APPLICATIONS OF QUANTUM FIELD THEORY IN CURVED SPACETIMES

by

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APPROVAL

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This dissertation has been read by each member of the dissertation committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the Division of Graduate Education.

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STATEMENT OF PERMISSION TO USE

DEDICATION

To Ruth,

Your love has given me support and courage to keep working. Your faith in me has allowed me to grow in ways I couldn't think possible. Thank you for unselfishly putting your dreams on hold and making mine your own. You are my muse.

To Carolina and Ricardo,

You have taught me to value new things in life. Your innocent laugh is among the more precious gifts I have received.

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ABSTRACT

While there is as yet no full theory of Quantum Gravity, some computations can still be performed in the regime where both gravitational and quantum effects are appreciable. These types of calculations, all of them perturbations, are performed in the hope they would provide guidance for the development of the full theory.

This dissertation presents work related to three calculations using Quantum Field Theory in Curved Spacetimes. Primarily, the stress energy tensor of vacuum states is computed near Big Rip singularities, sudden singularities and in presence of a Schwarzschild-(anti) de Sitter black hole.

Big Rip and sudden singularities are examples of future singularities that have been attracting interest because of their relation to the latest measurements that detected accelerated expansion of the universe. A Schwarzschild-de Sitter black hole is the ultimate compact object that forms in the presence of a cosmological constant.

This dissertation also contains a theorem linking the appearance of sudden singularities with the type of fluid that would drive them.

CHAPTER 1

INTRODUCTION

On the road to quantum gravity, physicist have devised several methods to compute sensible quantities in the realm where Quantum Mechanics and General Relativity overlap. The first steps were taken with calculations that would keep the background gravitational field as a fixed classical field as described by General Relativity and other fields would be quantized in this background. This is known as Quantum Field Theory In Curved Spacetimes (QFTCS). While enthusiasm in the Physics community has shifted towards String Theory and Loop Gravity, some predictions of QFTCS are still regarded as one of the limits that any Quantum Theory of Gravity must reproduce. This is so mainly because calculations in QFTCS are very conservative, in the number of extra assumptions, before running into renormalization problems. Nowadays, it is broadly accepted that QFTCS yields reasonable results as long as the calculations are limited to the first-order (in $\hbar$) quantum corrections [**BD84**].

In the early stages of QFTCS, physicist labored to give some meaning to calculations for which no intuition was available. Many important issues had to be settled before the methods of QFTCS could be considered mature. Among them, the notion of particle [**Ful73**] had an unexpected outcome. In a similar way as the passage from Galilean relativity to Einstenian relativity rendered the concept of simultaneity meaningless, because of inconsistencies with requirements from curved spacetimes, QFTCS abandons the notion of particle: "As its name suggests, quantum field theory is truly the quantum theory of fields, not particles" [**Wal06**]. Other related casualties are the uniqueness of the vacuum state and normal and time ordering. The discourse moved from particles to fields and QFTCS focuses primarily on quantities that are both local and covariant [**HW01**]. Whenever particles are mentioned in QFTCS literature, as well as in this dissertation, there are implicit symmetries that provide a preferred interpretation of particle.

Among the predictions of QFTCS is that strong gravitational fields can produce "particles". This is realized through several effects: Hawking radiation [**Haw75**], Unruh effect [**Unr76**], expanding mass shell [**HWE82, HSG03**]. It is an impressive result considering that the mass for the created field excitations comes from the energy of the gravitational field, yet within General Relativity the status of the gravitational energy is an unresolved issue [**Sza04**]. Other applications include the gravitational Casimir effect [**DC79**] and cosmological inflation [**MC81, PS96**]. Exploiting the equivalence principle, strong acceleration fields should also yield similar effects. This is the subject of Quantum Field Theory In Accelerated Frames. We have, for example, proton decay [**Mül97**] or modified Lamb Shift [**AM95**].

In QFTCS, strong gravitational fields are those where the curvature approaches the Planck value. That is 10^{-33}cm in length or 10^{93}g/cm^3 in density. Traditionally, the two situations where these values were expected to happen were: (i) near the cosmological Big Bang singularity, and (ii) near the singularity inside a black hole. New cosmological models [**Cal02, Bar04**] following the discovery of the accelerated expansion of the universe deliver an exciting third possibility: (iii) cosmological future singularities. Notice that "singularity" is a recurrent adjective here.

If the fixed gravitational field is allowed to change under the backreaction from the quantized fields, we have Semiclassical Gravity. This dissertation mainly studies three metrics in the Semiclassical Gravity approximation: Big Rip singularities, sudden singularities, and Schwarzschild-de Sitter black holes. Big Rip and sudden singularities are examples of future cosmological singularities driven by dark energy fluids. They happen without collapse of the universe, as opposed to the Big Crunch. Schwarzschild-de Sitter metrics correspond to uncharged, non-rotating black holes immersed in a non-vanishing cosmological constant.

One of the natural questions within Quantum Gravity is whether quantum effects might eliminate the spacetime singularities thus producing a theory without such pathological features. Although Semiclassical Gravity, because of the perturbational nature of the calculations, cannot offer full positive answers, the negative answers it offers are outstanding. In other words, if a quantum perturbation is found to weaken a singularity then we cannot conclude that Quantum Gravity is free of spacetime singularities; but if a quantum perturbation is found to enhance the classical singularity then Quantum Gravity is more likely to posses spacetime singularities as well.

The organization is as follows. Chapter 2 contains relevant background material. A brief introduction to cosmology, emphasizing spatially flat cosmological models with barotropic fluids, will allow us to define the future singularities mentioned above. The review continues with an exposition on classical Schwarzschild-de Sitter black holes. The chapter concludes with a quick look at the semiclassical formulas used in latter chapters. No new results are included in this introduction. The third chapter presents a theorem that links the development of a sudden singularity and the equation of state of the fluid causing it. It also explicitly develops some useful monotonicity results in flat Robertson-Walker (RW) cosmology. The fourth and fifth chapters concentrate on the vacuum contributions near the Big Rip and sudden singularities respectively. The final chapter presents QFTCS calculations in Schwarzschild-de Sitter and some thermodynamical considerations therein.

CHAPTER 2

REVIEWS

Cosmology

In the first part of this section, there is a critical review of the standard cosmological model. Far from being a comprehensive review, it only touches on those points relevant to understanding the ambit of phantom energy theory. In the second part, the review is specialized to the case where all cosmological fluids are considered barotropic and finite-time future singularities and their classification are introduced.

