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Dickerson, David S., and Damien J. Pitman. "Advanced college-level students' categorization and use of mathematical definitions." Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education 2 (2012): 187-193.

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This qualitative study of five undergraduate mathematics majors found that some students, (even students at an advanced level of undergraduate mathematical study) have a mathematician's perspective neither on the concept of mathematical definition nor on the structure of mathematics as a whole. Participants in this study were likely to reason from incomplete concept images rather than from concept definitions and were likely to perceive that definitions (like theorems) need to be verified. The results of this study have implications for college-level mathematics instruction.

INTRODUCTION AND FRAMEWORK

There is a large body of literature that documents students' misconceptions and difficulties with mathematical proofs. Some of these difficulties have to do with their perceptions of the nature and logical structure of proof, some with students' inadequate problem-solving skills, and some having to do with mathematical communication and concept understanding (Moore, 1994). He identified seven difficulties that students typically have with proofs of which Knapp (2006) points out that all but one are related in some way to students' facility with definitions.

The Nature of Mathematical Definitions

Definitions play a central role in mathematics. Mathematicians and students of mathematics use definitions routinely but seldom think about the nature of mathematical definition (Wilson, 1990). The process of defining in mathematics is the process of giving names to mathematical objects. In natural language, most definitions are *extracted*; that is they describe how a word is used and what is meant by it. In mathematics, definitions are *stipulated*; that is created on the advice of experts. "Extracted definitions report usage, while stipulative definitions create usage, indeed create concepts, by decree" (Edwards & Ward, 2004, p. 412).

Definitions are arbitrary (Winicki-Landman & Leikin, 2000). There may be many ways to define an object and ultimately one must be selected as *the* definition. A square can be defined to be a regular quadrilateral, or it can be defined to be a polygon whose diagonals are equal and perpendicular, or it can be defined in some other way. Once a definition is selected, then all other equivalent, biconditional statements become theorems that need to be proved. A definition is neither true nor false; is merely accepted or rejected (Wilson, 1990).

Student Understanding of the Concept of Mathematical Definition

One does not learn mathematics as quickly and easily as it can be presented. Tall and Vinner (1981) state that the human brain is neither that efficient nor logical in its operations and suggested that how students think about mathematical concepts may be quite different than how the concepts are formally defined. Students sometimes argue from their concept images rather than the concept definitions and in so doing they are not using definitions the way a mathematician would (Edwards and Ward, 2004). But further, students' concept images of *mathematical definition* might be faulty or incomplete. Students who perceive that mathematical definitions are no different than other mathematical statements that require justification are not categorizing definitions the way a mathematician would (Edwards and Ward, 2004).

Students do not use definitions the way mathematicians do.

One way students misuse definitions stems from their perception that mathematical ideas are extracted rather than stipulated. In a study of 14 undergraduate mathematics majors, Edwards and Ward (2004) report that even when students can correctly state definitions, sometimes they abandon the definitions and argue from their personal concept images. When for one participant, a concept definition conflicted with her concept image she seemed to think that the definition had not been extracted correctly and argued (incorrectly) from her concept image instead.

Another way student misuse definitions stems from a poor intuitive understanding of the concept in question. Knapp (2006) provides a framework for understanding how students' use definitions in proofs. Her participants were 10 undergraduate students in a first course in real analysis. She parses knowing a definition into *ventriloquating* (reciting without fully understanding) and *appropriating* (being able to use a definition). "Appropriating a definition requires students' personally meaningful understanding to match the culturally meaningful understanding" (Knapp, 2006, p. 18). Students who can state a concept definition but who revert to a faulty or incomplete concept image when making arguments are not appropriating that definition.

Students do not categorize mathematical definitions the way mathematicians do.

In a study of 251 college mathematics majors, Vinner (1977) reported that students frequently mis-categorize mathematical definitions as theorems and axioms, or even as non-mathematical statements such as facts and laws. This could be because teachers try to justify definitions (e.g., $x^{-a} = 1/x^{a}$) and in so doing, they might give students the impression that they are proving these definitions. Further, he states that some familiar definitions are introduced to students in middle school before the definitional structure of mathematics is made clear to students. These early impressions of certain specific definitions, of mathematical definition in general, or of the overall structure of mathematics may have lasting effects.

Edwards and Ward (2004) found that students of advanced undergraduate mathematics (abstract algebra) had difficulties "arising from the students' understanding of the very nature of mathematical definitions" (p. 412) not merely the content of the definitions or the loaded nature of certain terms with their non-mathematical usage. Similar to Vinner's (1977) study, Edwards and Ward (2004) reported that one of their participants believed that once a theorem "is proven, it becomes a definition." Some participants in their study viewed mathematical definitions as extracted rather than stipulated. One participant believed that mathematicians are not free to create definitions, but "have to make the definitions from what something actually is" (p. 415) and another believed that when a definition is created, that it must pass peer review to be sure that it is error free.

