Extensions to the development of the Sinc-Galerkin method for parabolic problems

by Randy Ross Doyle

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics
Montana State University

Abstract:
A Galerkin method in both spatial and temporal domains is developed for the parabolic problem. The development is carried out in detail in the case of one spatial dimension. The discretization in the case of two spatial dimensions along with numerical implementation is also carried out. The basis functions for the Galerkin methods are the sinc functions (composed with conformal maps) which, in conjunction with the highly accurate sinc function quadrature rules, form the foundation of a very powerful numerical method for the parabolic problem. The spectral analysis of the discrete sinc system receives close attention and, in particular, is shown to be uniquely solvable. This is the case for either of the conformal mappings used in the temporal domain. A consequence of this spectral analysis provides the motivation for the iterative method of solution of the discrete system. This iterative solution method is numerically tested not only on linear problems, but also on Burgers' (a nonlinear) problem.

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ABSTRACT

A Galerkin method in both spatial and temporal domains is developed for the parabolic problem. The development is carried out in detail in the case of one spatial dimension. The discretization in the case of two spatial dimensions along with numerical implementation is also carried out. The basis functions for the Galerkin methods are the sinc functions (composed with conformal maps) which, in conjunction with the highly accurate sinc function quadrature rules, form the foundation of a very powerful numerical method for the parabolic problem. The spectral analysis of the discrete sinc system receives close attention and, in particular, is shown to be uniquely solvable. This is the case for either of the conformal mappings used in the temporal domain. A consequence of this spectral analysis provides the motivation for the iterative method of solution of the discrete system. This iterative solution method is numerically tested not only on linear problems, but also on Burgers' (a nonlinear) problem.

CHAPTER 1

INTRODUCTION AND SUMMARY

Parabolic partial differential equations arise in many applications of the physical and biological sciences. The equations for fluid flow in the boundary layer along a wall and for axi-symmetric flow in a channel give rise to parabolic partial differential equations. A description of bacterial population is given by Fisher's equation, a nonlinear parabolic partial differential equation, while another such equation models neutron population in a nuclear reactor. Both the advection-diffusion equation and Burgers' equation, used as model equations of the vorticity transport arising from the vorticity-stream function formulation of the incompressible Navier-Stokes equations, are nonlinear parabolic differential equations.

All of these parabolic problems in one spatial dimension are encompassed in the formulation

$$u_t(x,t) - \mu u_{xx}(x,t) = f(x,t,u) \quad , \tag{1.1}$$

where $\mu > 0$.

A standard approach in obtaining a numerical solution of the linear parabolic partial differential equation

$$\begin{aligned} &V_t(x,t) = \mu V_{xx}(x,t) + f(x,t,V) \ ; \ (x,t) \in \mathcal{D} \\ &V(0,t) = g_0(t) \\ &V(1,t) = g_1(t) \\ &V(x,0) = V_0(x) \\ &\mathcal{D} = \{(x,t) : 0 < x < 1; t > 0\} \end{aligned} \tag{1.2}$$

where

$$V_0(0) = g_0(0)$$

$$V_0(1) = g_1(0)$$

begins with a discretization of the spatial domain using a finite difference, finite element, collocation, or Galerkin method [1], [3], [4], [5] and [6]. The result of this approach is a system of ordinary differential equations in the time variable. One generally uses a time differencing scheme to complete the approximation. Low order methods with small time steps are required to obtain accurate approximations due to stability constraints on the discrete time evolution operator [6].

A great deal of effort has been expended in the development of methods which increase the accuracy of the temporal approximation. Seward, Fairweather, and Johnson [15] have classified methods of order greater than two for the time integration. Fairweather and Saylor [5] give methods which have up to fourth order accuracy in time. Each of these techniques possess certain advantages. However, each of these methods has a finite order of accuracy in time and so, when used with an infinite order spatial approximation (spectral, for example), the overall result is a method unbalanced in total order. Space-time finite element methods have been developed [4] which avoid time differencing, and yet when piecewise polynomial elements are used, again a finite order method results.

A method that is infinite in order in the time domain has been developed by Lewis, Lund, and Bowers [9] for the linear parabolic partial differential equation. There the fully Sinc-Galerkin discretization of (1.1) has the matrix form

$$\mathcal{A}\vec{u} = \vec{f} \tag{1.3}$$

where $\mathcal{A}$ is a tensor product of the spatial and temporal matrices associated with a Sinc-Galerkin approach to the one-dimensional problems taken from (1.1) by fixing t and x, respectively. Due to the manner in which the discrete system in

(1.3) was developed in [9], the authors there could only numerically justify the invertibility of the matrix $\mathcal{A}$. The work in this thesis removes this defect from the fully Sinc-Galerkin method. By rewriting the system (1.3) in a slightly different form, the invertibility of the coefficient matrix $\mathcal{A}$ will be established in Corollary 5.5.

The present work extends the results of Lewis, Lund, and Bowers in two other complementary directions. First is the development of a methodology to handle the more general problem (1.1) when f represents heat loss or gain

$$f = -cu + g$$

or convection

$$f = -cu_x \quad .$$

Secondly, it is shown in the linear case

$$f(x,t,u) = g(x,t) - cu(x,t) \quad ,$$

that the method, which is iterative, must converge, and an upper bound for the rate of convergence is exhibited. Finally, motivated by the convergence results for the iterative method on the linear problem, a first approach to nonlinear problems is developed using Burgers' equation as the prototype.

In the present work an infinite order, multiple space dimension, fully Galerkin method using sinc basis functions in all dimensions is developed. The problem is transformed as follows: If $V(x,t)$ solves system (1.2), let $U(x,\tau)$ be defined by

$$U(x,\tau) = V(x,t) - G(x,t) \tag{1.4}$$

where

$$\begin{aligned} G(x,t) &= (1-x)g_0(t) + xg_1(t) \\ &+ (V_0(x) - (1-x)V_0(0) - xV_0(1)) \exp(-\tau) \end{aligned}$$

and

$$\tau = \mu t \quad .$$

Then $U(x,\tau)$ solves the differential equation

$$U_\tau(x,\tau) + G_t(x,t)/\mu = U_{xx}(x,\tau) + G_{xx}(x,t)$$
$$+ f\left(x,t,U(x,\tau) + G(x,t)\right)/\mu \quad ,$$

where

$$t = \tau/\mu \quad .$$

This reduces to

$$\begin{aligned} &U_\tau(x,\tau) = U_{xx}(x,\tau) + F(x,\tau,U) \\ &U(0,\tau) = U(1,\tau) = U(x,0) = 0 \qquad (1.5) \\ &0 < x < 1 \quad , \quad \tau > 0 \end{aligned}$$

where

$$F(x,\tau,U) = G_{xx}(x,t) - G_t(x,t)/\mu$$
$$+ f\left(x,t,U(x,\tau) + G(x,t)\right)/\mu \quad .$$

The function $U(x,\tau)$ solving (1.5) is equivalent to the function $V(x,t)$ solving (1.2) via the change of variable in (1.4). This transformation of the problem allows for the use of sinc basis functions without the need to add special functions to handle initial and boundary conditions.

The organization of this thesis is summarized in the following remarks. Chapter 2 contains a brief review of some analytic and numerical properties of the sinc function. The former is included so as to assemble in one convenient source various results that are scattered in the literature. The approximate properties of the sinc function are largely taken from the extensive work in [17] but have here been limited and restricted to include the results germaine to the present thesis. Included, in particular, is the development of the sinc quadrature rules which play such an important role in the approximation of the inner products for the Sinc-Galerkin method.

These inner product approximations are developed in Chapter 3. Not only is the time domain mapping function which was used in [9] examined, but also a different choice of mapping function is considered. This latter choice has provided a complementary alternative to numerical procedures for Sturm-Liouville eigenvalue problems [3], quadratures for various integral transforms [12] and the numerical solution of a (scalar) nonlinear ordinary differential equation [16]. In the context of the present thesis, this choice of mapping in the temporal domain also gives rise to a discrete system of the form (1.3) which is shown to be uniquely solveable.

In Chapter 4 the assembly of the inner product development for the problem (1.1) is carried out where the function f can represent simple non-homogeneous, heat loss (gain), or linear or nonlinear convection. In all cases the discrete system takes the form

$$\mathcal{A}\vec{u} = \vec{f} + m(\vec{u}) \tag{1.6}$$

where m is a linear or nonlinear (in the case of Burgers' equation) function of the vector $\vec{u}$. Whereas the former may be solved by direct methods an iterative procedure is defined and carried out in this thesis.

