Asymptotic and oscillatory solutions of N-th order linear differential equations
by Gerald Edwin Bendixen

A Thesis submitted to the Graduate Faculty in partial sulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Mathematics
Montana State University

Abstract:
In this thesis asymptotic solutions of the n-th order, linear, homogeneous differential equation (x^n)(t) + A1(t)(x^(n-1))(t)+...+An(t)x(t)=0 are obtained, where the Aj(t) are continuous complex-valued functions on [a ,∞). The solutions are found by transforming the given equation into a vector-matrix differential system for which the result of N. Levinson, Duke Math. J. 15 (1948), pp. 111 - 126, is applicable. To apply his result, asymptotic estimates for the zeros of related characteristic polynomials are obtained. The special case where n = 3 is treated in detail. The oscillatory nature and boundedness properties of solutions of the given equation are also investigated. The results generalize those of G. W. Pfeiffer, J. Diff. Equations 11 (1972), pp. 138 - 144 and pp. 145 - 155.

ASYMPTOTIC AND OSCILLATORY SOLUTIONS OF N-TH ORDER LINEAR DIFFERENTIAL EQUATIONS

by

GERALD EDWIN BENDIXEN

A Thesis submitted to the Graduate Faculty in partial
sulfillment of the requirements for the degree

of

DOCTOR OF PHILOSOPHY

in

Mathematics

Approved:

Robert D. Engle
Head, Major Department

Louis C. Barrett
Chairman, Examining Committee

Henry L Parsons
Graduate Dean

MONTANA STATE UNIVERSITY
Bozeman, Montana

June, 1973

ACKNOWLEDGMENT

The author wishes to express his sincere appreciation to his major professor, Dr. Louis C. Barrett, for his continued interest and helpful suggestions during the preparation of this dissertation.

TABLE OF CONTENTS

ABSTRACT

In this thesis asymptotic solutions of the n-th order, linear, homogeneous differential equation

$$x^{(n)}(t) + A_1(t)x^{(n-1)}(t) + \ldots + A_n(t)x(t) = 0$$

are obtained, where the $A_i(t)$ are continuous complex-valued functions on $[a, \infty)$. The solutions are found by transforming the given equation into a vector-matrix differential system for which the result of N. Levinson, Duke Math. J. 15 (1948), pp. 111 - 126, is applicable. To apply his result, asymptotic estimates for the zeros of related characteristic polynomials are obtained. The special case where n = 3 is treated in detail. The oscillatory nature and boundedness properties of solutions of the given equation are also investigated. The results generalize those of G. W. Pfeiffer, J. Diff. Equations 11 (1972), pp. 138 - 144 and pp. 145 - 155.

INTRODUCTION

In this thesis we investigate the asymptotic behavior of solutions of ordinary linear homogeneous differential equations

$$x^{(n)}(t) + A_1(t)x^{(n-1)}(t) + \ldots + A_n(t)x(t) = 0 \qquad (1)$$

where the $A_j(t)$ are continuous complex-valued functions on $[a,\infty)$. Several authors have considered particular cases of this problem. Coppel [2] determined asymptotic solutions of $(ry')' + py = 0$ under certain restrictions on p and q. To obtain his solutions, he transformed both the dependent and the independent variables so that the resulting equation could be compared with either $\ddot{z}(t) + z(t) = 0$ or $\ddot{z}(t) - z(t) = 0$. He used a fundamental result of Levinson [5] to determine the asymptotic solutions.

In a similar fashion, Hinton [4] studied the two-term differential equation $(ry^{(m)})^{(k)} \pm qy = 0$. His conclusions are based upon a comparison of the transformed system with a system with constant coefficients whose characteristic polynomial is given by either $\lambda^n + \lambda = 0$ or $\lambda^n - \lambda = 0$. Pfeiffer [6] discusses solutions of $y''' + py' + qy = 0$ by following the same procedures and ultimately comparing the transformed equation with one whose characteristic polynomial is either $\lambda^3 + \lambda = 0$ or $\lambda^3 - \lambda = 0$.

To study the solutions of equation (1), we employ the same techniques as those of the previously mentioned authors. However, we allow the transformed equation to be compared with a system whose characteristic polynomial is given by $\lambda^n + a_1\lambda^{n-1} + \ldots + a_n$, where the a_j are complex-valued constants. In order to apply the fundamental result of Levinson, we derive asymptotic estimates for the zeros of the characteristic polynomial. Lemma 1.2 and Lemma 1.5 of Chapter 1 include these estimates. The other lemmas of Chapter 1 are used to prove the central theorems of Chapter 2. These theorems present specific asymptotic solutions of equation (1). The case $n = 3$ is treated in detail because the algebraic calculations are not so tedious.

Chapter 3 contains several results describing the oscillatory nature and boundedness properties of solutions of (1). Some illustrative examples of the theory appear in Chapter 4.

CHAPTER 1

FUNDAMENTAL LEMMAS

The primary purpose of this thesis is to investigate the asymptotic behavior of solutions of ordinary, linear, homogeneous differential equations. The present chapter is devoted to a derivation of the transformations and lemmas used to prove the central theorems given in Chapter 2.

We begin with the n-th order equation

$$x^{(n)}(t) + A_1(t)x^{(n-1)}(t) +\dots+ A_n(t)x(t) = 0 \tag{1}$$

and assume that the coefficient functions $A_j(t)$ $j=1,2,\dots,n$, are continuous complex-valued functions of the real variable t on an interval $a \leq t < \infty$. Equation (1) may be replaced by the equivalent first order vector-matrix differential system

$$y'(t) = \mathcal{A}\, y(t) \tag{2}$$

where $y(t) = [x, x', \dots, x^{(n-1)}]^T$ and

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -A_n & -A_{n-1} & -A_{n-2} & \dots & -A_1 \end{bmatrix}$$

Upon changing the dependent variable from y to z(t), by setting $z(t) = T(t)y(t)$, where T is the nonsingular diagonal matrix $T = \mathrm{dia}[\varphi(t), 1, \varphi^{-1}(t), \ldots, \varphi^{2-n}(t)]$, and $\varphi(t)$ is any positive, continuously differentiable function on $a \leq t < \infty$, system (2) becomes

$$z'(t) = [T\mathcal{A}T^{-1} + T'T^{-1}]\, z(t). \qquad (3)$$

We transform this system, in turn, by introducing a new independent variable s given by

$$s(t) = \int_a^t \varphi(\tau)\, d\tau \qquad (4)$$

To ensure that the function s(t) has a unique continuous inverse $t = \psi(s)$, defined on $0 \leq s < \infty$, we further require $\varphi(t)$ to be non-integrable on $a \leq t < \infty$. In terms of the new variable s, (3) may be written as

$$\frac{dz(\psi(s))}{ds} = \mathcal{B}(\psi(s))\; z(\psi(s)) \qquad (5)$$

where $\mathcal{B}(\psi(s)) = [\varphi^{-1}T\mathcal{A}T^{-1} + \varphi^{-1}T'T^{-1}]$. It is important to note that $\psi(s)$, and not s, is the argument of each of the functions φ^{-1}, T, $\mathcal{A}$, T^{-1}, and T' occuring in this expression. The explicit structure of the matrix $\mathcal{B}$ is indicated by

$$\begin{bmatrix} \varphi'\varphi^{-2} & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & -\varphi'\varphi^{-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -A_n\varphi^{-n} & -A_{n-1}\varphi^{1-n} & -A_{n-2}\varphi^{2-n} & \cdots & -A_1\varphi^{-1} + (2-n)\varphi'\varphi^{-2} \end{bmatrix} \qquad (6)$$

The central theorems of Chapter 2 are applications of a fundamental result of Levinson [5] to system (5). For convenience of reference this result, taken without change of notation from Coddington and Levinson [1], page 92, is reproduced next as Theorem 1.1. Observe that the dependent variable z, as well as φ^{-1}, T, α, T^{-1} and T', are all composite functions of s in system (5). On the other hand, the dependent variable x and the $n \times n$ matrices V and R, in system (7) have the independent variable t as their arguments. The notation $|V|$ is used to denote the norm of V, and E is the $n \times n$ identity matrix.

Theorem 1.1 Consider the linear system

$$x'(t) = (A + V(t) + R(t))\, x(t) \tag{7}$$

Let A be a constant matrix with characteristic roots μ_j, $j=1,2,\ldots,n$, all of which are distinct. Let the matrix V be differentiable and satisfy

$$\int_0^\infty |V'(t)|\, dt < \infty$$

and let $V(t) \to 0$ as $t \to \infty$. Let the matrix R be integrable and let

$$\int_0^\infty |R(t)|\, dt < \infty$$

Let the roots of $\det(A + V(t) - \lambda E) = 0$ be denoted by $\lambda_j(t)$, $j=1,2,\ldots,n$. Clearly, by reordering the μ_j if necessary, $\lim_{t\to\infty} \lambda_j(t) = \mu_j$. For a given k, let

$$D_{kj}(t) = \mathrm{Re}(\lambda_k(t) - \lambda_j(t))$$

Suppose all j, $1 \leq j \leq n$, fall into one of two classes I_1 and I_2,

$j \in I_1$ if $\int_0^t D_{kj}(\tau)\, d\tau \to \infty$ as $t \to \infty$ and

$$\int_{t_1}^{t_2} D_{kj}(\tau)\, d\tau > -K \quad (t_2 \geq t_1 \geq 0)$$

$j \in I_2$ if $\int_{t_1}^{t_2} D_{kj}(\tau)\, d\tau < K \quad (t_2 \geq t_1 \geq 0)$

where k is fixed and K is a constant. Let p_k be a characteristic vector of A associated with μ_k, so that

$$A\, p_k = \mu_k\, p_k$$

Then there is a solution ϕ_k of (7) and a t_o, $0 \le t_o < \infty$, such that

$$\lim_{t\to\infty} \phi_k(t) \exp\left[- \int_{t_o}^{t} \lambda_k(\tau)\, d\tau\right] = p_k$$

If the hypothesis is satisfied for all k, $1 \le k \le n$, and if Φ is the matrix with columns $\phi_1, \phi_2, \ldots, \phi_n$, then Φ is a fundamental matrix because $\det\Phi(t) \ne 0$ for large t since the p_j are independent.

The lengthy form of the hypothesis of Theorem 1.1, concerning the real part of the difference of the eigenvalues, suggests the following convention. We shall, henceforth, say that afunction D(t), defined on $a \le t < \infty$, satisfies Condition I iff either

(i) $\int_a^{\infty} D(t)\, dt = \infty$ and

$$\int_{t_1}^{t_2} D(t)\, dt > -K \;,\; a \le t_1 \le t_2 \quad \text{for some constant } K$$

or else

(ii) $\int_{t_1}^{t_2} D(t)\, dt < K \;,\; a \le t_1 \le t_2$ for some constant K

By the same token, a function $D(s)$, defined on $0 \leq s < \infty$, satisfies Condition I iff (i) or (ii) is fulfilled with $a = 0$ and $t = s$. Once the domain of D is understood, we shall simply say that D satisfies Condition I.

To motivate much of what follows, let us examine, for a moment, Theorem 1.1. Since $\lambda_j(t)$ denotes a zero of an n-th degree polynomial in λ, if $n \geq 5$ it will usually be impossible to express $\lambda_j(t)$ explicitly in terms of the coefficients of the polynomial. Fortunately, exact expressions for these zeros are not needed to make use of Theorem 1.1. For, suppose each $\lambda_j(t)$ admits of an approximate representation of the form $\lambda_j(t) = \tilde{\lambda}_j(t) + \eta_j(t)$, where $\eta_j(t)$ is integrable on $[0,\infty)$. Then $\mathrm{Re}[\lambda_k(t) - \lambda_j(t)] = \mathrm{Re}[\tilde{\lambda}_k(t) - \tilde{\lambda}_j(t)] + \mathrm{Re}[\eta_k(t) - \eta_j(t)]$. It follows that $D_{kj}(t) = \mathrm{Re}[\lambda_k(t) - \lambda_j(t)]$ satisfies Condition I, with k fixed, and for every j, iff $\tilde{D}_{kj}(t) = \mathrm{Re}[\tilde{\lambda}_k(t) - \tilde{\lambda}_j(t)]$ does also.

As for the conclusion of the theorem, it entails the asymptotic relation $\lim_{t \to \infty} \varphi_k(t) \exp \int_{t_o}^{t} -\lambda_k(\tau)\, d\tau = p_k$ which becomes

$$M \cdot \lim_{t \to \infty} \varphi_k(t) \exp \int_{t_o}^{t} -\tilde{\lambda}_k(\tau)\, d\tau = p_k$$ where M is a nonzero constant.

