ALLOWABLE ROTATION NUMBERS FOR SIEGEL DISKS OF RATIONAL MAPS

by

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Abstract

The results presented here answers in part a conjecture of Douady about sharpness of the Brjuno condition. Douady hypothesized that a Siegel disk exists for a rational function if and only if the Brjuno condition is satisfied by the rotation number. It is known that the Brjuno condition is sharp for quadratic polynomials and many special families. This thesis focuses on a class of rational functions, many of which have not been considered previously.

Specific examples of maps for which these results apply include quadratic rational maps with an attracting cycle. Also included are those rational functions arising of Newton's method on cubic polynomials with distinct roots.

Introduction

Complex dynamics is the study of dynamical behavior of holomorphic or meromorphic maps in one complex variable. Given such a map f, this makes sense if the sequence of iterates $f^n = f \circ \ldots \circ f$ is well-defined in some sufficiently large set. Most of the theory has been developed for polynomials as holomorphic self-maps of the plane and rational functions as meromorphic self-maps of the Riemann sphere. For general background see [Bea91], [CG93] and [Mil06].

One of the first questions asked in any dynamical system is the local behavior near fixed points and periodic points. By passing to an appropriate iterate and conjugating, f may be assumed to have a fixed point at 0 with

$$f(z) = \lambda z + O(z^2).$$

The constant $\lambda = f'(0)$ is the **multiplier** of f at the fixed point. If $|\lambda| < 1$, then it is easy to see that the fixed point is locally attracting, and if $|\lambda| > 1$, then it is equally easy to see that it is locally repelling.

The case where $|\lambda| = 1$ has two very different subcases, depending on whether λ is a root of unity or not. If $\lambda = e^{2\pi i p/q}$, then the fixed point is called **rationally indifferent**, and unless f^q is the identity, the local behavior is complicated with attracting and repelling directions. Since rationally indifferent periodic points will not be considered in this thesis, the reader may refer to the literature for the detailed description. Specifically, the Leau-Fatou Flower Theorem is of interest.

If $\lambda = e^{2\pi i \alpha}$ with $\alpha \in \mathbb{R} \backslash \mathbb{Q}$, the fixed point is called **irrationally indifferent**, and the question whether f is analytically conjugate to the linear rotation $z \mapsto e^{2\pi i \alpha} z$ is called the **center problem**. It turns out that this question of linearizability is equivalent to (Lyapunov) stability of the fixed point.

Following Milnor, here is a short history of the problem: Kasner is said to have conjectured in 1912 [Kas12] that linearization is always possible, but some of the literature [AIR12] suggests that he may not have been entirely convinced. In 1917, Pfeiffer [Pfe17] gave examples of non-linearizable functions. In 1919, Julia [Jul19] gave an incorrect proof that for rational functions of degree $d \geq 2$ a linearization is never possible. In 1927, Cremer [Cre28] showed that given $|\lambda| = 1$ and $d \geq 2$, if

$$\liminf_{q \to \infty} |\lambda^q - 1|^{d^{-q}} = 0,$$

then no fixed point of multiplier λ for a rational function of degree d can be linearizable. It turns out that the set of λ for which Cremer's condition holds is topologically generic, i.e., the intersection of a countable number of open dense subsets of the unit circle.

In 1942, Siegel [Sie42] proved that if there exists constants $c, \tau > 0$ with

$$|\lambda^q - 1| \geq c q^{-\tau}$$

for all $q \geq 1$, then every local holomorphic map with a fixed point with multiplier λ is linearizable. It turns out that this condition is equivalent to $\lambda = e^{2\pi i \alpha}$ with α being a Diophantine irrational, and it is known that the set which satisfies this condition for every $\tau > 2$ is a full measure subset of the unit circle.

Siegel's result has subsequently been improved by Brjuno, Rüssmann and Yoccoz [Brj65], [R̈67], [Yoc95]. In order to state the result, let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ with sequence of convergents (p_n/q_n) (i.e., approximations given by the truncated continued fraction expansion, written in reduced form). Then we define the **Brjuno function**

$$B(\alpha) = \sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n}$$

and the set of **Brjuno numbers** as

$$\mathcal{B} = \{\alpha \in \mathbb{R} \setminus \mathbb{Q} : B(\alpha) < \infty\}$$

With these definitions, the local linearization problem is now completely settled by the following theorems.

Theorem 1.1 (Brjuno, Rüssmann). *If $\alpha \in \mathcal{B}$ and $\lambda = e^{2\pi i\alpha}$, then every local holomorphic function $f(z) = \lambda z + O(z^2)$ is linearizable.*

Theorem 1.2 (Yoccoz). *If $\alpha \notin \mathcal{B}$ and $\lambda = e^{2\pi i\alpha}$, then the quadratic polynomial $f(z) = \lambda z + z^2$ is not linearizable.*

The big outstanding conjecture is the following one stated by Douady in 1986 [Dou87].

Conjecture 1.3 (Douady). If $\alpha \notin \mathcal{B}$ and $\lambda = e^{2\pi i\alpha}$, and if $f(z) = \lambda z + O(z^2)$ is a rational function of degree $d \geq 2$, then f is not linearizable.

This conjecture is open even for cubic polynomials, but there are several partial results showing that the conjecture is "generically true". Pérez-Marco showed in [PM93] that Douady's conjecture holds for every $d \geq 2$ in an open and dense subset of the parameter space of rational maps of degree d, and in [PM01] he showed that the conjecture holds outside of a pluripolar subset of the same parameter space. Geyer [Gey] showed it for certain classes of polynomials, and Cheraghi, using Inou and

Shishikura's renormalization technique, showed in [Che10] that Douady's conjecture holds if the continued fraction coefficients are sufficiently large.

In this thesis, Geyer's approach is generalized to certain families of rational functions and it is shown that Douady's conjecture holds for these. Specifically treated are the quadratic rational maps with a super attracting cycle and the class of rational functions arising as implementations of Newton's method on cubic polynomials with distinct roots.

Chapter 2 will discuss the necessary dynamical and complex analysis background material. Chapter 3 provides the background on quasiconformal mappings and the techniques of quasiconformal surgery that will prove necessary. The definitions and results in this chapter are somewhat modified to suit this context, but are otherwise standard. Most of the novel results and definitions are contained in Chapter 4. The final two chapters concentrate on establishing the main result. Chapter 5 is the easiest new example that this result applies to. It can be understood independently of the algebraic geometry techniques in Chapter 4. The final chapter (Chapter 6) combines and generalizes the techniques of Chapters 4 and 5 to prove the main result. Those readers wishing only to understand the statement of the main result will find the needed definitions in Sections 4.2 and 4.3.

Background

This first section is a crash course in complex dynamics. A better and more thorough exposition is given in the readable [Mil06] or [CG93]. Any results in this section stated without proof can be found in one of these two resources. Throughout, assume that d is a fixed value with $d \in \mathbb{N}$ and $d > 1$.

As is usual in dynamics, consider the family of functions

$$F = \{f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n} : n \in \mathbb{N}\}$$

for a fixed rational function f.

2.1 Holomorphic Functions on the Riemann Sphere

The Riemann Sphere, $\widehat{\mathbb{C}}$, is a complex manifold with two charts. On the subset $\mathbb{C}$ the chart is $\varphi_1(z) = z$, while on the subset $\widehat{\mathbb{C}} - \{0\}$ it is $\varphi_2(z) = \frac{1}{z}$. A function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is holomorphic (on $\widehat{\mathbb{C}}$) if $\varphi_j \circ f \circ \varphi_k^{-1}$ is holomorphic (on $\mathbb{C}$) where φ_k and φ_j are chosen appropriately. For instance if $f(\infty) \neq \infty$, f is said to be holomorphic at ∞ if $f(\frac{1}{z})$ is holomorphic at 0. A standard exercise in power series expansions is to demonstrate that the only holomorphic functions from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$ are rational.

Definition 2.1. The **degree** of a rational function $f(z) = \frac{P(z)}{Q(z)}$, where P and Q have no common factor, is given by

$$\deg(f) = \max\{\deg(P), \deg(Q)\}.$$

From the references, the Riemann-Hurwitz Formula is known to hold in this case. In other words, the number of critical points of f is $2d - 2$ counted with multiplicity.

Definition 2.2. The **forward orbit** of a point z_0 is $\{f^n(z_0) : n \in \mathbb{N}\}$.

Definition 2.3. A set of points, $\{z_0, z_1, \ldots, z_{n-1}\}$ is a **cycle of period n** if $f(z_j) = z_{j+1}$ for $j = 0, 1, \ldots, n-2$ and $f(z_{n-1}) = z_0$.

Later, the following counting argument will come into play.

Lemma 2.4. *If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a degree $d > 1$ rational function and P is a finite, non-empty set of points with $f(P) = P = f^{-1}(P)$, then P consists of at most two points. Furthermore, f is conjugate to either a polynomial or to $z \to \frac{1}{z^d}$.*

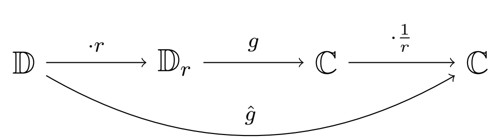
The deep result now called de Branges's Theorem (formerly the Bieberbach Conjecture) will be used to avoid some technical estimates. A quick rescaling will be convenient in this context.

Theorem 2.5 (de Branges)**.** *If $f : \mathbb{D} \to \mathbb{C}$ is injective and holomorphic with $f(z) = z + \sum a_n z^n$, then $|a_n| \leq n$.*

Corollary 2.6. *If $g : \mathbb{D}_r \to \mathbb{C}$ is injective and holomorphic with $f(z) = z + \sum b_n z^n$, then $|b_n| \leq nr^{-n+1}$.*

Proof. Conjugate g to $\hat{g}$ by letting $\hat{g}(z) := \frac{1}{r}g(rz)$; then

$$\hat{g} := \frac{1}{r}(rz + \sum b_n(rz)^n) = z + \sum b_n r^{n-1} z^n$$



Application of de Branges's Theorem to $\hat{g}$ yields the desired estimate. □

2.2 Fatou, Julia, and Classification

Definition 2.7. Let U be an open subset of $\widehat{\mathbb{C}}$. A family, F, of functions $f : U \to \widehat{\mathbb{C}}$ is **normal** on U if any infinite sequence $\{f_n\} \subset F$ has a subsequence that converges locally uniformly (in the spherical metric) to some limit function $g : U \to \widehat{\mathbb{C}}$.

Definition 2.8. The **Fatou set (of f)** is the maximal open set on which the family of functions f^n is a normal family. Its connected components are called **Fatou components (of f)**.

Intuitively, one can think of the Fatou set as the set on which nearby points behave similarly under iteration. Sometimes this is called stability.

Definition 2.9. f has an **attracting fixed point** at z_0 if $f(z_0) = z_0$ and $f'(z_0) \in \mathbb{D} - \{0\}$. f has a **super-attracting fixed point** at z_0 if $f(z_0) = z_0$ and $f'(z_0) = 0$.

Definition 2.10. f has an **indifferent fixed point** at z_0 if $f(z_0) = z_0$ and $f'(z_0) = e^{2\pi i\beta}$ with $\beta \in \mathbb{R}$. The indifferent fixed point z_0 is **rationally indifferent** if $\beta \in \mathbb{Q}$ and **irrationally indifferent** if $\beta \in \mathbb{R}\backslash\mathbb{Q}$.

Definition 2.11. f has an **repelling fixed point** at z_0 if $f(z_0) = z_0$ and $|f'(z_0)| > 1$

Let the set $Z = \{z_0, \dots, z_{n-1}\}$ be a cycle with period n. (This means that $f^k(z_j) \neq z_j$ for $1 < k < n$.)

Definition 2.12. A cycle of period n is **attracting** if $(f^n)'(z_0) \in \mathbb{D} - \{0\}$. It is **super attracting** if $(f^n)'(z_0) = 0$.

Definition 2.13. A cycle of period n is **indifferent** if $(f^n)'(z_0) = e^{2\pi i\beta}$ with $\beta \in \mathbb{R}$. It is **irrationally indifferent** if $\beta \in \mathbb{R}\backslash\mathbb{Q}$ and **rationally indifferent** if $\beta \in \mathbb{Q}$.

Definition 2.14. A cycle of period n is **repelling** if $|(f^n)'(z_0)| > 1$.

Definition 2.15. A cycle or fixed point is **non-repelling** if it is super-attracting, attracting, or indifferent.

The fact that this definition does not depend on the choice of z_0 is a consequence of the chain rule.

Definition 2.16. The **Julia set (of f)** is the complement of the Fatou set.

Intuitively, the Julia set can be thought of as the set on which points display sensitive dependence on initial condition under iteration. This is sometimes referred to as chaos. [SC93]

There will be occasion to utilize the following theorem which can be found in [Ahl53].

Theorem 2.17 (Montel). *Any family F of holomorphic functions $\{f : D \to \widehat{\mathbb{C}}\}$ on a domain $D \subset \widehat{\mathbb{C}}$, that omits three common points is normal.*

A classic result, the Classification Theorem indicates that for an invariant Fatou Component there are only four possible behaviors. Another powerful result is the No Wandering Domains Theorem [Sul85] which indicates that all Fatou components are preperiodic. The combination of these two results yields the conclusion that all the Fatou components are members of cycles of the four types or their preimages. Of these four types the primary focus here will be on the Siegel disk and the attracting basin.