General Overview

Physical cosmology undertakes the study of the observable universe mainly by applying Einstein's theory of gravity (General Relativity) to the evolution of the universe. The scales are so vast, above 10^{30} meters, that galaxies are considered small wrinkles on the otherwise smooth, homogeneous, and isotropic universe. Homogeneity and isotropy are based on large scale structure observations. The largest known structure, as of November 2007, is the Sloan Great Wall at 10^{25} meters [**GIJS+05**], five orders of magnitude larger than our Milky Way. Smoothness is evidenced by the Cosmological Microwave Background (CMB). Additionally, the universe is known to be dynamic. As early as 1916, de Sitter [**dS16**] wrote "It should be pointed out that all stars appear to show a slight systematic displacement towards the red." Hubble takes the credit for gathering enough data [**Hub29**] to propose the eponymous law. The Supernova Cosmology Project [**P**$^+$**99**] and the Supernova Search Team Collaboration [**R**$^+$**98**] showed that the expansion of the universe is accelerating.

Homogeneity, isotropy, smoothness and dynamics were accommodated by the model proposed, in 1922, by Friedman [**Fri99**], who extended the solutions of the Einstein equations found, in 1917, by Einstein [**Ein52**] and de Sitter [**dS17**] and included matter with the mentioned symmetries. Later, Robertson [**Rob35**] generalized this model to include the three possible types of spatial curvature: flat, hyperbolic and spherical. Robertson also proved the uniqueness of the metric. Because of historical reasons, the resulting models are said to be described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric and Einstein's equations specialized to these models are known as the Friedman equations. These equations describe the evolution of the "shape" of the universe –technically, its curvature– in terms of the so-called scale parameter but they leave the matter-energy content unknown. This last is considered, in what is called the Big Bang model, as a remnant from the explosion at the genesis of the universe. While the Big Bang model explains Hubble's law, the existence of the CMB, and the proportion of light elements in the universe, it poses several problems regarding the content of the universe. The problems will be detailed below along with the placeholders that parameterize our ignorance: inflation field, dark matter and dark energy. For each of these, we shall mention the original motivation, some experimental facts, and (unsolved) issues. A reader in need of further initiation is referred to mainstream articles on the subject (consult for example [**BOPS99**]).

The inflation field was proposed [**Gut81**] to alleviate the horizon, flatness, and relic problems. The horizon problem has to do with establishing a very isotropic and homogeneous universe at the onset of the classical gravity regime, after the initial singularity, when the universe is already made out of causally disconnected regions. The flatness problem is related to the density of the universe needing to be extremely close to the density required, in the Friedman equations, for a spatially flat universe. This last is the so called critical density. The difficulty resides in the critical density being unstable. Solving the flatness problem without inflation would require a fine tuning of the density to about one part in 10^{50} to 10^{60} at the time when gravity separated from the other fundamental forces. The relic problem is associated with some Grand Unification Theories and it corresponds to the observed absence of magnetic monopoles and other strange particles that would have survived from the Big Bang.

On the theoretical side, inflation posed new problems: the “graceful exit” has already been sorted out [**Lin82, AS82**], but the new “incompatibility with string theory” seems to have been born [**HTK$^+$07**], although this might be a problem for string theory rather than a problem for inflation. On the experimental side, the lack of detection of inflatons, the particles of the inflation field, is explained by their convenient decay during the reheat epoch [**HM82**]. A thorough review on inflation can be found in Bassett *et al.* [**BTW06**].

Without dark matter, there would be multiple crises in understanding the motion of galaxies in clusters, structure formation in the early universe, galactic rotation curves, and gravitational lensing of colliding clusters.

In order to analyze the motion of galaxies in clusters [**Zwi37**], the virial theorem is used to estimate the mass of a cluster of galaxies. This is then compared to the mass estimated from the luminosity of the galaxies in the cluster. The kinematic (virial theorem) mass was found to be much greater than its luminous counterpart.

In modeling structure formation in the early universe [**BE84**], dark matter provides the extra gravitational pull needed to aggregate matter in a time frame smaller than the age of the universe. Baryonic matter couldn't have done the job by itself because it was coupled to radiation which prevented it from collapsing by itself.

In galactic rotation curves [**RF70**], the discrepancy between the optically measured mass of galaxies, calculated via the so-called luminosity function, and the kinematical one, calculated using the Doppler effect and Kepler's law, is filled by dark matter.

Last year, it was reported that dark matter was detected [**CBG+06**] spaced apart from luminous matter. Using weak gravitational lensing theory, it is possible to determine the profile of the gravitational well of the lens. If the lens is provided by colliding clusters of galaxies, such profile can be tested for matches against two profiles determined by direct observation. One is the distribution of galaxies in the cluster and the other is the concentration of X-ray-emitting intracluster plasma. This last is the dominant component of luminous matter; but during the collision, it experiences ram pressure which causes it to decouple from dark matter. On the other hand, galaxies behave like collisionless particles. Since dark matter is assumed to have the same collisionless behavior, the observed match of the gravitational well profile with the galaxy distribution constitutes evidence in favor of dark matter.

One issue still remains unsolved: the angular momentum transfer problem [**NB91, DB04**]. The basic idea is that the interaction between dark matter and baryonic matter also implies the transfer of angular momentum. The original angular momentum of galaxies comes from tidal torques between neighboring protogalaxies during early evolution. In current models, too much of that initial angular momentum is lost by the baryonic component during the collapse into galaxies and therefore rotationally supported disks cannot form in galaxies.

The status of dark energy is even more interesting. The inflation hypothesis leaves the universe with a density *very* close to the critical value, but dark matter, baryonic matter and radiation would account for only 30% of the total. Some alternatives have been explored and the discovery of the accelerated expansion of the universe [**R+98, P+99**] led the Physics community to embrace the proposal [**TW97, CDS98**] that an unknown fluid with negative pressure, referred to as dark energy, makes up for the remaining 70% (see [**PR03, CST06**] for thorough reviews). The accelerated expansion of the universe (and indirectly dark matter) has been confirmed through several paths: Wilkinson Microwave Anisotropy Probe measurements of CMB [**S+06**], Sloan Digital Sky Survey measurements of Large Scale Structure [**T+06**], high-redshift type Ia supernovae measurements done by the Supernova Cosmology Project [**K+03**] and by the Supernova Search Team [**R+04**], dependence of the baryonic mass fraction on the angular diameter of clusters [**A+04**], etc.

