = (dRez)(dImz)/\pi$.

$$\int \Pi\, dz_n^*dz_n/2\pi i\, \exp\{-z^*.A.z + \mathbf{a}^*.z + z^*.\mathbf{b}\}$$
$$= (\det A)^{-1}\, \exp\{\mathbf{a}^*.A^{-1}.\mathbf{b}\}. \tag{B.6}$$

This integral also admits a continuum generalization.

In one dimension we have

$$\int dz^* dz/2\pi i \ \exp(-az^2 - bz^*z - cz^{*2} + dz + ez^*)$$

$$= (b^2 - 4ac)^{-1/2} \exp\{\frac{bde-ae^2-cd^2}{b^2-4ac}\}. \qquad (B.7)$$

$$\int dz^* dz/2\pi i \ z^*z \ \exp(-z^*z + a^*z + z^*b)$$

$$= (1 + a^*b) \exp(a^*b). \qquad (B.8)$$

Here is a concatenation formula for Gaussian integrals

$$\int_{-\infty}^{\infty} dr \ \exp\{\frac{ia}{2}(\frac{(t-r)^2}{(c-d)} + \frac{(r-s)^2}{(d-e)})\}$$

$$= (\frac{2\pi i(c-d)(d-e)}{a(c-e)})^{1/2} \exp\{\frac{ia(t-s)^2}{2(c-e)}\}. \qquad (B.9)$$

$$\int_{-\infty}^{\infty} dx \ x^{-(\rho+1)} \exp\{-\frac{ia}{2}(x + r^2/x)\}$$

$$= -2\pi i \ r^{-\rho} I_\rho(-iar)\Theta(\mathrm{Re}a). \qquad (B.10)$$

In the above equation, I_ρ is the modified Bessel function of order ρ and Θ is the unit step function. The Bessel functions satisfy a Gaussian concatenation formula

$$\int_0^\infty rdr \ e^{i\alpha r^2} I_\mu(-iar) I_\mu(-ibr) \qquad (B.11)$$

$$= \frac{i}{2\alpha} \exp\{-i(\frac{a^2+b^2}{4\alpha})\} I_\mu\{-i\frac{ab}{2\alpha}\} \qquad \mathrm{Re}\mu > -1, \ \mathrm{Re}\alpha > 0.$$

They also satisfy

$$\int_0^\infty dxdy \ e^{-ax-by} (xy)^{\mu/2} I_\mu(-ic(xy)^{1/2})$$

$$= 4\int_0^\infty drds \ (rs)^{\mu+1} e^{-as^2-br^2} I_\mu(-icrs)$$

$$= 4(-2ic)^{\mu} (4ab+c^2)^{-(\mu+1)} \Gamma(\mu+1) \qquad (B.12)$$

where $\Gamma(x)$ is the Gamma function.

The following formulae for bigenerating functions are useful in determining wave functions from propagators.

The Hille-Hardy formula:

$$z^{-\mu/2} (1-z)^{-1} \exp\{-\frac{1}{2}(x+y)\frac{1+z}{1-z}\} I_{\mu}\{\frac{2}{1-z}(xyz)^{1/2}\} \qquad (B.13)$$
$$= \Sigma z^n n!/\Gamma(n+\mu+1) e^{-(x+y)/2} (xy)^{\mu/2} L_n^{\mu}(x)L_n^{\mu}(y).$$

Mehler's formula:

$$(1-z^2)^{-1/2} \exp\{y^2 - (y-zx)^2/(1-z^2)\}$$
$$= \Sigma z^n H_n(x)H_n(y)/(2^n n!). \qquad (B.14)$$

APPENDIX C

LINEAR MOMENTUM TRANSFORMATIONS

Finding the linear transformation that reduces a quadratic Hamiltonian to one linear in position can be troublesome so we list here some useful transformations for standard Hamiltonians.

Time-dependent Harmonic Oscillator:

$$S = \int p\dot{q} - a^2(t)p^2/2 - b^2(t)q^2/2 - c(t)pq + d(t)p + F(t)q - E(t)\ dt. \quad (C.1)$$

Let $p = P - i\Omega Q$, $q = Q$ and define Ω so that it is a particular solution to $i\dot{\Omega} = b^2 - a^2\Omega^2 - 2ic\Omega$. Then

$$S = (-i\Omega Q^2/2| + \int P\dot{Q} - a^2P^2/2 - (c - ia^2\Omega)PQ + Pd + (F - i\Omega d)Q - E\ dt. \quad (C.2)$$

Constant $\mathbb{E}$ and $\mathbb{B}$ Fields:

$$S = \int \mathbf{p}.\dot{\mathbf{q}} - (\mathbf{p} - m\omega(\omega x\mathbf{q}))^2/2m + e\mathbf{q}.\mathbf{E}\ dt \quad (C.3)$$

with $w = \hat{B}$ and $w = eB/2m$.