Theorem 2.18 (Classification). *Let f be a rational map of degree $d > 1$, and let U be a Fatou component of f. If $f(U) \subseteq U$, then U is one of:*

- ***Attracting Basin -*** *A domain on which iterates of f converge to a constant function whose image is inside the set. (This will be discussed in further detail in section 2.5.)*

- · ***Petal of a Parabolic Point -*** *A domain on which iterates of f converge to a constant function whose image is on the boundary. (This case will be ignored completely in this text.)*

- · ***Siegel disk -*** *A simply connected rotation domain, i.e. a region on which the action of f is conjugate to irrational rotation. (This will be discussed in detail in section 2.4.)*

- · ***Herman ring -*** *A topological annulus on which f is conjugate to irrational rotation. (This case will be ignored completely in this text.)*

Theorem 2.19 (No Wandering Domains)**.** *If f is a rational map of degree ≥ 2 then every Fatou component of f is pre-periodic.*

Theorem 2.20 (Fatou-Shishikura Inequality)**.** *The number of non-repelling cycles of a rational function f is less than or equal to the number of critical points of f.*

2.3 A Notion of Radius

There will be a need at some point for a notion of radius to ensure that a family of Siegel disks does not degenerate. What 'degenerates' means in this context is not particularly obvious. In the interest of being clear without precision, this picture illustrates degenerating Siegel disks.

To this end, recall the Riemann Mapping theorem, which can be found in [Ahl53].

Theorem 2.21. *For any non-empty, simply connected, open, and properly contained subset D of $\mathbb{C}$, there exists a conformal homeomorphism $\phi : D \to \mathbb{D}$. The map ϕ is called a* ***Riemann map****. Further for any fixed $z \in D$, the map ϕ is uniquely determined by requiring that $\phi(z) = 0$, and $\arg(\phi'(0)) = 0$. The map ϕ with these properties is called the* ***normalized Riemann map at z****.*

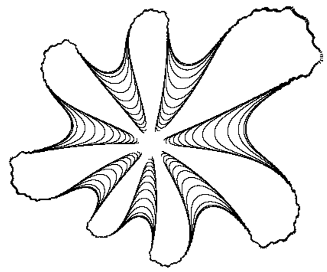


Figure 2.1: Degenerating Siegel Disks

Definition 2.22. For any non-empty, simply connected, open, and proper subset D of $\mathbb{C}$ and $z \in D$, the **conformal radius** of D at z is given by $\frac{1}{\phi'(z)}$ where ϕ is the normalized Riemann map.

Notice, that for any non-empty, simply connected, open, and proper subset D of $\mathbb{C}$ and $z \in D$, the conformal radius of D at z is given by $(\phi^{-1})'(0)$ where ϕ is the normalized Riemann map. This last lemma will be the most commonly applied notion of conformal radius.

Lemma 2.23. *The conformal radius of a non-empty, simply connected, open, and proper subset D of $\mathbb{C}$ at $z \in D$ is given by the maximal δ such that there exists a bijective, conformal map $h : \mathbb{D}_\delta \to D$ with $h(0) = z$, $h'(0) = 1$.*

Proof. A conformal mapping as above is given by $\phi^{-1}(\frac{z}{(\phi^{-1})'(0)})$ where again ϕ is the normalized Riemann map. Application of the Schwarz Lemma demonstrates that this is the largest possible δ. □

The classic Koebe $\frac{1}{4}$ Theorem is due to Bieberbach [Bie16].

Theorem 2.24 (Koebe $\frac{1}{4}$)**.** *If f is an injective analytic function from $\mathbb{D}$ to $\mathbb{C}$ with $f(0) = 0$, $f'(0) = 1$ then $f(\mathbb{D})$ contains the disk about zero of radius $\frac{1}{4}$.*

Corollary 2.25. *If f is an injective analytic function from $\mathbb{D}_r$ to $\mathbb{C}$ with $f(0) = 0$ and $|f'(0)| > k_0$ for some $k_0 > 0$, then $\mathbb{D}_{\frac{k_0 r}{4}} \subset f(\mathbb{D}_r)$*

Proof. Let $k = f'(0)$. Apply the Koebe $\frac{1}{4}$ Theorem to the rescaling $g(z) = \frac{1}{kr} f(rz)$. $\square$

Definition 2.26. Let D be a domain. For a fixed $z_0 \in D$, the **inner radius** of D at z is the radius of the largest open disc centered at z_0 contained in D.

Lemma 2.27. *Let D be a domain and let $z_0 \in D$. Let r be the inner radius of D at z_0 and R be the conformal radius of D at z_0. Then*

$$r \leq R \leq 4r$$

Proof. Without loss of generality assume $z_0 = 0$. From Lemma 2.23, there is a conformal $h : \mathbb{D}_R \to D$ with $h(0) = 0$ and $h'(0) = 1$. From Corollary 2.25, $\mathbb{D}_{\frac{R}{4}} \subset h(\mathbb{D}_R) = D$. So

$$r \geq \frac{R}{4}.$$

Let ϕ be the normalized Riemann map at 0. Then $\phi|_{\mathbb{D}_r} : \mathbb{D}_r \to \mathbb{D}$ with $\phi(0) = 0$, so with a rescaling and the Schwarz Lemma, $\phi'(0) \leq \frac{1}{r}$. Therefore,

$$R = \frac{1}{\phi'(0)} \geq r.$$

$\square$

2.4 Linearizability

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function with an irrationally indifferent fixed point at z_0. Throughout this section, assume that f is given by a power series of the form

$$f(z) = z_0 + \lambda(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + ...,$$

where $\lambda = e^{2\pi i\alpha}$, with $\alpha \in \mathbb{R}\backslash\mathbb{Q}$.

Definition 2.28. For the fixed point z_0 and the map f as above, α is the **rotation number**.

Definition 2.29. A rational function f of degree d is said to be **linearizable** on a domain D containing z_0 if there is an $r > 0$ and a conformal function $h : \mathbb{D}_r \to D$ of the form $h(w) = z_0 + w + O(w^2)$ such that for all $w \in \mathbb{D}_r$

$$f(h(w)) = h(\lambda w).$$

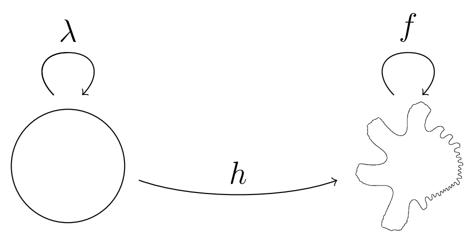


Figure 2.2: f is linearizable.

Definition 2.30. A maximal domain in $\widehat{\mathbb{C}}$ on which f is linearizable is called a **Siegel disk**. A **Siegel cycle** is a maximal open set in $\widehat{\mathbb{C}}$ whose components are a periodic cycle of period $n \in \mathbb{N}$, with f^n linearizable on each component. The image of 0 under the linearizing map is called a **Siegel point**.

Lemma 2.31. *If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map with $f(0) = 0$, $f'(0) = e^{2\pi i\alpha}$ for α irrational, and D is a bounded simply connected open neighborhood of 0 which is invariant under f, then f is linearizable on D.*

Proof. Since D is bounded and invariant, the set $f^n(D) = D$ must omit three points, so Montel's Theorem gives that D is in the Fatou set. Then the Classification Theorem indicates that D must be contained in a Siegel disk, as it contains a fixed point with irrational rotation number. □

The following lemma will demonstrate that any holomorphic function which conjugates f to an irrational rotation is in fact a linearization. This will be utilized to obtain a linearization from a formal conjugacy.

Lemma 2.32. *If $f(z) = \lambda z + O(z^2)$ and $h(z) = z + O(z^2)$ are holomorphic with $f(h(z)) = h(\lambda z)$ for $z \in \mathbb{D}_r$ with $r > 0$, then $h(z)$ is conformal.*

Proof. Since h is holomorphic, all that needs to be shown is that h is injective. Choose $r_1 < r$. Because the series for $h(z)$ converges on $\mathbb{D}_r$, it converges on $\overline{\mathbb{D}}_{r_1}$. So $h(\overline{\mathbb{D}}_{r_1})$ is compact, hence bounded. Further $h(\mathbb{D}_{r_1})$ is also bounded and must be open by the open mapping theorem. Now, because

$$f(h(\mathbb{D}_{r_1})) = h(\lambda\mathbb{D}_{r_1}) = h(\mathbb{D}_{r_1}),$$

Lemma 2.31 implies $h(\mathbb{D}_{r_1})$ is an f-invariant subset of a Siegel disk.

Notice that h is conformal on some neighborhood of 0 because $h'(0) = 1$. Then, on this neighborhood of 0, h is a linearizing map. The Schwarz Lemma implies that the linearizing map is unique up to rotation, therefore from the Identity Theorem, h must be conformal because the linearizing map is conformal.

□

2.5 Cycle Phenomena

Some results about local behavior near attracting fixed points and cycles will be needed to ensure the success of surgery techniques. They can be found in any complex dynamics text (for example [Mil06], [CG93]), but are presented here because of the resemblance to certain more involved techniques employed further on.

Lemma 2.33. *If f is a map with an (super) attracting fixed point, then there exist $r > \rho > 0$ with $f(\overline{\mathbb{D}}_r) \subset \mathbb{D}_\rho$.*

Proof. Without loss of generality, let 0 be a fixed point of the map f, with $f'(0) = \lambda$ and $|\lambda| < 1$. Then,

$$|\lambda| = \lim_{z \to 0} \frac{|f(z)|}{|z|}.$$

So there exists r such that for $z \in \overline{\mathbb{D}}_r$ implies $\frac{|f(z)|}{|z|} < \frac{1+|\lambda|}{2}$. Therefore,

$$|f(z)| < \frac{1+|\lambda|}{2}|z| \leq \frac{1+|\lambda|}{2}r.$$

Then, with $\rho := \frac{1+|\lambda|}{2}r$,

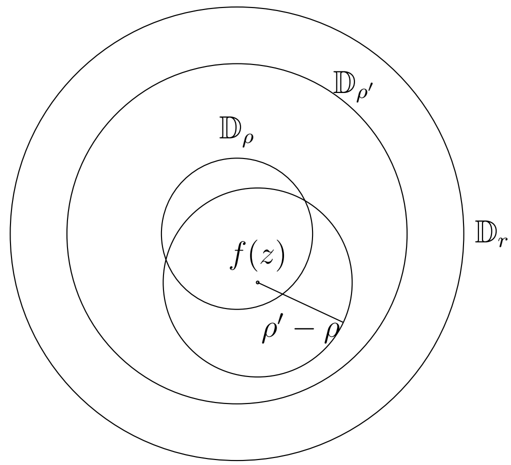
$$f(\overline{\mathbb{D}}_r) \subset \mathbb{D}_\rho.$$

□

A similar result for families will prove to be necessary.

Lemma 2.34. *If f is a map with an (super) attracting fixed point, and f_ϵ is a family which converges locally uniformly to f as $\epsilon \to 0$, then there exist $\epsilon_0 > 0$ and $r > \rho' > 0$ with $f_\epsilon(\overline{\mathbb{D}}_r) \subset \mathbb{D}_{\rho'}$ for all $|\epsilon| < \epsilon_0$.*

Proof. It is known that $f(\overline{\mathbb{D}}_r) \subset \mathbb{D}_\rho$. Now, let $\rho' \in (\rho, r)$. Then because $f_\epsilon \to f$ locally uniformly, $|f_\epsilon(z) - f(z)|$ can be made as small as needed on any compact set, for instance $\overline{\mathbb{D}}_r$. Choose ϵ_0 such that $|f_\epsilon(z) - f(z)| < \rho' - \rho$ for all $\epsilon < \epsilon_0$ and $z \in \overline{\mathbb{D}}_r$



$$|f_\epsilon(z)| = |f_\epsilon(z) - f(z) + f(z)| \leq |f_\epsilon(z) - f(z)| + |f(z)| \leq \rho' - \rho + \rho = \rho'$$

then observe that $f_\epsilon(\overline{\mathbb{D}}_r) \subset \mathbb{D}_{\rho'}$. □

Obviously, it would be nice to have a similar result for cycles. Here, the symbol "$\Subset$" means "compactly contained".