The measurements above essentially fix the present value of the density of dark energy. In order to completely define the stress-energy tensor, the pressure of the dark energy is set via the equation of state (EOS)

$$w := p/\rho \tag{2.1}$$

Here, w is dubbed the equation of state parameter. The effective value of the equation of state parameter, this is the value after including all contributions, must be smaller than

$-1/3$ to ensure accelerated expansion of the universe. Dark energy corresponds to negative values of w. Dark energy with $w_\Lambda = -1$, when constant, corresponds to a cosmological constant. It is a threshold below which a fluid is classified as phantom energy. The name was proposed by Caldwell [**Cal02**]: "A phantom is something which is apparent to the sight or other senses but has no corporeal existence – an appropriate description for a form of energy necessarily described by unorthodox physics..." Current observations [**S$^+$06**] are compatible with phantom energy, $w = -0.967 \pm 0.073$, albeit marginally. Phantom energy has several problems: the dominant energy condition cannot be satisfied cosmologically [**CHT03**] (causing instability troubles), yet more fine tuning [**WC06**], among others.

Notice that $w \approx -1$ describes exotic energy/matter. By comparison, w for mass in the center of the sun is of the order 10^{-6}. In fact, the difference between dark matter and dark energy is their equation of state. Dark matter is a fluid with insignificant pressure and dark energy is a fluid whose pressure is negative and comparable to its energy content.

The fact that the energy density of phantom energy increases as the universe expands causes a runaway situation that can end in singularities. Near the singularities, quantum effects related to gravity are expected to become noticeable.

It is remarkable that none of the main contributors of these fluids (inflation, dark matter and dark energy) have secured, by themselves, a place in the Standard Model of Particle Physics.

Flat RW with Barotropic Fluids

As mentioned above, observations support a dynamical universe that is homogeneous and isotropic at large scales. It can be proven [**Rob35**] that there are three types of geometries corresponding to non-static solutions of the Einstein equations with cosmological constant and fluids satisfying such high symmetry. The metric for the spatially flat case is known as the Robertson-Walker metric:

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) \tag{2.2}$$

where a is the so-called scale factor and the coordinates (t, x, y, z) have the usual meaning of their Minkowski counterparts. As a evolves, say between initial time t_0 and final time t_1, the distance between two points at some fixed coordinates is multiplied by a factor $a(t_1)/a(t_0)$. Hence the name scale factor. The expansion of the universe implied by Hubble law is encoded by $\dot{a} > 0$; and the accelerated expansion of the universe is conveyed by $\ddot{a} > 0$.

The Einstein equations for the metric Eq. (2.2) are

$$3H^2 = \kappa^2 \sum \rho_i\,, \tag{2.3a}$$

$$2\dot{H} = -\kappa^2 \sum (\rho_i + p_i)\,. \tag{2.3b}$$

where $H := \dot{a}/a$ is the Hubble expansion rate, ρ_i and p_i are the energy density and the pressure of the i-th fluid, and Newton's gravitational constant is set to 1 ($\kappa := \sqrt{8\pi}$).

Note in Eq. (2.3b) that both the pressure and the energy density play a role in the determination of $\ddot{a}$ (since $\dot{H} = (\ddot{a}\,a - \dot{a}^2)/a^2$). This means that both $\sum \rho_i$ and $\sum p_i$ decide the sign of acceleration of the expansion of the universe. Because of the negative sign in the front of the right hand side, only negative pressures large enough to overcome the always-positive energy density can account for positive acceleration of a.

If the fluids can be modeled as barotropic perfect fluids, then, by definition, their pressures are functions only of their corresponding energy densities. It is customary [**NOT05**] to define an alternate parameter for the equation of state:

$$f_i(\rho_i) := -(\rho_i + p_i)\ . \tag{2.4}$$

These are functions of ρ_i. In terms of f_i, the equation of state parameters w_i can be written as

$$w_i = -1 - \frac{f_i}{\rho_i}\ . \tag{2.5}$$

Note that the i-th fluid describes phantom energy as long as f_i is positive definite. Using Eq. (2.4), we can remove the pressures from Einstein's equations

$$3H = \frac{\dot{\rho}_i + Q_i}{f_i}\,, \tag{2.6a}$$

and from the conservation of energy equations of each fluid

$$\sum Q_i = 0\,. \tag{2.6b}$$

Table 2.1. Some cosmological fluids contributing to the Friedman equations. The third column shows the behavior of the density as a changes in time. The fourth column presents the pressure as function of the energy density.

Fluid	Symbol	Density	Pressure
Cosmological constant	Λ	a^0	$-\rho_\Lambda$
Curvature (effective)	K	a^{-2}	0
Baryonic and dark matter	m	a^{-3}	0
Dark energy (const. w)	ϕ	$a^{-3(1+w)}$	$w\,\rho_\phi$
Radiation	R	a^{-4}	$\frac{1}{3}\rho_R$

The Q_i account for the possible transfer of energy between fluids. As usual, conservation of stress-energy and Einstein's equations are not independent.

Now, we will make the assumption that there is no direct interaction amongst the fluids, i.e. $Q_i = 0$. The justification for this step is in the results. If at the end of our calculations there is only one dominant fluid, phantom energy near the singularity, whose energy density is much larger than the energy densities of all other fluids combined, then we can neglect the interaction between fluids. In this case, we can integrate Eq. (2.6a) to find ρ_i as functions of a, and Eq. (2.3a) to find the evolution of a in time:

$$3\log\left(\frac{a}{a_0}\right) = \int_{\rho_{i0}}^{\rho_i} \frac{\mathrm{d}\rho_i}{f_i(\rho_i)}\,, \tag{2.7}$$

$$\frac{\kappa}{\sqrt{3}}(t-t_0) = \int_{a_0}^{a} \frac{\mathrm{d}a}{a\sqrt{\sum \rho_i(a)}}\,. \tag{2.8}$$

The subindex zero indicates the values of the quantities today.