But, a nonzero constant times a characteristic vector is also a characteristic vector. Hence, the conclusion of the theorem remains valid if $\lambda_j(t)$ is replaced by any approximation $\tilde{\lambda}_j(t)$ such that the difference $\lambda_j(t) - \tilde{\tilde{\lambda}}_j(t)$ is integrable, on $[t_o,\infty)$.

Now, let us return to the matrix $\mathcal{B}(\psi(s))$ of system (5). Suppose its characteristic polynomial is given by $P(\lambda) = \lambda^n + B_1(s)\lambda^{n-1} + \ldots + B_n(s)$ having $\lambda_k(s)$, $k=1,2,\ldots,n$, as its zeros. Suppose that the $B_j(s)$ are continuous and that $\lim_{s\to\infty} B_j(s) = a_j$, for $j=1,2,\ldots,n$, where the a_j are complex-valued constants. Set $p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n$ and impose the condition that the zeros μ_k, $k=1,2,\ldots,n$, of this polynomial shall be distinct. By Rouché's Theorem, and a reordering of the indices if necessary, we have that $\lambda_k(s)$ is continuous on $[0,\infty)$, and we may take $\lim_{s\to\infty} \lambda_k(s) = \mu_k$, $k=1,2,\ldots,n$.

The rate at which each difference $\lambda_k(s) - \mu_k$ converges to zero is dependent upon how fast the differences $B_j(s) - a_j$ of the respective coefficients in $P(\lambda)$ and $p(\lambda)$ converge to zero. Under our assumptions, $B_j(s) - a_j$ and $\lambda_k(s) - \mu_k$ belong to the class of complex-valued functions $f(s)$ that are continuous on $[0,\infty)$, and which converge to zero as s becomes infinite. It can be readily verified that (Lemma 1, Appendix 1) any such function that is a member of $L^q[0,\infty)$ is also a member of $L^r[0,\infty)$ for $r \geq q$. Furthermore, if f(s) and g(s) are two such functions with $f(s) \in L^q[0,\infty)$ and

$g(s) \in L^r[0,\infty)$, an application of Holder's inequality (Lemma 2, Appendix 1) shows that the product $f(s)\cdot g(s)$ of these functions is a member of $L^m[0,\infty)$ where $m = \max\,(1,\, qr/(q+r))$.

As has been mentioned, exact formulas for the zeros of an arbitrary n-th degree polynomial cannot be found, in general. Thus, it becomes necessary to resort to methods that yield approximations to such zeros. We now proceed to establish five important lemmas. Two of these give useful approximations to $\lambda_k(s)$. The other three set forth sufficient conditions for $\mathrm{Re}[\lambda_k(s) - \lambda_j(s)]$, defined on $[0,\infty)$, to satisfy Condition I.

<u>Lemma 1.2</u> Let $P(\lambda) = \lambda^n + B_1(s)\lambda^{n-1} + \ldots + B_n(s)$ where the $B_j(s)$ are continuous complex-valued functions defined on $0 \le s < \infty$. Suppose that

(i) $\lim_{s\to\infty} B_j(s) = a_j$. $j=1,2,\ldots,n$, where the a_j are complex-valued constants,

(ii) $p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n$ has distinct zeros μ_j, $j=1,2,\ldots,n$,

(iii) $(a_j - B_j(s)) \in L^2[0,\infty)$, $j=1,2,\ldots,n$.

Then the zeros $\lambda_k(s)$, $k=1,2,\ldots,n$, of $P(\lambda)$ are given by

$$\lambda_k(s) = \mu_k + (p'(\mu_k))^{-1} \sum_{j=1}^{n} [a_j - B_j(s)]\mu_k^{n-j} + \eta_k(s)$$ where $\eta_k(s)$ is some integrable function on $[0,\infty)$, and where $\lambda_k(s)$ is that zero of $P(\lambda)$ which converges to μ_k as s becomes infinite. To avoid ambiguity of notation, let it be understood that

$\lambda_k(s) = (p'(0))^{-1}[a_n - B_n(s)] + \eta_k(s)$, if $\mu_k = 0$.

Proof: As has been pointed out, hypothesis (i) makes it possible to ensure that $\lambda_k(s) - \mu_k$ is continuous on $[0,\infty)$ and converges to zero as s becomes infinite. Hence, we may set $\lambda_k(s) = \mu_k + \alpha_k(s)$ where $\lim_{s \to \infty} \alpha_k(s) = 0$. Since $\lambda_k(s)$ is a zero of $P(\lambda)$, $P(\mu_k + \alpha_k(s)) = 0$.

The left member of this equation may be viewed as a polynomial of degree n in $\alpha_k(s)$. When written as such, we obtain

$$P(\mu_k) + P'(\mu_k)\alpha_k + \frac{P''(\mu_k)}{2!}\alpha_k^2 + \ldots + \alpha_k^n = 0$$

which gives $P(\mu_k) + P'(\mu_k)\alpha_k[1 + o(1)] = 0$, where $o(1)$ designates a function of s that converges to zero as s becomes infinite. Solving for α_k, and noting that $1/[1 + o(1)] = [1 + o(1)]$, we have

$$\alpha_k(s) = -\frac{P(\mu_k)}{P'(\mu_k)}[1 + o(1)]$$

Now, $p(\mu_k) = 0$, and $P'(\mu_k) = [p'(\mu_k) + o(1)]$, because the differences $B_j(s) - a_j$, $j=1,2,\ldots,n$, in $P'(\mu_k) - p'(\mu_k) =$

$\sum_{j=1}^{n-1} (n-j)[B_j(s) - a_j]\mu_k^{n-j-1}$ are all continuous on $[0,\infty)$ and converge to zero as s becomes infinite. Moreover, $[1 + o(1)]/[p'(\mu_k) + o(1)] = [1 + o(1)]/p'(\mu_k)$, and $p'(\mu_k) \neq 0$ since, by (ii), the μ_k are distinct. Hence

$$\alpha_k(s) = \frac{-P(\mu_k) + p(\mu_k)}{p'(\mu_k)}\,[1 + o(1)] = \varepsilon_k(s) + \mu_k(s),$$

where $\varepsilon_k(s) = (p'(\mu_k))^{-1} \sum_{j=1}^{n} [a_j - B_j(s)]\mu_k^{n-j} =$ $(p'(\mu_k))^{-1}[p(\mu_k) - P(\mu_k)]$ and $\lim_{s\to\infty} \eta_k(s) = 0$. Note that s appears in the coefficients of $P(\mu_k)$. Since $\lambda_k(s) = \mu_k + \varepsilon_k(s) + \eta_k(s)$, it is evident that $\eta_k(s)$ has the form given in the conclusion of the theorem.

We next prove that $\eta_k(s)$ is integrable. Starting with the equation $P(\mu_k + \varepsilon_k(s) + \eta_k(s)) = 0$, we express its left hand member as a polynomial of degree n in $\eta_k(s)$. This gives $P(\mu_k + \varepsilon_k) + P'(\mu_k + \varepsilon_k)\ {}_k + \ldots + \eta_k^n = 0$, which may be expressed as $P(\mu_k + \varepsilon_k) + P'(\mu_k + \varepsilon_k)\ {}_k[1 + o(1)] = 0$. For reasons that are already familiar, we have $P'(\mu_k + \varepsilon_k) = p'(\mu_k) + o(1)$. Thus, we find that $P(\mu_k + \varepsilon_k) + p'(\mu_k)\eta_k[1 + o(1)] = 0$.

Now, expand $P(\mu_k + \varepsilon_k)$ as a polynomial in ε_k, to obtain

$$P(\mu_k) + P'(\mu_k)\varepsilon_k + \ldots + \varepsilon_k^n + p'(\mu_k)\eta_k[1 + o(1)] = 0 \qquad (8)$$

Then use the definition of $\varepsilon_k(s)$ to verify that

$$P(\mu_k) + P'(\mu_k)\varepsilon_k = [P'(\mu_k) - p'(\mu_k)]\varepsilon_k$$

Equation (8) becomes

$$[P'(\mu_k) - p'(\mu_k)]\varepsilon_k + \frac{1}{2} P''(\mu_k)\varepsilon_k^2 + \ldots + \varepsilon_k^n = -p'(\mu_k)\eta_k[1 + o(1)]$$

Each term on the left hand side of this last equation consists of bounded functions times products of two or more factors of the type $B_j(s) - a_j$, $j-1,2,\ldots,n$. From hypothesis (iii), it follows that each term is an integrable function on $[0,\infty)$ so that $\eta_k[1 + o(1)]$ is integrable. Therefore, $\eta_k(s)$ is integrable, and the proof is complete.

In stating our next lemma, it will be convenient if we first define $D_{kj}(s) = \mathrm{Re}[\lambda_k(s) - \lambda_j(s)]$, on $0 \leq s < \infty$. The lemma specifies sufficient conditions for each $D_{kj}(s)$ to satisfy Condition I when k is fixed, and $j=1,2,\ldots,n$.

Lemma 1.3 Let $P(\lambda) = \lambda^n + B_k(s)\lambda^{n-1} + \ldots + B_n(s)$ where the $B_j(s)$ are continuous, complex-valued functions defined on $0 \leq s < \infty$. Suppose that the zeros of $P(\lambda)$ are denoted by $\lambda_k(s)$, k=1,2,...,n, and that

(i) $\lim_{s \to \infty} B_j(s) = a_j$, j=1,2,...,n, where the a_j are complex-valued constants,

(ii) the zeros μ_j, j=1,2,...,n, of $p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n$ are distinct,

(iii) $a_j - B_j(s)$, j=1,2,...,n, are members of $L^2[0,\infty)$,

(iv) for fixed k, $\mathrm{Re} \sum_{m=1}^{n} [a_m - B_m(s)] \left[(p'(\mu_k))^{-1}\mu_k^{n-m} - (p'(\mu_j))^{-1}\mu_j^{n-m} \right]$

satisfies Condition I if j is an index such that $\mathrm{Re}[\mu_k - \mu_j] = 0$.

Then, $D_{kj}(s) = \mathrm{Re}[\lambda_k(s) - \lambda_j(s)]$ satisfies Condition I for the fixed k and all j-1,2,...,n.

Proof: This lemma will be proved by considering the separate cases when $\mathrm{Re}[\mu_k - \mu_j] \neq 0$ and when $\mathrm{Re}[\mu_k - \mu_j] = 0$. First, suppose that k is fixed and j is an index for which $\mathrm{Re}[\mu_k - \mu_j] \neq 0$. Then $\lim_{s\to\infty} \mathrm{Re}[\lambda_k(s) - \lambda_j(s)] = L \neq 0$ because $\lambda_k(s)$ and $\lambda_j(s)$ are continuous on $[0,\infty)$ and converge to μ_k and μ_j, respectively. If L is positive, $D_{kj}(s)$ satisfies part (i) of Condition I. If L is negative, part (ii) of Condition I holds. Consequently, $\mathrm{Re}[\lambda_k(s) - \lambda_j(s)]$ satisfies Condition I for fixed k, and any j such that $\mathrm{Re}[\mu_k - \mu_j] \neq 0$.

Next, assume that for some index j, $\mathrm{Re}[\mu_k - \mu_j] = 0$. Then, by Lemma 1.2, $\mathrm{Re}[\lambda_k(s) - \lambda_j(s)]$ is equal to

$$\mathrm{Re} \sum_{m=1}^{n} [a_m - B_m(s)]\Big[(p'(\mu_k))^{-1}\mu_k^{n-m} - (p'(\mu_j))^{-1}\mu_j^{n-m}\Big] + \mathrm{Re}[\eta_k(s) - \eta_j(s)],$$

where both η_k and η_j are integrable on $0 \leq s < \infty$. (Note that this expression vanished if j=k, so there is no need to assume that j≠k.) In virtue of hypothesis (iv), $\mathrm{Re}[\lambda_k(s) - \lambda_j(s)]$ fulfills Condition I. Hence, the conclusion of the lemma is established.

An alternative, and more easily applied version of Lemma 1.3 is given by:

Lemma 1.4 With $P(\lambda)$, $B_j(s)$, and $\lambda_k(s)$ the same as in Lemma 1.3, suppose that

(i) $\lim_{s\to\infty} B_j(s) = a_j$, $j=1,2,\ldots,n$, where the a_j are complex-valued constants,

(ii) the zeros μ_j, $j=1,2,\ldots,n$, of $p(\lambda) = \lambda^n + a_1\lambda^{n-1} +\ldots+ a_n$ are distinct,

(iii) $a_j - B_j(s)$, $j=1,2,\ldots,n$, are integrable on $0 \leq s < \infty$,

Then $\mathrm{Re}[\lambda_k(s) - \lambda_j(s)]$ satisfies Condition I for $k=1,2,\ldots,n$, and $j=1,2,\ldots,n$.