Lemma 2.35. *If f_ϵ is a family of maps which converges locally uniformly to f and f has an (super) attracting cycle, then for all sufficiently small ϵ, there exists an open set E with $f_\epsilon(E) \Subset E$.*

Proof. Without loss of generality, assume that 0 is in the cycle. Passing to the first iterate f^n for which 0 is fixed, there exists r and ρ with $r > \rho > 0$ and $f_\epsilon^n(\mathbb{D}_r) \Subset \mathbb{D}_\rho$ (Lemma 2.34). Let D_j be a nested sequence of discs with

$$\mathbb{D}_\rho = D_1 \Subset D_2 \Subset \cdots \Subset D_{n-1} \Subset D_n = \mathbb{D}_r.$$

Define, for $j = 1, \ldots, n$

$$E_j = f_\epsilon^j(D_j).$$

Then $f_\epsilon^n(\mathbb{D}_r) \Subset \mathbb{D}_\rho$ implies

$$f_\epsilon(E_n) = f_\epsilon(f_\epsilon^n(D_n)) = f_\epsilon(f_\epsilon^n(\mathbb{D}_r)) \Subset f_\epsilon(\mathbb{D}_\rho) = f_\epsilon(D_1) = E_1$$

and $D_j \Subset D_{j+1}$ for $j = 1, \ldots, n-1$ implies

$$f_\epsilon(E_j) = f_\epsilon(f_\epsilon^j(D_j) = f_\epsilon^{j+1}(D_j) \Subset f_\epsilon^{j+1}(D_{j+1}) = E_{j+1}.$$

Let $E := \bigcup_{j=1}^n E_j$. Now, $f_\epsilon(E) \Subset E$. (Recalling that E must be open from the Open Mapping Theorem.) □

Naishul's theorem, stated and reproven in this context by Perez Marco in [PM97], demonstrates that the multipliers of indifferent cycles are preserved by homeomorphic conjugacy. Let f and g be rational maps.

Theorem 2.36 (Naishul)**.** *If $f = \phi^{-1} \circ g \circ \phi$ with ϕ an orientation preserving homeomorphism, $f(z_0) = z_0$, and $f'(z_o) = e^{2\pi i\beta}$, then $g'(\phi(z_0)) = e^{2\pi i\beta}$.*

2.6 Perturbation Families

The perturbation families presented in this section can be found in their original form in [Gey]. In this text, they are presented with slightly modified notation. Fix $\lambda = e^{2\pi i\alpha}$ with $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. The subject of discussion will be a family of maps given locally by

$$f_\epsilon(z) = \lambda z + \sum_{k=2}^{\infty} g_k(\epsilon) z^k$$

where $g_k(\epsilon)$ are polynomials in $\epsilon \in \mathbb{C}$. Definitions 2.37, 2.38 and 2.39 will in general be applied to maps which are not holomorphic on $\widehat{\mathbb{C}}$, but are in a neighborhood of 0.

Definition 2.37. A family of maps $\{f_\epsilon\}$ is said to be **uniformly linearizable** at 0 for $|\epsilon| \leq \hat{\epsilon}$ if there exists $\delta > 0$ and a conformal family of maps $h_\epsilon(z) = z + O(z^2)$ such that $f_\epsilon(h_\epsilon(z)) = h_\epsilon(\lambda z)$ for $|\epsilon| \leq \hat{\epsilon}$ and $|z| < \delta$.

In other words, a uniformly linearizable family is a family with Siegel disks (all with the same rotation number) whose conformal radii are uniformly bounded away from zero.

Definition 2.38. f_ϵ is an **essentially quadratic** family at 0 if $f_\epsilon(z) = \sum_{n=0}^{\infty} g_n(\epsilon) z^n$ with $g_n(\epsilon)$ polynomials with $\deg(g_0) = 0,\ \deg(g_1) = 0,\ \deg(g_2) = 1$, and $\deg(g_n) < n - 1$ for $n > 2$.

Perhaps the simplest example of an essentially quadratic family is $f_\epsilon(z) = \lambda z + \epsilon z^2$.

Definition 2.39. f_ϵ is an **subquadratic** family at 0 if $f_\epsilon(z) = \sum_{n=0}^{\infty} g_n(\epsilon) z^n$ with $g_n(\epsilon)$ polynomials with $\deg(g_0) = 0,\ \deg(g_1) = 0,\ \deg(g_2) = 0$, and $\deg(g_n) < n - 1$ for $n > 2$.

Perhaps the simplest example of a subquadratic family is $f_\epsilon(z) = \lambda z + \epsilon z^3$.

The next lemma, due to Geyer [Gey], will be utilized later to pass from a cycle to a fixed point. Composing families $\{f_\epsilon\}_{\epsilon \in \mathbb{D}_r}$ and $\{g_\epsilon\}_{\epsilon \in \mathbb{D}_\rho}$ will mean $\{f_\epsilon \circ g_\epsilon\}_{\epsilon \in \mathbb{D}_r \cap \mathbb{D}_\rho}$.

Lemma 2.40 (Geyer)**.** *Composition of an essentially quadratic family and a subquadratic family is an essentially quadratic family.*

The following is a generalization due to Geyer [Gey] of a lemma due to Pérez-Marco [PM97]. The proof is provided because of its interesting technique and the illustration it provides.

Lemma 2.41 (Geyer)**.** *If an essentially quadratic family $f_\epsilon(z) = \lambda z + \sum_{k=2}^{\infty} f_k(\epsilon) z^k$ is uniformly linearizable for $|\epsilon| \leq \epsilon_0$ and $|z| < \delta_0$, then the quadratic polynomial $P(z) = \lambda z + z^2$ is linearizable for $|z| < \delta_1(\delta_0)$.*

Proof. The idea behind the proof is to start with f_ϵ, flip parameter space over to get F_η and pull the linearization down with the Maximum Principle to P_λ. In spirit when $\eta \to 0$, $\epsilon \to \infty$.

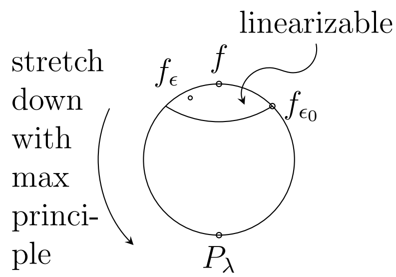


Figure 2.3

Let F_η and g_k be such that

$$\begin{aligned}
F_\eta(z) &= \frac{1}{\eta} f_{\frac{1}{\eta}}(\eta z) \\
&= \lambda z + \sum_{k=2}^{\infty} \frac{1}{\eta} f_k(\frac{1}{\eta})(\eta z)^k \\
&= \lambda z + \sum_{k=2}^{\infty} \eta^{k-1} f_k\left(\frac{1}{\eta}\right) z^k \\
&= \lambda z + \sum_{k=2}^{\infty} g_k(\eta) z^k.
\end{aligned}$$

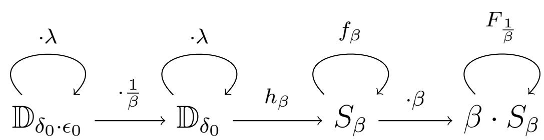
Notice that $F_0(z) = \lambda z + z^2$. Because f_ϵ are essentially quadratic, g_k are polynomials with $g_2(0) \neq 0$, and $g_k(0) = 0$ for $k > 2$. Let $H_\eta(z)$ be a formal linearizing map, a formal power series solution to

$$F_\eta(H_\eta(z)) = H_\eta(\lambda z).$$

Assume H_η is given by

$$H_\eta(z) = z + \sum_{k=2}^{\infty} H_k(\eta) z^k.$$

By induction it can be seen that the $H_k(\eta)$ are polynomial (hence analytic) in η. Now, it is necessary to show that $H_\eta(z)$ converges for $\eta = 0$. Let $\beta \in \mathbb{C}$ with $|\beta| = \epsilon_0$. H_η converges for $|\eta| = \frac{1}{\epsilon_0}$ on $|z| < \delta_1 = \delta_0 \cdot \epsilon_0$ because $F_{\frac{1}{\beta}}$ is conjugate to f_β by $z \to \beta z$.



The map $H_{\frac{1}{\beta}}$ is injective by Lemma 2.32. By de Branges's Theorem, $|H_k(\beta)| \leq k\delta_1^{-k+1}$. Because the H_k are polynomial in η, application of the Maximum Principle yields $|H_k(0)| \leq k\delta_1^{-k+1}$. Finally, by an easy geometric series argument, H_η converges for $\eta = 0$ on $|z| < \delta_1$. Since $F_0(z) = \lambda z + z^2$, the quadratic polynomial is linearizable.

□

2.7 Known Results

Milnor [Mil06] summarizes the known main known results about rotation numbers with the following theorems.

Definition 2.42. A **holomorphic germ** is a power series at a point that converges on some open neighborhood of that point.

Theorem 2.43 (Cremer, [Cre28])**.** *For a topologically generic choice of $\alpha \in S^1$, any rational map of degree 2 or more with rotation number α is not linearizable.*

In contrast the following is known:

Theorem 2.44 (Siegel, [Sie42])**.** *There is a set of full measure in S^1 so that for α in this set, the holomorphic germ $f(z) = e^{2\pi i\alpha}z + a_2z^2 + ...$ is linearizable.*

Definition 2.45. The **continued fraction expansion** of an irrational number α, is given by

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

where $a_j \in \mathbb{N}$.

Definition 2.46. The **n-th convergent** to α is (in reduced form)

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

Definition 2.47. A (rotation) number α is said to be a **Brjuno number** if:

$$\sum_n \frac{\log(q_{n+1})}{q_n} < \infty$$

where q_n is the denominator of the n-th convergent to α.

Theorem 2.48 (Brjuno [Brj65], Rüssman[R̈67]). *If α is Brjuno then any holomorphic germ with rotation number α is linearizable.*

Theorem 2.49 (Yoccoz [Yoc95]). *If a number α is not Brjuno, then the quadratic map $f(z) = e^{2\pi i\alpha}z + z^2$ has a fixed point at 0 which is not linearizable.*

From the combined work of Cremer, Siegel, Rüssman, Brjuno, and Yoccoz, there is good reason to believe that the following conjecture holds. In fact, the main result is that this holds under certain conditions. It was first conjectured by Douady in a slightly different form in [Dou87].

Conjecture 2.50 (The Douady Conjecture)**.** A rational function of degree $d > 1$ has a Siegel disk (or Siegel cycle), if and only if the rotation number is Brjuno.

There have already been some inroads made in proving this conjecture, some of which are paraphrased below.

Theorem 2.51 (Perez-Marco [PM93])**.** *Let $\lambda = e^{2\pi i\beta}$, $\beta \in \mathbb{R}\backslash\mathbb{Q}$, and β not a Brjuno number. Then the set of non-linearizable polynomials is topologically generic in*

$$\{P(z) = \lambda z + a_2 z^2 + \cdots + a_d z^d : a_2, \ldots, a_d \in \mathbb{C}\}.$$

Theorem 2.52 (Geyer [Gey01])**.** *Let $\lambda = e^{2\pi i\alpha}$ and $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Then the function $E_\lambda(z) = \lambda z e^z$ is linearizable if and only if α is Brjuno.*

Theorem 2.53 (Geyer [Gey01])**.** *Let $\lambda = e^{2\pi i\alpha}$, $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, and $d > 1$. Then the polynomial $P_{\lambda,d} = \lambda z(1 + z/d)^d$ is linearizable if and only if α is Brjuno.*

Definition 2.54. The polynomial P is **saturated** if the number of infinite orbits in the Julia set of P equals the number of irrationally indifferent cycles of P.

Theorem 2.55 (Geyer [Gey])**.** *Let P be a saturated polynomial of degree $d > 1$ with connected Julia set. Then every Siegel cycle has Brjuno rotation number.*

Pérez-Marco showed in [PM01] he showed that the conjecture holds outside of a pluripolar subset of the same parameter space.

Definition 2.56. A real number α is of **high type** if

$$\alpha \in \mathrm{HT}_N := \{[0; a_1, a_2, \ldots] : a_1, a_2, \cdots \geq N\}$$

for some $N \in \mathbb{N}$.

Theorem 2.57 (Cheraghi [Che10])**.** *There is an $N \in \mathbb{N}$ such that for β non-Brjuno in HT_N no rational function with $f(0) = 0$ and $f'(0) = e^{2\pi i\beta}$ is linearizable.*

Quasiconformal Surgery

This section will dwell on the application of quasiconformal surgery to this problem. Lemmas 3.13 and 3.15 are due to Shishikura [Shi87]. The specific versions of Theorems 3.8 and 3.11 appear in Bodil Branner and Nuria Fagella's excellent new book [BF14]. Lemma 3.16 does not appear in the literature.

3.1 Quasiconformal Mappings

This will be a necessarily short introduction to quasiconformal mappings, intended to provide only the minimum working material and impart intuition. There are several good resources for the full depth of the subject available. A great introduction is in [MS07] or chapter one of [BF14]. The classic text is [Ahl66].

Definition 3.1. The symbol ∂ indicates the operator $\frac{\partial}{\partial z}$.

$$\partial f := \frac{\partial f}{\partial z} = \frac{f_x - i f_y}{2}$$

Definition 3.2. The symbol $\overline{\partial}$ indicates the operator $\frac{\partial}{\partial \overline{z}}$.

$$\overline{\partial} f := \frac{\partial f}{\partial \overline{z}} = \frac{f_x + i f_y}{2}$$

If unfamiliar with this notation, one should note that the statement $\overline{\partial} f = 0$ is equivalent to the Cauchy-Riemann equations.

Definition 3.3. The **dilatation** of a real differentiable map is

$$\mu_f = \frac{\overline{\partial} f}{\partial f}$$

A conformal map g therefore has $\mu_g \equiv 0$.