Table 2.2. Classification of finite-time future singularities ([**CST06, NOT05**]). Entries with a dash mean unspecified behavior, and entries not showing divergence are to be considered finite.

	a	$\rho \sim \lvert\dot{a}\rvert$	$\lvert p \rvert \sim \lvert\ddot{a}\rvert$	$\lvert\dddot{a}\rvert$ and higher
I - big rip	∞	∞	∞	-
III	a_s	∞	∞	-
II - sudden	a_s	ρ_s	∞	-
IV	a_s	0	0	∞
V	∞	ρ_s	p_s	∞

The fluids in Eq. (2.3) can be combinations of curvature effects, radiation, dark matter, and dark energy among other things (see Table (2.1)). The critical density ρ_c is defined as the total density of fluids that would account for today's value of the Hubble parameter. From Eq. (2.3a), we get

$$\rho_c := \frac{3}{\kappa^2} {H_0}^2 \, . \tag{2.9}$$

It is sometimes convenient to work with the ratios Ω_i of the energy density of each component with respect to the critical density

$$\Omega_i := \frac{\rho_i}{\rho_c} \, . \tag{2.10}$$

Replacing κ in Eq. (2.8), we get the more familiar equation

$$t - t_0 = \frac{1}{H_0} \int_{a_0}^{a} \frac{\mathrm{d}a}{a\sqrt{\sum \Omega_i(a)}} \, . \tag{2.11}$$

Finite-time future singularities will appear if for some reason the integral in Eq. (2.8) cannot keep increasing. A classification of finite-time future singularities was introduced by [**NOT05**] and completed in [**CST06**]. It is summarized in Table (2.2). This classification is based roughly on the possible types of divergence of the scale factor and its derivatives. Divergence of the energy density while the scale factor remains finite implies, via Eq. (2.3a), divergence of the first derivative of the scale factor. Similarly, divergence of the pressure while the energy density remains finite involves, via Eq. (2.3b), divergence of the second derivative of the scale factor.

Not all of the singularities listed in Table (2.2) are singularities in the sense that geodesics cannot be extended. In Big Rip singularities [**CKW03**], the different bindings –gravitational, electromagnetic, chromodynamical– disassociate successively. For sudden singularities, it has been proven [**FJL04**] that tidal forces are not strong enough to break all finite-sized bodies.

In this dissertation, the lack a of subindex on ρ or p will denote either total values or values corresponding to phantom energy. In many cases, the difference between the total value and the phantom value will be neglected as we will be working on models where phantom energy is the dominant fluid. Context will provide further refinement.

Classical Schwarzschild - de Sitter Black Holes

In this section we review both Schwarzschild-de Sitter and Schwarzschild-anti de Sitter black holes. They correspond to spherically symmetric, non-rotating, uncharged back holes with positive and negative cosmological constants, respectively. Their metric is given by

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \tag{2.12}$$

with

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \ . \tag{2.13}$$

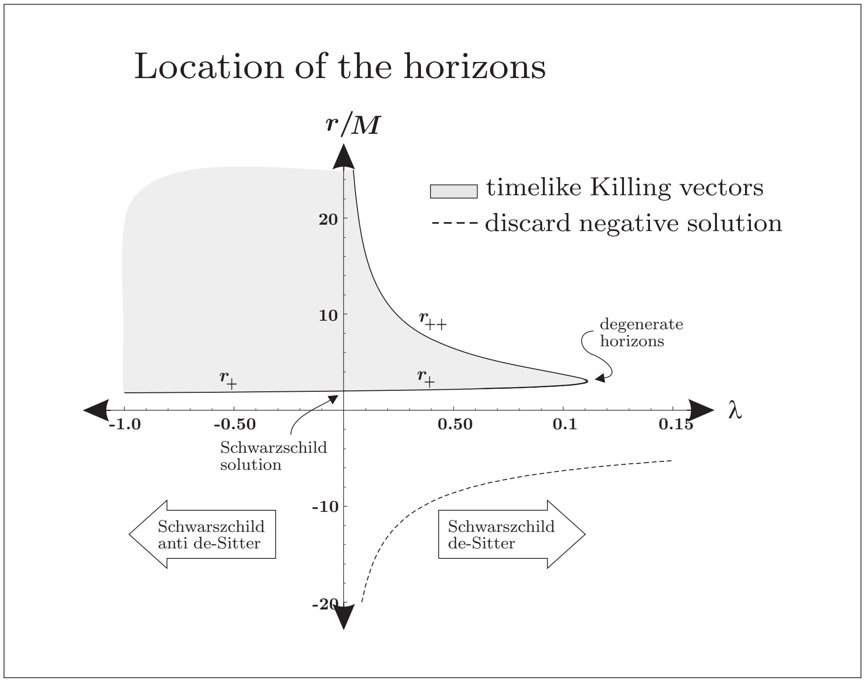
where $M > 0$ is the mass of the black hole and Λ is the cosmological constant. Regarding the coordinates, t is the time coordinate measured by a stationary clock at spatial infinity, the area of the sphere generated by the spherical symmetry of the black hole is $4\pi r^2$, and θ and ϕ are the zenithal and azimuthal angles, respectively.

These spacetimes are not asymptotically flat (unless Λ vanishes); they instead asymptotically approach the form of de Sitter spacetime (if $\Lambda > 0$) or anti-de Sitter spacetime (if $\Lambda < 0$). We define λ, a dimensionless measure of the cosmological constant, to be

$$\lambda = \Lambda M^2 \ . \tag{2.14}$$

The horizons of the metric Eq. (2.12) are located at $f(r) = 0$. The zeros of this equation (see Fig. (2.1)) are those of a cubic polynomial on r/M; and therefore, it must have one or three real roots. Negative solutions are discarded because we required $M > 0$ in order to avoid a spacetime with a naked singularity.

Figure 2.1. Position of the zeros of $g^{rr} = 0$ in units of the mass of the black hole as function of the cosmological constant. The negative solution must be discarded because of the definition of r.



Positive Cosmological Constant.

The equation $f(r) = 0$ yields no positive roots when $\lambda > \frac{1}{9}$, see Fig. (2.1). If $0 < \lambda < \frac{1}{9}$, there are two horizons, located at radii

$$r_+ = \frac{2M}{\sqrt{\lambda}} \cos\left(\frac{\pi + \cos^{-1}(3\sqrt{\lambda})}{3}\right) , \tag{2.15a}$$

$$r_{++} = \frac{2M}{\sqrt{\lambda}} \cos\left(\frac{\pi - \cos^{-1}(3\sqrt{\lambda})}{3}\right) . \tag{2.15b}$$

The inner horizon, at r_+, is the black hole horizon; it's position is bounded by $2M < r_+ < 3M$; the outer horizon, r_{++}, is the cosmological horizon, bounded by $r_{++} > 3M$. As $\lambda \to 1/9$, the two horizons become degenerate: $r_+ = r_{++} = 3M$.