The foundation for this iterative procedure is a consequence of the spectral study of the matrix $\mathcal{A}$ which is the subject of Chapter 5. This chapter not only puts the fully Sinc-Galerkin method on a firm analytic footing (unique solvability of the discrete system), but also points (via the above mentioned iterative scheme) to a future direction for an efficient computational procedure in the case of nonlinear problems.

The final Chapter 6 gives the numerical results for seven examples. The examples are selected to illustrate the choice of the temporal domain mapping function, the various parameter selections and the exponential convergence rate of the method. Specifically, examples are included which are computed directly (a direct method is used to solve (1.6)) or iteratively (an iterative method is used to solve (1.6)). To illustrate the method's applicability in the case of more than one spatial variable, a two-dimensional example is also included. The final example is Burgers' well-known sine initial condition problem.

CHAPTER 2

SINC INTERPOLATION AND QUADRATURE

This thesis addresses the development and extension of the fully Sinc-Galerkin method which began in [16] and was refined in [9]. The building blocks of the method for (1.1) consist of the Galerkin approach coupled with the sinc function. Numerical sinc function methods are based on E.T. Whittaker's work [19] concerning interpolation of functions at the integers. The foundation of this work is the sinc function

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \quad , \quad x \in \mathbb{R} \quad .$$

The resulting formal cardinal expansion for a function $f(x)$ is defined by

$$\sum_{k=-\infty}^{\infty} f(k)\mathrm{sinc}(x-k) \quad .$$

In order to generalize the expansion to handle interpolation at any evenly spaced grid, define for $h > 0$

$$S(k,h)(x) = \mathrm{sinc}\left[\frac{x-kh}{h}\right] \tag{2.1}$$

and denote the Whittaker cardinal function of f by

$$C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh)S(k,h)(x) \tag{2.2}$$

whenever this series converges.

With regard to using (2.2) as an interpolation tool, it is important to know for which functions the right-hand side of (2.2) converges, and when convergent, to what it converges. In this direction, two important classes of functions will be identified. The first is the somewhat restrictive class for which (2.2) is exact. The second is the class of functions where the difference between $f(x)$ and $C(f,h)(x)$ is "small". J.M. Whittaker [20] and McNamee, Stenger and Whitney [13] identified these classes by displaying a natural link between the Whittaker series of f and the Fourier transform of f.

The Fourier transform of a function g is defined by

$$\mathcal{F}(g)(x) = \int_{-\infty}^{\infty} g(t)\exp(ixt)dt \quad . \tag{2.3}$$

A fundamental result of Fourier analysis is the inversion theorem. Specifically, if $g \in L^2(\mathbb{R})$ then there exists $\mathcal{F}(g) \in L^2(\mathbb{R})$ whereby g can be recovered from $\mathcal{F}(g)$ by the Fourier inversion integral

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}(g)(x)\exp(-ixt)dx \quad .$$

If the analytic structure of g is taken into account, it is to be expected that more can be said about the Fourier transform of g. When g is an entire function this is a fundamental result due to Paley and Wiener [14, page 375].

Theorem 2.1 (The Paley-Wiener Theorem): *If $g \in L^2(\mathbb{R})$, is entire, and there are positive constants A and C so that for all $w \in \mathbb{C}$*

$$|g(w)| \le C\exp(A|w|) \quad ,$$

then

$$g(w) = \frac{1}{2\pi}\int_{-A}^{A} \mathcal{F}(g)(x)\exp(-iwx)dx \quad .$$

To show that the sinc function obeys the hypotheses of the Paley-Wiener Theorem with $A = \pi$ and $C = 1$ is straightforward. A simple integration yields the identity

$$\begin{aligned}\text{sinc}(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(-ixt)dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi_{(-\pi,\pi)}(x)\exp(-ixt)dx \quad .\end{aligned}$$

That is, the characteristic function $\chi_{(-\pi,\pi)}(x)$ for the interval $(-\pi,\pi)$ is the inverse Fourier transform of $\text{sinc}(t)$. From this, a consequence of the inversion theorem is

$$\mathcal{F}(\text{sinc})(x) = \chi_{(-\pi,\pi)}(x) = \int_{-\infty}^{\infty} \text{sinc}(t)e^{ixt}dt \quad . \tag{2.4}$$

Hence, using (2.1), a change of variables, and (2.4) gives

$$\begin{aligned}\mathcal{F}(S(k,h))(x) &= \int_{-\infty}^{\infty} \text{sinc}\left[\frac{t-kh}{h}\right]\exp(ixt)dt \\ &= h\exp(ixkh)\chi_{(-\pi/h,\pi/h)}(x) \quad .\end{aligned} \tag{2.5}$$

Hence, using the inversion theorem yields the useful identity

$$S(k,h)(x) = \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} \exp[-i(x-kh)t]dt \quad . \tag{2.6}$$

Definition 2.2 (The Paley-Wiener Class of Functions): *Let $B(h)$ be the set of functions g such that g is entire, $g \in L^2(\mathbb{R})$, and for all $w \in \mathbb{C}$,*

$$|g(w)| \leq C\exp\left(\frac{\pi}{h}|w|\right) \quad .$$

This is precisely the class where the representation of g by its cardinal series (2.2) is exact. To see this a few preliminary results are needed.

Theorem 2.3: *If $g \in B(h)$ then*

$$g(w) = \frac{1}{h}\int_{-\infty}^{\infty} g(t)\text{sinc}\left[\frac{t-w}{h}\right]dt \quad .$$

Theorem 2.4: *If $g \in L^2(\mathbb{R})$ then*

$$k(w) = \frac{1}{h}\int_{-\infty}^{\infty} g(t)\operatorname{sinc}\left[\frac{t-w}{h}\right] dt \quad .$$

is in $B(h)$.

A proof of Theorems 2.3 and 2.4 can be found in [17]. The above can now be combined in the following theorem.

Theorem 2.5: *If $g \in B(h)$ then*

$$g(w) = \sum_{k=-\infty}^{\infty} \alpha_k S(k,h)(w)$$

where

$$\alpha_k = \frac{1}{h}\int_{-\infty}^{\infty} g(t)\operatorname{sinc}\left[\frac{t-kh}{h}\right] dt = g(kh) \quad .$$

The result can be stated in terms of the class of functions where the Whittaker cardinal expansion (2.2) is exact: If $g \in B(h)$ then $g(w) = C(f,h)(w)$ for all $w \in \mathbb{C}$. This result is stronger than that which was sought, and can be strengthened further by a result derived from Theorem 2.4 and the identity

$$\frac{1}{h}\int_{-\infty}^{\infty} S(k,h)(t)S(p,h)(t)dt = \delta_{pk} = \begin{cases} 1 & , \ p = k \\ 0 & , \ p \neq k \end{cases} \quad .$$

This identity follows upon replacing g in Theorem 2.3 by $S(k,h)(t)$ and recalling that $S(k,h)$ is in $B(h)$.

Theorem 2.6: *The set*

$$\left\{\frac{1}{\sqrt{h}}\, S(k,h)\right\}_{k=-\infty}^{\infty}$$

is a complete orthonormal set in $B(h)$ ([17]).

The class of functions $B(h)$ is quite restrictive in the sense that entirety cannot be expected of a function arising in applications. Hence, a larger class of functions is sought whereby the Whittaker cardinal expansion (2.2), while not exact, provides an accurate interpolatory series. A class of functions with this property is identified in [17].

Definition 2.7: *Let*

$$D_S = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < d,\ 0 < d \leq \pi/2\}$$

and denote by $B^p(D_S)$ the set of functions f such that f is analytic in D_S, and

$$\int_{-d}^{d} |f(t+iy)|dy = \mathcal{O}(|t|^{\alpha}) \quad \text{as} \quad t \to \pm\infty$$

for some α, $0 \leq \alpha < 1$, and for $p = 1$ or 2

$$N_p(f, D_S) = \lim_{y \to d^-} \left[\left\{ \int_{-\infty}^{\infty} |f(t+iy)|^p\, dt \right\}^{1/p} + \left\{ \int_{-\infty}^{\infty} |f(t-iy)|^p\, dt \right\}^{1/p} \right] < \infty \quad .$$

For $p = 1$ let $N(f, D_S) = N_1(f, D_S)$ and $B(D_S) = B^1(D_S)$.

In this set of functions the error between f and $C(f,h)$ is given by the following theorem.