Definition 3.4. Given a pair of domains $U, V \subseteq \widehat{\mathbb{C}}$, a homeomorphism $f : U \to V$ is **K-quasiconformal** if it is absolutely continuous on almost all lines parallel to either the x or y axis and $|\mu_f(z)| \leq \frac{K-1}{K+1}$ almost everywhere.

Conformal maps are 1-quasiconformal.

Definition 3.5. A homeomorphism which is K-quasiconformal for some K is said to be **quasiconformal**.

Definition 3.6. A function g is **quasiregular** if there is a rational function f and a quasiconformal map ϕ so that

$$g = f \circ \phi.$$

Definition 3.7. A complex valued function μ on $\widehat{\mathbb{C}}$ is a **Beltrami Coefficient** if it is Lebesgue measurable with $||\mu||_\infty < 1$.

Theorem 3.8 (Weyl's Lemma)**.** *If ϕ is quasiconformal with $\mu_\phi = 0$ a.e., then ϕ is conformal.*

Definition 3.9. The **pullback** of a Beltrami coefficient μ by a function f (f must be absolutely continuous with respect to Lebesgue measure and a.e. real differentiable) is given by

$$f^*(\mu)(u) = \frac{\partial_{\bar{z}} g(u) + \mu(g(u))\overline{\partial_z g(u)}}{\partial_z g(u) + \mu(g(u))\overline{\partial_{\bar{z}} g(u)}}.$$

The function $g = \varphi \circ f \circ \varphi$ for $\varphi(z) = \begin{cases} z & : z \neq \infty \\ \frac{1}{z} & : z = \infty \end{cases}$

In the case of a holomorphic f (and $u, f(u) \neq \infty$) this simplifies to

$$f^*(\mu)(u) = \mu(f(u))\frac{\overline{\partial_z f(u)}}{\partial_z f(u)}.$$

This shows that the pullback by a holomorphic function does not increase the L_∞ norm of a Beltrami coefficient.

The following results are stated in this specific form in [BF14].

Theorem 3.10 (Measurable Riemann Mapping Theorem)**.** *For any Beltrami coefficient μ there exists a unique quasiconformal mapping $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $\mu_f = \mu$ a.e. fixing 0,1,∞. A quasiconformal mapping fixing $0, 1, \infty$ is **normalized**.*

Theorem 3.11 (Parameterized Measurable Mapping Theorem)**.** *Suppose $\{\mu_\epsilon\}_{\epsilon \in \mathbb{D}_r}$ is a family of Beltrami coefficients on $\widehat{\mathbb{C}}$, and $\epsilon \to \mu_\epsilon(z)$ is continuous for each fixed $z \in \widehat{\mathbb{C}}$. Assume there is a uniform $k < 1$ with $||\mu_\epsilon|| \leq k$ for all $|\epsilon| < r$. Let ϕ_ϵ be the normalized quasiconformal mapping given by Theorem 3.10. Then for any fixed $z \in \widehat{\mathbb{C}}$ the map $\epsilon \to \phi_\epsilon(z)$ is continuous.*

Theorem 3.12. *Let $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a quasiregular map. Let μ be a Beltrami Coefficient such that $g^*(\mu) = \mu$, then there exists a rational function f and a quasiconformal mapping φ such that*

$$\varphi \circ g \circ \varphi^{-1} = f.$$

3.2 The Cartoon Lemma

The following theorem is a corollary to Shishikura's Fundamental Lemma on Quasiconformal Surgery which can be found in [Shi87]. A proof of this lemma is included, not because of its difficulty or originality but because it allows for a clearer exposition of the surgery performed in the 'Example' and 'Main Result' sections to follow. It also meshes nicely with computer generated pictures, allowing for an explanatory cartoon. For this reason, it will be referred to as the "Cartoon Lemma".

Lemma 3.13 (The Cartoon Lemma)**.** *Let g_ϵ be a quasiregular map and E be an open subset of $\widehat{\mathbb{C}}$ with*

- $g_\epsilon(E) \subset E$
- *g_ϵ analytic on $E \cup \widehat{\mathbb{C}} \backslash g_\epsilon^{-1}(E)$*

then there exists a quasiconformal mapping φ_ϵ *of* $\widehat{\mathbb{C}}$ *such that* $\varphi_\epsilon \circ g_\epsilon \circ \varphi_\epsilon^{-1}$ *is rational. Moreover* φ_ϵ *is conformal on*

$$E \cup \widehat{\mathbb{C}} \backslash \overline{\bigcup_{n \geq 1}^{\infty} g_\epsilon^{-n}(E)}.$$

The proof presented here, which is modeled after that found in [MS07], will be split into two main components. First, it will be shown that such a map has an invariant Beltrami coefficient σ, then application of the Measurable Riemann Mapping Theorem to σ will conclude the proof.

Proof. The aim is to define a Beltrami Coefficient σ on $\widehat{\mathbb{C}}$ so that $g_\epsilon^* \sigma = \sigma$. Set $\sigma = 0$ on E. Next pull σ back by g_ϵ on $g_\epsilon^{-1}(E)$. This makes sense because for $z \in E \cap g_\epsilon^{-1}(E)$, g_ϵ is analytic, so $g_\epsilon^* \sigma(z) = 0 = \sigma(z)$. Assign $\sigma = g_\epsilon^* \sigma$ on $g_\epsilon^{-1}(E) \backslash E$. Continuing inductively, extend σ to all of $\bigcup_{n \geq 1} g_\epsilon^{-n}(E)$ by repeatedly pulling back by g_ϵ. Define $\sigma = 0$ on the complement.

Now, because σ is a Beltrami coefficient on E, and g_ϵ is quasiregular σ is a Beltrami Coefficient on $g_\epsilon^{-1}(E)$. At this point the concern is that repeated pullbacks may have inflated the L_∞ norm of σ. In other words, σ might be a complex valued function, but not a Beltrami Coefficient. Utilizing the fact that $\overline{\partial} g_\epsilon = 0$ on $\widehat{\mathbb{C}} \backslash g_\epsilon^{-1}(E)$ and the chain rule for differentials

$$\mu_{h \circ f}(z) = (\mu_h \circ f(z)) \cdot \frac{\overline{\partial f(z)}}{\partial f(z)}$$

and taking absolute values see

$$|\mu_{h \circ f}(z)| = |\mu_h \circ f(z)|.$$

This means σ as constructed by repeated pullback by g_ϵ has $||\sigma||_\infty < 1$. In other words, σ is a Beltrami Coefficient. By construction σ also has $g_\epsilon^*(\sigma) = \sigma$. From Theorem 3.12, there is a quasiconformal map φ_ϵ with $\varphi_\epsilon \circ g_\epsilon \circ \varphi_\epsilon^{-1}$ rational. Moreover, because $\sigma = 0$ on $E \cup \widehat{\mathbb{C}} \setminus \bigcup_{n\geq 1} g_\epsilon^{-n}(E)$, by Theorem 3.8 φ_ϵ is analytic on

$$E \cup \widehat{\mathbb{C}} \setminus \overline{\bigcup_{n\geq 1}^{\infty} g_\epsilon^{-n}(E)}.$$

□

Figure 3.1: cartoon of a quadratic rational map with attracting cycle

For illustrative purposes consider the above cartoon of a quadratic map with an attracting fixed point. Each cone represents a component of the inverse image of the attractive basin. E is taken to be a neighborhood of the attracting fixed point. In E define σ to be 0, following the arrows backwards, define σ to be the pullback by g_ϵ.

3.3 Shishikura's Lemma

Before proceeding, the following classification of quasiregular mappings will be needed. This form appears in [BF14].

Theorem 3.14. *A continuous mapping* $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *is quasiregular if and only if* g *is locally quasiconformal except at a discrete set of points.*

The following lemma due to Shishikura has been modified slightly to ease exposition. The change is in the argument of the function ρ. In the original, Shishikura includes a dependence on ϵ. Since parabolic points will not be considered, this dependence will not be necessary in this context.

Lemma 3.15 (Shishikura)**.** *Let* $h(z)$ *be a polynomial and let* $R > 0$*. Define* H_ϵ : $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *for* $\epsilon \in \mathbb{C}$*, by*

$$H_\epsilon(z) = z + \epsilon \cdot h(z) \cdot \rho(\tfrac{|z|}{R}) \text{ for } z \in \mathbb{C},$$
$$H_\epsilon(\infty) = \infty$$

where ρ *is a* C^∞*-function on* $\mathbb{R}$ *such that* $0 \leq \rho \leq 1$*,* $\rho = 1$ *on* $[0, 1]$*, and* $\rho = 0$ *on* $[2, \infty)$*. Then for small* $|\epsilon|$*,* H_ϵ *is quasiconformal. Furthermore,* $H_\epsilon \to id_{\widehat{\mathbb{C}}}$ *uniformly (with respect to the metric on* $\widehat{\mathbb{C}}$*) and* $||\mu_{H_\epsilon}||_\infty \to 0$*, when* $\epsilon \to 0$*.*

Proof. Let $A_{R,2R} := \{z \in \mathbb{C} : R \leq |z| \leq 2R\}$. Let $\phi(z) := h(z) \cdot \rho(\frac{|z|}{R})$, then by compactness, for $z \in A_{R,2R}$ there exists M with:

$$|\partial\phi(z)| \leq M \text{ and } |\overline{\partial}\phi(z)| \leq M.$$

Then because $H_\epsilon(z) = z + \epsilon\phi(z)$,

$$\overline{\partial} H_\epsilon(z) = \epsilon \overline{\partial}\phi(z)$$

$$\partial H_\epsilon(z) = 1 + \epsilon\partial\phi(z).$$

Therefore,

$$|\mu_{H_\epsilon}| = \left|\frac{\overline{\partial} H_\epsilon}{\partial H_\epsilon}\right| \leq \frac{|\epsilon| M}{1 - |\epsilon| M}.$$

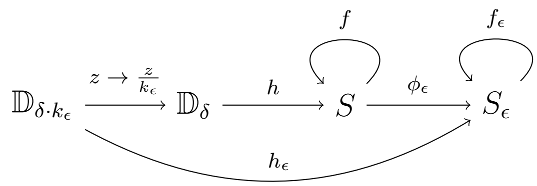
Choosing $|\epsilon| < \frac{1}{3M}$ see $|\mu_{H_\epsilon}| < \frac{1}{2}$ i.e. H_ϵ is locally quasiconformal. Theorem 3.14 implies that it must be quasiregular, so $H_\epsilon = g_\epsilon \circ \phi_\epsilon$ with g_ϵ rational, ϕ_ϵ quasiconformal. At (or near) ∞, H_ϵ is the identity, which is degree one. Therefore g_ϵ must be degree one. In addition, H_ϵ fixes ∞, so g_ϵ must be a polynomial of degree one, i.e. linear, therefore H_ϵ is a homeomorphism, hence quasiconformal. Convergence to the identity is obvious. □

Lemmas 3.13 and 3.15 suggest a scheme. If $f_\epsilon(z) = f(H_\epsilon(z))$ with H_ϵ as in Lemma 3.15, also satifies the hypothesis of the Cartoon Lemma (Lemma 3.13), then f_ϵ is quasiconformally conjugate to a rational map r_ϵ by quasiconformal maps φ_ϵ converging to the identity as $\epsilon \to 0$.

3.4 Sufficient Conditions for Uniform Linearizability

Lemma 3.16. *If $\{f_\epsilon\}_{|\epsilon| \leq \hat{\epsilon}}$ is a family of quasiregular maps such that f_ϵ is conformally conjugate to f on S by ϕ_ϵ with $|\phi'_\epsilon(0)| \geq \frac{1}{2}$, then the family $\{f_\epsilon\}$ is uniformly linearizable.*

Proof. Set $|\phi'_\epsilon(0)| = k_\epsilon$. Let $h(z) = z + O(z^2)$ be a normalized linearizing map for f on S. Also let δ be the conformal radius of S.



One quickly observes that normalized linearizing maps for f_ϵ on S_ϵ are given by

$$\phi_\epsilon \circ h \left(\frac{z}{k_\epsilon} \right).$$

Therefore, $\{f_\epsilon\}$ is uniformly linearizable, because these maps all act on disks of radius $\delta \cdot k_\epsilon \geq \frac{\delta}{2}$. In fact, this is equivalent to the observation that the conformal radius of S_ϵ is bounded below. $\square$

What it Means to Share Dynamics

The tools in the first section are standard to Algebraic Geometry, while those in the second section are new. For a standard introduction to Algebraic Geometry, see [Sha74].

4.1 Algebraic Geometry

Remember that $d > 1$ is a fixed degree. Let Rat_d be the space of rational functions of degree d. Recall that a degree d rational function f corresponds to a point in $\mathbb{CP}^{2d+1}$ by the relationship:

$$\mathrm{Rat}_d \ni f = \frac{a_d z^d + \cdots + a_0}{b^d z^d + \cdots + b_0} \sim [a_d : \cdots : a_0 : b_d : \cdots : b_0] = ((a_k), (b_k)) \in \mathbb{CP}^{2d+1}.$$

Definition 4.1. The **resultant** of a point $((a_k), (b_k)) \in \mathbb{CP}^{2d+1}$ is given by:

$$\mathrm{Res}((a_k), (b_k)) = \det \begin{bmatrix} a_d & a_{d-1} & \dots & a_0 & 0 & \dots & 0 \\ 0 & a_d & \dots & a_1 & a_0 & \dots & 0 \\ \vdots & & & \vdots & & & \vdots \\ 0 & \dots & 0 & a_d & a_{d-1} & \dots & a_0 \\ b_d & b_{d-1} & \dots & b_0 & 0 & \dots & 0 \\ 0 & b_d & \dots & b_1 & b_0 & \dots & 0 \\ \vdots & & & \vdots & & & \vdots \\ 0 & \dots & 0 & b_d & b_{d-1} & \dots & b_0 \end{bmatrix}$$

The space Rat_d is a subset of $\mathbb{CP}^{2d+1}$, in fact $\mathrm{Rat}_d \sim \{x \in \mathbb{CP}^{2d+1} : \mathrm{Res}(x) \neq 0\}$.