Let $\mathbf{p} = \mathbf{P} - im\omega(\omega x\mathbf{q})x\omega$, $\mathbf{q} = \mathbf{Q}$, then

$$S = (-imw(wx\mathbf{Q})^2/2| + \int \mathbf{P}.\dot{\mathbf{Q}} - \mathbf{P}^2/2m + \omega\mathbf{P}.(\omega x\mathbf{Q} + i(\omega x\mathbf{Q})x\omega) + e\mathbf{Q}.\mathbf{E}\ dt. \quad (C.4)$$

Rotating Coordinate System about $\boldsymbol{\omega}$:

$$S = \int \mathbf{p}.\dot{\mathbf{q}} - p^2/2m + \mathbf{p}.\boldsymbol{\omega}x\mathbf{q}\ dt. \quad (C.5)$$

APPENDIX D

TIME-DEPENDENT HARMONIC OSCILLATOR

In this appendix we collect several useful facts and formulae concerning the time-dependent harmonic oscillator (TDHO). The TDHO is any system that obeys the differential equation

$$(D^2 + \omega^2(t))f(t) = 0. \qquad (D.1)$$

We may formally solve this equation in terms of the initial data by constructing a Green's function that satisfies

$$(D^2 + \omega^2)G(t,s) = \delta(t,s). \qquad (D.2)$$

For the retarded Green's function we have the boundary conditions $G_R(t,s) = 0$ for $t < s$, lim $t \to s^+$ $G_R(\dot{t},s) = 1$, and $G_R(t,t) = 0$.

We may express $f(t)$ in terms of $f(s)$, $\dot{f}(s)$ at the earlier time s by

$$f(t) = G_R(t,s)\dot{f}(s) - G_R(t,\dot{s})f(s) \qquad (D.3)$$

or using the notation $\overleftrightarrow{\partial}$ (see Eq. (4.31))

$$f(t) = G_R(t,s)\overleftrightarrow{\partial}_s f(s). \qquad (D.4)$$

The advanced Green's function $G_A(s,r) = G_R(r,s)$.

The advanced, retarded, and causal Green's functions may be expanded in terms of the time-independent Green's functions. To do this let $\omega^2(t) = w^2 - F(t)$ and define G^o as the solution to

$$(D^2 + w^2)G^o = \delta. \qquad (D.5)$$

Then we may write G in terms of G^o as

$$G(s,r) = G^o(s,a)(\delta(a,r) - F(a)G^o(a,r))^{-1}. \qquad (D.6)$$

In the above equation we are using a continuous matrix notation where repeated indices (arguments of functions) from the beginning of the alphabet are to be summed (integrated) over. To make this more obvious we will write G(s,r) as the matrix Gsr. A term like GsaGar stands for $\int G(s,a)G(a,r)\,da$.

The variational derivative of G with respect to F is

$$\delta Gsr/\delta Ft = GstGtr. \quad \text{(D.7)}$$

It is important to note that the t in the above equation is not summed over. Summed indices are from the beginning of the alphabet.

We may construct an explicit solution for G_R by defining a new function $\Omega(t)$ by requiring it to be any particular solution of

$$i\dot{\Omega} + \Omega^2 = \omega^2 = w^2 - F. \quad \text{(D.8)}$$

Then we may write (D.9)

$$G_R sr = \theta(s-r)\int_r^s dt\, \exp\{i\int_s^t \Omega(u)\,du\}\exp\{i\int_r^t \Omega(v)\,dv\}.$$

Notice that G is the product of a regular function $\mathcal{G}$ (the integral) and a distribution (the step function). Using this expression we find for the regular $\mathcal{G}$

$$\mathcal{G}_R\dot{s}r = -i\Omega s\mathcal{G}_R sr + \exp\{i\int_r^s \Omega\}, \quad \text{(D.10)}$$

$$\mathcal{G}_R s\dot{r} = -i\Omega r\mathcal{G}_R sr - \exp\{-i\int_r^s \Omega\}, \quad \text{(D.11)}$$

$$\mathcal{G}_R\dot{s}\dot{r} = -\Omega s\Omega r\mathcal{G}_R sr + i\Omega s e^{-i\int \Omega} - i\Omega r e^{i\int \Omega}. \quad \text{(D.12)}$$

The regular Wronskian is $W = \mathcal{G}_R\dot{s}\dot{r}\mathcal{G}_R sr - \mathcal{G}_R\dot{s}r\mathcal{G}_R s\dot{r} = 1$.