The Killing vector ∂_t is timelike only within the region between the horizons. Since spacetimes with $\lambda > \frac{1}{9}$ have no horizons and no timelike Killing vector, we will not consider them further here.

The surface gravities of the horizons are given by

$$\kappa_{+} = \frac{M}{r_{+}^2} - \frac{\lambda\, r_{+}}{3M^2} \ , \tag{2.16a}$$

$$\kappa_{++} = -\frac{M}{r_{++}^2} + \frac{\lambda\, r_{++}}{3M^2} \ . \tag{2.16b}$$

The black hole horizon surface gravity decreases smoothly from the Schwarzschild value of $1/(4M)$ to zero as λ is increased from zero to $1/9$, while the cosmological horizon surface gravity increases from zero to a maximum value of $\kappa_{++} = 1/(12M)$ at $\lambda = 1/18$, and then decreases to zero as the horizons become degenerate, see Fig. (2.2). The black hole horizon surface gravity is always greater than the surface gravity of the cosmological horizon for all $\lambda < 1/9$.

Negative Cosmological Constant.

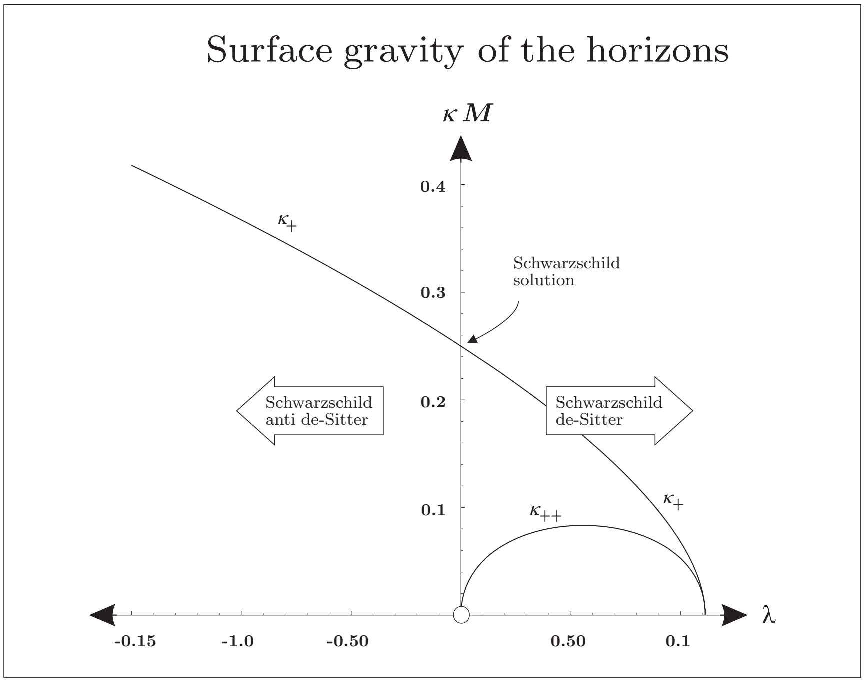
In this case, there exist only one horizon, the black hole horizon r_+, located at

$$r_{+} = \frac{2M}{\sqrt{|\lambda|}} \sinh\left(\frac{\sinh^{-1}(3\sqrt{|\lambda|}\,)}{3}\right) \ , \tag{2.17}$$

see Fig. (2.1). As λ becomes more negative, the horizon radius decreases from the Schwarzschild value of $2M$ down to zero; asymptotically like

$$r_{+} \simeq \left(\frac{6}{|\lambda|}\right)^{\frac{1}{3}} M \ . \tag{2.18}$$

Figure 2.2. Surface gravity in units of inverse mass of the black hole as function of the cosmological constant.



The region where ∂_t is timelike now extends outside the black hole horizon outwards to infinity. The surface gravity, κ_+, is computed by the same equation as for the $\lambda > 0$ case, Eq. (2.16a), and increases asymptotically, as $|\lambda| \to \infty$, like

$$\kappa_+ \simeq \frac{(6\lambda^2)^{\frac{1}{3}}}{2M} \tag{2.19}$$

for large $|\lambda|$.

A thorough review of classical Schwarzschild-de Sitter and Schwarzschild-anti de Sitter black holes can be found in [**SH99**].

Semiclassical Fields

Computing the expectation value of the vacuum stress-energy tensor of a quantized field in an arbitrary spacetime has not been accomplished yet. In presence of symmetries, such a daunting task simplifies and becomes more tractable. In this section we will review some subtleties involved in the specific case of calculating the vacuum expectation value of the stress-energy tensor for a conformally invariant quantized field in a conformally invariant flat spacetime. The basic ideas are relatively simple [**BD77, BC77**].

The stress-energy tensor of matter can be obtained from the matter action S by a functional derivation respect to the metric:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} S \ . \tag{2.20}$$

This formula is valid for both classical and quantum theories. The "in" and "out" states are determined by the boundary conditions of the integral that defines S. When those states correspond to vacuum states of a quantum theory, $T^{\mu\nu}$ is referred to as the vacuum expectation value of the stress-energy tensor; and, it is usually adorned with the bracket notation: $\langle T^{\mu\nu}\rangle$. In this dissertation, the bracket will be implicit whenever it is needed.

The matter action S we will consider is that of free, conformally-invariant, quantized fields and contains renormalizing counterterms corresponding to one-loop linearized gravitons propagating in some given background metric. Therefore, $T^{\mu\nu}$ is finite. We will not use any marker to remind us that $T^{\mu\nu}$ is the renormalized value. Instead, the bare and divergent values will have subscripts: $T^{\mu\nu} = {T_{\text{bare}}}^{\mu\nu} - {T_{\text{div}}}^{\mu\nu}$.

Following [**CD79**], consider a manifold M that can be obtained via a conformal transformation ω from the flat manifold M_0 such that (i) there are intermediate manifolds corresponding to intermediate values of ω, and (ii) they have corresponding Cauchy surfaces to determine the positivity of frequencies. These premises ensure that we "preserve" the vacuum defined in M_0 while the conformal transformation is applied.