Definition 4.2. A **projective variety** is the zero set of some finite set of polynomial equations on projective space.

Definition 4.3. The **Zariski topology** on $\mathbb{CP}^{2d+1}$ is the topology in which the closed sets are the projective varieties.

Definition 4.4. A **quasiprojective variety** is a relatively open subset of a projective variety.

For example, in $Rat_2 \subset \mathbb{CP}^3$,

$$\left\{ f(z) = \frac{a_2 z^2 + a_1 z + a_0}{b_2 z^2 + b_1 z + b_0} : a_1 b_0 - a_0 b_1 = 0 \right\}$$

is the quasiprojective variety corresponding to those maps with a critical point at the origin.

Definition 4.5. A quasiprojective variety X is **irreducible** if for any quasiprojective varieties X_1 and X_2 with $X = X_1 \cup X_2$ either $X_1 = X$ or $X_2 = X$.

From [Sha74] it is known that any quasiprojective variety can be decomposed into finitely many irreducible components.

4.2 Conflux

Recall that the number of critical points of f is $2d - 2$ counted with multiplicity and the forward orbit of a point z_0 is $\{f^n(z_0) : n \in \mathbb{N}\}$.

Definition 4.6. Points c_1 and c_2 are said to be **forward orbit equivalent** if their forward orbits eventually coincide. i.e. there are $n, m \in \mathbb{N}$ with

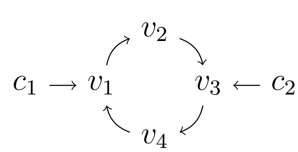
$$f^n(c_1) = f^m(c_2).$$

Definition 4.7. A point is said to be **preperiodic** if its forward orbit is finite.

The set of preperiodic critical points is denoted $\mathbf{P_f^0}$. The set of critical points with infinite forward orbits is denoted $\mathbf{P_f^\infty}$.

Definition 4.8. An equivalence class $[c_j] \subset P_f^\infty$ (i.e. a collection of critical points $c_1, c_2, \ldots, c_k \in P_f^\infty$ which are forward orbit equivalent) has a **conflux** at a if there are $n_1, n_2, \ldots, n_k \in \mathbb{N}$ with $f^{n_1}(c_1) = f^{n_2}(c_2) = \cdots = f^{n_k}(c_k) = a$ and there are not $j_1, j_2, \ldots, j_k \in \mathbb{N}$ with $j_1 < n_1, j_2 < n_2, \ldots, j_k < n_k$ with $f^{j_1}(c_1) = f^{j_2}(c_2) = \cdots = f^{j_k}(c_k)$.

The notion of conflux is well defined for equivalence classes in P_f^∞, but extension of the definition to equivalence classes in P_f^0 is not well defined. Therefore, the term conflux will only be used with regard to equivalence classes in P_f^∞. To see that conflux isn't well defined for equivalence classes in P_f^0, notice in the following diagram the conflux could be taken to be either v_1 or v_3.



The following theorem is a rephrasing of slightly coarser version of Epstein's main result from [Eps99] in the present context.

Theorem 4.9 (Fatou-Shishikura)**.** *The total number of indifferent and attracting cycles is less than or equal to the number of confluxes.*

Definition 4.10. The **pointshed** of $[c_j] = \{c_1, \ldots, c_k\}$, an equivalence class of critical points in P_f^∞ with conflux a and n_j's as in Definition 4.8, is

$$\bigcup_{j=0}^{n_1-1} f^j(c_1) \cup \bigcup_{j=0}^{n_2-1} f^j(c_2) \cup \cdots \cup \bigcup_{j=0}^{n_k-1} f^j(c_k).$$

Definition 4.11. The **eddy** of $[c_j] = \{c_1, \ldots, c_k\}$, an equivalence class in P_f^0, is

$$\bigcup_{j=0}^{\infty} f^j(c_1) \cup \bigcup_{j=0}^{\infty} f^j(c_2) \cup \cdots \cup \bigcup_{j=0}^{\infty} f^j(c_k).$$

Because the points in P_f^0 are preperiodic, the eddy of an equivalence class in P_f^0 consists of finitely many points.

Notice that the conflux of an equivalence class with a single non-preperiodic critical point is its critical value and the pointshed of such an object is just the critical point.

4.3 Momentous Graphs

The motivation for this section is to give as simple as possible a meaning to "f and f_ϵ share dynamics". Let G be a directed graph. Let N_G be the set of nodes of G and C_G be the set of connected components of G (weakly connected in the di-graph sense). The plan will be to associate to a function f a graph which keeps track of the number of infinite tails of critical points.

Definition 4.12. A **weight function** on G is a function $w : N_G \to \mathbb{N} \cup \{0, -1\}$.

In the construction to follow, weight will represent (roughly) the multiplicity of the critical point of f associated to a node of G. Confluxes will be associated to a node with weight -1.

Definition 4.13. A **mass function** on G is a function $m : C_G \to \overline{\mathbb{D}} \cup \{\infty\}$.

In the construction to follow, mass will represent (roughly) the multiplier of the cycle of f associated to a connected component of G. Only components that contain non-repelling cycles will be assigned finite mass.

Definition 4.14. A **momentous graph** is a directed graph equipped with both a weight function and a mass function, with the restriction that each node with weight in $\mathbb{N} \cup \{0\}$ have exactly one outgoing edge and each node with weight -1 have no outgoing edge.

An arbitrary momentous graph will not in general correspond to the dynamics of a degree d rational function, at least because of the Fatou-Shishikura Inequality (Theorem 2.20) and Epstein's refinements (Theorem 4.9).

Fix a degree d rational function f. Denote by T_f the union of all the pointsheds, eddies, confluxes, and non-repelling cycles of f. Construction 1 will be utilized to generate a graph G.

Construction 1 (Making G)**.** Consider T_f as a set, let T_f be the nodes of G. There is an edge from t_1 to t_2 if and only if $f(t_1) = t_2$. Define the weight function $w : T_f \to \mathbb{N} \cup \{0, -1\}$ on t_0 to be the multiplicity of t_0 as a critical point of f unless t_0 is a conflux for f, in that case $w(t_0) = -1$. Define the mass function on a component $C_G \subset G$ to be the multiplier of the non-repelling cycle if C_G contains one and ∞ if it does not.

Definition 4.15. A function f is said to have **G-dynamics** if the graph generated by Construction 1 on f is isomorphic to G including mass and weight functions.

Definition 4.16. The **inertia** of a momentous graph with components K_i and weights w_j is

$$I(G) := \#\{K_i : mass(K_i) \notin \{0, \infty\}\} + \sum_{n \in N_f} w(n).$$

The five components below comprise a momentous graph G with inertia $I(G) = 8$

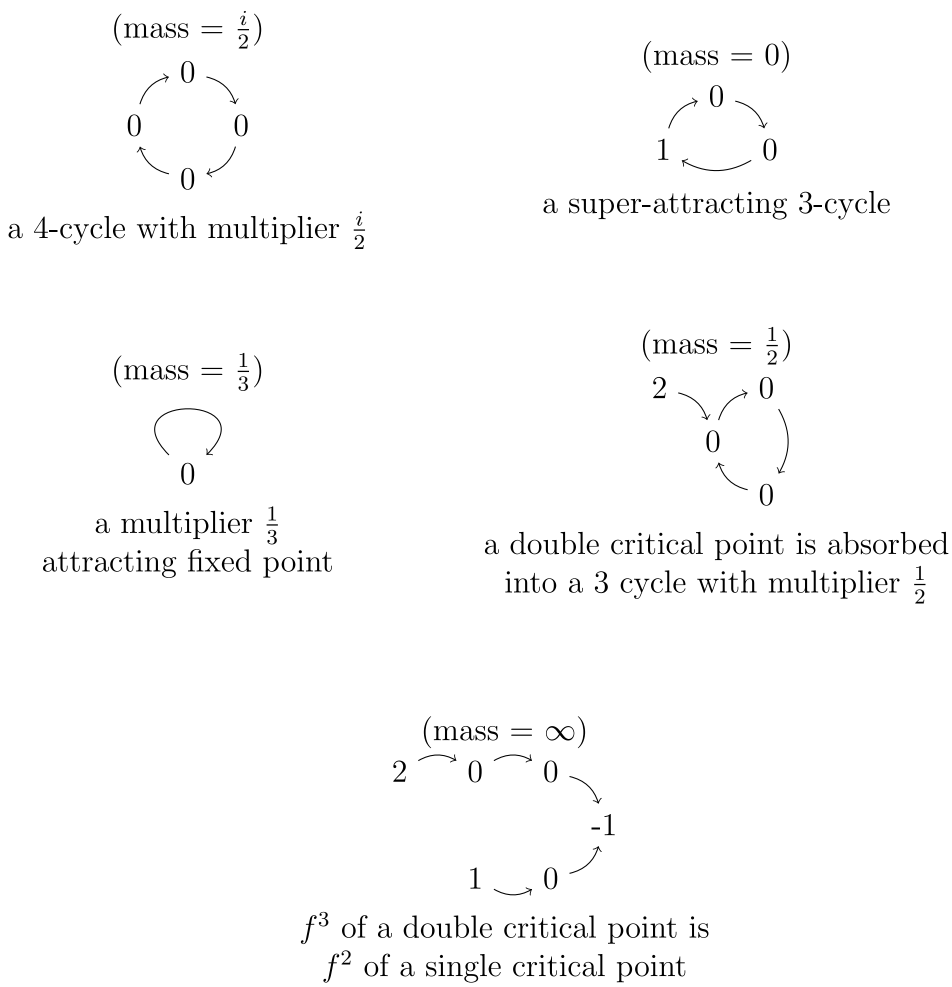


Figure 4.1: G with $I(G) = 8 = 3 + 5$.

One element of the class of maps which will be discussed in Section 5 is given by

$$f(z) = \frac{(1+\lambda)z^2 - \lambda z}{(1+\lambda)^2 z^2 + (2-\lambda^2)z - (1+\lambda)}.$$

It has associated momentous graph, G_f, with $I(G_f) = 2$.

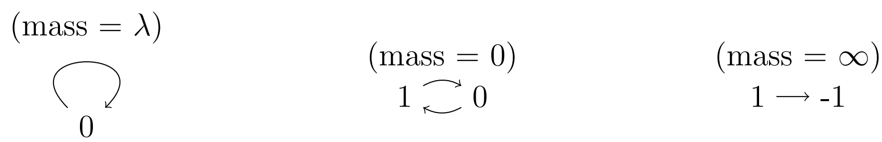


Figure 4.2: G_f from Section 5

From Theorem 4.9, it is known that a rational map of degree d can have G-dynamics only if $I(G) \leq 2d-2$.

Definition 4.17. For a fixed degree d, an **extremely momentous graph** or **EMG** is a momentous graph in which at least one of the components has mass $\lambda = e^{2\pi i\alpha}$ for α irrational, at least one component has mass in $\mathbb{D}$, $I(G) = 2d-2$, and no component has mass $e^{\frac{2\pi ip}{q}}$ for $p, q \in \mathbb{Z}$.

Definition 4.18. A rational function f (of degree d) is said to have **EMG dynamics** if G generated by Construction 1 is an EMG.

When a function f has EMG dynamics, it cannot have any other non-repelling cycles. In this sense f is fully specified by its EMG dynamics. It will be seen later that in fact these dynamics dictate the form of f up to finitely many Möbius conjugation equivalence classes.

4.4 Algebraic Families of Rational Maps

Definition 4.19. An **algebraic family of rational maps** is a rational mapping of an irreducible (quasi) projective variety V into the space of rational functions of degree d, Rat_d.

Earlier, Rat_d, the $(2d+1)$-dimensional space of rational functions, was identified with a subset of the space $\mathbb{CP}^{2d+1}$. Because of the strong relationship between critical points and momentous graphs it seems reasonable to attempt to keep track of the critical points of a rational function. For instance, one could consider $f(z) = \frac{P(z)}{Q(z)}$ with ∞ not a critical point. Then it would be discovered that the critical points of f are the zeros of the polynomial

$$P'(z)Q(z) - P(z)Q'(z) = (a_d b_{d-1} - b_d a_{d-1}) \prod_{k=1}^{2d-2} (z - c_k).$$

(This is polynomial in the coefficients of f as well as in z.) "Keeping track" of the critical points of f would then amount to consideration of a subset of

$$\mathrm{Rat}_d \times \widehat{\mathbb{C}} \times \cdots \times \widehat{\mathbb{C}}.$$

Given by the points

$$(f, c_1, \ldots, c_{2d-2})$$

where the c_j's are the critical points of f repeating values as often as multiplicity requires. Assuming ∞ isn't critical is problematic, but there is a way to recover.