Proof: Since, the $a_j - B_j(s)$ are continuous on $[0,\infty)$, and tend to zero as s becomes infinite, they are all members of $L^2[0,\infty)$. Hence, the zeros $\lambda_k(s)$ $k=1,2,\ldots,n$, may be represented as in Lemma 1.2. For any particular k, $\mathrm{Re}[\lambda_k(s) - \lambda_j(s)] = \mathrm{Re}[\mu_k - \mu_j] +$

$$\mathrm{Re}\sum_{m=1}^{n}[a_m - B_m(s)]\left[(p'(\mu_k))^{-1}\mu_k^{n-m} - (p'(\mu_j))^{-1}\mu_j^{n-m}\right] + \mathrm{Re}[\eta_k(s)-\eta_j(s)]$$

As in the proof of Lemma 1.3, we consider when $\mathrm{Re}[\mu_k - \mu_j] \neq 0$, and when $\mathrm{Re}[\mu_k - \mu_j] = 0$. If $\mathrm{Re}[\mu_k - \mu_j] \neq 0$, it follows as before that $\mathrm{Re}[\lambda_k(s) - \lambda_j(s)]$ satisfies Condition I. In case $\mathrm{Re}[\mu_k - \mu_j] = 0$, (iii) and Lemma 1.2 ensure that $\mathrm{Re}[\lambda_k(s) - \lambda_j(s)]$ is equal to a finite sum, each term of which is integrable on $[0,\infty)$. Since any integrable function satisfies part (ii) of Condition I, $\text{Re}[\lambda_k(s) - \lambda_j(s)]$ satisfies Condition I for $k=1,2,\ldots,n$, and $j=1,2,\ldots,n$, as was to be shown.

To illustrate how Lemma 1.2 may be replaced by an analogous result, involving a more general requirement than we present the following lemma. Further extensions are possible at the expense of considerable additional algebraic manipulations and attending analytical difficulties.

Lemma 1.5 Let $P(\lambda) = \lambda^n + B_1(s)\lambda^{n-1} + \ldots + B_n(s)$ where the $B_j(s)$ are continuous complex-valued functions on $0 \leq s < \infty$. Suppose that

(i) $\lim_{s \to \infty} B_j(s) = a_j$, $j=1,2,\ldots,n$, where the a_j are complex-valued constants,

(ii) $p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n$ has distinct zeros μ_j, $j=1,2,\ldots,n$,

(iii) $a_j - B_j(s)$, $j=1,2,\ldots,n$, are members of $L^3[0,\infty)$,

Then, the zeros $\lambda_k(s)$, $k=1,2,\ldots,n$, of $P(\lambda)$ are given by

$$\lambda_k(s) = \mu_k + \varepsilon_k(s) + \eta_k(s)$$

where $\eta_k(s)$ is integrable on $0 \leq s < \infty$, and

$$\varepsilon_k(s) = (p'(\mu_k))^{-1} \sum_{m=1}^{n} [a_m - B_m(s)]\mu_k^{n-m} +$$

$$(p'(\mu_k))^{-2} \left[\sum_{m=1}^{n} [a_m - B_m(s)]\mu_k^{n-m}\right]\left[\sum_{m=1}^{n} [a_m - B_m(s)](n-m)\mu_k^{n-m-1}\right] +$$

$$\frac{1}{2} (p'(\mu_k))^{-3}\, p''(\mu_k) \left[\sum_{m=1}^{n} [a_m - B_m(s)]\mu_k^{n-m}\right]^2 .$$

Proof: The proof of this lemma follows along the same lines as that of Lemma 1.2. Set $\lambda_k(s) = \mu_k + \varepsilon_k(s) + \eta_k(s)$, where $\varepsilon_k(s)$ is that function already defined in the statement of the lemma. Since $\lambda_k(s)$ satisfies the equation $P(\lambda_k) = 0$, for every s, we have $P(\mu_k + \varepsilon_k + \eta_k) = 0$. The left side of this equation may be viewed as a polynomial in $\eta_k(s)$, so that we have $P(\mu_k + \varepsilon_k) + P'(\mu_k + \varepsilon_k)\eta_k + \ldots + \eta_k^n = 0$. By hypotheses (i) and (ii), the term $\varepsilon_k(s)$ and the difference $\lambda_k(s) - \mu_k$ both converge to zero as s tends to infinity. Hence, $\eta_k(s)$ also converges to zero, and so

$$P(\mu_k + \varepsilon_k) + P'(\mu_k + \varepsilon_k)\eta_k[1 + o(1)] = 0.$$

As s becomes infinite, $P'(\mu_k + \varepsilon_k)$ converges to the nonzero value $p'(\mu_k)$. Using this fact, we obtain

$$P(\mu_k + \varepsilon_k) + p'(\mu_k)\eta_k[1 + o(1)] = 0.$$

Expansion of $P(\mu_k + \varepsilon_k)$, as a polynomial in ε_k, yields

$$P(\mu_k) + P'(\mu_k)\varepsilon_k + \ldots + \varepsilon_k^n + p'(\mu_k)\eta_k[1 + o(1)] = 0 \qquad (9)$$

Hypotheses (i) and (iii), ensure that $\varepsilon_k(s)$ is a member of $L^3[0,\infty)$. Therefore, $\varepsilon_k^m(s)$ is integrable on $0 \leq s < \infty$, for $3 \leq m \leq n$. (Lemma 2, Appendix 1). Equation (9) may be written as

$$P(\mu_k) + P'(\mu_k)\varepsilon_k + \frac{1}{2} P''(\mu_k)\varepsilon_k^2 + f_k(s) + p'(\mu_k)\eta_k[1 + o(1)] = 0 \qquad (10)$$

where $f_k(s)$ is some integrable function on $[0,\infty)$. Defining $\tilde{P}(\lambda) = P(\lambda) - p(\lambda)$, we transform equation (10) into

$$\tilde{P}(\mu_k) + [p'(\mu_k) + \tilde{P}(\mu_k)] + \frac{1}{2} p''(\mu_k)\varepsilon_k^2 + g_k(s) =$$

$$-p'(\mu_k)\eta_k[1 + o(1)] \qquad (11)$$

where $g_k(s) = f_k(s) + \frac{1}{2} \tilde{P} (\mu_k)\varepsilon_k^2$. From the definition of $\tilde{P}(\lambda)$, together with (iii), it follows that $\tilde{P}(\mu_k)$ is a member of $L^3[0,\infty)$ also. Consequently, $\tilde{P}'(\mu_k)$, and $\tilde{P}''(\mu_k)$, are members of $L^3[0,\infty)$. The function $g_k(s)$ is therefore integrable on $0 \leq s < \infty$. Replacing ε_k by its expression

$$\varepsilon_k = -(p'(\mu_k))^{-1} \tilde{P}(\mu_k) + (p'(\mu_k))^{-2} \tilde{P}(\mu_k) \tilde{P}'(\mu_k) +$$

$$\frac{1}{2} (p'(\mu_k))^{-3} (\tilde{P}(\mu_k))^2 p''(\mu_k),$$

we conclude that the left side of equation (11) is integrable. Hence η_k is an integrable function, and the lemma holds.

In order to introduce our final lemma, we make two important restrictions in the hypotheses (i) and (ii) of Lemma 1.5. First, we require that all of the coefficients a_j of $p(\lambda)$ be real, instead of allowing them to be complex-valued. Since all of the μ_j are still to be distinct, complex-valued zeros of $p(\lambda)$ must now occur in conjugate pairs, if they occur at all. Secondly, we impose the condition that no more than two of the μ_j are to have real parts equal to the same real number. In particular, if $p(\lambda)$ admits of distinct conjugate pairs of complex zeros, the real part of one pair is not equal to the real part of any other pair. Neither is a real zero to equal the real part of a complex zero. These alterations in Lemma 1.5, lead to the following result.

Lemma 1.6 Let $P(\lambda) = \lambda^n + B_1(s)\lambda^{n-1} + \ldots + B_n(s)$ where the $B_j(s)$ are continuous, complex-valued functions defined on $0 \le s < \infty$. Let the zeros of $P(\lambda)$ be denoted by $\lambda_k(s)$, $k=1,2,\ldots,n$, and suppose that

(i) $\lim_{s \to \infty} B_j(s) = a_j$, $j=1,2,\ldots,n$, where the a_j are real-valued constants,

(ii) the zeros μ_j, $j=1,2,\ldots,n$, of $p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n$ are distinct; furthermore, $\text{Re}\mu_j = \text{Re}\mu_k$ implies that $\mu_j = \overline{\mu}_k$,

(iv) for fixed k,

$$- \sum_{m=1}^{n} \text{Im}[B_m(s)]\text{Im}\left[\frac{\mu_k^{n-m}}{p'(\mu_k)}\right] -$$

$$\sum_{m=1}^{n} \sum_{r=1}^{n} \text{Im}[(a_m - B_m(s))(a_r - B_r(s))]\text{Im}\left[\frac{(n-m)\mu_k^{2n-m-r-1}}{(p'(\mu_k))^2}\right] -$$

$$\frac{1}{2} \sum_{m=1}^{n} \sum_{r=1}^{n} \text{Im}[(a_m - B_m(s))(a_r - B_r(s))]\text{Im}\left[\frac{p''(\mu_k)\,\mu_k^{2n-m-r}}{(p'(\mu_k))^3}\right]$$

satisfies Condition I for any index j such that $\text{Re}[\mu_j - \mu_k] = 0$.

Then, $\text{Re}[\lambda_k(s) - \lambda_j(s)]$ satisfies Condition I for the fixed k and all $j=1,2,\ldots,n$.

Proof: As in the proof of Lemma 1.3, we consider the separate cases: when $\mathrm{Re}[\mu_k - \mu_j] \neq 0$, and when $\mathrm{Re}[\mu_k - \mu_j] = 0$. If j is an index for which $\mathrm{Re}[\mu_k - \mu_j] \neq 0$, the proof that $\mathrm{Re}[\lambda_k - \lambda_j]$ satisfies Condition I is just the same as in the proof of Lemma 1.3.

So suppose j is an index such that $\mathrm{Re}[\mu_k - \mu_j] = 0$. Then, since the coefficients of $p(\lambda)$ are real numbers, hypothesis (ii) implies that $\mu_k = \overline{\mu_j}$, where $\overline{\mu_j}$ denotes the complex conjugate of μ_j. Under our present restrictions on the a_j and the zeros of $p(\lambda)$, Lemma 1.5 still applies. Thus, we have $\lambda_k(s) = \mu_k + \varepsilon_k(s) + \eta_k(s)$ and $\lambda_j(s) = \mu_j + \varepsilon_j(s) + \eta_j(s)$, where $\eta_k(s)$ and $\eta_j(s)$ are integrable, and $\varepsilon_k(s)$ and $\varepsilon_j(s)$ have the form given in the conclusion of that lemma. Using these expressions, and the fact that $\mathrm{Re}[\mu_k - \mu_j] = 0$, we obtain $\mathrm{Re}[\lambda_k(s) - \lambda_j(s)] = \mathrm{Re}[\varepsilon_k(s) - \varepsilon_j(s)] + \mathrm{Re}[\eta_k(s) - \eta_j(s)]$. As was inferred earlier, when $\eta_k(s)$ and $\eta_j(s)$ are integrable on $[0,\infty)$, $\mathrm{Re}[\lambda_k(s) - \lambda_j(s)]$ satisfies Condition I iff $\mathrm{Re}[\varepsilon_k(s) - \varepsilon_j(s)]$ does. Hence, our proof will be complete if we can show that $\mathrm{Re}[\varepsilon_k(s) - \varepsilon_j(s)]$ satisfies Condition I with $\mu_k = \overline{\mu_j}$.