METHODS

The purpose of our study was to understand how students of advanced undergraduate-level mathematics perceived the concept of *mathematical definition* and how they use definitions to verify simple conjectures. We know from Edwards and Ward (2004) and from Vinner (1977) that college students do not categorize or use definitions the way mathematicians do but we wanted to probe more deeply into how this group fit definitions into the structure of mathematics.

The data from this study comes from semi-structured, task-based interviews (Goldin, 2000) with five undergraduate mathematics majors. Our participants were juniors or seniors currently enrolled in a course in real analysis. Each was interviewed once for approximately 45 minutes. The interviews were audio-recorded and transcribed. The transcripts were coded separately by each of the two authors. At the end of this coding phase, the two authors met with each other and compared and refined their codes, and grouped the codes into broader categories.

There were several factors that informed our choice of tasks in our interview protocol. Each task was intended to elicit a discussion regarding some aspect of definition and its place in mathematics. Participants were asked to select one of among several definitions and to discuss what made it preferable to the others, to use two competing definitions for *even number* to determine the parities of certain integers, and to discuss whether definitions needed to be justified or proved. These tasks elicited the discussions that became the data for this study.

RESULTS

The two findings of this study largely confirm previous studies in the area. Our findings are (1) participants did not make clear distinctions between definitions and theorems; and (2) participants were also likely to argue from their concept images rather than from the concept definitions. While both Vinner (1977) and Edwards and Ward (2004) report that students sometimes perceive that definitions need some kind of justification to be accepted, our evidence suggests a different possible source for students' failure to categorize definitions the way mathematicians do.

Students' Categorization of Definitions

We found that our participants did not separate definitions (whose meanings are stipulated) from other mathematical statements whose validity must be verified with a proof. We presented our participants with the definition $x^{-a} = 1/x^a$ and so that there would be no doubt, they were told that this was the definition for x^{-a}. They were then asked, "Does this definition need to be proved?" All five participants indicated that this definition (and others too) needed to be proved.

> Alan: I definitely could prove this because I had to do a proof just this semester on why 0 is less than 1. I didn't think that needed to be proved but apparently… I'd have to stare at for a while before [I got it] in my head of how to start to work it out but I just like defining it better personally [because] if somebody already proved it then there's a definition from that [and] then you don't need to… If it's already been proved… somebody else has already done the work… So it's already been proved. I think a definition is fine.

Alan has two ideas here. First, that in upper division college mathematics, students are frequently asked to prove things that are obvious and that a possible way around these difficult proofs is to define things. That his ideas are not well formed is apparent when he discusses defining as merely relying on a theorem that someone else proved. Other participants share his ideas about his experiences in upper division mathematics classes. Fredrick for example discussed the necessity of a proof based upon the mathematical level of the audience. "It depends on who you're saying this to. If you're talking to high schoolers, then I would say 'no.' But if it was like college or something and you're doing abstract algebra or something… I guess it's necessary." Fredrick and Alan both perceived that they had been asked to prove intuitively obvious facts that do not need justification outside of upper division mathematics classrooms, and at least some definitions might fall into this category.

Other participants discussed the reasoning behind the definitions and why they have been defined in a particular way. Colleen, for example discusses the justification of the definition of x^{-a}. "There's a reason why $x^{-a} = 1/x^a$. So I guess because there's a reason, that it probably would be a good idea to be proved. [But] I get lost when you try and prove it to me because some brilliant, crazy mathematician proved it… If you just tell me $x^{-a} = 1/x^a$, I'm good with that. Somebody already did all the legwork." Similar to Alan, Colleen saw a definition as an end run around a difficult proof and believed that some mathematician had to prove it sometime in the past.

When asked about the definition $x^{-a} = 1/x^a$, Dori said, "I would say you should prove it. That's not obvious to people who are just seeing it [for the first time], so yeah." She went on to discuss the necessity of proving that all squares are regular quadrilaterals. "I feel like at some point we [proved] that… So, I say, 'prove everything.'" At first, she said that definitions that are not immediately obvious to someone seeing them for the first time should be justified, but then followed by saying that all mathematical statements needed justification and that at some point she proved the definition "A square is a regular quadrilateral."

Wendy believed that definitions have to be justified in order to be incorporated into the structure of mathematics.

> Wendy: They *are* [proved]. It's not *should they* [be proved?]. Definitions have to come from somewhere. We learned there's different forms like lemmas and stuff like that… I don't

> remember the order; I used to. I know the lemma is the least right thing and the theorem is like the top or something… [I don't] remember the order of them but I know there's different degrees, I think.

Unlike Alan and Colleen, Wendy did not seem to resent being asked to prove things she thought were obvious. To her, proving definitions was just part of the work of a mathematician. In her view, once a definition has been proved, it becomes something similar to (but possibly not the same thing as) a theorem.

All of the participants believed definitions needed to be proved. Colleen wanted a justification for why a definition was the way it was. Alan, Dori and Fredrick said that all mathematical statements required proof. And Wendy described how a proved definition might fit into the structure of mathematics. To varying degrees they tended to confound definitions with other mathematical statements requiring justification and most indicated they had seen proofs for definitions in some of their college-level mathematics courses.