Consider instead,

$$P'(z)Q(z) - P(z)Q'(z) = \sum_{k=0}^{2d-2} t_k z^k.$$

Homogenize, to obtain

$$S(z, w) = \sum_{k=0}^{2d-2} t_k z^k w^{2d-2-k}.$$

Factoring up to constants, the critical points of f are the zeros of

$$S(z,w) = \prod_{k=1}^{2d-2} (w_k z - z_k w).$$

Therefore, the critical points are $c_k = [z_k : w_k]$. The space under consideration now will be

$$\mathbb{CP}^{2d+1} \times \mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1.$$

From the Segre embedding, it is known that this space is a projective variety. The subset of this space where the first coordinate, $((a_k),(b_k))$, corresponds to a degree d rational map, is the subset where $\mathrm{Res}((a_k),(b_k)) \neq 0$. So the space $\mathrm{Rat}_d \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ is a quasiprojective variety. Furthermore, because the critical points of a rational function f can be expressed as polynomial equations in the coordinates of f the space

$$\mathrm{Rat}_d^{\mathrm{crit}} := \{(f, c_1, \ldots, c_{2d-2}) : c_1, \ldots, c_{2d-2} \text{ are critical points of } f\}$$

is a quasiprojective variety. This quasiprojective variety is often referred to as Rat_d with marked critical points.

In a similar fashion, the periodic points up to some maximal period n could also be marked. The added difficulty in the case of periodic points is twofold. First, the points of period n are also of period kn for $k \in \mathbb{N}$. Consider $f^n(z) = z$ as $\frac{P_n(z)}{Q_n(z)} = z$ which becomes $P_n(z) - zQ_n(z) = 0$. Assuming that $f^n(\infty) \neq \infty$, then $P_n(z) - zQ_n(z)$ can be factored as $-b_{d^n,n} \cdot \prod_{m|n} \phi_m(z)$ where $b_{d^n,n}$ is the leading coefficient of $Q_n(z)$ and $\phi_m(z)$ is the polynomial whose roots have minimal period m. Dealing with $f^n(\infty) = \infty$ can be done as above by homogenization. (For details, see [Sil07].) The other difficulty is that the coefficients of f^n need to be polynomial in the coefficients of f. This can be seen with induction. The space

$$\mathrm{Rat}_{d,n}^{\mathrm{crit,per}}$$

with marked critical and periodic points of period less than or equal to n is therefore a quasiprojective variety.

In Construction 1, a method was developed to obtain from a rational function f a momentous graph G. This construction will provide a somewhat dual concept. Here a method will be outlined for converting a momentous graph into a set of equations that capture the essence of the graph. To this end, consider the space

$$\mathbb{CP}^{2d+1} \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$$

with coordinates $(f, c_1, \ldots, c_{2d-2},\ {}^1p_1^1,\ {}^2p_1^1, \ldots,\ {}^kp_n^n)$. The later coordinates need a little explaining. Because they will in application represent periodic points, they are marked as follows:

$${}^cP_m^n$$

where "c" is used to distinguish between cycles of the same period, "m" denotes which element of a particular cycle is represented, and "n" denotes the period. In other words

$$f({}^cp_m^n) = {}^cp_{(m+1) \mod n}^n.$$

Of course, in this space f might not be a rational function of degree d, but these points will be excluded later on in application.

Construction 2. Let G be a momentous graph, let k be the sum of the weights on the non-conflux nodes in G. Assign labels $c_1, \ldots, c_k$ to each node, assigning without replacement one label for each unit of weight. (There will be many possible ways to do this, any one will work.) Periodic points will also need labels, for the jth periodic cycle of period n assign labels ${}^{j}p_1^n, \ldots, {}^{j}p_n^n$. This gives rise to the first set of equations:

$$f({}^{j}p_l^n) = {}^{j}p_{l+1}^n \text{ for } l = 1, \ldots, n-1 \text{ and } f({}^{j}p_n^n) = {}^{j}p_1^n. \tag{4.1}$$

The reason for this somewhat obtuse labeling is to ensure multiple cycles of the same period are adequately represented.

The next obvious set of equations is for those nodes which are assigned multiple labels. For instance, if a node has weight l and has labels $c_{k+1}, \ldots, c_{k+l}$, then the associated equations are:

$$c_{k+1} = \cdots = c_{k+l} \tag{4.2}$$

If in the pointshed of a conflux node there are nodes assigned labels, equations can be generated by counting the number of edges that must be traversed backwards to reach a label. Suppose that there are l labels, $c_{k_1}, \ldots, c_{k_l}$, in the pointshed, and n_j is the number of edges from c_{k_j} to the conflux. The equations generated by such a relation are:

$$f^{n_1}(c_{k_1}) = f^{n_2}(c_{k_2}) = \cdots = f^{n_l}(c_{k_l}) \tag{4.3}$$

If there are nodes assigned labels connected to a periodic node with label ${}^{j}p_k^n$ (meaning traversing edges in their direction gives a path from a labeled node to the periodic node), then equations are generated in almost the same way as for those in the pointshed of a conflux. Suppose that there are l labels, $c_{k_1}, \ldots, c_{k_l}$, connected to the node with label ${}^{j}p_k^n$, and n_j is the number of edges from c_{k_j} to the node labeled

${}^{j}p_k^n$. Then the equations are:

$$f^{n_1}(c_{k_1}) = f^{n_2}(c_{k_2}) = \cdots = f^{n_l}(c_{k_l}) = \ {}^{j}p_k^n \tag{4.4}$$

The final equations will deal with the masses of cycles. Only those cycles with mass in $\overline{\mathbb{D}}$ will contribute equations. The equations of cycles with mass 0, i.e. periodic nodes with weight > 1, are easy. Once these have been labeled by both $c_{k_1}, \ldots, c_{k_l}$ and ${}^{j}p_l^n$, the equations are:

$$c_{k_1} = \cdots = c_{k_l} = \ {}^{j}p_m^n. \tag{4.5}$$

Suppose that a cycle labeled ${}^{j}p_1^n, \ldots, \ {}^{j}p_n^n$ has mass $\lambda \in \overline{\mathbb{D}} - \{0\}$, then the associated equation is:

$$\lambda = (f^n)'({}^{j}p_1^n) \tag{4.6}$$

It should be noted that similar, but not identical graphs may yield identical equations. This construction could be modified to ensure that the only identical graphs gave identical equations, but this would only serve to add confusion.

Example. Construction 2 clearly needs an example. Let G be the following momentous graph:

On component C_3, label the two nodes in C_3 with weight one from left to right c_1 and c_2. Label the periodic points starting at the bottom and proceeding clockwise

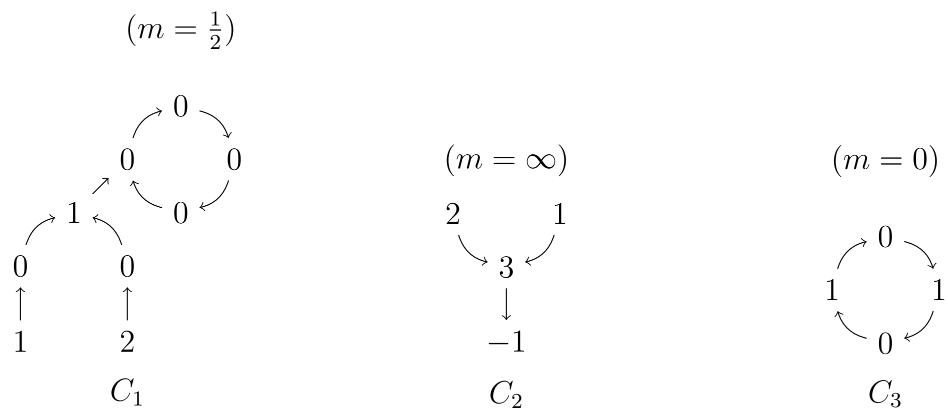


Figure 4.3

${}^1p_1^4,\ {}^1p_2^4,\ {}^1p_3^4,\ {}^1p_4^4$. Therefore (from Equations 4.1 and 4.5) the equations from C_3 are:

$$
\begin{aligned}
f({}^1p_1^4) &= {}^1p_2^4 \\
f({}^1p_2^4) &= {}^1p_3^4 \\
f({}^1p_3^4) &= {}^1p_4^4 \\
f({}^1p_4^4) &= {}^1p_1^4 \\
c_1 &= {}^1p_2^4 \\
c_2 &= {}^1p_4^4
\end{aligned}
$$

For component C_2, label the top left c_3 and c_4, the top right c_5, the node with weight 3 by c_6, c_7, and c_8. The resulting equations from (Equations 4.2 and 4.3) are:

$$c_3 = c_4$$

$$c_6 = c_7 = c_8$$

$$f^2(c_3) = f^2(c_5) = f(c_6)$$

Finally on component C_1, label the node with weight two by c_9 and c_{10}. Label the lower left node by c_{11} and the node with three edges by c_{12}. Next, label the periodic cycle starting at the bottom and proceeding clockwise by ${}^2p_1^4$, ${}^2p_2^4$, ${}^2p_3^4$, ${}^2p_4^4$. Then (from Equations 4.1, 4.4, and 4.6) the equations are:

$$\begin{aligned} f({}^2p_1^4) &= {}^2p_2^4 \\ f({}^2p_2^4) &= {}^2p_3^4 \\ f({}^2p_3^4) &= {}^2p_4^4 \\ f({}^2p_4^4) &= {}^2p_1^4 \\ f^3(c_{11}) &= f^3(c_9) = f(c_{12}) = {}^2p_2^4 \\ (f^4)'({}^2p_1^4) &= \frac{1}{2} \end{aligned}$$

Definition 4.20. For a momentous graph G and n the maximum length of a cycle in G, define

$$\mathbf{V_G} := \{f \in \mathrm{Rat}_{d,n}^{\mathrm{crit,per}} : f \text{ has } G\text{-dynamics}\}$$

Theorem 4.21. *If G is a momentous graph, then V_G is a quasiprojective variety in $Rat_{d,n}^{crit,per}$.*

Proof. From Construction 2, V_G is the set of common zeros of a set of equations. To demonstrate that V_G is a quasiprojective variety it must only be seen that these equations are polynomials. The equations generated as in Equation 4.1 and 4.5 are polynomial because f is rational. The equations generated as in Equation 4.2 are polynomial because the critical points are polynomial is the coefficients of f. Those equations generated as Equation 4.3 and 4.4 are polynomial because the coefficients f^n are polynomial in the coefficients of f. Finally, those equations arising as Equation 4.6 are polynomial because the coefficients of $(f^n)'$ are polynomial in the coefficients of f. □

Lemma 4.22. *[Sha74] A projection map π on a quasiprojective variety is a regular mapping, hence rational.*

Corollary 4.23. *In particular, $\pi : Rat_{d,n}^{crit,per} \to Rat_d$ is a rational map.*

The following is apparent from the definitions and the Constructions 1 and 2.

Lemma 4.24. *If G is a momentous graph, and a rational function f has G-dynamics then there is a point $(f, c_1, \dots) \in Rat_{d,n}^{crit,per}$ with G-dynamics and*

$$\pi((f, c_1, \dots)) = f.$$

Since V_G is a quasiprojective variety, it has finitely many irreducible components. Let N be the number of irreducible components. Label these components V_G^j for $j = 1, \dots, N$.

Corollary 4.25. *$\pi(V_G^j)$ is an algebraic family of rational maps.*

Definition 4.26. An algebraic family of rational maps is **stable** if there is a uniform upper bound on the period of the attracting cycles of rational maps occurring in the family.

Lemma 4.27. *If G is an EMG, then $\pi(V_G^j)$ is stable.*

Proof. Let $f \in \pi(V_G^j)$ and let N be the longest cycle in G. Then because $I(G) = 2d-2$, the number of confluxes is equal to the number of cycles with multipliers in $\overline{\mathbb{D}} - \{0\}$. Therefore, from Theorem 4.9 it is known that f cannot have another attracting cycle. Finally, because the lengths of the super-attracting and attracting cycles in G are fixed, f has no attracting cycle of length greater than N. $\square$

Definition 4.28. A family is **trivial** if all of its members are conjugate by Möbius transformations.

Definition 4.29. A rational map is **critically finite** if the forward orbit of each critical point is finite.

The following crucial theorem is due to McMullen [McM87].

Theorem 4.30. *[McMullen] A stable family of rational maps is either trivial or all of its members are critically finite.*

Theorem 4.31. *The set of rational functions which have the same EMG dynamics as f is a union of finitely many trivial families.*

Proof. Let G be the EMG given by Construction 1. Then from Theorem 4.21, V_G is a quasiprojective variety. Let V_G^j for $j = 1, \ldots, n$ be the irreducible components of V_G. By Theorem 4.9, functions with EMG dynamics cannot be critically finite, so Lemma 4.27 and Theorem 4.30 imply that $\pi(V_G^j)$ is trivial. $\square$

Example

This example is the easiest case of the original results included here. The broad strokes of the technique here were outlined in [Gey] and most will carry directly to Section 6. In this section, fix $d = 2$.