To this end, denote the imaginary unit by i and the complex conjugates of ε_k and ε_j by $\overline{\varepsilon_k}$ and $\overline{\varepsilon_j}$, respectively. Also observe that $\mathrm{Re}[\varepsilon_k - \varepsilon_j] = (1/2)[\varepsilon_k + \overline{\varepsilon_k} - \varepsilon_j - \overline{\varepsilon_j}]$, and that $p'(\overline{\mu_j}) = \overline{p'(\mu_k)}$. By using familiar conjugation laws, we may then verify that

$$\varepsilon_k - \varepsilon_k = 2i \sum_{m=1}^{n} [a_m - B_m(s)] \mathrm{Im} \left[\frac{\mu_k^{n-m}}{p'(\mu_k)} \right] +$$

$$2i \sum_{m=1}^{n} \sum_{r=1}^{n} [a_m - B_m(s)][a_r - B_r(s)] \mathrm{Im} \left[\frac{(n - m)\mu_k^{2n-m-r-1}}{(p'(\mu_k))^2} \right] +$$

$$i \sum_{m=1}^{n} \sum_{r=1}^{n} [a_m - B_m(s)][a_r - B_r(s)] \mathrm{Im} \left[\frac{p''(\mu_k)\mu_k^{2n-m-r}}{(p'(\mu_k))^3} \right]$$

Finally, add to the forgoing expression its complex conjugate. Except for a factor of 4, this results in the expression given in hypothesis (iv), which is presumed to fulfill Condition I. We conclude that, for fixed k, $\mathrm{Re}[\lambda_k(s) - \lambda_j(s)]$ satisfies Condition I in case j is any index such that $\mathrm{Re}[\mu_k - \mu_j] = 0$. This concludes the proof of the lemma.

CHAPTER 2

ASYMPTOTIC SOLUTIONS

In this chapter we present five theorems and one corollary describing asymptotic solutions of linear differential equations. The proofs of these results are based on the preliminary lemmas and Theorem 1.1, of Chapter 1. The first two theorems, as well as the final theorem, are expressed in rather general terms. Therefore, they appear to be quite complicated. The other theorems deal with a third order linear differential equation.

In the case of the third order equation, it is possible to make a number of simplifications in the general theory. For then, the order of the differential equation is no longer an indefinite fixed "n". Thus, with considerably less effort, we are able to obtain sharper results when $n=3$.

Oscillatory and boundedness properties of our asymptotic solutions are examined in Chapter 3.

Theorem 2.1 Consider the linear differential equation

$$x^{(n)}(t) + A_1(t)x^{(n-1)}(t) + \ldots + A_n(t)x(t) = 0 \qquad (1)$$

Suppose that

(i) $A_j(t)$, $j=1,2,\ldots,n$, are continuous complex-valued functions defined on $[a,\infty)$,

(ii) $\varphi(t)$ is a positive, continuously differentiable function defined on $[a,\infty)$,

(iii) $\int_a^\infty \varphi(t)\, dt = \infty$,

(iv) $\lim_{t\to\infty} \varphi'(t)\varphi^{-2}(t) = 0$,

(v) $\lim_{t\to\infty} A_j(t)\varphi^{-j}(t) = a_j$, $j=1,2,\ldots,n$, where the a_j are complex valued constants,

(vi) $p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_j$ has distinct zeros μ_j, $j=1,2,\ldots,n$,

(vii) $(\varphi'\varphi^{-2})'$ and $(A_j\varphi^{-j})'$, $j=1,2,\ldots,n$, are integrable on $[a,\infty)$,

(viii) $\varphi'\varphi^{-3/2}$ and $[a_j - A_j\varphi^{-j}]\varphi^{1/2}$, $j=1,2,\ldots,n$, are all members of $L^2[a,\infty)$,

(ix) For fixed k,

$$\mathrm{Re}\left\{\sum_{m=1}^{n}\left[a_m - \frac{(n-m+1)^2-3(n-m+1)}{2} A_{m-1}\varphi^{-m-1}\varphi' - A_m\varphi^{-m}\right] \cdot \right.$$

$$\left.\left[(p'(\mu_k))^{-1}\mu_k^{n-m} - (p'(\mu_j))^{-1}\mu_j^{n-m}\right] \cdot \varphi\right\}$$

satisfies Condition I if j is an index for which $\mathrm{Re}\mu_j = \mathrm{Re}\mu_k$. (See Page 5 for the definition of Condition I).

Then, there exists a solution $x_k(t)$ of (1) and a number t_o, $a \leq t_o < \infty$, such that

$$x_k(t) = \varphi^{-1}(t)\left[\exp \int_{t_o}^{t} [\mu_k + \varepsilon_k(\tau)]\varphi(\tau)\, d\tau\right][1 + o(1)] \tag{12}$$

where

$$\varepsilon_k(t) = \sum_{m=1}^{n}(p'(\mu_k))^{-1}\Big[a_m - A_m\varphi^{-m} -$$

$$\frac{(n-m+1)^2-3(n-m+1)}{2} A_{m-1}\varphi'\varphi^{-m-1}\Big] \cdot \mu_k^{n-m}$$

Furthermore, the first n-1 derivatives of $x_k(t)$ have the asymptotic form

$$x_k^{(j)}(t) = \varphi^{j-1}(t)\left[\exp \int_{t_o}^{t} [\mu_k + \varepsilon_k(\tau)]\varphi(\tau)\, d\tau\right][\mu_k^j + o(1)]$$

for j=1,2,...,n-1.

If hypothesis (ix) is satisfied for k=1,2,...,n, then there will exist n linearly independent solutions of the type just described.

Proof: In the previous chapter, we transformed Equation (1) into the system

$$\frac{dz(\psi(s))}{ds} = \mathcal{B}(\psi(s)) \ z(\psi(s)) \qquad (5)$$

The argument $\psi(s)$ was obtained as the inverse of the function $s(t) = \int_a^t \varphi(\tau)d\tau$. Expression (6), Page 2, indicates the explicit form of the matrix $\mathcal{B}(\psi(s))$. It was noted that hypothesis (i), (ii), and (iii) ensure that $t = \psi(s)$ is uniquely defined, and that the elements of $\mathcal{B}(\psi(s))$, as well as $\psi(s)$, are continuous on $0 \leq s < \infty$. Thus, the transformation of (1) into (5) is possible. Moreover, t tends to infinity with s.

With the addition of hypothesis (iv) and (v), it follows that $\mathcal{B}(\psi(s))$ converges to the constant matrix C whose elements are identified by

$$C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

A direct expansion of $\det[\mathcal{B}(\psi)(s)) - \lambda E]$, reveals that the coefficients of the characteristic polynomial

$P(\lambda) = \lambda^n + B_1(s)\lambda^{n-1} + \ldots + B_n(s)$, associated with $\mathcal{B}(\psi(s))$, may be written as

$$B_j(s) = A_j\varphi^{-j} + \frac{(n-j+1)^2-3(n-j+1)}{2} A_{j-1}\varphi'\varphi^{-j-1} + b_j(s) \qquad (13)$$

wherein $\psi(s)$ is the argument of A_j, φ, and A_{j-1}, and $b_j(s)$ is a sum of terms each involving $\varphi'\varphi^{-2}$ raised to a power $m \geq 2$, and $A_0 = 1$.

Again, let $\lambda_j(s)$ and μ_j denote the zeros of $P(\lambda)$ and $p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n$, respectively. As has already been pointed out on page 7 , we may assume that $\lim_{s\to\infty} \lambda_j(s) = \mu_j$, $j=1,2,\ldots,n$. Because of (vi), each μ_j is a simple zero of $p(\lambda)$; therefore, $p'(\mu_k) \neq 0$. The rest of our proof consists of a detailed explanation of how Theorem 1.1 may be applied to system (5).

With s, rather than t, viewed as the independent variable, set $A = C$, $V(s) = \mathcal{B}(\psi(s)) - C$, and $R(s) = 0$. Hypothesis (vii) then guarantees that $\int_0^\infty |V'(s)|ds = \int_a^\infty |B'(t)|dt < \infty$, $\lim_{s\to\infty} V(s) = 0$.

Of course, $\int_0^\infty |R(s)|ds = 0$. Hence, all of the hypotheses of Theorem 1.1 will be satisfied, provided $D_{kj}(s) = \text{Re}[\lambda_k(s) - \lambda_j(s)]$ fulfills Condition I with k fixed and for all $j=1,2,\ldots,n$.

To ascertain the behavior of the $D_{kj}(s)$, we show that the hypotheses of Lemma 1.3 are fulfilled. Returning to Equation (13) and using (iv) and (v) we first verify that $\lim_{s\to\infty} B_j(s) = a_j$. As has already been mentioned, the μ_j are distinct. Next, employ the transformation $t = \psi(s)$ to change from the variable t to s in hypothesis (viii). This reveals that $\varphi'(\psi(s))\ \varphi^{-2}(\psi(s))$ and $[a_j - A_j(\psi(s))\ \varphi^{-j}(\psi(s))]$ are members of $L^2[0,\infty)$. These functions are also continuous on $0 \leq s < \infty$, and they converge to zero as s becomes infinite. Consequently, $b_j(s)$ is integrable on $[0,\infty)$. Thus, $a_j - B_j(s)$ is a member of $L^2[0,\infty)$ for $j=1,2,\ldots,n$. We have shown that every hypothesis of Lemma 1.3 holds, except the last one. But, it also holds, in virtue of (iv). Therefore, Lemma 1.3 is in force. It follows that, for fixed k, Condition I is satisfied by each $D_{kj}(s)$, $j=1,2,\ldots,n$.

With this result established, Theorem 1.1 may be applied. We conclude that there exists a solution $z_k(\psi(s))$ of system (5), and a number s_o, $0 \leq s_o < \infty$, such that

$$z_k(\psi(s) = [\exp \int_{s_o}^{s} \lambda_k(\xi)d\xi][p_k + \mathbf{o}(1)] \tag{14}$$

where p_k is the eigenvector of C corresponding to the eigenvalue μ_k. The symbolism $\mathbf{o}(1)$ denotes a vector whose components tend to zero (but not necessarily at the same rate) as s becomes infinite.

Let us center our attention, for a moment, on Equation (14). By Lemma 1.2, we have

$$\lambda_k(s) = \mu_k + (p'(\mu_k))^{-1} \sum_{j=1}^{n} [a_j - B_j(s)]\mu_k^{n-j} + \eta_k(s)$$

where $\eta_k(s)$ is integrable on $[0,\infty)$. Replacing $B_j(s)$ by its equivalent expression (13), we obtain

$$\lambda\ (s) = \mu_k + (p'(\mu_k))^{-1} \sum_{j=1}^{n} [a_j - A_j\varphi^{-j} -$$

$$\frac{(n-j+1)^2-3(n-j+1)}{2} A_{j-1}\varphi'\varphi^{-j-1} - b_j(s)]\mu_k^{n-j} + \eta_k(s)$$

where $b_j(s)$ is integrable on $[0,\infty)$. Collecting integrable terms, we may write this equation as $\lambda_k(s) = \mu_k + \varepsilon_k^*(\psi(s)) + \eta_k^*(s)$ where

$$\varepsilon_k^* = (p'(\mu_k))^{-1} \sum_{j=1}^{n} [a_j - A_j\varphi^{-j} - \frac{(n-j+1)^2 - 3(n-j+1)}{2} A_{j-1}\varphi'\varphi^{-j-1}]\mu_k^{n-j}$$

and

$$\eta_k^*(s) = \eta_k(s) - (p'(\mu_k))^{-1} \sum_{j=1}^{n} b_j(s)\mu_k^{n-j}$$

is integrable on $[0,\infty)$.

Using this result, in (14), we have

$$z_k(\psi(s)) = [\exp \int_{s_o}^{s} [\mu_k + \varepsilon_k^*(\psi(\xi))]d\xi][\exp \int_{s_o}^{s} \eta_k^*(\xi)d\xi][p_k + o(1)].$$

Now set $d_k = \exp \int_{s_o}^{\infty} \eta_k^*(s)ds$, and observe that

$\exp \int_{s_o}^{s} \eta_k^*(\xi)d\xi = d_k \exp[o(1)] = d_k[1 + o(1)]$. What is more,

$d_k[1 + o(1)][p_k + o(1)] = [d_k p_k + o(1)]$; consequently,

$$z_k(\psi(s)) = [\exp \int_{s_o}^{s} [\mu_k + \varepsilon_k^*(\psi(\xi)) d\xi][d_k p_k + o(1)]$$

Note that d_k is a scalar, whereas p_k is an n-component vector. Recalling that $t = \psi(s)$, and that t becomes infinite with s, we rewrite this result as

$$z_k(t) = [\exp \int_{t_o}^{t} [\mu_k + \varepsilon_k(\tau)]\varphi(\tau)d\tau][d_k p_k + o(1)]$$

where $\varepsilon_k(t)$ is the same function of t that ε_k^* is of $\psi(s)$. Without loss of generality, we may take $d_k p_k = [1, \mu_k, \mu_k^2, \ldots, \mu_k^{n-1}]^T$. For, with $d_k p_k$ thus defined, it follows by direct expansion, and the fact that $p(\mu_k) = 0$, that $[C - \mu_k E]\, d_k p_k = 0$. Since $z(t) = T(t)y(t)$, we have

$$\begin{bmatrix} \varphi(t)\; x_k(t) \\ x_k'(t) \\ \varphi^{-1}(t)\; x_k''(t) \\ \vdots \\ \varphi^{2-n}(t)\; x_k^{(n-1)}(t) \end{bmatrix} = [\exp \int_{t_o}^{t} [\mu_k + \varepsilon_k(\tau)]\varphi(\tau)d\tau] \begin{bmatrix} 1 + o(1) \\ \varphi_k + o(1) \\ \varphi_k^2 + o(1) \\ \vdots \\ \varphi_k^{n-1} + o(1) \end{bmatrix}$$

From this we verify that $x_k(t)$ and its first n-1 derivatives have the properties described in the conclusion of our theorem. If hypothesis (ix) is satisfied for k=1,2,...,n, then by Theorem 1.1, we have n linearly independent solutions of the type described above.