Theorem 2.8: *If for $p = 1$ or 2, $f \in B^p(D_S)$ and $C(f,h)(x)$ is convergent then*

$$\begin{aligned} E(f)(x) &:= f(x) - C(f,h)(x) \\ &= \frac{\sin(\pi x/h)}{2\pi i} \int_{-\infty}^{\infty} \left[\frac{f(t-id)}{(t-x-id)\sin[\pi(t-id)/h]} - \frac{f(t+id)}{(t-x+id)\sin[\pi(t+id)/h]} \right] dt \quad . \end{aligned} \tag{2.7}$$

Further, if $f \in B(D_S)$ then

$$\|E(f)\|_\infty \leq \frac{N_1(f, D_S)}{2\pi d \sinh(\pi d/h)} \tag{2.8}$$

and if $f \in B^2(D_S)$ then

$$\|E(f)\|_\infty \leq \frac{N_2(f, D_S)}{2\sqrt{\pi d}\,\sinh(\pi d/h)} \quad . \tag{2.9}$$

A proof of the above theorem can be found in Stenger [17]. A corollary to Theorem 2.8 is that if f is in $B^p(D_S)$ and g is a function analytic in D_S that is bounded there independent of h, then the product function fg is also in $B^p(D_S)$.

The error statement in Theorem 2.8 is valid only on the real line. While there are extensions to regions in $\mathbb{C}$, the goal of this work is to approximate on subsets of the real line and so the theorem is sufficiently general for the purposes of the thesis.

An order statement can be made using (2.8) and (2.9). As $h \to 0$, $\sinh(\pi d/h) \to \infty$. Hence $\|E(f)\|_{2,\infty} \to 0$ for either $f \in B(D_S)$ or $f \in B^2(D_S)$, and the rate of this convergence is dependent on $\sinh(\pi d/h)$. Note that $1/\sinh(\pi d/h)$ is $\mathcal{O}(\exp(-\pi d/h))$ as $h \to 0$.

For the purpose of practical approximation the infinite sum defining $C(f,h)$ in (2.2) is truncated to

$$C_{M,N}(f,h)(x) = \sum_{k=-M}^{N} f(kh)\text{sinc}\left[\frac{x-kh}{h}\right] \quad .$$

It cannot, in general, be expected that the error introduced by truncation will maintain the exponential rate of convergence of the previous paragraph. However, in the case where f decreases sufficiently rapidly the exponential error rate (as a function of the number of retained interpolation points) is maintained.

Theorem 2.9: *If $f \in B^p(D_S)$ for $p = 1$ or 2, $d > 0$, and there exist positive constants α, β and L so that*

$$|f(x)| \leq L \begin{cases} \exp(\alpha x) & , \ x \in (-\infty, 0) \\ \exp(-\beta x) & , \ x \in [0, \infty) \end{cases} \tag{2.10}$$

then given a positive integer M, choose N and h by the relations

$$N = \left[\left|\frac{\alpha}{\beta} M + 1\right|\right] \tag{2.11}$$

and

$$h = \left(\frac{\pi d}{\alpha M}\right)^{1/2} \quad .$$

This leads to

$$\|f - C_{M,N}(f,h)\|_\infty \leq \kappa_1 \sqrt{M} \exp\left(-\sqrt{\pi d \alpha M}\right)$$

where κ_1 is a constant dependent only on f.

Proof: From Theorem 2.8 there exists a constant κ_2 so that

$$|f(x) - C(f,h)(x)| \leq \kappa_2 \exp(-\pi d/h)$$

for all $x \in \mathrm{I\!R}$. By the triangle inequality and the geometric series it follows that

$$\begin{aligned}
&|f(x) - C_{M,N}(f,h)(x)| \\
&\leq \kappa_2 \exp(-\pi d/h) + \sum_{j=M+1}^{\infty} |f(-jh)| + \sum_{j=N+1}^{\infty} |f(jh)| \\
&\leq \kappa_2 \exp(-\pi d/h) + L \left\{ \sum_{j=M+1}^{\infty} \exp(-\alpha j h) + \sum_{j=N+1}^{\infty} \exp(-\beta j h) \right\} \\
&\leq \kappa_2 \exp(-\pi d/h) + L \left\{ \frac{\exp(-\alpha M h)}{\alpha h} + \frac{\exp(-\beta N h)}{\beta h} \right\} \\
&\leq \left\{ \kappa_2 + L \left[\frac{1}{\alpha} + \frac{1}{\beta}\right] \frac{\sqrt{\alpha \pi d M}}{\pi d} \right\} \exp\left(-\sqrt{\alpha \pi d M}\right) \\
&\leq \left\{ \kappa_2 + L \left(\frac{\alpha + \beta}{\alpha \beta}\right) \frac{\sqrt{\alpha \pi d}}{\pi d} \right\} \sqrt{M} \exp\left(-\sqrt{\alpha \pi d M}\right) \quad .
\end{aligned}$$

The proof is completed by defining the constant κ_1

$$\kappa_1 = \left\{ \kappa_2 + L \, \frac{\alpha+\beta}{\alpha\beta} \, \frac{\sqrt{\alpha\pi d}}{\pi d} \right\} \quad .$$

The interpolation results above show that the use of sinc functions as an interpolatory basis results in very accurate approximations of a function f on the real line if f is in $B^p(D_S)$ and satisfies the decay condition (2.10).

The primary use of Theorems 2.8 and 2.9 in this thesis is to develop the sinc quadrature rule. A few preliminary results are needed to develop this integration rule.

Substituting $x = 0$ in (2.5) shows that

$$\frac{1}{h} \int_{-\infty}^{\infty} S(k,h)(x)dx = 1$$

for all integers k. Hence if the Whittaker cardinal expansion of f in (2.2) is convergent for all x then

$$\int_{-\infty}^{\infty} C(f,h)(x)dx = h \sum_{k=-\infty}^{\infty} f(kh) \quad .$$

Combining this identity with the result of Theorem 2.8 leads to the quadrature rule

$$\int_{-\infty}^{\infty} f(x)dx = h \sum_{k=-\infty}^{\infty} f(kh) + \Theta(f) \quad , \tag{2.12}$$

where

$$\Theta(f) = \int_{-\infty}^{\infty} E(f)(x)dx \quad .$$

For real positive α, the Laplace integral

$$\int_{-\infty}^{\infty} \frac{\exp(i\alpha x)}{x-z} dx = \begin{cases} 2\pi i \exp(i\alpha z) & , \ \mathrm{Im}\{z\} > 0 \\ 0 & , \ \mathrm{Im}\{z\} < 0 \end{cases}$$

may be used in $\Theta(f)$ along with the definition of $N(f, D_S)$ to show that

$$|\Theta(f)| \leq \frac{\exp(-\pi d/h) N(f, D_S)}{2\sinh(\pi d/h)} \quad .$$

This result shows that the sinc quadrature on $\mathbb{R}$ is identical to the standard trapezoidal rule, but that the restriction of f to $B(D_S)$ gives an error of the order of $\mathcal{O}\left(\exp(-2\pi d/h)\right)$ as opposed to the trapezoidal error of order $\mathcal{O}(h^2)$ when f'' is bounded.

A restriction on the growth rate of f must be imposed to guarantee an exponential rate of convergence for (2.12) when the sum in (2.12) is truncated. This is similar to the restrictions outlined in Theorem 2.9 for interpolation.

Theorem 2.10: *Assume $f \in B(D_S)$, $d > 0$, and for positive constants α, β and L, f satisfies (2.10). Given an integer M, choose N by (2.11) and h by*

(2.13) $$h = \left(\frac{2\pi d}{\alpha M}\right)^{1/2} \quad .$$

Then

(2.14) $$\left| \int_{-\infty}^{\infty} f(x)dx - h \sum_{k=-M}^{N} f(kh) \right| \leq \kappa_3 \exp\left(-\sqrt{2\alpha\pi d M}\right) \quad ,$$

where κ_3 depends only on f.

The domains of the present thesis are different than the entire real line so that the development of an accurate quadrature rule based on (2.14) requires the following definitions.