5.1 The Setup

Fix an $n \in \mathbb{N}$ and an $\alpha \in \mathbb{R} - \mathbb{Q}$. Let $\lambda = e^{2\pi i\alpha}$ and let G be an EMG given by:

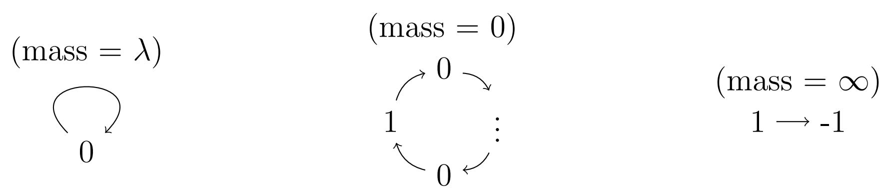


Figure 5.1: f's momentous graph

A map with G-dynamics has a super-attracting n cycle and a fixed point with rotation number α.

5.2 Discreteness

In this section, it will be seen that the set of quadratic rational maps with a super-attracting orbit and a Siegel point with rotation number α at zero is discrete (up to conjugacy by Möbius transformation). In Chapter 6, the calculations performed here will be replaced with the more sophisticated results from Chapter 4.

Let f be such a map. Then it is known [Mil06] that the image of a critical point must be dense on the boundary of the Siegel disk and f has $2 \cdot deg(f) - 2 = 2$ critical points. Therefore f cannot exchange the critical points and have a Siegel disk. In fact, if f exchanged critical points, Lemma 2.4 would give the exact form of f and neither possibility has a Siegel disk.

The function f has two critical points, one of which is in the super-attracting cycle, this critical point is periodic, while the other necessarily has an infinite forward orbit. With this knowledge make the following normalization conditions, require (by Möbius transformation) that the Siegel point be at 0, 1 be the periodic critical point, and $f(\infty) = 1$. The reason for this normalization is to simplify explanations. Other reasonable normalizations would be equally effective.

Therefore, after conjugation, the following has been required:

$$f(0) = 0,\ f(\infty) = 1,\ f'(0) = e^{2\pi i\alpha},\ f'(1) = 0,\ f^n(1) = 1$$

Assuming f does not fix ∞, f has the form

$$f(z) = \frac{a_2 z^2 + a_1 z + a_0}{z^2 + b_1 z + b_0}.$$

Because $f(\infty) = 1$, $a_2 = 1$.

$$f(z) = \frac{z^2 + a_1 z + a_0}{z^2 + b_1 z + b_0}$$

Now because $f(0) = 0$, $\frac{a_0}{b_0} = 0$, so $a_0 = 0$.

$$f(z) = \frac{z^2 + a_1 z}{z^2 + b_1 z + b_0}$$

Next, calculate $f'(z)$.

$$f'(z) = \frac{(2z + a_1)(z^2 + b_1 z + b_0) - (z^2 + a_1 z)(2z + b_1)}{(z^2 + b_1 z + b_0)^2}$$

Requiring that f has rotation number α and defining $\lambda = e^{2\pi i\alpha}$, find

$$f'(0) = \frac{a_1 b_0}{b_0^2} = \frac{a_1}{b_0} = \lambda.$$

See that

$$a_1 = \lambda b_0.$$

Because it was required that $f'(1) = 0$, see

$$b_1 = \lambda b_0 - \lambda b_0^2 - 2b_0.$$

Therefore,

$$f(z) = \frac{z^2 + \lambda b_0 z}{z^2 + (\lambda b_0 - \lambda b_0^2 - 2b_0)z + b_0}$$

Finally, because $f^n(1) = 1$, the solutions of a degree 2^n polynomial for b_0 in terms of λ give all the possible quadratic rational maps with the specified properties. Therefore there are only finitely many such maps. Going forwards, assume that f is one of these normalized rational functions.

5.3 Perturbation

The treatment of perturbations follows the construction in [Shi87]. Label the orbit of 1 as $1 = z_0, z_1 \ldots, z_n$. Let $\rho(x)$ be a decreasing C^∞ function from $\mathbb{R}^+$ to $[0, 1]$ with $\rho(x) = 1$ if $x < 1$ and $\rho(x) = 0$ if $x > 2$. From Lemma 2.35, let E be such that $f(E) \subset E$ and let R be large enough so that $f(\widehat{\mathbb{C}} \backslash \mathbb{D}_R) \subset E$. Define

$$h(z) := z^2 \prod_{j=1}^{n} (z - z_j)^2$$

$$H_\epsilon(z) := z + \epsilon h(z)\rho\left(\frac{|z|}{R}\right)$$

Now perturb f to f_ϵ as in Lemma 3.15, by letting

$$f_\epsilon(z) := f(H_\epsilon(z))$$

5.3.1 Essential Quadratitude

Lemma 5.1. *f_ϵ is an essentially quadratic family at 0.*

Proof. Let a power series for $f(z)$ at 0, and $h(z)$ at be given by

$$f(z) = \sum_{n=1}^{\infty} a_n z^n$$
$$h(z) = h_2 z^2 + h_3 z^3 + \dots h_d z^d$$

so that

$$g(z) = z + \epsilon h(z) = z + \epsilon h_2 z^2 + \epsilon h_3 z^3 + \cdots + \epsilon h_d z^d$$

Recalling Faa di Bruno's Formula, the coefficients c_n of the power series

$$f(g(z)) = f_\epsilon(z) = \sum_{n=0}^{\infty} c_n z^n$$

are given by

$$c_n = \sum_{C_n} a_k b_{i_1} b_{i_2} \dots b_{i_k}$$

where $C_n = \{(i_1, i_2, \dots, i_k) : 1 \le k \le n, i_1 + i_2 + \cdots + i_k = n\}$. Because $\deg_\epsilon(b_1) = 0$ and $\deg_\epsilon(b_k) = 1$ for $k \ge 2$ the maximum degree in ϵ of c_n is given by $\left\lfloor \frac{n}{2} \right\rfloor$.

Specifically $\deg(c_n) < n - 1$ for $n > 2$, $\deg(c_0) = \deg(c_1) = 0$. By inspection, $c_2 = a_1\epsilon + a_2$, but $a_1 \neq 0$, since 0 is an irrationally indifferent fixed point, so f_ϵ is essentially quadratic. $\square$

5.3.2 Conjugation to Rational Map

Lemma 5.2. *There exists $\epsilon_0 > 0$ such that for all $|\epsilon| < \epsilon_0$, the map f_ϵ is quasiconformally conjugate to a rational map r_ϵ and $r_\epsilon \to f$ uniformly as $\epsilon \to 0$. Furthermore, this conjugacy is conformal on the Siegel disk.*

Proof. Let E with $f_\epsilon(E) \subset E$ be obtained from application of Lemma 2.35 to $\{f_\epsilon\}$. Application of Lemma 3.13 to f_ϵ gives ϕ_ϵ a conjugation to a rational map and this is a conformal conjugacy on the Siegel disk. □

5.3.3 Preservation of Dynamics

Lemma 5.3. *$r_\epsilon(z)$ has G-dynamics .*

Proof. The rational function r_ϵ is of the form

$$r_\epsilon = \varphi_\epsilon \circ f \circ H_\epsilon \circ \varphi_\epsilon^{-1}.$$

The functions φ_ϵ and H_ϵ are quasiconformal, meaning they are also homeomorphisms. This implies that r_ϵ has the same topological properties as f. Specifically, $\deg(r_\epsilon) = \deg(f)$ and if z_0 is critical for f, then $w_0 = \varphi \circ H_\epsilon^{-1}$ is critical for r_ϵ. The forward orbit of the critical point must also have the same cardinality. Therefore, r_ϵ has a period n super-attracting orbit and the other critical point has a conflux at its critical value. Because the conjugacy φ is conformal on the Siegel disk, r_ϵ must also have a Siegel disk. □

5.4 Uniform Linearizability

It has been established that each f_ϵ is quasiconformally conjugate to a (possibly different) rational map, with conformal conjugacy on the Siegel disk. However, in light of the discreteness of the set of rational maps with G-dynamics and because of the convergence of f_ϵ to f these rational maps must in fact be f for small enough ϵ. Also $|\phi'_\epsilon(0)| \to 1$ because $\phi_\epsilon \to id$ uniformly. This can be seen using the Cauchy Integral Formula or examination of the method for solving the Beltrami Equation. Therefore, from Lemma 3.16, f_ϵ is uniformly linearizable.

5.5 Conclusion

In light of the sections above and Lemma 2.41 the quadratic polynomial $P(z) = e^{2\pi i\alpha}z + z^2$ is linearizable. Yoccoz's result (Theorem 2.49) then implies that the rotation number α must satisfy the Brjuno condition. This in turn, confirms the Douady conjecture for rational maps of degree two with a super-attracting orbit. Finally, the following theorem has been proven.

Theorem 5.4. *If f is a quadratic rational map with a super-attracting orbit and a fixed Siegel disk with rotation number α, then α must be a Brjuno number.*

This case is indeed new, for if the period of the super-attracting orbit is strictly greater than one, f is not conjugate to a polynomial.

Main Result

In this section, the following will be proven:

Theorem 6.1 (Main Result)**.** *If f a degree $d > 1$ rational map has EMG dynamics and a Siegel cycle, then the rotation number of the Siegel cycle is Brjuno.*

First, it will be shown that up to Möbius transformation there are only finitely functions sharing dynamics with f. Next, a perturbation family will be created. It will then be shown that the perturbation family is essentially quadratic. It will also be shown that members of this family are quasiconformally conjugate to rational functions that share dynamics with f. Then, it will be seen that this implies that the perturbation family is uniformly linearizable. Finally, the rotation number of the Siegel cycle will be seen to be Brjuno.

6.1 Discreteness

Suppose that f has EMG dynamics and a Siegel cycle. Construct from f, following Construction 1, an EMG G.

First, note that in light of Theorem 4.31 the set of all rational maps with G-dynamics is composed of finitely many trivial families. In particular this implies that up to Möbius transformation there are finitely many rational functions with G-dynamics. In other words, $\pi(V_G)$ is discrete as a subset of the orbifold, Rat_d/M, the degree d rational maps modulo Möbius conjugation.

Because G is an EMG, f has an attracting cycle Z. From Lemma 2.4, it can be seen that there is a point in $f^{-1}(Z)\backslash Z$. If not then f would be a conjugate to a polynomial or $z \to \frac{1}{z^d}$. The polynomial case is dealt with in [Gey] and $z \to \frac{1}{z^d}$ cannot have a Siegel disk. After conjugation by a Möbius transformation, it can be assumed that 0 is in the Siegel cycle, 1 is in the (super) attracting cycle, $f(\infty) = 1$, and $\infty \notin Z$.

6.2 Perturbation

Remember that T_f is the set of points in $\widehat{\mathbb{C}}$ that correspond to the nodes of G. Let $h(z)$ be a polynomial with double zeros at all points $p \in T_f$. Furthermore, on the Siegel cycle $h(z)$ should have a double zero at 0, and a triple zero at all other points in the cycle. Let R be large enough so that $f(\widehat{\mathbb{C}} \backslash \mathbb{D}_R)$ is contained in a set E as given by Lemma 2.35 in the basin of 1. Let $A_{R,2R} := \{z \in \mathbb{C} : R \leq |z| \leq 2R\}$. If needed, further enlarge R so that $T_f \cap A_{R,2R}$ is empty. Consider the perturbation family

$$f_\epsilon := f \circ H_\epsilon$$

with

$$H_\epsilon := z + \epsilon \cdot h(z) \cdot \rho\left(\frac{|z|}{R}\right).$$

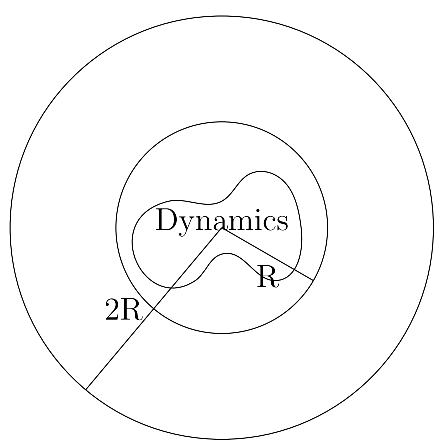


Figure 6.1

From Lemma 3.15, it is known that there exists ϵ_0 such that for all $|\epsilon| < \epsilon_0$ the map H_ϵ is quasiconformal. By Lemma 3.13, there are rational maps r_ϵ and quasiconformal maps φ_ϵ such that $r_\epsilon = \varphi_\epsilon \circ f_\epsilon \circ \varphi^{-1}$. Furthermore, the conjugacy is conformal on

$$E \cup \widehat{\mathbb{C}} \backslash \overline{\bigcup_{n \geq 1}^{\infty} f_\epsilon^{-n}(E)}$$

. Remember, f is a degree d rational function with G-dynamics.