The only purpose for which hypothesis (ix) was used, in the proof of the preceding theorem, was to confirm the final hypothesis of Lemma 1.3. The conclusion of Theorem 1.1 is still true, and, in fact, it is true for k=1,2,...,n, if (ix) is replaced by the alternative hypothesis of our next theorem. When applicable, Theorem 2.2 is generally easier to apply than Theorem 2.1.

Theorem 2.2 Suppose hypotheses (i) through (viii) of Theorem 2.1 continue to hold. In addition, let

$$[A_j(t)\varphi^{-j}(t) + \frac{(n-j+1)^2-3(n-j+1)}{2} A_{j-1}\varphi'\varphi^{-j-1} - a_j] \cdot \varphi(t)$$

be integrable on $[a,\infty)$ for j=1,2,...,n.

Then, there exist n linearly independent solutions $x_k(t)$, k=1,2,...,n, of (1) and a number t_o, $a \leq t_o < \infty$, such that

$$x_k(t) = \varphi^{-1}[\exp \int_{t_o}^{t} \mu_k \varphi(\tau)d\tau][1 + o(1)].$$

Moreover, the first n-1 derivatives of $x_k(t)$ have the asymptotic form

$$x_k^{(j)}(t) = \varphi^{j-1}[\exp \int_{t_o}^{t} \mu_k \varphi(\tau)d\tau][\mu_k^j + o(1)] \quad j=1,2,\ldots,n.$$

Proof: This theorem can be established by repeating that part of the proof given for Theorem 1.2 which used hypotheses (i) through (viii). Having found $b_j(s)$ to be integrable, continue the proof by showing that $a_j - B_j(s)$ is also integrable on $[0,\infty)$, under the additional hypothesis of the present theorem. Then, invoke Lemma 1.4, instead of Lemma 1.3, to verify that every $D_{kj}(s)$ conforms to Condition I. Finally, apply Theorem 1.1 using the fact that for $k=1,2,\ldots,n$, $\lambda_k(s) = \mu_k + \eta_k^{**}(s)$ where $\dot{\eta}_k^{**}(s)$ is integrable on $0 \leq s < \infty$.

As with the foregoing theorem, the rest of the theorems in this chapter are obtained by making a variety of changes in the hypothesis of Theorem 2.1. Three of the next four theorems are stated for the third order differential equation, rather than for the general equation of order "n". Under this restriction on order, it is much easier to find explicit formulas for the coefficients of the characteristic polynomial $P(\lambda)$. Consequently, there is no need to approximate the $B_j(s)$, $j=1,2,3$. In our fourth theorem, we are again concerned with the n=th order differential equation.

Our initial theorem, pertaining to the third order equation, comes about when hypotheses (iv), (v), (viii), and (ix) of Theorem 2.1 are revised as follows. The product $\varphi'\varphi^{-2}$ in (iv) is allowed to converge to any constant c; the a_j of (v) are required to be real; the new functions described by (viii) need only be members of $L^3[0,\infty)$; and finally, since the a_j are all real numbers, we may algebraically simplify hypothesis (ix).

Specifically, this yields:

<u>Theorem 2.3</u> Consider the linear differential equation

$$x'''(t) + A_1(t)\, x''(t) + A_2(t)\, x'(t) + A_3(t)\, x(t) = 0 \qquad (15)$$

Suppose

(i) $A_j(t)$, $j=1,2,3$, are continuous complex-valued functions defined on $a \le t < \infty$,

(ii) $\varphi(t)$ is a positive, continuously differentiable function defined on $a \le t < \infty$,

(iii) $\int_a^\infty (t)dt = \infty$,

(iv) $\lim_{t\to\infty} \varphi' \varphi^{-2} = c$, where c is a constant,

(v) Define $\hat{B}_1(t) = A_1(t)\varphi^{-1}(t)$

$$\hat{B}_2(t) = A_2(t)\varphi^{-2}(t) - A_1(t)\varphi'(t)\varphi^{-3}(t) - (\varphi'(t)\varphi^{-2}(t))^2$$

$$\hat{B}_3(t) = A_3(t)\varphi^{-3}(t) - A_2(t)\varphi'(t)\varphi^{-4}(t)$$

$\lim_{t\to\infty} \hat{B}_j(t) = a_j$, $j=1,2,3$, where the a_j are real-valued constants,

(vi) $p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$ has distinct zeros μ_1, μ_2, μ_3, and $\mathrm{Re}\mu_j = \mathrm{Re}\mu_k$ implies $\mu_j = \bar{\mu}_k$,

(vii) $(\varphi'\varphi^{-2})'$ and $(A_j\varphi^{-j})'$, j=1,2,3, are integrable on $[a,\infty)$,

(viii) $[a_j - \hat{B}_j(t)]\varphi^{(1/3)}(t)$, j=1,2,3, are members of $L^3[a,\infty)$,

(ix) For fixed k,

$$\varphi\sum_{m=1}^{3}\mathrm{Im}\,\hat{B}_m\left[\mathrm{Im}\,\frac{\mu_k^{3-m}}{p'(\mu_k)}\right] -$$

$$\varphi\sum_{m=1}^{3}\sum_{r=1}^{3}\mathrm{Im}[(a_m - \hat{B}_m)(a_r - \hat{B}_r)]\left[\mathrm{Im}\,\frac{(3-m)\mu_k^{5-m-r}}{(p'(\mu_k))^2}\right] +$$

$$\frac{1}{2}\quad\varphi\sum_{m=1}^{3}\sum_{r=1}^{3}\mathrm{Im}[(a_m - \hat{B}_m)(a_r - \hat{B}_r)]\left[\mathrm{Im}\,\frac{p''(\mu_k)\mu_k^{6-m-r}}{(p'(\mu_k))^3}\right]$$

satisfies Condition I if j is an index for which $\mathrm{Re}[\mu_k - \mu_j] = 0$. It is to be understood that $\mu_k^0 = 1$ if $\mu_k = 0$.

Then there exists a solution $x_k(t)$ of (15) and a number t_o, such that

$$x_k(t) = \varphi^{-1}(t)[\exp\int_{t_o}^{t}[\mu_k + \varepsilon_k(\tau)]\varphi(\tau)d\tau][1 + o(1)]$$

where

$$\varepsilon_k = (p'(\mu_k))^{-1}\left[\sum_{m=1}^{3}[a_m - \hat{B}_m]\mu_k^{3-m}\right] +$$

$$(p'(\mu_k))^{-2}\left[\sum_{m=1}^{3}[a_m - \hat{B}_m]\mu_k^{3-m}\right]\left[\sum_{m=1}^{2}[a_m - \hat{B}_m](3 - m)\mu_k^{2-m}\right] - \qquad (16)$$

$$\frac{1}{2}(p'(\mu_k))^{-3}p''(\mu_k)\left[\sum_{m=1}^{3}[a_m - \hat{B}_m]\mu_k^{3-m}\right]^2$$

Furthermore, the first and second derivatives of $x_k(t)$ have the asymptotic forms

$$x_k'(t) = [\exp\int_{t_o}^{t}[\mu_k + \varepsilon_k(\tau)]\varphi(\tau)d\tau\ [\mu_k - c + o(1)]$$

and

$$x_k''(t) = \varphi(t)\ [\exp\int_{t_o}^{t}[\mu_k + \varepsilon_k(\tau)]\varphi(\tau)d\tau][\mu_k^2 - \mu_k c + o(1)]$$

Proof: The proof of this theorem follows the line of proof given for Theorem 2.1. Hypotheses (i), (ii), and (iii) allow to be transformed into the system

$$\frac{dz(\psi(s))}{ds} = \mathcal{B}(\psi(s))z(\psi(s)) \tag{17}$$

where

$$\mathcal{B}(\psi(s)) = \begin{bmatrix} \varphi'\varphi^{-2} & 1 & 0 \\ 0 & 0 & 1 \\ -A_3\varphi^{-3} & -A_2\varphi^{-2} & -A_1\varphi^{-1}-\varphi'\varphi^{-2} \end{bmatrix}$$

and $z = [\varphi x, x', \varphi^{-1}x'']^T$

The argument $\psi(s)$ is obtained by letting $s(t) = \int_a^t \varphi(\tau)\, d\tau$ and denoting the inverse transformation by $t = \psi(s)$. The elements of $\mathcal{B}(\psi(s))$, and $\psi(s)$ are continuous on $0 \leq s < \infty$. Moreover, t tends to infinity with s. From hypothesis (iv) and (v), we observe that as s tends to infinity, or equivalently as t tends to infinity, $\mathcal{B}(\psi(s))$ approaches the constant matrix C displayed below.

$$C = \begin{bmatrix} c & 1 & 0 \\ 0 & 0 & 1 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

where b_1, b_2, b_3 are constants.

By direct expansion of $\det[\mathcal{B}(\psi(s)) - \lambda E]$, we find that the coefficients of the characteristic polynomial $P(\lambda) = \lambda^3 + \hat{B}_1(\psi(s))\lambda^2 + \hat{B}_2(\psi(s))\lambda + \hat{B}_3(\psi(s))$ associated with $\mathcal{B}(\psi(s))$ are those given in hypothesis (iv) with the argument t replaced by $\psi(s)$. Let $\lambda_j(s)$ denote the zeros of $P(\lambda)$.

Again, utilizing Rouché's Theorem and renaming subscripts if necessary, we may take $\lim_{s\to\infty} \lambda_j(s) = \mu_j$, j=1,2,3, in virtue of (vi). We shall now apply Theorem 1.1 to system (17).

Letting $A = C$, and $V(s) = \mathcal{B}(\psi(s)) - C$, hypothesis (vi) shows that $\int_0^\infty |V'(s)|ds < \infty$, furthermore, $\lim_{s\to\infty} V(s) = 0$. In our case, the matrix R(s) mentioned in Theorem 1.1, is identically zero. Hence, the hypotheses of Theorem 1.1 will be satisfied provided the real part of $\lambda_k(s) - \lambda_j(s)$ satisfies Condition I.

Hypothesis (viii) implies that the functions $(a_j - \hat{B}_j(\psi(s))$ are members of $L^3[0,\infty)$. Now applying Lemma 1.6, we may verify that $\mathrm{Re}[\lambda_k(s) - \lambda_j(s)]$ satisfies Condition I. Hence, we may conclude by Theorem 1.1 that there exists a solution $z_k(\psi(s))$ and a number s_o such that

$$z_k(\psi(s)) = [\exp \int_{s_o}^{s} \lambda_k(\xi)\, d\xi][p_k + o(1)],$$

where p_k is the eigenvector of C corresponding to the eigenvalue μ_k. No loss in generality results in assuming that p_k is of the form $p_k = [1, \mu_k - c, \mu_k^2 - \mu_k c]^T$. Since $z(t) = T(t)\, y(t)$, and $s(t) = \int_a^t \varphi(\tau)\, d\tau$, we may retrace the transformations back to the variables x and t, obtaining

$$x_k(t) = \varphi^{-1}(t)[\exp \int_{t_o}^{t} \lambda_k(s(\tau))\varphi(\tau)\, d\tau][1 + o(1)].$$

Now using the asymptotic form of $\lambda_k(s)$ given by Lemma 1.5, we conclude that the asymptotic form of $x_k(t)$ as that given in the conclusion of the theorem. In a similar fashion, we transform the other two components of the asymptotic form of $z_k(\psi(s))$ to obtain the expressions for the drivatives of $x_k(t)$ given in the conlcusion of this theorem.

Corollary 2.4 Suppose that hypotheses (i) through (viii) of Theorem 2.3 hold and, in addition, suppose that $\text{Im}[A_j \varphi^{-j+1}]$, $j=1,2,3$, are integrable on $a \leq t < \infty$. Then the conclusion of Theorem 2.3 holds.