Definition 2.11: *Let D be a simply connected domain and D_S as in Definition 2.7. Given distinct a, b on the boundary of D, let χ be a conformal map from D onto D_S satisfying*

$$\Gamma = \chi^{-1}(\mathbb{R})$$

and

$$\lim_{\substack{z \to a \\ z \in \Gamma}} \chi(z) = -\infty \quad , \quad \lim_{\substack{z \to b \\ z \in \Gamma}} \chi(z) = +\infty \quad .$$

Let $B(D)$ denote the family of functions analytic in D that satisfy

$$\int_{\chi^{-1}(x+L)} |F(z)dz| \to 0 \;\text{ as }\; x \to \pm\infty$$

$$L = \{iy : |y| < d\}$$

and

$$N(F,D) = \lim_{\substack{C \subset D \\ C \to \partial D}} \int_C |F(z)dz| < \infty$$

where C is a simple closed curve in D.

If $f \in B(D)$ then g defined by

$$g(x) = f\left(\chi^{-1}(x)\right)\left(\chi^{-1}\right)'(x)$$

is in $B(D_S)$ of Definition 2.7. The growth restriction in (2.10) for an arbitrary simply connected domain D is characterized by the following definition.

Definition 2.12: *A function $f \in B(D)$ is said to decay exponentially with respect to the conformal mapping χ of D onto D_S if there exist positive constants L, α and β so that*

$$\left|\frac{f(x)}{\chi'(x)}\right| \le L \begin{cases} \exp\left(-\alpha|\chi(x)|\right) & , \; x \in \Gamma_L \\ \exp\left(-\beta|\chi(x)|\right) & , \; x \in \Gamma_R \end{cases} \tag{2.15}$$

where

$$\Gamma_L = \{z : \chi(z) \in (-\infty, 0)\} \quad , \quad \Gamma_R = \{z : \chi(z) \in [0, \infty)\} \quad . \tag{2.16}$$

With this definition, the following theorems are the direct analogues of Theorems 2.9 and 2.10 for an arc $\Gamma \subset D$.

Theorem 2.13: *Assume $f \in B(D)$ and f decays exponentially with respect to the conformal map $\chi : D \to D_S$. Given a positive integer M select*

$$N = \left[\left|\frac{\alpha}{\beta} M + 1\right|\right]$$

and

$$h = \sqrt{\pi d/(\alpha M)} \quad .$$

Then upon setting

$$z_k = \chi^{-1}(kh)$$

it follows that for all $z \in \Gamma$

$$\left|\frac{f(z)}{\chi'(z)} - C_{M,N}\left[\frac{f}{\chi'}\ ,\ h\right](\chi(z))\right|$$
$$= \left|\frac{f(z)}{\chi'(z)} - \sum_{k=-M}^{N} \frac{f(z_k)}{\chi'(z_k)}\, S(k,h)\,(\chi(z))\right|$$
$$\leq \kappa_4 \exp\left(-\sqrt{\alpha\pi d M}\right) \quad ,$$

where the constant κ_4 depends only on f.

Theorem 2.14: *Assume $f \in B(D)$ and f decays exponentially with respect to the conformal map $\chi : D \to D_S$. Given a positive integer M define*

$$N = \left[\left|\frac{\alpha}{\beta} M + 1\right|\right] \tag{2.17}$$

and

$$h = \sqrt{2\pi d/(\alpha M)} \quad . \tag{2.18}$$

Upon recalling $\Gamma = \chi^{-1}(\mathbb{R})$ *and setting*

$$z_k = \chi^{-1}(kh)$$

it follows that

$$\left| \int_\Gamma f(z)dz - h \sum_{k=-M}^{N} \frac{f(z_k)}{\chi'(z_k)} \right| \leq \kappa_5 \exp\left(-\sqrt{2\pi d \alpha M}\right) \tag{2.19}$$

where κ_5 *depends only on* f.

There is an important special case of the previous result that plays a fundamental role in the construction and evaluation of the inner products defined in the next chapter. This special case uses the interpolatory identity

$$S(j,h)\left(\chi(z_k)\right) = \delta_{jk}$$

and takes the form for $f \in B(D)$

$$\int_\Gamma f(z)S(k,h)(\chi(z))dz = h\,\frac{f(z_k)}{\chi'(z_k)} + \mathcal{O}\left(\exp(-\pi d/h)\right) \quad . \tag{2.20}$$

That is, the reproduction property of the sinc kernel in Theorem 2.3 remains "approximately" true for $f \in B(D)$.

CHAPTER 3

INNER PRODUCT APPROXIMATIONS

The quadrature rule (2.19) is readily adaptable to the numerical approximation of the inner products arising in the Sinc-Galerkin discretization of (1.1). In this chapter, the focus is on the discretization of

$$Pu(x,t) \equiv u_t(x,t) - u_{xx}(x,t) = f(x,t,u) \tag{3.1}$$

where

$$(x,t) \in (a,b) \times (0,\infty) \quad .$$

The emphasis will be on the discretization of Pu but also included is the discretization of $f(x,t,u)$ where f takes any one of the forms

$$f(x,t,u) = f(x,t) \quad , \tag{3.2}$$

$$f(x,t,u) = u(x,t) \quad , \tag{3.3}$$

$$f(x,t,u) = u_x(x,t) \tag{3.4}$$

and

$$f(x,t,u) = u(x,t)u_x(x,t) = \frac{1}{2}\,\frac{\partial}{\partial x}\left[(u(x,t))^2\right] \quad . \tag{3.5}$$

The following definition, which specializes Definition 2.11, establishes the notation that will be adhered to throughout the remainder of this thesis.

Definition 3.1: *Let ϕ be an invertible conformal map in the simply connected domain D with the property that $\phi : (a,b) \to (-\infty,\infty)$. Let $\{x_k\}$ be implicitly defined by the relationship $kh = \phi(x_k)$, for all $k \in \mathbb{Z}$. Let $\hat{\phi}$ be an invertible conformal map in the simply connected domain D_t with the property that $\hat{\phi}$: $(0,\infty) \to (-\infty,\infty)$. Let $\{t_j\}$ be implicitly defined by the relationship $j\hat{h} = \hat{\phi}(t_j)$, for all $j \in \mathbb{Z}$.*

For the following, assume that the parameters h, $\hat{h}$, M_t, N_t, M_x and N_x have been assigned. In the ensuing work, error terms are developed as functions of these assignments. These error terms are used to determine optimal choices for the parameters, and this is all sorted out in detail in Chapter 6. Towards simplifying notation and the convenient delineation of spatial versus temporal domain variables the following definition is useful.

Definition 3.2: *Let $w(x)$ and $\hat{w}(t)$ represent positive weight functions on (a,b) and $(0,\infty)$, respectively. The Galerkin inner product of f is defined by*

$$
\begin{aligned}
< f >_{kj} &\equiv \frac{1}{h\hat{h}} \int_a^b \int_0^\infty \left\{ f(x,t) S(k,h) \circ \phi(x) S(j,\hat{h}) \circ \hat{\phi}(t) w(x) w(t) \right\} dt dx \\
&\equiv \frac{1}{h\hat{h}} \int_a^b \int_0^\infty \left\{ f(x,t) S_k(x) \hat{S}_j(t) w(x) w(t) \right\} dx dt \quad .
\end{aligned}
\tag{3.6}
$$

In (3.6) the sinc function (2.1) when composed with $\phi(x)$ will be denoted by $S_k(x)$ (similarly for $\hat{S}_j(t) \equiv S(j,\hat{h}) \circ \hat{\phi}(t)$). The convention of a hat denoting a function in the temporal domain will be adhered to throughout the remainder of the thesis. In like manner this applies to parameters where appropriate, e.g. $\hat{h}$ is the "mesh size" in the temporal domain. As a final convention, it is typographically convenient to set

$$
\frac{d}{d\phi} S_k(x) \equiv S_k'(x) \quad , \quad \frac{d^2}{d\phi^2} S_k(x) = S_k''(x) \tag{3.7}
$$

and

$$\frac{d}{d\hat{\phi}}\,\hat{S}_j(t) \equiv \hat{S}_j'(t) \quad . \tag{3.8}$$

This replaces the somewhat unwieldly notation $S_{k\phi}(x)$, $S_{k\phi\phi}(x)$ or $S_{j\hat{\phi}}(t)$.