Lemma 6.2. *r_ϵ is a degree d rational function and has G-dynamics.*

Proof. First, remember that H_ϵ and φ are quasiconformal (in particular homeomorphisms) with:

$$r_\epsilon = \varphi_\epsilon \circ f \circ H_\epsilon \circ \varphi_\epsilon^{-1}.$$

Therefore, the degree of $r_\epsilon = d$ because degree is a topological invariant. Furthermore, if f has a critical point at z_0, then r_ϵ has a critical point of the same multiplicity at $\varphi_\epsilon \circ H_\epsilon^{-1}(z_0)$. If f has an attracting cycle Z, then because $H_\epsilon'(p) = 1$ and $H_\epsilon(p) = p$ for all $p \in Z$ and because the conjugacy is conformal near Z it must be that r_ϵ has a cycle with the same multiplier at $\varphi(Z)$. If f has an indifferent cycle W, then from Naishul's Theorem (Theorem 2.36), r_ϵ has an indifferent cycle $\varphi(W)$.

This implies that the momentous graph associated to r_ϵ is isomorphic to G preserving weights and masses except possibly at confluxes. From Theorem 4.9, r_ϵ cannot have fewer confluxes than f. Therefore, r_ϵ is degree d and has G-dynamics.

□

Corollary 6.3. *f_ϵ is quasiconformally conjugate to f.*

Proof. Because φ_ϵ converges to the identity as $\epsilon \to 0$, $r_\epsilon \to f$. Because the set of functions with G-dynamics is discrete up to Möbius conjugacy, for small enough $|\epsilon|$ there must be a Möbius transformation M_ϵ with $M_\epsilon \circ r_\epsilon \circ M_\epsilon^{-1} = f$. Then f_ϵ is quasiconformally conjugate to f by $M_\epsilon \circ \varphi_\epsilon$. □

6.3 Essential Quadratitude

Lemma 6.4. *f_ϵ is essentially quadratic at 0, and subquadratic at all the other points in the Siegel cycle.*

This is a slight modification on the technique used to prove Lemma 5.1.

Proof. Let a power series for $f(z)$ at 0, and $h(z)$ at be given by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$
$$h(z) = h_2 z^2 + h_3 z^3 + \dots h_d z^d$$

so that

$$g(z) = z + \epsilon h(z) = z + \epsilon h_2 z^2 + \epsilon h_3 z^3 + \cdots + \epsilon h_d z^d$$

Recalling Faa di Bruno's Formula, the coefficients c_n of the power series

$$f(g(z)) = f_\epsilon(z) = \sum_{n=0}^{\infty} c_n z^n$$

are given by

$$c_n = \sum_{C_n} a_k b_{i_1} b_{i_2} \dots b_{i_k}$$

where $C_n = \{(i_1, i_2, \dots, i_k) : 1 \leq k \leq n, i_1 + i_2 + \cdots + i_k = n\}$. Because $\deg_\epsilon(b_1) = 0$ and $\deg_\epsilon(b_k) = 1$ for $k \geq 2$ the maximum degree in ϵ of c_n is given by $\left\lfloor \frac{n}{2} \right\rfloor$.

Specifically $\deg(c_n) < n - 1$ for $n > 2$, $\deg(c_0) = \deg(c_1) = 0$. By inspection, $c_2 = a_1\epsilon + a_2$, but $a_1 \neq 0$, since 0 is in the Siegel cycle, so f_ϵ is essentially quadratic.

An almost identical argument applied at each of the other points in the Siegel cycle yields the subquadratic result, since at those points $\deg_\epsilon(b_1) = \deg_\epsilon(b_2) = 0$ and $\deg_\epsilon(b_k) = 1$ for $k \geq 3$, so that the maximum degree in ϵ of c_n is given by $\left\lfloor \frac{n}{3} \right\rfloor$, so that $\deg(c_n) < n - 1$.

□

6.4 Uniform Linearizability and Conclusion

Corollary 6.3, demonstrates that f_ϵ is in fact quasiconformally conjugate to f. Specifically, observe that f_ϵ must have a Siegel cycle for sufficiently small ϵ. Therefore to ease discussion of rotation number, pass to a suitable (let n be the length of the Siegel cycle) iterate of f_ϵ. In fact the quasiconformal conjugacy is preserved in other words, f_ϵ^n is quasiconformally conjugate to f^n. From Theorem 3.16, the family f_ϵ^n is uniformly linearizable. As the composition of an essentially quadratic family and subquadratic families (Lemma 6.4), from Lemma 2.40 it must be that f_ϵ^n is an essentially quadratic family. Therefore from Lemma 2.41, the quadratic polynomial $\lambda z + z^2$ is linearizable, and the rotation number must be Brjuno. This completes the proof of:

Theorem (Main Result)**.** *If f a degree $d > 1$ rational map has EMG dynamics and a Siegel cycle, then the rotation number of the Siegel cycle is Brjuno.*

6.5 Corollaries

This corollary is a slightly more general version of the result in Chapter 5. There the result assumed a fixed Siegel disk, here the result is expanded to encompass cycles.

Corollary 6.5. *If f is a quadratic rational map with a Siegel cycle and a super attracting cycle, then the rotation number of the Siegel cycle must be Brjuno.*

Proof. Because the degree of f is 2 and the momentous graph, G, associated to f has $I(G) = 2 = 2(2) - 2$, an irrationally indifferent cycle, and a super attracting cycle. Then G is an EMG, so from Theorem 6.1, the rotation number of the Siegel cycle is Brjuno. □

Corollary 6.6. *Suppose* $N(z) = z - \frac{p(z)}{p'(z)}$ *is Newton's Method on any cubic polynomial* p *with distinct roots. Then* $N(z)$ *has a Siegel cycle if and only if* $N(z)$ *has an irrationally indifferent cycle with a Brjuno rotation number.*

Proof. $N(z)$ is degree 3, and therefore has 4 critical points. $N(z)$ has super-attracting fixed points at each zero of the cubic polynomial. If $N(z)$ also has a Siegel disk, then it has EMG dynamics. Therefore, from Theorem 6.1 the rotation number is Brjuno. The opposite implication follows from Theorem 2.48. □

Bibliography

[Ahl53] Lars V. Ahlfors. *Complex Analysis*. McGraw-Hill, 1953.

[Ahl66] Lars V. Ahlfors. *Lectures on Quasiconformal Mappings*. University Lecture Series. American Mathematical Society, 1966.

[AIR12] Daniel S. Alexander, Felice Iavernaro, and Alessandro Rosa. *Early days in complex dynamics*. Vol. 38. History of Mathematics. A history of complex dynamics in one variable during 1906–1942. American Mathematical Society, Providence, RI; London Mathematical Society London, 2012, pp. xviii+454. ISBN: 978-0-8218-4464-9.

[Bea91] Alan F. Beardon. *Iteration of rational functions*. Vol. 132. Graduate Texts in Mathematics. Complex analytic dynamical systems. Springer-Verlag, New York, 1991, pp. xvi+280. ISBN: 0-387-97589-6. DOI: `10.1007/978-1-4612-4422-6`. URL: `http://dx.doi.org/10.1007/978-1-4612-4422-6`.

[Bie16] L. Bieberbach. “Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln”. In: *Sitzungsberichte Preussische Akademie der Wissenschaften* (1916), pp. 940–955.

[BF14] Bodil Branner and Núria Fagella. *Quasiconformal surgery in holomorphic dynamics*. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2014, p. 428. ISBN: 9781107042919.

[Brj65] A. D. Brjuno. “On convergence of transforms of differential equations to the normal form”. In: *Dokl. Akad. Nauk SSSR* 165 (1965), pp. 987–989. ISSN: 0002-3264.

[CG93] Lennart Carleson and Theodore W. Gamelin. *Complex dynamics*. Universitext: Tracts in Mathematics. New York: Springer-Verlag, 1993, pp. x+175. ISBN: 0-387-97942-5.

[Che10] D. Cheraghi. “Typical orbits of quadratic polynomials with a neutral fixed point: non-Brjuno type”. Preprint, `arXiv:1001.4030 [math.DS]`. Jan. 2010.

[Cre28] H. Cremer. “Zum Zentrumproblem”. ger. In: *Mathematische Annalen* 98 (1928), pp. 151–163. URL: `http://eudml.org/doc/159207`.

[Dou87] Adrien Douady. “Disques de Siegel et anneaux de Herman”. In: *Astérisque* 152-153 (1987). Séminaire Bourbaki, Vol. 1986/87, 4, 151–172 (1988). ISSN: 0303-1179.

[Eps99] A. Epstein. “Infinitesimal Thurston Rigidity and the Fatou-Shishikura Inequality”. Preprint, `arXiv:math/9902158 [math.DS]`. Feb. 1999.

[Gey] Lukas Geyer. “Linearizability of saturated polynomials”. preprint.

[Gey01] Lukas Geyer. “Siegel discs, Herman rings and the Arnold family”. In: *Trans. Amer. Math. Soc.* 353.9 (2001), pp. 3661–3683. ISSN: 0002-9947. DOI: `10.1090/S0002-9947-01-02662-9`. URL: `http://dx.doi.org/10.1090/S0002-9947-01-02662-9`.

[Jul19] G Julia. “Sur quelques problèmes relatifs à l’itération des fractions rationnelles”. In: *CR Acad. Sci. Paris* 168 (1919), pp. 147–149.

[Kas12] E Kasner. “Conformal geometry”. In: *Proc Fifth Internat. Congress Math.* Vol. 2. 1912, p. 83.

[McM87] Curt McMullen. “Families of Rational Maps and Iterative Root-Finding Algorithms”. English. In: *Annals of Mathematics*. Second Series 125.3 (1987), pp. 467–493. ISSN: 0003486X. URL: `http://www.jstor.org/stable/1971408`.

[MS07] Nathan Mercer and Rich Stankewitz. *Introduction to Quasiconformal Mappings in the Plane with an Application to Quasiconformal Surgery*. 2007. URL: `http://rstankewitz.iweb.bsu.edu/QCMapsIntro.pdf`.

[Mil06] John Milnor. *Dynamics in One Complex Variable*. Annals of Mathematics Studies. Princeton University Press, 2006. ISBN: 0-691-12488-4.

[PM01] R. Pérez-Marco. “Total convergence or general divergence in small divisors”. In: *Comm. Math. Phys.* 223.3 (2001), pp. 451–464. ISSN: 0010-3616. DOI: `10.1007/s002200100457`. URL: `http://dx.doi.org/10.1007/s002200100457`.

[PM93] Ricardo Pérez Marco. “Sur les dynamiques holomorphes non linéarisables et une conjecture de V. I. Arnol′d”. In: *Ann. Sci. École Norm. Sup. (4)* 26.5 (1993), pp. 565–644. ISSN: 0012-9593. URL: `http://www.numdam.org/item?id=ASENS_1993_4_26_5_565_0`.

[PM97] Ricardo Pérez-Marco. "Fixed points and circle maps". In: *Acta Math.* 179.2 (1997), pp. 243–294. ISSN: 0001-5962. DOI: 10.1007/BF02392745. URL: http://dx.doi.org/10.1007/BF02392745.

[Pfe17] GA Pfeiffer. "On the Conformal Mapping of Curvilinear Angles. The Functional Equation $\varphi[f(x)] = a_1\varphi(x)$". In: *Transactions of the American Mathematical Society* 18.2 (1917), pp. 185–198.

[R̈67] Helmut Rüssmann. "Über die Iteration analytischer Funktionen". In: *J. Math. Mech.* 17 (1967), pp. 523–532.

[Sha74] I. R. Shafarevich. *Basic algebraic geometry*. Translated from the Russian by K. A. Hirsch, Die Grundlehren der mathematischen Wissenschaften, Band 213. Springer-Verlag, New York-Heidelberg, 1974, pp. xv+439.

[Shi87] Mitsuhiro Shishikura. "On the quasiconformal surgery of rational functions". In: *Ann. Sci. École Norm. Sup. (4)* 20.1 (1987), pp. 1–29. ISSN: 0012-9593. URL: http://www.numdam.org/item?id=ASENS_1987_4_20_1_1_0.

[Sie42] Carl Ludwig Siegel. "Iteration of analytic functions". In: *Ann. of Math. (2)* 43 (1942), pp. 607–612. ISSN: 0003-486X.

[Sil07] Joseph H. Silverman. *The arithmetic of dynamical systems*. Vol. 241. Graduate Texts in Mathematics. Springer, New York, 2007, pp. x+511. ISBN: 978-0-387-69903-5. DOI: 10.1007/978-0-387-69904-2. URL: http://dx.doi.org/10.1007/978-0-387-69904-2.

[SC93] Steven Spielberg and Michael Crichton. Universal Studios. 1993. URL: https://www.youtube.com/watch?v=VfjjSeE-D4c.

[Sul85] Dennis Sullivan. "Quasiconformal Homeomorphisms and Dynamics I. Solution of the Fatou-Julia Problem on Wandering Domains". English. In: *Annals of Mathematics*. Second Series 122.2 (1985), pp. 401–418. ISSN: 0003486X. URL: http://www.jstor.org/stable/1971308.

[Yoc95] Jean-Christophe Yoccoz. "Théorème de Siegel, nombres de Bruno et polynômes quadratiques". In: *Astérisque* 231 (1995). Petits diviseurs en dimension 1, pp. 3–88. ISSN: 0303-1179.