Proof: If the $\text{Im}[A_j\varphi^{-j+1}]$, j=1,2,3, are integrable, then so are $\text{Im}[B_j\varphi]$ j=1,2,3. Since the $\text{Im}[B_j]$ converge to zero as s tends to infinity, and are continuous, products of them when multiplied by $\varphi(t)$ are also integrable functions of t on $a \leq t < \infty$. Hence hypothesis (ix) of Theorem 2.3 holds and the corollary is proved.

Looking back, the additional constraint in hypothesis (vi) of Theorem 2.3 which is not present in hypothesis (vi) of Theorem 2.1 was used to show that the real part of the difference of $\lambda_k(s)$ and $\lambda_j(s)$ may be expressed as the function given in hypothesis (ix) of Theorem 2.3 (except for a multiplier of φ). If one dispenses with this additional constraint, there is usually no such simplification for $\text{Re}[\lambda_k(s) - \lambda_j(s)]$. It is then necessary to use the approximate expressions for λ_k and λ_j given by Lemma 1.5. The following theorem illustrates this change. The proof of this theorem follows that of Theorem 2.1 and is not given here.

Theorem 2.5 Consider the linear differential equation

$$x'''(t) + A_1(t)\, x''(t) + A_2(t)\, x'(t) + A_3(t)\, x(t) = 0 \qquad (15)$$

Suppose

(i) $A_j(t)$, $j=1,2,3$, are continuous complex-valued functions defined on the interval $[a,\infty)$,

(ii) $\varphi(t)$ is a positive, continuously differentiable function defined on $a \leq t < \infty$,

(iii) $\int_a^\infty \varphi(t)dt = \infty$,

(iv) $\lim_{t\to\infty} \varphi'\varphi^{-2} = c$, where c is a constant,

(v) Define $\hat{B}_1(t) = A_1(t)\varphi^{-1}(t)$

$$\hat{B}_2(t) = A_2(t)\varphi^{-2}(t) - A_1(t)\varphi'(t)\varphi^{-3}(t) - (\varphi'(t)\varphi^{-2}(t))^2$$

$$\hat{B}_3(t) = A_3(t)\varphi^{-3}(t) - A_2(t)\varphi'(t)\varphi^{-4}(t)$$

and suppose $\lim_{t\to\infty} \hat{B}_j(t) = a_j$, $j=1,2,3$, where the a_j are complex-valued constants,

(vi) $p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$ has distinct zeros μ_1, μ_2, μ_3,

(vii) $(\varphi'\varphi^{-2})'$ and $(A_j\varphi^{-j})'$, $j=1,2,3$, are integrable on $[a,\infty)$,

(viii) $[a_j - \hat{B}_j(t)]\varphi^{(1/3)}(t)$, $j=1,2,3$, are members of $L^3[a,\infty)$,

(ix) Define $\tilde{P}(\lambda) = (B_1-a_1)\lambda^2 + (B_2-a_2)\lambda + (B_3-a_3)$ and suppose the real part of

$$-\frac{\tilde{P}(\mu_k)}{p'(\mu_k)} + \frac{\tilde{P}(\mu_j)}{p'(\mu_j)} + \frac{\tilde{P}(\mu_k)\tilde{P}'(\mu_k)}{(p'(\mu_k))^2} - \frac{\tilde{P}(\mu_j)\tilde{P}'(\mu_j)}{(p'(\mu_j))^2} -$$

$$\frac{1}{2}\frac{p''(\mu_k)(\tilde{P}(\mu_k))^2}{(p'(\mu_k))^3} - \frac{1}{2}\frac{p''(\mu_j)(\tilde{P}(\mu_j))^2}{(p'(\mu_j))^3}$$

multiplied by $\varphi(t)$, satisfies Condition I.

Then there exists a solution $x_k(t)$ of (15) and a number t_o, such that

$$x_k(t) = \varphi^{-1}(t)[\exp \int_{t_o}^{t} [\mu_k + \varepsilon_k(\tau)]\varphi(\tau)d\tau][1 + o(1)]$$

where

$$\varepsilon_k = -\frac{\tilde{P}(\mu_k)}{p'(\mu_k)} + \frac{\tilde{P}(\mu_k)\tilde{P}'(\mu_k)}{(p'(\mu_k))^2} - \frac{1}{2}\frac{\tilde{P}''(\mu_k)(\tilde{P}(\mu_k))^2}{(p'(\mu_k))^3}$$

Furthermore, the first and second derivatives of $x_k(t)$ have the asymptotic forms

$$x_k'(t) = [\exp \int_{t_o}^{t} [\mu_k + \varepsilon_k(\tau)]\varphi(\tau)d\tau \; [\mu_k - c + o(1)]$$

and

$$x_k''(t) = \varphi(t)\ [\exp \int_{t_o}^{t} [\mu_k + \varepsilon_k(\tau)]\varphi(\tau)\,d\tau][\mu_k^2 - \mu_k c + o(1)].$$

We conclude this chapter with one more theorem. This theorem, as well as Theorem 2.1, is used in Chapter 3, to derive oscillatory and boundedness properties of solutions of n-th order linear differential equations. For these purposes, it is not always necessary to have asymptotic expressions available for the characteristic values of the related differential system. This fact is reflected in the hypotheses of the ensuing theorem. They are sufficiently strong that Theorem 1.1 may be applied, even in the absence of asymptotic representations of the characteristic values.

Theorem 2.6 Consider the linear differential equation

$$x^{(n)}(t) + A_1(t)\, x^{(n-1)}(t) + \dots + A_n(t)\, x(t) = 0. \qquad (1)$$

Suppose that

(i) $A_j(t)$, $j=1,2,\dots,n$, are continuous, real-valued function defined on $[a,\infty)$,

(ii) $\varphi(t)$ is a positive, continuously differentiable, function defined on $[a,\infty)$,

(iii) $\int_a^\infty \varphi(t)\,dt = \infty$,

(iv) $\lim_{t\to\infty} \varphi'\varphi^{-2} = 0$,

(v) $\lim_{t\to\infty} A_j(t)\varphi^{-j}(t) = a_j$, $j=1,2,\dots,n$, where the a_j are constants,

(vi) the zeros μ_j, $j=1,2,\dots,n$, of $p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \dots + a_n$ are distinct and $\mathrm{Re}\mu_k = \mathrm{Re}\mu_j$ implies $\mu_k = \bar{\mu}_j$,

(vii) $(\varphi'\varphi^{-2})'$ and $(A_j\varphi^{-j})'$, $j=1,2,\dots,n$, are integrable on $[a,\infty)$.

Define $P(\lambda) = \det[\mathcal{B}(t) - \lambda E]$ where $\mathcal{B}(t)$ has the form

$$\mathcal{B}(t) = \begin{bmatrix} \varphi'\varphi^{-2} & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & -\varphi'\varphi^{-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -A_n\varphi^{-n} & -A_{n-1}\varphi^{1-n} & -A_{n-2}\varphi^{2-n} & \cdots & -A_1\varphi^{-1}+(2-n)\varphi'\varphi^{-2} \end{bmatrix}$$

Then there exist n linearly independent solutions $x_k(t)$, $k=1,2,\ldots,n$, of (1) and a number t_o, $a \leq t_o < \infty$, such that

$$x_k(t) = \varphi^{-1}[\exp \int_{t_o}^{t} \lambda_k(\tau)\varphi(\tau)\,d\tau][1 + o(1)]$$

and

$$x_k^{(j)}(t) = \varphi^{j-1}[\exp \int_{t_o}^{t} \lambda_k(\tau)\varphi(\tau)\,d\tau][\mu_k^j + o(1)], \qquad j=1,2,\ldots,n-1$$

where $\lambda_k(t)$ is that zero of $P(\lambda)$ which converges to μ_k as t becomes infinite.

Proof: The proof of this theorem is essentially the same as that of Theorem 2.1. The most important difference is that all the coefficients of $P(\lambda)$ and $p(\lambda)$ are real-valued instead of complex-valued. Hence, any complex-valued zeros of $P(\lambda)$ or $P(\lambda)$ occur as conjugate pairs. This observation, together with hypothesis (vi), allow us to verify that $Re[\lambda_k - \lambda_j]$ satisfies Condition I (as functions of s), for any indices k and j, $1 \leq k, j \leq n$. For, if $Re[\mu_k - \mu_j] \neq 0$, the condition holds as before. If $Re[\mu_k - \mu_j] = 0$, then $Re[\mu_k - \mu_j] = 0$ identically for all s.

CHAPTER 3

OSCILLATORY AND BOUNDEDNESS PROPERTIES OF SOLUTIONS

In this chapter we study the oscillatory and boundedness properties of solutions of the differential equation

$$x^{(n)}(t) + A_1(t)x^{(n-1)}(t) + \ldots + A_n(t)x(t) = 0 \qquad (1)$$

where the coefficients $A_j(t)$, $j=1,2,\ldots,n$, are assumed to be continuous real-valued functions defined on $[a,\infty)$. We shall call a nontrivial solution $x(t)$ of (1) oscillatory if and only if it has an infinite number of zeros in the interval $[a,\infty)$, and we shall call it nonoscillatory otherwise. Equation (1) will be called oscillatory if and only if it admits of some nontrivial oscillatory solution. Otherwise, it will be called nonoscillatory. Our results are based upon the theorems developed in Chapter 2 concerning asymptotic solutions of (1).

Although asymptotic solutions only provide information on some interval $[t_o,\infty)$ where $t_o \geq a$; nevertheless, is readily verified, as is done in Pfeiffer [7], that a solution is oscillatory on $[a,\infty)$ iff it is oscillatory on $[t_o,\infty)$.

Having derived asymptotic forms for the derivatives of solutions of (1) in Chapter 2, we also present boundedness properties for these derivatives by employing our earlier results.

In addition to the assumption that the coefficients $A_j(t)$ are all real-valued, we suppose that all the hypotheses of Theorem 2.1 hold. Under these assumptions, each a_j is necessarily real. The following corollaries of Theorem 2.1 describe the oscillatory nature and boundedness properties of solutions of Equation (1).

Corollary 3.1 If $\mathrm{Im}\mu_k \neq 0$, then Equation (1) is oscillatory.

Corollary 3.2 If for $k=1,2,\ldots,n$, hypothesis (ix) of Theorem 2.1 is satisfied and all the μ_k are real, then Equation (1) is non-oscillatory.

Corollary 3.3 If $\mathrm{Re}\mu_k < 0$, then the modulus of the asymptotic solution $x_k(t)$ of (1), described in the conclusion of Theorem 2.1, increases without bound as t becomes infinite. The same is true of the first n-1 derivatives of $x_k(t)$.

Corollary 3.4 If $\mathrm{Re}\mu_k < 0$, then the modulus of the asymptotic solution $x_k(t)$ of (1), described in the conclusion of Theorem 2.1, converges to zero as t becomes infinite. The same is true of the first n-1 derivatives of $x_k(t)$.

Corollary 3.5 If $\mathrm{Re}\mu_k = 0$ and $\int_a^\infty \mathrm{Re}[\varepsilon_k - \varphi'\varphi^{-2}]\, dt < \infty$, then the modulus of the solution described by the conclusion of Theorem 2.1 is bounded on $[a,\infty)$.

Corollary 3.6 If $\text{Re}\mu_k = 0$ and $\int_a^\infty \text{Re}[\varepsilon_k - \varphi'\varphi^{-2}]\, dt = \infty$, then the modulus of the solution of (1), described by the conclusion of Theorem 2.1 is unbounded on $[a,\infty)$.

To illustrate how these Corollaries may be established, we shall prove the first three.

Proof of Corollary 3.1: Both the real part and the imaginary part of the asymptotic solution $x_k(t)$, given by Equation (12) on Theore m 2.1, are solutions of (1). Taking the real part, we have the solution

$$\tilde{x}_k(t) = \varphi^{-1}[\exp \int_{t_o}^{t} \text{Re}[\mu_k + \varepsilon_k]\varphi\, d\tau][\cos \int_{t_o}^{t} \text{Im}[\mu_k + \varepsilon_k]\varphi\, d\tau\, [1 + o(1)]$$

Since $\varepsilon_k = o(1)$ and $\varphi(t)$ is nonintegrable, it is clear that if $\text{Im}\mu_k \neq 0$, the absolute value of the argument of the cosine function increases without bound as t becomes infinite. Hence, $x_k(t)$ is oscillatory and, thus, equation (1) is oscillatory.

Proof of Corollary 3.2: If for k=1,2,...,n, hypothesis (ix) is satisfied, and all the μ_k are real, then there are n linearly independent solutions of (1) specified by (12) of Theorem 2.1. Each of these is nonoscillatory. For the $\varepsilon_k(t)$ are now real-valued, and the factor $[1 + o(1)]$ of the solution $x_k(t)$ becomes and remains positive. We still must show that every solution is nonoscillatory.