For $u_t(x,t)$, the Galerkin inner product with sinc basis elements in both the space and time domains is given by

$$< u_t >_{kj} \equiv \frac{1}{h\hat{h}} \int_a^b \int_0^\infty \left[u_t(x,t)S_k(x)w(x)\hat{S}_j(t)\hat{w}(t)\right] dtdx \quad .$$

This expression contains the t derivative of u, but the desired result is interpolation of the variable u with no derivatives. Theorem 2.13 guarantees an accurate interpolation of $u(x,t)$ but does not guarantee an accurate interpolation of $u_t(x,t)$. Integrating by parts to remove the t derivative from the dependent variable u leads to the equality

$$\begin{aligned} < u_t >_{kj} &= \frac{1}{h\hat{h}} \int_a^b \left[u(x,t)S_k(x)w(x)\hat{S}_j(t)\hat{w}(t)\right]\Bigg|_{t=0}^{t=\infty} dx \\ &- \frac{1}{h\hat{h}} \int_a^b \int_0^\infty \left[u(x,t)S_k(x)w(x)\left(\hat{S}_j(t)\hat{w}(t)\right)_t\right] dtdx \quad . \end{aligned} \tag{3.9}$$

While there are several possible methods for dealing with the first integral in (3.9) (the boundary integral), for the present it is convenient to assume that for each fixed x,

$$\lim_{t\to 0^+} u(x,t)\hat{w}(t)/\hat{\phi}(t) = 0 \tag{3.10}$$

and

$$\lim_{t\to\infty} u(x,t)\hat{w}(t)/\hat{\phi}(t) = 0 \quad . \tag{3.11}$$

The immediate consequence of the assumptions in (3.10) and (3.11), coupled with the fact that

$$\hat{S}_j(t) = \frac{\sin\left[\frac{\pi}{\hat{h}}\left(\hat{\phi}(t) - j\hat{h}\right)\right]}{\frac{\pi}{\hat{h}}\left(\hat{\phi}(t) - j\hat{h}\right)} \quad ,$$

is that the boundary integral in (3.9) vanishes. The rationale of these assumptions is related to the boundary behaviors of $u(*, t)$ and $\hat{w}(t)$, and partially motivates the selection of $\hat{w}(t)$ in the present development. Under the assumption that (3.10) and (3.11) are satisfied, then (3.9) may be written as

$$\text{(3.12)} \quad \begin{aligned} < u_t >_{kj} &= -\frac{1}{h\hat{h}} \int_a^b \int_0^\infty \left[u(x,t)S_k(x)w(x)\left(\hat{S}_j(t)\hat{w}(t)\right)_t\right] dxdt \\ &= -\frac{1}{h\hat{h}} \int_a^b \int_0^\infty u(x,t)S_k(x)w(x)\left(\hat{S}_j'(t)\hat{\phi}'(t)\hat{w}(t) + \hat{S}_j(t)\hat{w}'(t)\right) dtdx \quad , \end{aligned}$$

where the primes denote differentiation with respect to t and the convention in (3.8) is used.

In order to apply the quadrature rule (2.19) to the right-hand side of (3.12) assume that for each fixed x

$$u(x,t)\hat{w}'(t) \in B(D_t)$$

and

$$u(x,t)\hat{w}(t)\hat{\phi}'(t) \in B(D_t) \quad ,$$

where $B(D_t)$ is defined in Definition 2.11 with $D \equiv D_t$. If each of the two above functions decay exponentially with respect to $\hat{\phi}$ (Definition 2.12), then the application of the sinc quadrature rule (2.19) in the variable t gives

$$\text{(3.13)} \quad \begin{aligned} < u_t >_{kj} = -\frac{1}{h} \int_a^b S_k(x)w(x) \Bigg\{ & u(x,t_j)\hat{w}'(t_j)/\hat{\phi}'(t_j) \\ & + \sum_{i=-M_t}^{N_t} u(x,t_i)\hat{w}(t_i)\hat{S}_j'(t_i) \Bigg\} dx \quad . \end{aligned}$$

The error in this approximation is of the order $\mathcal{O}\left(\exp\left(-\gamma_t\sqrt{M_t}\right)\right)$ where γ_t depends on the exponential decay of u and the region D_t. If in addition it is assumed that for each fixed t

$$u(x,t)w(x) \in B(D)$$

then the special case of the sinc quadrature (2.20) in the x-variable can be applied to (3.13) resulting in the expression

$$\begin{aligned} < u_t >_{kj} &= -\sum_{i=-M_t}^{N_t} \left[u(x_k,t_i)\hat{w}(t_i)\hat{S}_j'(t_i)w(x_k)/\phi'(x_k)\right] \\ &\quad - u(x_k,t_j)w(x_k)\hat{w}'(t_j)/\left(\hat{\phi}'(t_j)\phi'(x_k)\right) \end{aligned} \tag{3.14}$$

with the addition of an error term of the order $\mathcal{O}\left(\exp(-\pi d/h)\right)$.

The Galerkin inner product for $u_{xx}(x,t)$ may be handled in a similar manner upon integrating by parts twice. This gives the computation

$$\begin{aligned} < u_{xx} >_{kj} &= \frac{1}{h\hat{h}}\int_a^b\int_0^\infty u_{xx}(x,t)w(x)S_k(x)\hat{w}(t)\hat{S}_j(t)dtdx \\ &= \frac{1}{h\hat{h}}\int_0^\infty \left[u_x(x,t)w(x)S_k(x)\right]\Big|_{x=a}^{x=b}\hat{w}(t)\hat{S}_j(t)dt \\ &\quad - \frac{1}{h\hat{h}}\int_0^\infty \left[u(x,t)\{w(x)S_k(x)\}'\right]\Big|_{x=a}^{x=b}\hat{w}(t)\hat{S}_j(t)dt \\ &\quad + \frac{1}{h\hat{h}}\int_a^b\int_0^\infty u(x,t)\{w(x)S_k(x)\}''\hat{w}(t)\hat{S}_j(t)dtdx \ . \end{aligned} \tag{3.15}$$

In the above calculation the primes denote differentiation with respect to x. As before, the boundary integrals in (3.15) will be assumed to be zero; that is, for each fixed t, assume

$$\lim_{x\to b^-} u_x(x,t)w(x)/\phi(x) = 0$$

and

$$\lim_{x\to a^+} u_x(x,t)w(x)/\phi(x) = 0$$

as well as

$$\lim_{x \to b^-} u(x,t)\{\phi'(x)w(x) + w'(x)\}/\phi(x) = 0$$

and

$$\lim_{x \to a^+} u(x,t)\{\phi'(x)w(x) + w'(x)\}/\phi(x) = 0 \quad .$$

Continuing with the only remaining integral in (3.15) and expanding the derivative results in

$$\begin{aligned}
< u_{xx} >_{kj} &= \frac{1}{h\hat{h}} \int_a^b \int_0^\infty u(x,t)\{w(x)S_k(x)\}''\hat{w}(t)\hat{S}_j(t)dtdx \\
&= \frac{1}{h\hat{h}} \int_a^b \int_0^\infty u(x,t)\hat{w}(t)\hat{S}_j(t)S_k(x)w''(x)dtdx \\
&+ \frac{1}{h\hat{h}} \int_a^b \int_0^\infty u(x,t)\hat{w}(t)\hat{S}_j(t)S_k'(x)\{\phi''(x)w(x) + 2\phi'(x)w'(x)\}dtdx \\
&+ \frac{1}{h\hat{h}} \int_a^b \int_0^\infty u(x,t)\hat{w}(t)\hat{S}_j(t)S_k''(x)[\phi'(x)]^2 w(x)dtdx \quad .
\end{aligned}$$

Under the assumptions that for each fixed t

$$u(x,t)w''(x) \quad ,$$

$$u(x,t)\{\phi''(x)w(x) + 2w'(x)\phi'(x)\}$$

and

$$u(x,t)w(x)[\phi'(x)]^2$$

are each in $B(D)$ and each decays exponentially with respect to ϕ, then the sinc quadrature rule (2.19) in the variable x applied to the above integral yields the approximation

$$
\begin{aligned}
< u_{xx} >_{kj} &= \frac{1}{\hat{h}} \int_0^\infty u(x_k,t)\hat{w}(t)\hat{S}_j(t)w''(x_k)/\phi'(x_k)dt \\
&+ \frac{1}{\hat{h}} \int_0^\infty \sum_{i=-M_x}^{N_x} u(x_i,t)\hat{w}(t)\hat{S}_j(t)S_k'(x_i)\left\{\phi''(x_i)w(x_i)/\phi'(x_i)\right. \\
&\qquad \left. + 2w'(x_i)\right\} dt \\
&+ \frac{1}{\hat{h}} \int_0^\infty \sum_{i=-M_x}^{N_x} u(x_i,t)\hat{w}(t)\hat{S}_j(t)S_k''(x_i)\phi'(x_i)w(x_i)dt
\end{aligned}
$$