Since the metric changes as $g_{\mu\nu} \mapsto \omega g_{\mu\nu}$, we can compute the functional derivative with respect to variations of ω for functionals of the metric:

$$\frac{\delta}{\delta\omega} = 2\left(\frac{\delta}{\delta\omega} g_{\mu\nu}\right)\frac{\delta}{\delta g_{\mu\nu}} \,. \tag{2.21}$$

It is also trivial to prove that

$$g_{\alpha\beta}\frac{\delta}{g_{\alpha\beta}}\left(g_{\mu\gamma}\frac{\delta}{g_{\nu\gamma}}F[g_{\lambda\kappa}]\right) = g_{\mu\gamma}\frac{\delta}{g_{\nu\gamma}}\left(g_{\alpha\beta}\frac{\delta}{g_{\alpha\beta}}F[g_{\lambda\kappa}]\right) \tag{2.22}$$

for any functional F of the metric. In particular, we will use the above identities on the renormalized matter action.

The bare action of conformally-coupled, massless fields is invariant under conformal transformations. Moreover, it vanishes. However, the renormalization process separates the divergent part which is not conformally invariant. Since ${T_{\text{div}}}^{\mu\nu}$ then has a trace, $T^{\mu\nu}$ acquires a trace. When a quantum quantity lacks some symmetry that its classical counterpart possesses, the asymmetric quantum contribution is called an anomaly. Thus, $T^{\mu\nu}$ has a trace anomaly. For a historical account of the discovery of the trace anomaly, see [**Duf94**].

Using the identities from Eq. (2.21) and Eq. (2.22) on the renormalized matter action we obtain the following equation linking the renormalized expectation value of the stress energy tensor and the trace anomaly:

$$\frac{\delta}{\delta\omega}\left(\sqrt{-g}\,T^{\mu}{}_{\nu}\right) = 2\,g_{\nu\alpha}\frac{\delta}{\delta g_{\mu\alpha}}\left(\sqrt{-g}\,T^{\beta}{}_{\beta}\right) \tag{2.23}$$

This is a differential functional equation that can be integrated after the trace anomaly is found. This last calculation depends on the renormalization method used to find S_{div}. Fortunately, (see references in [**Duf94**]) all three standard renormalization methods (point splitting, zeta function, and dimensional regularization) lead to the same answer. The fact that all standard renormalization methods coincide gives this result as solid a footing as we can get in QFTCS (the best confirmation would be experimental). In the case that the initial manifold M_0 is flat and the vacuum expectation value of the stress-energy tensor vanishes there, the formula reduces to (see [**CD79**]):

$$\begin{aligned} T_{\mu\nu} = \frac{\alpha}{3}\left(g_{\mu\nu}R^{;\sigma}{}_{;\sigma} - R_{;\mu\nu} + RR_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R^2\right) \\ + \beta\left(\frac{2}{3}RR_{\mu\nu} - R^{\sigma}_{\mu}R_{\nu\sigma} + \frac{1}{2}g_{\mu\nu}R_{\sigma\tau}R^{\sigma\tau} - \frac{1}{4}g_{\mu\nu}R^2\right)\ , \end{aligned} \tag{2.24}$$

where R is the Ricci scalar, $R_{\mu\nu}$ is the Ricci tensor, and the spin-dependent coefficients α and β are given in Table (2.3). The usage of this formula is subject not only to the existence of the conformal mapping of the metric, but also to the appropriate mapping of the Cauchy surface to either Minkowski or Rindler spacetimes.

Table 2.3. Spin-dependent coefficients in Eq. (2.24).

spin	α	β
0	$\frac{1}{2800\pi^2}$	$\frac{1}{2800\pi^2}$
$\frac{1}{2}$	$\frac{3}{2800\pi^2}$	$\frac{11}{5600\pi^2}$
1	$-\frac{9}{1400\pi^2}$	$\frac{31}{1400\pi^2}$

CHAPTER 3

SUDDEN SINGULARITIES

This chapter contains some classical results on spatially flat RW cosmology, as opposed to the remaining chapters, where the studies are semiclassical. One main assumption posed in this chapter is that of one dominant fluid. It is justified by the fact that dark energy is already the major contributor to the energy budget of the universe and its share is expected to grow because of two reasons. One, the expansion of the universe dilutes the contribution from other fluids; and two, phantom energy and the expansion of the universe feedback positively on each other.

The first section proves that energy densities of cosmological fluids and the scale factor evolve monotonically under rather weak assumptions. These monotonicity results are used to argue against models producing type-IV singularities via a single barotropic non-interacting fluid. The cases presented in the literature have at least two interacting fluids (e.g. [**Per05**]) with one notable exception [**NOT05**].

The second section presents a theorem that links the existence of sudden singularities to the equation of state of the fluid driving them.

In the third section, a model (defined by equations (32) and (35) in Nojiri *et al.* [**NOT05**]) proposed as an example that encompass several types of future singularities as particular cases is revisited. Henceforth, that model will be noted as the NOT model. It is shown that it ends in sudden singularities, and no other.

Monotonicities in Flat RW and Type IV Singularities

It is possible to extract some information from the equations governing the flat RW spacetime with barotropic fluids (see chapter 2)

$$3\log\left(\frac{a}{a_0}\right) = \int_{\rho_{i0}}^{\rho_i} \frac{\mathrm{d}\rho_i}{f_i(\rho_i)}\,, \tag{2.7}$$

$$\frac{\kappa}{\sqrt{3}}(t-t_0) = \int_{a_0}^{a} \frac{\mathrm{d}a}{a\sqrt{\sum\rho_i(a)}}\,, \tag{2.8}$$

without knowing the explicit form of the equations of state. It is important to note that these integrals have been written in definite form. This will keep us from making unphysical assumptions about the integration constants that could occur if the integrals were cast in indefinite form.

Regarding the monotonicity of a respect to t, first note that flat RW doesn't have curvature terms in the right hand side of Eq. (2.3a). This means that the scale factor, as determined implicitly by Eq. (2.8), fails to be monotonic only if the total energy density somehow vanishes or crosses infinity. Remember that Eq. (2.8) was obtained under the assumption of non-interacting fluids and that the regime when dark energy is the dominant fluid is compatible with such assumption. Hence, in models where the sum in the denominator of the integrand of Eq. (2.8) is approximated, $\sum\rho_i \approx \rho_\phi$, we find that a grows monotonically as time passes by.

The spacetime manifold can be extended in time if the integrand in Eq. (2.8) can be evaluated as a real number. That is, $a(t+\delta t)$ is defined implicitly by Eq. (2.8) as long as the integrand yields a real number that can be added to the integral up to $a(t)$. If the integrand fails to be a real number, say for $a(t_s) = a_s$, then we might have a sudden or type-III singularity at a_s (see Table (2.2)). On the other hand, if a, the upper limit of Eq. (2.8), can go to infinity, then we have a big-rip singularity if the integral converges; and there is no singularity if the integral diverges so that $t \to \infty$ as $a \to \infty$.