To this end, let $x(t) = \sum_{k=1}^{n} c_k x_k(t)$ denote an arbitrary nontrivial solution of (1). Let α denote the index of the greatest zero μ_k for which $c_k \neq 0$. Note that

$$\lim_{t\to\infty} \frac{x_k(t)}{x_\alpha(t)} = \lim_{t\to\infty} \frac{\varphi^{-1}[\exp \int_{t_o}^{t} [\mu_k + \varepsilon_k]\varphi \, d\tau][1 + o(1)]}{\varphi^{-1}[\exp \int_{t_o}^{t} [\mu_\alpha + \varepsilon_\alpha]\varphi \, d\tau][1 + o(1)]}$$

$$= \lim_{t\to\infty} [\exp \int_{t_o}^{t} [\mu_k - \mu_\alpha + \varepsilon_k - \varepsilon_\alpha]\varphi \, d\tau][1 + o(1)] = 0$$

if $k \neq \alpha$.

Therefore, $x(t) = c_\alpha x_\alpha(t)[1 + o(1)]$ and, for sufficiently large t_1, $x(t) \neq 0$ on $[t_1,\infty)$.

Proof of Corollary 3.3: The modulus of the solution given by (12) os Theorem 2.1 has the form

$$|x_k(t)| = [\exp \int_{t_o}^{t} \mathrm{Re}[\mu_k + \varepsilon_k - \varphi'\varphi^{-2}]\varphi \, d\tau][1 + o(1)]$$

Recalling that $\varphi'\varphi^{-2} = o(1)$, $\varepsilon_k(t) = o(1)$, and $\varphi(t)$ is nonintegrable on $[a,\infty)$, it is clear that the modulus from hypothesis (iv) $|x_k(t)|$ of the solution $x_k(t)$ increases without bound as t becomes infinite, if $\mathrm{Re}\mu_k > 0$.

To draw the same conclusion for the derivatives, express the modulus of the j-th derivative of $x_k(t)$ as

$$|x_k^{(j)}(t)| = [\exp \int_{t_o}^{t} \mathrm{Re}[\mu_k + \varepsilon_k + (j-1)\varphi'\varphi^{-2}]\varphi \, d\tau]|\mu_k^j + o(1)|$$

for $j=1,2,\ldots,n-1$. For the same reasons as before, the modulus of the j-th derivative of the solution $x_k(t)$ increases without bound as t becomes infinite.

The last three corollaries of the foregoing list may be established by making only slight modifications in the above proof.

By using Theorem 2.5, instead of Theorem 2.1, it is possible to state and prove another set of corollaries pertaining to the oscillatory nature and boundedness properties of solutions of the third order linear differential equation. In this instance, the oscillatory nature may be characterized in terms of the behavior of the discriminant of the polynomial $p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$.

The following corollaries are obtained when Theorem 2.6 applies. The proof of these corollaries may be constructed using the same techniques employed in the proofs of the previous corollaries of this chapter.

Corollary 3.7 If one of the zeros μ_k of $p(\lambda)$ has nonzero imaginary part, then equation (1) is oscillatory.

Corollary 3.8 If all of the zeros of $p(\lambda)$ are real, then equation (1) is nonoscillatory.

Corollary 3.9 If $\mathrm{Re}\mu_k > 0$, then the modulus of the solution $x_k(t)$ of (1) described by the conclusion of Theorem 2.6 increases without bound as t becomes infinite. The same is true for the first n-1 derivatives of $x_k(t)$.

<u>Corollary 3.10</u> If $\text{Re}\mu_k < 0$, then the modulus of the solution $x_k(t)$ described by the conclusion of Theorem 2.6 converges to zero as t becomes infinite. The same is true for the first n-1 derivatives of $x_k(t)$.

CHAPTER 4

SUMMARY AND EXAMPLES

In this chapter we discuss briefly the results of the first three chapters of this thesis. Chapter 1 contains the preliminary lemmas used in the proof of the theorems given in Chapter 2. In Chapter 2, several theorems were presented which gave the asymptotic form of solutions of certain linear differential equations. These theorems are applications of Theorem 1.1 to a first order differential system. The differences between the various theorems have already been pointed out.

In each of these theorems an unspecified function $\varphi(t)$ appears. One may question whether a method exists which will determine an applicable function $\varphi(t)$. The answer to this question is still unresolved. However, there are certainly some likely candidates for this function. Among these candidates are the functions $|A_j(t)|^{1/j}$, $|ReA_j(t)|^{1/j}$, and $|ImA_j(t)|^{1/j}$, $j=1,2,\ldots,n$. Of course, any such choice must still be governed by the requirements that $\varphi(t)$ be continuously differentiable, nonintegrable on $[a,\infty)$ and satisfy the other relevant hypotheses of the applicable theorem. Using two of these suggested possibilities, we generalize the results of Pfeiffer [6].

Let $n = 3$, $A_1 = 0$, $A_2 = q$, $A_3 = r$, and $\varphi(t) = r^{1/3}$. Then Theorem 2.1 generalizes Theorem 4 of Pfeiffer. In this case, the hypotheses of Theorem 2.1 are less restrictive than those of Pfeiffer's theorem. For instance, Theorem 2.1 does not require A_3 to be a real-valued function nor $p(\lambda)$ to be $\lambda^3 + 1$. In fact, $p(\lambda)$ may have complex-valued coefficients. It is worth mentioning that Theorem 2.1 corrects an oversight by Pfeiffer. He should have included the requirement that $\text{Im}[q\, r^{-1/3}]$ satisfy Condition I among his other hypotheses. It then becomes possible to prove his theorem by constructing a proof similar to that of Theorem 2.1.

If $n = 3$, $A_1 = 0$, $A_2 = q$, $A_3 = r$, and $\varphi(t) = q^{1/2}$, Theorem 2.1 generalizes Theorems 6 and 8 of Pfeiffer 6 .

If $A_1 = A_2 = \ldots = A_{n-1} = 0$, $A_n = q$, $\varphi = q^{1/n}$, Theorem 2.1 generalizes a corollary of Hinton [4, p. 594].

At the outset of Chapter 3 a number of corollaries of Theorem Theorem 2.1 are given which describe oscillation and boundedness properties of solutions of linear differentail equations. These corollaries generalize Theorems 4 and 5 of Pfeiffer [7].

Theorem 2.1 may not be applicable to a differential equation of the form $[r\, y^{(m)}]^{(k)} + qy = 0$, unless r possesses a continuous k-th derivative. This equation is treated by Hinton [4], using methods analogous to those of Pfeiffer. If r does not have a continuous k-th derivative, Hinton's transformation coupled with the asymptotic form of the zeros of the characteristic polynomial found in Chapter 1, may be employed to generalize his results.

Pfeiffer [6] also considers an equation of the form $y + \alpha_2 t^{\beta_2} y' + \alpha_2 t^{\beta_2} y = 0$ as an example. He obtained results describing the asymptotic form of its solutions for a broad range of values of β_1 and β_2. However, if $(2/5)(2\beta_2 + 1) > \beta_1 > (1/2)(\beta_2 - 1)$, he could draw no conclusions using his theorems. We shall describe the asymptotic form of the solutions for values of β_1 and β_2 which include those that satisfy the preceding inequalities. This example, stated as Theorem 4.1, also illustrates the manner in which one may apply Theorem 1.1 without determining close approximations of the characteristic values of the system.

Theorem 4.1 Consider the equation

$$x'''(t) + \alpha_1 t^{\beta_1} x'(t) + \alpha_2 t^{\beta_2} x(t) = 0 \qquad (18)$$

where α_1, α_2, β_1, β_2 are real constants. Suppose one of the following hypotheses holds.

(i) $\alpha_2 \neq 0$, $\beta_2 > -3$, and $\beta_2 > \frac{3}{2}\alpha_1$. Define $\varphi(t) = t^{(\beta_2/3)}$,

$$\mu_1 = -\sqrt[3]{\alpha_2}\ ,\ \mu_2 = \sqrt[3]{\alpha_2}\ \frac{1 + \sqrt{3}\, i}{2}\ ,\ \mu_3 = \sqrt[3]{\alpha_2}\ \frac{1 - \sqrt{3}\, i}{2}$$

(i)' $\alpha_1 \neq 0$, $\beta_1 > -2$, and $\beta_1 > \frac{2}{3}\beta_2$. Define $\varphi(t) = t^{(\beta_1/2)}$,

$$\mu_1 = 0,\ \mu_2 = \sqrt{\alpha_1}\, i,\ \mu_3 = -\sqrt{\alpha_1}\, i$$

If α_1 is equal to zero, then that part of the relevant hypotheses concerning β_1 is assumed to hold. Similarly, if $\alpha_2 = 0$, that part of the hypothesis concerning β_2 is assumed to hold. Then there exist three solutions of (18) and a number t_o such that

$$x_k(t) = \varphi^{-1}(t)\left[\exp \int_{t_o}^{t} \lambda_k(\tau)\varphi(\tau)\, d\tau\right][1 + o(1)] \qquad k=1,2,3$$

where $\lambda_k(t)$ is that root of

$$\lambda^3 + [\alpha_1 t^{\beta_1}\varphi^{-2}(t) - (\varphi'(t)\varphi^{-2}(t))^2]\lambda + \alpha_2 t^{\beta_2}\varphi^{-3}(t) -$$

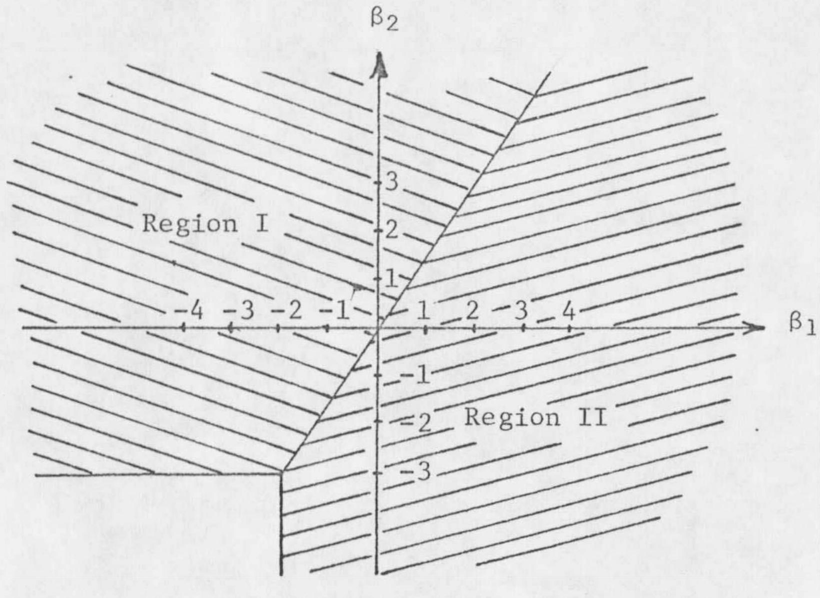
$$\alpha_1 t^{\beta_1}\varphi'(t)\varphi^{-4}(t) = 0$$

which converges to μ_k as t tends to infinity. Moreover, the first two derivatives of $x_k(t)$ have the following asymptotic forms:

$$x_k'(t) = [\exp \int_{t_o}^{t} \lambda_k(\tau)\varphi(\tau)\, d\tau][\mu_k + o(1)]$$

and

$$x_k''(t) = \varphi(t)[\exp \int_{t_o}^{t} \lambda_k(\tau)\varphi(\tau)\, d\tau][\mu_k^2 + o(1)]$$



Given that $\alpha_1 \neq 0$, and $\alpha_2 \neq 0$, Region I of the above diagram depicts those values of β_1 and β_2 where hypothesis (i) holds. Region II depicts those values of β_1 and β_2 where hypothesis (i)' holds. The unlined region has been considered by Ghizzetti [3]. His results may be found in Coppel [2, Page 92].

Proof: For brevity, we shall merely sketch the proof of this theorem.

Suppose hypothesis (i) holds. By following the method of proof of Theorem 2.3, we obtain a system much like (17). In this case, the matrix has only real elements. Hence the characteristic polynomial has real coefficients and $\lambda_2(\psi(s)) = \overline{\lambda}_3(\psi(s))$. So the real difference of any two zeros of $P(\lambda)$ is easily shown to satisfy Condition I. Now apply Theorem 1.1 to the system. The reaminder of the proof consists of transforming the result to the variables t and x.