If in addition, for each fixed x, the function

$$u(x,t)\hat{w}(t) \in B(D_t)$$

then the sinc quadrature rule (2.20) in the variable t results in the approximation

$$
\begin{aligned}
< u_{xx} >_{kj} &= \frac{\hat{w}(t_j)u(x_k,t_j)w''(x_k)}{\hat{\phi}'(t_j)\phi'(x_k)} \\
&+ \hat{w}(t_j) \sum_{i=-M_x}^{N_x} u(x_i,t_j)S_k'(x_i)\{\phi''(x_i)w(x_i)/\phi'(x_i) + 2w'(x_i)\}/\hat{\phi}'(t_j) \\
&+ \hat{w}(t_j) \sum_{i=-M_x}^{N_x} u(x_i,t_j)S_k''(x_i)\phi'(x_i)w(x_i)/\hat{\phi}'(t_j) \quad .
\end{aligned}
\tag{3.16}
$$

As in the development from (3.13) to (3.14) the error incurred in the approximation in (3.16) is of the order $\mathcal{O}\left(\exp\left(-\gamma_x\sqrt{M_x}\right) + \exp(-\pi\hat{d}/\hat{h})\right)$. Before specifying the weight functions as well as the conformal maps, it is convenient to collect the above development into a theorem.

Theorem 3.3: *Let $u(x,t)$ be a function of two variables, and let*

$$D_E = \{z : -d < \arg[(z-a)/(b-z)] < d\} \quad , \tag{3.17}$$

$$D_W = \{z : -d < \arg(z) < d\} \tag{3.18}$$

and

$$D_B = \{z : -d < \arg(\sinh(z)) < d\} \quad . \tag{3.19}$$

Given w, $\hat{w}$, ϕ, and $\hat{\phi}$, and for $q = W$ or B, assume that

$$u\hat{w}\,,\ u\hat{w}\hat{\phi}'\,,\ u\hat{w}' \in B(D_q) \tag{3.20}$$

and each decays exponentially with respect to $\hat{\phi}$ (Definition 2.12) and that

$$uw(\phi')^2\,,\ uw\,,\ uw''\,,\ u(\phi'' w + 2w'\phi') \in B(D_E) \tag{3.21}$$

and each decays exponentially with respect to ϕ. Assume further that

$$\lim_{\substack{x\to b^-\\ x\to a^+}} u_x(x,*)w(x)/\phi(x) = 0 \quad , \tag{3.22}$$

$$\lim_{\substack{x\to b^-\\ x\to a^+}} u(x,*)\{\phi'(x)w(x) + w'(x)\}/\phi(x) = 0 \tag{3.23}$$

and

$$\lim_{\substack{t\to 0^+\\ t\to\infty}} u(*,t)\hat{w}(t)/\hat{\phi}(t) = 0 \quad . \tag{3.24}$$

Then the approximation

$$\begin{aligned} < u_t >_{kj} = &- \sum_{i=-M_t}^{N_t} \left[u(x_k,t_i)\hat{w}(t_i)\hat{S}_j'(t_i)w(x_k)/\phi'(x_k)\right] \\ &- u(x_k,t_j)w(x_k)\hat{w}'(t_j)/\left(\hat{\phi}'(t_j)\phi'(x_k)\right) \end{aligned} \tag{3.25}$$

is of order $\mathcal{O}\left(\exp\left(-\gamma_t\sqrt{M_t}\right) + \exp(-\pi d/h)\right)$ and the approximation

$$(3.26)\quad \begin{aligned} < u_{xx} >_{kj} &= \frac{\hat{w}(t_j)u(x_k,t_j)w''(x_k)}{\hat{\phi}'(t_j)\phi'(x_k)} \\ &+ \hat{w}(t_j)\sum_{i=-M_x}^{N_x} u(x_i,t_j)S_k'(x_i)\{\phi''(x_i)w(x_i)/\phi'(x_i)+2w'(x_i)\}/\hat{\phi}'(t_j) \\ &+ \hat{w}(t_j)\sum_{i=-M_x}^{N_x} u(x_i,t_j)S_k''(x_i)\phi'(x_i)w(x_i)/\hat{\phi}'(t_j) \end{aligned}$$

is of the order $\mathcal{O}\left(\exp\left(-\gamma_x\sqrt{M_x}\right)+\exp(-\pi\hat{d}/\hat{h})\right)$.

In a similar fashion, the Galerkin inner products for the function $f(x,t,u)$ on the right-hand side of (3.1) may be derived. The inner products of each of these will be dealt with separately, but in little detail as the development is for the most part a reproduction of the above arguments. For f in (3.2), the inner product is, from (3.6) with (2.20)

$$(3.27)\quad \begin{aligned} < f >_{kj} &= \frac{1}{h\hat{h}}\int_a^b\int_0^\infty \left\{f(x,t)w(x)S_k(x)\hat{w}(t)\hat{S}_j(t)\right\}dtdx \\ &= \frac{\hat{w}(t_j)f(x_k,t_j)w(x_k)}{\hat{\phi}'(t_j)\phi'(x_k)} \end{aligned}$$

with error order $\mathcal{O}\left(\exp(-\pi d/h)+\exp(-\pi\hat{d}/\hat{h})\right)$ under the assumption that

$$f(x,*)w(x)$$

is in $B(D_E)$ and

$$f(*,t)\hat{w}(t)$$

is in $B(D_q)$, $q=W$ or B. In a similar fashion the inner product for (3.3) is

$$(3.28)\quad \begin{aligned} < u >_{kj} &= \frac{1}{h\hat{h}}\int_a^b\int_0^\infty \left\{u(x,t)w(x)S_k(x)\hat{w}(t)\hat{S}_j(t)\right\}dtdx \\ &= \frac{\hat{w}(t_j)f(x_k,t_j)w(x_k)}{\hat{\phi}'(t_j)\phi'(x_k)} \end{aligned}$$

with error order the same as for f, under the assumption that

$$u(x,*)w(x)$$

is in $B(D_E)$ and

$$u(*,t)\hat{w}(t)$$

is in $B(D_q)$, $q = W$ or B. Notice that these two conditions are already included in Theorem 3.3.

The u_x development reproduces the u_t development in (3.14), and so, without proof,

$$\begin{aligned} < u_x >_{kj} = &- \sum_{i=-M_x}^{N_x} \left[u(x_i,t_j)w(x_i)S_k'(x_i)\hat{w}(t_j)/\hat{\phi}'(t_j) \right] \\ &- u(x_k,t_j)w'(x_k)\hat{w}(t_j)/\left(\hat{\phi}'(t_j)\phi'(x_k)\right) \end{aligned} \tag{3.29}$$

under the assumptions that the functions

$$u(x,*)w'(x) \quad , \quad u(x,*)w(x)\phi'(x) \tag{3.30}$$

are in $B(D_E)$ and decay exponentially with respect to ϕ while

$$u(*,t)\hat{w}(t)$$

is in $B(D_q)$, $q = W$ or B. The last condition is part of the hypotheses of Theorem 3.3, while (3.30) is new.

In the case of (3.5), the identity $uu_x = \frac{1}{2}\frac{\partial}{\partial x}(u^2)$, shows that the development in (3.29) is applicable. That is, at every step of (3.29) u is replaced by u^2. Hence, for Burgers' equation the approximation reads

$$\begin{aligned} 2 < uu_x >_{kj} &= < \frac{\partial}{\partial x}(u^2) >_{kj} \\ &= - \sum_{i=-M_x}^{N_x} \left[u^2(x_i,t_j)w(x_i)S_k'(x_i)\hat{w}(t_j)/\hat{\phi}'(t_j) \right] \\ &\quad - u^2(x_k,t_j)w'(x_k)\hat{w}(t_j)/\left(\hat{\phi}'(t_j)\phi'(x_k)\right) \end{aligned} \tag{3.31}$$

under the assumptions that

$$u^2(x,*)w'(x) \quad , \quad u^2(x,*)w(x)\phi'(x)$$

are in $B(D_E)$ and decay exponentially with respect to ϕ while

$$u^2(*,t)\hat{w}(t)$$

is in $B(D_q)$, $q = W$ or B. Also the boundary conditions

$$\lim_{\substack{x\to a^+ \\ x\to b^-}} u^2(x,*)w(x)/\phi(x) = 0$$

must be met.