From Eq. (2.7), we can reason that ρ_i are monotonic functions of a as long as the f_i don't cross zero, cross infinity or jump branches (f_i could be multivalued). Using the monotonicity results for a, we can conclude that ρ_i are monotonic functions of time until f_i or $\sum \rho_i$ vanish or diverge or f_i change branches.

One corollary resulting from these statements is that a type-IV singularity (see Table (2.2)) cannot be produced by a barotropic non-interacting fluid. This case, although mathematically possible, is physically impossible. A realistic cosmological model would have to include other fluids, say baryonic matter. Baryonic-matter would then contribute to the right hand side Eq. (2.3a). This contribution, which evolves as a^{-3}, does not vanish because a_s remains finite (as required by the definition of type IV singularity). If the energy density of phantom-energy vanishes then the energy density of normal matter takes over and drives the evolution of a (without any singularity whatsoever).

EOS Parameter and Sudden Singularities

In this section, it will be shown that, in the approximation of dominant barotropic phantom energy, a sudden singularity develops only in a very restricted set of models and the specific behavior of the EOS parameter near the singularity will be calculated. In fact, such behavior is necessary and sufficient for the existence of the singularity.

We will begin by assuming the existence of a future sudden singularity. This type of singularity occurs when the pressure diverges but the energy density and the scale factor remain finite (Table (2.2)). By Eq. (2.3), the first derivative of the scale factor also remain finite but the second derivative diverges. Hence, a can be written, near the singularity, by

$$a(t) \approx a_s - A(t_s - t) + B(t_s - t)^{1+\frac{1}{1+\delta}} + \mathcal{O}((t_s - t)^{1+\frac{1}{1+\delta}+\epsilon}) \tag{3.1}$$

with definite positive both δ and ϵ and non vanishing a_s, A and B. With the chosen form, the exponent of the B term satisfies

$$1 + \frac{1}{1+\delta} \in (1,2]$$

when $\delta \geq 0$. Possible higher order terms in $t_s - t$ have been omitted ($\epsilon > 0$) because we only need to depict the divergence of the second derivative while keeping the first derivative finite. The scale factor in Eq. (3.1) has then the most general behavior near a sudden singularity.

By substituting this form of the scale factor into Eqs. (2.3), it can be proven that the behaviors of the total energy density and total pressure near the singularity are given by

$$\sum \rho_i \approx \frac{3A^2}{{a_s}^2 \kappa^2} - \frac{6AB(\delta+2)(t_s - t)^{\frac{1}{1+\delta}}}{{a_s}^2(\delta+1)\kappa^2} + \mathcal{O}((t_s - t)^{\frac{1}{1+\delta}+\epsilon}) \,, \tag{3.2}$$

$$\sum p_i \approx -\frac{2B(\delta+2)}{a_s(\delta+1)^2}(t_s - t)^{-1+\frac{1}{1+\delta}} + \mathcal{O}((t_s - t)^{-1+\frac{1}{1+\delta}+\epsilon}) \,. \tag{3.3}$$

The total energy density converges and the pressure diverges as expected.

Note that the signs in front of A and B in Eq. (3.1) have been chosen in such a way that if A and B are both positive then

- a approaches a_s from below,
- $\sum \rho_i$ also approaches $\rho_s := \frac{3A^2}{{a_s}^2 \kappa^2}$ from below, and
- $\sum p_i$ diverges to $-\infty$.

If A was negative, then $\dot{a}$ would be negative. But this cannot happen because of the monotonicity of a found in the previous section and the observed fact that $\dot{a}$ is positive today (the universe is expanding).

Now, let us analyze the conditions under which the contributions from fluids like dark matter or electromagnetic radiation would not be significant near the singularity. This would allow us to claim that phantom energy is indeed the dominant fluid driving the singularity. We need the total energy density to be much larger than the dark-matter energy density. This is, the constant term in the behavior of $\sum \rho_i$ (see Eq. (3.2)) must be much larger than $\rho_{m0} {a_0}^3 / {a_s}^3$, where ρ_{m0} is the energy density of dark matter measured today. Therefore, a_s and A must be big enough to satisfy

$$A^2 a_s \gg \frac{\kappa^2}{3} {a_0}^3 \rho_{m0} \, .$$

If A vanished, then this inequality could not be satisfied. That is, only positive definite A is compatible with the assumption of non-interacting fluids.

If B were negative, the dominant fluid causing the singularity would have both energy density and pressure positive. Thus, such a fluid would, near the singularity, simply not be phantom energy. If B was zero, the corresponding term in the scale factor (Eq. (3.1)) would vanish and the singularity, if any, would not be a sudden singularity.

Assuming that only one fluid, dark energy, contributes significantly near the singularity, the sums in Eqs. (3.2) and (3.3) can be replaced by this single contribution. Then, Eq. (2.4) has the form

$$f \approx \operatorname{sign}(B)\,\frac{(3\,|A|)^{\delta}(2\,|B|\,(\delta+2))^{1+\delta}}{\kappa^{2\delta}\,(\delta+1)^{2+\delta}\,{a_s}^{1+2\delta}}\left|\frac{3A^2}{{a_s}^2\kappa^2}-\rho\right|^{-\delta}. \tag{3.4}$$

Although it has been argued that only positive A and B are physically relevant for phantom-energy driven future singularities, the above formula shows the correct sign dependence should A or B be negative. Note that the sign of f depends only on the sign of B.

Conversely, if phantom energy is modeled by

$$f = \frac{C}{(\rho_s-\rho)^{\delta}} + \mathcal{O}((\rho_s-\rho)^{1-\delta}), \tag{3.5}$$

with positive C, ρ_s, and δ, then the evolution is such that the scale factor near the singularity will be of the form of Eq. (3.1) with

$$\epsilon = \min\left\{2, \frac{\delta+3}{\delta+1}\right\} - 1 - \frac{1}{1+\delta},$$

which, as required, is positive for $\delta > 0$. The proof of this statement, although cumbersome because of the several power series involved in the general case, follows the lines of the next section. While Copeland *et. al.* showed that the first term of Eq. (3.5) yields a sudden singularity (see Eq. (461) in [**CST06**]), the calculation shown here is more general in that it only analyzes the behavior near the singularity (hence the operator $\mathcal{O}$ and the need to keep track of ϵ). Thus, it encompasses other models, *e.g.* model (32) in [**NOT05**], that might behave differently far from the singularity. An example of this universality will be found at the end of next section.