Suppose hypothesis (i)' holds. By again following the method of the proof of Theorem 2.3, we obtain a system much like (17). The characteristic polynomial has only real coefficients. If $\alpha_1 < 0$, the limit of the zeros of the characteristic polynomial are real numbers. So the real part of the difference of any two characteristic values satisfies Condition I. If $\alpha_1 > 0$, we have by direct calculation, that

$$\lambda_1(t) = [-\frac{\alpha_2}{\alpha_1} t^{(\beta_2-(3/2)\beta_1)} + \frac{\beta_1}{2} t^{-(\beta_1+2)/2}][1 + o(1)]$$

and that $\mathrm{Re}\lambda_2(t) = \mathrm{Re}\lambda_3(t) = -(1/2)\lambda_1(t)$. Observe that if one o(1) term is dropped from the above expression, the difference between the resulting approximate value of $\lambda_1(\psi(s))$ and the exact value of $\lambda_1(\psi(s))$ may not be integrable. In this respect, this application of Theorem 1.1 is different than the theorems of Chapter 2. Now, use the nonotone properties of the expression above to show that in terms of s, $Re[\lambda_k(\psi(s)) - \lambda_j(\psi(s))]$ satisfies Condition I for any values of k and j. Theorem 1.1 then applies and the proof of this theorem follows.

Our final example is an application of Theorem 2.1 to a differential equation which is not treated by any other known theorem.

Example 4.2 Let $\varphi = 1$, and apply Theorem 2.1 to the equation

$$x'''(t) + t^{-1} x'(t) + (1 + i\, t^{-1})x(t) = 0$$

Then there are three solutions $x_k(t)$ having the asymptotic representations

$$x_k(t) = \exp\left\{ \mu_k t + \frac{1}{3}\left[\frac{-1}{\mu_k} + i\, \mu_k\right] \ln(t) \right\} [1 + o(1)]$$

where μ_k, k=1,2,3, are the three roots of -1.

APPENDICES

APPENDIX 1

In this appendix we state and prove two lemmas concerning integrability properties of the class of complex-valued functions of the real variable s that are continuous on $0 \le s < \infty$, and which converge to zero as s becomes infinite. These lemmas find frequent application in the main body of this thesis, and in Appendix 2.

<u>Lemma 1</u> Suppose that

(i) f(s) is a continuous complex-valued function, of a real variable s, defined on $0 \le s < \infty$,

(ii) $\lim_{s \to \infty} f(s) = 0$,

(iii) f(s) is a member of $L^q[0,\infty)$, $1 \le q < \infty$,

Then f(s) is a member of $L^r[0,\infty)$ for all r such that $q \le r < \infty$.

<u>Proof</u>: By hypotheses (i) and (ii), we may choose an s_o, $0 \le s_o < \infty$ such that $|f(s)| \le 1$ on $s_o \le s < \infty$. Hence,

$$\int_{s_o}^{\infty} |f(s)|^r ds \le \int_{s_o}^{\infty} |f(s)|^q ds$$

and the conclusion follows from (iii).

Lemma 2 Suppose that

(i) $f(s)$ and $g(s)$ are continuous complex-valued functions of a real variable s on $0 \leq s < \infty$,

(ii) $\lim_{s \to \infty} f(s) = 0$ and $\lim_{s \to \infty} g(s) = 0$,

(iii) $f(s) \in L^q[0,\infty)$ and $g(s) \in L^r[0,\infty)$ where $1 \leq q, r < \infty$,

Then the product function $f(s) \cdot g(s)$ is a member of $L^m[0,\infty)$ where $m = \max(1, qr/(q+r))$.

Proof: Let $\alpha = (q+r)/r$ and $\beta = (q+r)/q$. Applying Hölder's Inequality we have

$$\int_0^\infty |f \cdot g|^{qr/(q+r)} ds \leq \left[\int_0^\infty [|f|^{qr/(q+r)}]^\alpha ds\right]^{1/\alpha} \left[\int_0^\infty [|g|^{qr/q+r}]^\beta ds\right]^{1/\beta}$$

Since the right hand side of the inequality is finite, by hypothesis (iii), so is the left hand side. Thus, the lemma is proved if $qr/(q+r) \geq 1$.

On the other hand, if $qr/(q+r) < 1$, let s_o be chosen so that $|f \cdot g|^{qr/(q+r)} < 1$ when $s_o \leq s < \infty$. Since

$$\int_{s_o}^\infty |f \cdot g| ds \leq \int_{s_o}^\infty |f \cdot g|^{qr/(q+r)} ds$$

we have that the product function is a member of $L^1[0,\infty)$. This concludes the proof of the lemma.

APPENDIX 2

The purpose of this appendix is to present a method of generating approximate formulas for the zeros of a polynomial whose coefficients are functions of a real variable. The method is one which may be iterated to achieve closer approximations at each successive stage of the process.

Let $P(\lambda) = \lambda^n + B_1(s)\lambda^{n-1} + \ldots + B_n(s)$ where the $B_j(s)$ are continuous complex-valued functions of a real variable s defined on the interval $0 \leq s < \infty$. Suppose that $\lim_{t \to \infty} B_j(s) = a_j$, $j=1,2,\ldots,n$,

where the a_j are complex-valued constants. Let $p(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n$ and require the zeros μ_k, $k=1,2,\ldots,n$, of this polynomial to be distinct. Using Rouché's Theorem, it can be verified that the zeros $\lambda_k(s)$, $k=1,2,\ldots,n$, of $P(\lambda)$ are continuous functions of s. By reordering the indices if necessary, we may assume that $\lim_{s \to \infty} \lambda_k(s) = \mu_k$, $j=1,2,\ldots,n$.

The rate at which each difference $\lambda_k(s) - \mu_k$ converges to zero is dependent upon how fast the differences $a_j - B_j(s)$ of the respective coefficients in $p(\lambda)$ and $P(\lambda)$ converge to zero. In the general case of an $n \geq 5$, it is usually impossible to express $\lambda_k(s)$ explicitly as a function of the coefficients $B_j(s)$. Therefore, an exact determination of $\lambda_k(s) - \mu_k$, in terms of $a_j - B_j(s)$, $j=1,2,\ldots,n$, will usually be out of the question. A suitable approximation to $\lambda_k(s)$ may be readily obtained, however. For our purpose, we seek an approximate zero $\tilde{\lambda}_k(s)$ of $P(\lambda)$ such that the difference $\lambda_k(s) - \tilde{\lambda}_k(s)$ is integrable on $[0,\infty)$.

To this end, set $\lambda_k(s) = \mu_k + \alpha_k(s)$. Remembering that $\lambda_k(s)$ is a zero of $P(\lambda)$, and dropping the subscript for convenience, we have $P(\mu + \alpha) = 0$. Now, $P(\mu + \alpha)$ may be viewed as a polynomial of degree n in α. Writing it as such, we obtain $P(\mu) + P'(\mu)\alpha + \ldots + \alpha^n = 0$. Since $\alpha(s)$ converges to zero as s becomes infinite, this equation may be expressed as

$$P(\mu) + P'(\mu)\alpha[1 + o(1)] = 0 \qquad (19)$$

where o(1) designates a function which vanishes as s becomes infinite infinite. As s becomes infinite

$$P'(\mu) = n\mu^{n-1} + \sum_{j=1}^{n-1} B_j(s)(n-j)\mu^{n-j-1}$$

converges to $p'(\mu)$. We may assume that $P'(\mu)$ is nonzero on $s_o \leq s < \infty$, when s_o is sufficiently large, for the zeros of $p(\lambda)$ are distinct, and $p'(\mu) \neq 0$.

With the understanding that $s \geq s_o$, equation (19) may be solved for α. This yields

$$\alpha = -\frac{P(\mu)}{P'(\mu)}[1 + o(1)] \qquad (20)$$

To make the dependence of α upon the differences $a_j - B_j(s)$ more explicit, recall that $p(\mu) = 0$, and write equation (19) as

$$\alpha = -\frac{P(\mu) - p(\mu)}{P'(\mu)}[1 + o(1)] \tag{21}$$

or

$$\alpha = (P'(\mu))^{-1}\sum_{j=1}^{n}[a_j - B_j(s)]\mu^{n-j}\,[1 + o(1)]$$

Since $P'(\mu) = p'(\mu)[1 + o(1)]$, and $1/[1 + o(1)] = [1 + o(1)]$, an alternative form of equation (20) is

$$\alpha = -\frac{P(\mu)}{p'(\mu)}[1 + o(1)] \tag{22}$$

From equations (20) and (22), it is evident that an approximation to $\lambda(s) = \mu + \alpha(s)$ is given by either

$$\mu - \frac{P(\mu)}{P'(\mu)} \quad \text{or} \quad \mu - \frac{P(\mu)}{p'(\mu)}$$

The second form would appear to be preferable to the first, because its denominator is independent of the variable s. However, as we shall see, the first form has some advantages when it comes to determining successively better approximations to λ.

Let us examine how close the approximate zero

$$\tilde{\lambda}(s) = \mu - \frac{P(\mu)}{P'(\mu)} \tag{23}$$

is to the exact zero $\lambda(s)$ of $P(\lambda)$. Set $\lambda(s) = \mu - \frac{P(\mu)}{P'(\mu)} + \beta(s)$.

We shall show that if $a_j - B_j(s)$ is a member of $L^q[0,\infty)$, then $\lambda(s) - \tilde{\lambda}(s) = \beta(s)$ is a member of $L^r[0,\infty)$, where $r = \max(1,q/2)$.

For simplicity, let

$$\varepsilon(s) = -\frac{P(\mu)}{P'(\mu)} = \frac{p(\mu) - P(\mu)}{P'(\mu)} = (P'(\mu))^{-1} \sum_{j=1}^{n} [a_j - B_j(s)]\mu^{n-j}.$$

Again, starting with $P(\lambda) = 0$, and employing the same kind of reasoning as before, we develop the succession of equations:

$$\begin{aligned} &P(\mu) = 0 \\ &P(\mu + \varepsilon + \beta) = 0 \\ &P(\mu + \varepsilon) + P'(\mu + \varepsilon)\beta\ [1 + o(1)] = 0 \\ &P(\mu + \varepsilon) + p'(\mu)\beta\ [1 + o(1)] = 0 \\ &P(\mu) + P'(\mu) \quad + (1/2)P''(\mu)\varepsilon^2 + \ldots + \varepsilon^n = -p'(\mu)\beta\ [1 + o(1)] \end{aligned} \tag{24}$$

Using the definition of ε, we obtain

$$(1/2)P''(\mu)\varepsilon^2 + (1/6)P'''(\mu)\varepsilon^3 + \ldots + \varepsilon^n = -p'(\mu)\beta\ [1 + o(1)] \tag{25}$$

By supposition, the $a_j - B_j(s)$ are continuous on $[0,\infty)$, converge to zero as s becomes infinite, and are members of $L^q[0,\infty)$. From Lemma 2, it follows that the left hand side of equation (25) is a member of $L^r[0,\infty)$, where $r = \max(1,q/2)$. As a special case, if

$q = 2$, we have shown that the difference $\beta(s) = \lambda(s) - \tilde{\lambda}(s)$, with $\lambda(s) = \mu - \frac{P(\mu)}{P'(\mu)}$ is integrable on $[0,\infty)$.

To continue the procedure, we could use equation (24) next to define

$$\delta(s) = -\frac{P(\mu + \varepsilon)}{P'(\mu + \varepsilon)}, \quad \text{where} \quad \varepsilon = -\frac{P(\mu)}{P'(\mu)}$$

It can be verified that if $a_j - B_j(s)$ is a member of $L^q[0,\infty)$, then $\lambda - \mu - \varepsilon - \delta$ is a member of $L^r[0,\infty)$ where $r = \max(1,q/3)$. This should suffice to indicate how these approximations may be extended.

REFERENCES

1. Coddington, E.A., and Levinson, N. Theory of Ordinary Differential Equations. New York: McGraw-Hill, 1955.

2. Coppel, W.A. Stability and Asymptotic Behavior of Differential Equations. Boston: D.C. Heath and Co., 1965.

3. Ghizzetti, A. "Un teorema sul comportamento asintotico degli equazioni differenziali lineari omogenee", Rend. Mat. e Appl. (5) 8 (1949), 28-42.

4. Hinton, D.B. "Asymptotic Behavior of Solutions of $(ry^{(m)})^{(k)} \pm qy = 0$", J. Diff. Equations 4 (1968), 590-596.

5. Levinson, N. "The Asymptotic Nature of Solutions of Linear Systems of Differential Equations", Duke Math. J. 15 (1948), 111-126.

6. Pfeiffer, G.W. "Asymptotic Solutions of $y''' + qy' + ry = 0$", J. Diff. Equations 11 (1972), 145-155.

7. Pfeiffer, G.W. "The Oscillatory Nature of the Equation $y''' + qy' + ry = 0$", J. Diff. Equations 11 (1972), 138-144.