A choice of weight and mapping functions will be made for the present discussion, although there are many possible choices. Let (a,b) be $(0,1)$, and define

$$\phi(x) = \ell n[x/(1-x)]$$

and

$$w(x) = 1/\sqrt{\phi'(x)} \quad .$$

As for the assumptions in (3.21)–(3.23), the boundary condition (3.23) is trivially satisfied. The assumptions in (3.21) take the following form. Assume the functions

$$\text{(3.32)} \qquad \frac{u}{(x(1-x))^{3/2}} \quad , \quad u\sqrt{x(1-x)} \quad , \quad \frac{-u}{4(x(1-x))^{3/2}} \quad , \quad 0$$

are in $B(D_E)$ and each decays exponentially with respect to ϕ (Definition 2.12). These assumptions on

$$\text{(3.33)} \qquad u(x,*)/(x(1-x))^{3/2}$$

imply the conditions in (3.32) since $x(1-x)$ is bounded on $[0,1]$. The function $u/(x(1-x))^{3/2}$ decays exponentially with respect to ϕ. The limiting value of

$\sqrt{x(1-x)}\,/\ell n[x/(1-x)]$ is 0 as $x \to 0^+$ or as $x \to 1^-$. These two conditions combined show that condition (3.22) is satisfied as well. This condition also implies condition (3.30). Thus (3.33) is a sufficient condition to meet all the required conditions of Theorem 3.3 in the variable x.

The mapping function used in [9] is defined by

$$\hat{\phi}(t) = \ell n(t)$$

with the weight function

$$\hat{w}(t) = \sqrt{\hat{\phi}'(t)} \quad .$$

A short calculation gives

$$\hat{w}(t) = 1/\sqrt{t}$$

$$\hat{w}'(t) = -1/(2t^{3/2})$$

and

$$\hat{w}(t)\hat{\phi}'(t) = 1/t^{3/2} \quad .$$

The assumptions in (3.20) reduce to one of (the latter two are the same)

$$u(*,t)/t^{3/2} \tag{3.34}$$

or

$$u(*,t)/\sqrt{t} \tag{3.35}$$

are in $B(D_W)$ and decay exponentially with respect to $\hat{\phi}$. Since $1/\hat{\phi}(t) = 1/\ell n(t) \to 0$ as $t \to 0^+$ or as $t \to \infty$ this verifies that condition (3.24) is satisfied. So the two conditions (3.34) and (3.35) are sufficient to meet all the required conditions of Theorem 3.3 in the variable t.

An alternative choice for the t domain mapping is defined by

$$\hat{\phi}(t) = \ell n(\sinh(t)) \quad .$$

Here the weight function selected is

$$\hat{w}(t) = \sqrt{\hat{\phi}'(t)}$$

as before. Then

$$\hat{w}(t) = \sqrt{\coth(t)} \quad ,$$

$$\hat{w}'(t) = -(\coth(t))^{3/2} / \left(2\cosh^2(t)\right)$$

and

$$\hat{w}(t)\hat{\phi}'(t) = (\coth(t))^{3/2} \quad ,$$

and condition (3.20) reduces to

$$u(*,t)(\coth(t))^{3/2}\text{sech}^2(t) \quad , \tag{3.36}$$

$$u(*,t)(\coth(t))^{3/2} \tag{3.37}$$

and

$$u(*,t)\sqrt{\coth(t)} \tag{3.38}$$

are in $B(D_B)$ and decay exponentially with respect to $\hat{\phi}$. Now (3.37) implies (3.36) and (3.38) as $\text{sech}(t)$ and $\tanh(t)$ are bounded on $(0,\infty)$. Also the exponential decay with respect to $\hat{\phi}$ implies that $u(x,t)\sqrt{\coth(t)}$ is bounded on $(0,\infty)$, and so as before condition (3.24) is necessarily satisfied. So the condition (3.37) is sufficient in the variable t for this choice of mapping function. The conditions of Theorem 3.3 for the two choices of mapping function are collected succinctly in Table 1. The constraints on the functions f in (3.4)–(3.5) are concisely tabulated in Table 2.

The particular choices of the parameters M_x, M_t, N_x, N_t, h and $\hat{h}$ which give rise to the exponential convergence rates of the above approximations will be spelled out in the examples of Chapter 6. The basic philosophy is to balance (with respect to order) the various error contributions arising from the different inner product approximations. The domains for the different variables and mapping functions are shown explicitly in Figures 1 and 2.

Variable and Subdomain	Growth Constraints
Mapping Function	Mesh points in the Domain
$x \in (0, .5)$	$\lvert u(x,*)\rvert < K\, x^{\alpha+1/2}$
$x \in [.5, 1)$	$\lvert u(x,*)\rvert < K(1-x)^{\beta+1/2}$
$\phi(x) = \ell n\left(\frac{x}{1-x}\right)$	$x_k = \frac{\exp(kh)}{1+\exp(kh)}$
$t \in (0,1)$	$\lvert u(*,t)\rvert < C\, t^{\delta+1/2}$
$t \in [1,\infty)$	$\lvert u(*,t)\rvert < C\, t^{-(\mu+1/2)}$
$\hat{\phi}(t) = \ell n(t)$	$t_j = \exp(j\hat{h})$
$t \in \left(0, \ell n(1+\sqrt{2})\right)$	$\lvert u(*,t)\rvert < C\, t^{\delta+1/2}$
$t \in \left[\ell n(1+\sqrt{2}), \infty\right)$	$\lvert u(*,t)\rvert < C\exp(-\mu t)$
$\hat{\phi}(t) = \ell n(\sinh(t))$	$t_j = j\hat{h} + \ell n\left[1 + \sqrt{1+\exp(-2j\hat{h})}\right]$

Table 1. Formulation of Definition 2.12 for the conformal maps of Chapter 3.

Conditions on the Function f in (3.4)–(3.5)	
$f(x,t,u)$ and Subdomain	Growth Constraints
$u_x(x,t)$	
$x \in (0,.5)$	$\lvert u(x,*)\rvert < K\, x^{\alpha+1/2}$
$x \in [.5,1)$	$\lvert u(x,*)\rvert < K(1-x)^{\beta+1/2}$
$\frac{1}{2}\left(u^2(x,t)\right)_x$	
$x \in (0,.5)$	$\lvert u^2(x,*)\rvert < K\, x^{\alpha+1/2}$
$x \in [.5,1)$	$\lvert u^2(x,*)\rvert < K(1-x)^{\beta+1/2}$

Table 2. Formulation of Definition 2.12 for the non-homogeneous terms in (3.1).

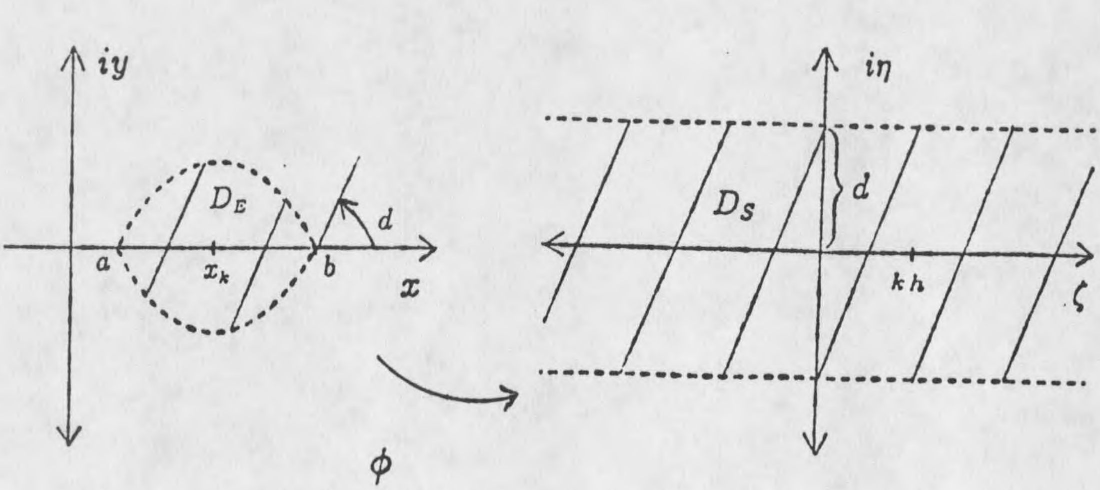

Figure 1. The spatial map ϕ.