Students' Use of Definitions

We found that our participants were more likely to argue from their concept images than from the concept definitions. After reading, and comparing and contrasting the two definitions for *even number* given below, four out of five of our participants indicated that zero was neither even nor odd. Only two recognized that there was a discrepancy between their concept image and the definition, of which only one determined (very tentatively) that zero is even.

1. *A number is called even provided it represents a number of objects that can be placed into two groups of equal size.*
2. *A number is called even provided it is an integer multiple of 2.*

Notice that under both definitions, zero is an even number. Zero objects can be placed into two piles of zero items each (perhaps a bit of a stretch for some), and zero is 2•0 which is an integer multiple of 2. The following excerpts all attempt to answer the question, "Is zero even, odd, or neither?"

Wendy's concept image of *even number* was that even numbers represent a collection of objects in which all objects can "pair up" simultaneously. She could talk about more formal definitions of *even* and explicitly mentioned both $n = 2a$ and $2|n$ from her mathematics classes but she kept coming back to her concept image of objects pairing up. In answering our question, she said, "It's neither because you're not starting with anything. It's not paired out or anything. Is that right?" Although Wendy understood the definitions on the paper, and even proposed alternative formulations, when it came to deciding if zero was even or odd, she reasoned *entirely* from her concept image rather than the concept definition to make that determination.

Colleen and Dori both recognized that their concept image that zero was not even was at odds with at least one of the definitions provided. Colleen said, "I don't remember. I think it's neither, personally. But wait! It can be divisible by 2. I don't know, I don't remember from [class] what we decided. Isn't there still a big argument about whether it is [even or not]? But technically, if you go by the definitions, it would be even. I don't know. I think it's neither. It's neither even nor odd." Although Colleen recognized that the definition demanded that zero be even, her concept image was strong enough for her to discard the definition. Dori, on the other hand tentatively discarded her concept image that zero was not even in favour of the definition. She said, "Neither. It definitely cannot be odd, but I'm torn between the neither and even. I would say 'neither' because you can't put it into two groups. [Under Definition #2], I would say 'even' but I'm sure, I'm positive that somebody could dispute me with 'neither' for the same reason I said [about Definition #1]. But I would say 'even.'" Similar to Colleen, Dori said that if she restricted herself only to the definitions (specifically Definition #2) then zero would be even but she still wasn't 100% convinced. Eventually, she decided it might be even although she was certain someone would have a problem with it. Still, she was uncomfortable with the notion that zero could be even so she offered her own addendum "zero doesn't count" to the definition of even to make it fit with her concept image that only positive and negative numbers could be even.

DISCUSSION AND CONCLUSION

We were interested in junior and senior mathematics majors' ideas about the concept of mathematical definition and found that at least some participants were still unclear as to the structure of mathematics as a whole despite the advanced level of their studies. They did not separate definitions from other kinds of statements that required justification and adhered very strongly to faulty or incomplete concept images. For example a concept image common to all of our participants excluded zero from the set of even numbers. Our participants were all familiar with the definition of even number and some suggested alternate definitions. But most of our participants seemed to be merely ventriloquating (Knapp, 2006) rather than appropriating the definition. Similar to Edwards and Ward (2004), one of our participants preferred to argue solely from her concept image rather than the concept definition, but two found that their concept image of even number differed from the concept definition and indicated that if they restricted themselves to only the concept definition, then zero would have to be even but neither were comfortable stating this claim with certainty. In this last case, it seems likely that these two participants perceived that the definitions had not been extracted properly.

Beyond corroborating the findings of previous studies, this study provides some evidence that students even at the advanced undergraduate levels are still developing an understanding not only of the concept of *mathematical definition*, but also of the mathematical system as a whole and their concept image of this entire system may not be fully formed. For example, all of our participants believed that they could prove a definition. Two possible reasons for this is given by Vinner (1977); first, students have seen their teachers motivate definitions before and so perceive that the definitions were proved, and second, certain familiar definitions are introduced to students before the structure of mathematics is made clear to them. We find something quite different; some of our participants perceived that in their advanced mathematics courses, they had frequently been engaged in proving completely obvious facts (e.g., $0 < 1$). In such courses, their notions of what did and did not require a proof were challenged to the point where they perceived that absolutely nothing, not even definitions could be taken for granted.

It may be only natural for students at this level to perceive that all basic information such as intuitively obvious theorems, definitions, and possibly even axioms must be verified because up to this point in their mathematical educations, they have been *learning* mathematics, but not really *doing* mathematics. It may be that they perceive that they have been asked to prove things solely for the purpose of demonstrating to their professors that they can reproduce some such verifications and do not see themselves as active participants within the mathematical system. Perhaps for some, the distinction between *definition* and *result* becomes clear only when one attempts to create one or the other. If so, it seems likely that engaging students in creating mathematics might help them better understand the mathematical system, and make the distinctions between definitions and theorems more apparent to them.

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