Another useful identity for $\mathcal{G}_R$ is (D.13)

$$\Omega s\Omega r\mathcal{G}_R sr - i\Omega r\mathcal{G}_R\dot{s}r + i\Omega s\mathcal{G}_R s\dot{r} + \mathcal{G}_R\dot{s}\dot{r} = -2i\Omega r e^{i\int \Omega}.$$

There is another explicit form for G_R that is used. It may be constructed by defining a new function N that is a particular solution to the TDHO and writing N in the form $N = S(t)\exp\{i\gamma(t)\}$. Then $\Omega = i\dot{N}/N$ and

$$D^2 S - C^2 S^{-3} + \omega^2 S = 0, \tag{D.14}$$

$$S^2\dot{\gamma} = C, \tag{D.15}$$

where C is some constant. If we define another function $\sigma(s,r) = \gamma(s) - \gamma(r)$ we may write G_R as

$$G_Rsr = \Theta sr\ \sin(\sigma sr)(\dot{\gamma}s\dot{\gamma}r)^{-1/2}. \tag{D.16}$$

Notice in the limit $\Omega \dashrightarrow \omega$ and $\dot{\gamma} \dashrightarrow -\omega$. Hence the explicit form for G_R when ω is a constant (G^o) is

$$G^o(T) = \Theta(T)\sin(\omega T)/\omega. \tag{D.17}$$

The function G_R may be canonically decomposed into positive ($\sim e^{-i\omega T}$) and negative ($\sim e^{i\omega T}$) parts,

$$G_R = G^{(+)} + G^{(-)}. \tag{D.18}$$

For example $G^o{}_R$ has the decomposition

$$\sin(\omega T)/\omega = e^{i\omega T}/2i\omega - e^{-i\omega T}/2i\omega. \tag{D.19}$$

Here $G^{(+)} = ie^{-i\omega T}/2\omega$ and $G^{(-)} = -ie^{i\omega T}/2\omega$. Notice that for any $\omega(t)$ we have

$$G^{(+)*} = G^{(-)}, \tag{D.20}$$

$$G^{(+)}sr = -G^{(-)}rs, \tag{D.21}$$

$$G^{(+)}\dot{s}s + G^{(-)}\dot{s}s = 1. \tag{D.22}$$

The causal Green's function G_C is constructed out of these functions by

$$G_Csr = G^{(+)}sr\Theta(s-r) - G^{(-)}sr\Theta(r-s). \tag{D.23}$$

For ω = constant, $G^o{}_C = i\exp\{-i\omega|T|\}/2\omega$. Notice that the causal or Feynman Green's function is symmetric, ie: $G_C(s,r) = G_C(r,s)$.

The generalized Wronskian of G_C is

$$Wsr = G_C\dot{s}\dot{r}G_Csr - G_C\dot{s}rG_Cs\dot{r} = -\delta srG_Css - 1/4. \qquad (D.24)$$

We now wish to show that the function $\exp\{i\int \Omega\}$ used in Eq. (D.10), etc. may be written as a continuous determinant. We do this so that we can explicitly display the functional dependence of $\exp\{i\int \Omega\}$ upon F(t). Once this is done we may find its functional derivatives with respect to F(t). This is of importance because $\exp\{i\int \Omega\}$ arises in connection with generating functionals of interest.

The generating functional of the TDHO was found in Chapter 7 to be

$$Z[F] = \exp\{-\frac{i}{2}\int_r^s \Omega(t) - w\, dt\}. \qquad (D.25)$$

Here s is strictly greater than r. In a time symmetric form we may write Z as

$$Z[F] = \exp\{-\frac{i}{2}\int \Omega - w\}\Theta sr + \exp\{\frac{i}{2}\int \Omega - w\}\Theta rs. \qquad (D.26)$$

However we may write Z as a path integral

$$Z = \int[dx]\ \exp\{-\frac{i}{2}\int x(D^2 + \omega^2)x\}. \qquad (D.27)$$

Using the continuum generalization of Eq. (A2.5) allows us to formally write Z as

$$Z = (\mathrm{Det}[D^2 + \omega(t)^2]/\mathrm{Det}[D^2 + w^2])^{-1/2} \qquad (D.28)$$

$$= (\mathrm{Det}[\delta - F(t)G^o{}_Cts])^{-1/2}. \qquad (D.29)$$

We have used the correct (causal) boundary conditions to pick out which Green's function should be used in the above expression. Hence we may equate Eqs. (D.26) and (D.29) to write

$$\exp\{i\int \Omega - w\}\Theta sr + \exp\{-i\int \Omega - w\}\Theta rs \qquad (D.30)$$

$$= \mathrm{Det}[\delta - FG^{o}{}_{C}].$$

The functional derivatives of the above expression are easily found to be

$$\delta/\delta F(t)\ \mathrm{Det}[\delta - FG^{o}{}_{C}] = -G_{C}tt\ \mathrm{Det}[\delta - FG^{o}{}_{C}]. \qquad (D.31)$$

X