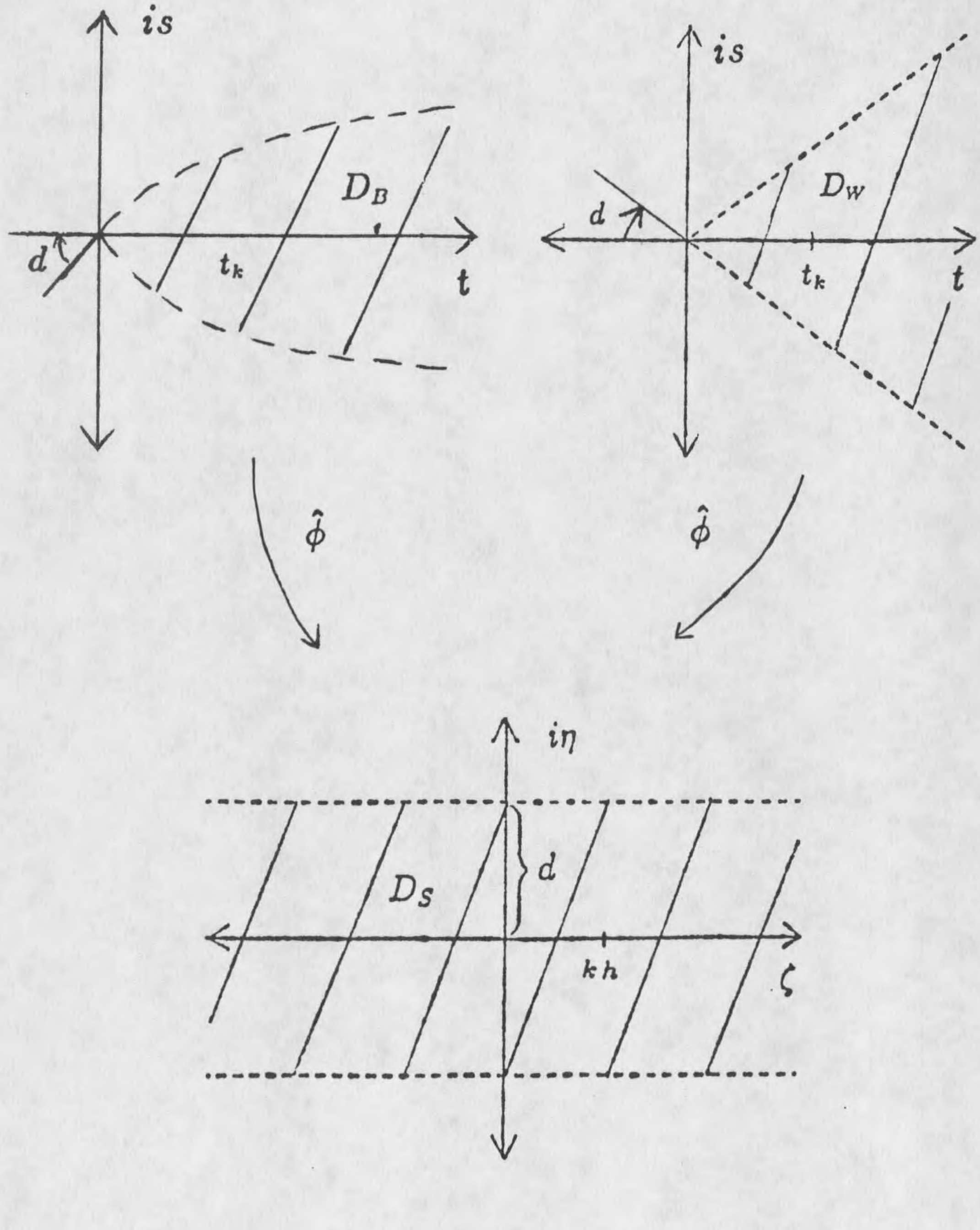

Figure 2. The temporal map $\hat{\phi}$.

CHAPTER 4

THE SINC-GALERKIN MATRIX SYSTEM

The matrix representation of the discrete system for the problem

$$(4.1) \qquad Pu(x,t) \equiv u_t(x,t) - u_{xx}(x,t) = f(x,t,u)$$

is obtained via the orthogonalization of the residual with respect to the basis elements $\{S_k \hat{S}_j\}$, that is

$$(4.2) \qquad < Pu_A - f >_{k,j} = 0 \quad ,$$

where the inner product is given in Definition 3.2. The assumed approximate solution u_A to the true solution u of (4.1) takes the form

$$(4.3) \qquad u_A(x,t) \equiv \sum_{\ell=-M_x}^{N_x} \sum_{p=-M_t}^{N_t} u_{\ell p} S_\ell(x) \hat{S}_p(t)$$

where the conventions laid down in Definition 3.2 of setting $S_p(x) \equiv S(p,h) \circ \phi(x)$ and hats for functions (or parameters) in the time domain remain in force. Also, throughout this chapter the integers $m_q = M_q + N_q + 1$, $q = x$ or t. That is, there are $m_x \cdot m_t$ unknown coefficients to determine in (4.3).

The basic methodology of the chapter is to use the inner product approximations of Chapter 3 to replace (4.2) by the Sinc-Galerkin system. In the course of this development it is notationally simpler to record the system as a tensor or Kronecker product. Besides a simplification, the tensor form has a twofold benefit. The first is in a convenient method to write down the discretization of (4.1) based on the discretization of related one-dimensional problems. This in turn yields an easy method to write down the discretization of (4.1) in the case that the independent variable x is not scalar; i.e. for multi-dimensional problems. A second benefit will emerge in Chapter 5 in the spectral analysis of the components comprising the tensor product.

The description of this matrix system is facilitated upon recalling that for a continuous function f on $[-\pi,\pi]$ the Fourier coefficients of f are given by the sequence

$$w_p = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\exp(-ipx)dx \quad , \quad p=0,\pm 1,\ldots \quad . \tag{4.4}$$

Lemma 4.1: *Let $\{w_{k-j}\}$ and $\{v_{k-j}\}$ denote the Fourier coefficients of $w(x) = -ix$ and $v(x) = -x^2$, respectively. Then*

$$h\,\frac{d}{d\chi}\,(S(j,h)\circ\chi(x))\Big|_{x=x_k} = w_{k-j} = \begin{cases} 0 & , \ j=k \\ \frac{(-1)^{k-j}}{k-j} & , \ j\neq k \end{cases} \tag{4.5}$$

and

$$h^2\,\frac{d^2}{d\chi^2}\,(S(j,h)\circ\chi(x))\Big|_{x=x_k} = v_{k-j} = \begin{cases} \frac{-\pi^2}{3} & , \ j=k \\ \frac{(-1)^{k-j}}{(k-j)^2} & , \ j\neq k \end{cases} \tag{4.6}$$

Proof: The representation

$$S(j,h)(x) = \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} \exp(-i(x-jh)t)dt$$

in (2.6) leads to the following derivative calculations

$$S'(j,h)(x) = \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} -it\exp(-i(x-jh)t)dt$$

and

$$S''(j,h)(x) = \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} -t^2\exp(-i(x-jh)t)dt \quad .$$

The substitutions $x = kh$ in the above derivatives, together with (4.4) lead to the identities (4.5) and (4.6), respectively.

Definition 4.2: *The matrix $I^{(p)}$, $p = 0, 1, 2$ is the $m \times m$ Toeplitz matrix of elements*

$$\delta_{j,k}^{(p)} = h^p S^{(p)}(j,h)(kh) \quad .$$

These are the $(k-j)$-th Fourier coefficients of the function $\omega(x) = (-ix)^p$, $x \in [-\pi, \pi]$. In particular, the matrix $I^{(0)}$ is the $m \times m$ identity matrix. The matrix $I^{(1)}$ is the $m \times m$, Toeplitz, skew symmetric matrix which has as its jk-th element $h\, S'(j,h)(kh)$. Explicitly,

$$I^{(1)} \equiv \begin{bmatrix} 0 & -1 & 1/2 & -1/3 & \cdots & \frac{(-1)^{m-1}}{m-1} \\ 1 & 0 & -1 & 1/2 & & \\ -1/2 & 1 & 0 & -1 & & \\ 1/3 & -1/2 & 1 & 0 & & \vdots \\ \vdots & & & & \ddots & -1 \\ \frac{(-1)^m}{m-1} & & \cdots & \cdots & 1 & 0 \end{bmatrix} .$$

Finally, the matrix $I^{(2)}$ is the $m \times m$, Toeplitz, symmetric matrix which has as its jk-th element $h^2 S''(j,h)(kh)$. Explicitly,