We reach then the following conclusion: a phantom-energy model where barotropic dark energy is the only significant fluid near the singularity will produce a sudden singularity, Eq. (3.1), if and only if its behavior near the singularity has the form of Eq. (3.5). The relationship between (A, B) and (C, ρ_s) can be read off from equations (3.4) and (3.5). One implication of this theorem is that sudden singularities cannot be achieved with a static equation-of-state parameter, it must have the form

$$w \approx -\frac{C}{\rho\,(\rho_s - \rho)^{\delta}} \sim \mathcal{O}((t_s - t)^{-1+\frac{1}{1+\delta}}) \tag{3.6}$$

near the singularity. Note that the -1 in Eq. (2.5) has been dropped because of the divergence of Eq. (3.5).

<u>Analysis of Model (32) of [**NOT05**]</u>

In this section we will follow the steps necessary to obtain Eq. (3.1) from Eq. (3.5). The general case is rather complicated because of the need to keep correct track of the orders of magnitude, and at one point it involves expanding a hypergeometric function composed with a logarithm evaluated at the singularity of the logarithm. So, instead of cluttering this section with long mathematical expressions, we will perform the basic steps while reviewing the NOT model ([**NOT05**]). Also, this derivation will expose some problematic points in case the reader is interested in reproducing the full calculations.

The NOT model studies dark energy as a fluid with

$$f(\rho) = \frac{b\,\rho^{1-\gamma}}{\gamma\,(\rho_s^\gamma - \rho^\gamma)} \tag{3.7}$$

where $\rho_s > 0$ is the dark energy density at the singularity. The parameters A, B, α, and β in equation (32) of [**NOT05**] are related to ρ_s, b, and γ in Eq. (3.7) by $A = -b/\gamma$, $B = b\,\rho_s^{-\gamma}/\gamma$, and $\beta = 1-\gamma$, and $\gamma \neq 0$; equation (35) of [**NOT05**] provides the relation between α, and β.

Using the monotonicity results from the first section of this chapter, it is trivial to show that Eq. (3.7) corresponds to phantom energy as long as and whenever $\rho_0 < \rho_s$. On the other hand, if $\rho_0 > \rho_s$, the fluid described by Eq. (3.7) might begin its evolution as non-phantom dark energy (for $\gamma > 0$ and $\rho_0 > \rho_s(1 + \sqrt{1 + 4\,b/\gamma\rho_s{}^{2\gamma}})/2$) or a normal fluid (i.e. both positive pressure and positive energy density) but it will become a normal fluid near the singularity.

With this model, Eq. (2.7) yields

$$\rho^\gamma = \rho_s{}^\gamma - \left(\rho_s{}^\gamma - \rho_0{}^\gamma\right)\sqrt{1 - b\frac{\log\left(\frac{a}{a_0}\right)}{\left(\rho_s{}^\gamma - \rho_0{}^\gamma\right)^2}} \tag{3.8}$$

Even though the above equation might have two signs at front of the radical sign, one of them is spurious. The sign shown yields $\rho \to \rho_0$ when $a \to a_0$. Compare this to equation (36) in [**NOT05**] where the radical sign is preceded by both signs.

As argued previously, the singularity occurs when the quantity inside the square root vanishes. This defines the value of a at the singularity:

$$a_s = a_0 \exp\left(\frac{\left({\rho_s}^{\gamma} - {\rho_0}^{\gamma}\right)^2}{6\,b}\right) \,. \tag{3.9}$$

If b is negative, then the scale factor at the singularity will be smaller than value today. In that case, the contribution from matter and dark matter is not diluted and a model without such contribution is unphysical (see the discussion at the end of the first section of this chapter). We will continue our analysis assuming $b > 0$.

Since we are interested in the behavior near the singularity, instead of Eq. (2.8), Eq. (2.3a) can be approximately written as

$$\frac{\kappa}{\sqrt{3}}(t_s - t) \approx \int_a^{a_s} \frac{\mathrm{d}a}{a\sqrt{\rho(a)}} \,. \tag{3.10}$$

The contributions from other fluids but dark energy have been discarded on the basis that a_s is much bigger than a_0. The integrand can be expanded around a_s, the integral can be evaluated, and the resulting series can be inverted. The first three terms of the inverted series are

$$a(t) = a_s\left(1 - \tau - \lambda\sqrt{\frac{2\,b}{3}}\,\frac{{\rho_s}^{-\gamma}}{|\gamma|}\,\tau^{\frac{3}{2}}\right) + \mathcal{O}\left(\tau^2\right) \tag{3.11}$$

where $\lambda := \operatorname{sign}\left(\log(\rho_0/\rho_s)\right)$ and $\tau := \kappa\sqrt{\rho_s/3}(t_s - t)$.

Note that a approaches a_s from below as required for compatibility with the assumption of one dominant fluid. Also, the second derivative of a is the first to diverge. Thus, the singularity is a sudden singularity, no matter the value of γ. Contrast this to the rich γ-dependent structure reported in [**NOT05**] and propagated in [**CST06**].

The independence of the type of singularity from γ can also be deduced from the behavior of Eq. (3.7) near the singularity:

$$f(\rho) \approx \frac{b\,{\rho_s}^{2(1-\gamma)}}{\gamma^2 \left|\rho_s - \rho\right|} + \mathcal{O}\left((\rho_s - \rho)^0\right) \tag{3.12}$$

which shows that ρ_s is a single pole and therefore falls within the theorem of the previous section.

CHAPTER 4

QUANTIZED FIELDS AND BIG RIP SINGULARITIES

In this chapter, we will study the effects of quantized conformally invariant fields on the evolution of cosmological models with future Big Rip Singularities. Being conformally invariant, the fields are necessarily massless. The cosmological model considered was first proposed by Caldwell [**Cal02**] and the singularity it produces has been described extensively [**CKW03**]. It features a fluid, the phantom energy, with constant EOS parameter w below the cosmological constant threshold $w < -1$. The energy density of this fluid ρ_ϕ is big enough so that the total energy density is the critical energy density

$$\rho_c := \frac{3 {H_0}^2}{8 \pi G} = 8 \times 10^{-30} g/cm^3 \tag{4.1}$$

needed for the cosmological model to have flat spatial sections. In what follows, the contribution from fluids other than dark energy and dark matter will be neglected. The neglected fluids constitute at most 4.5% [**S**$^+$**06**] of the energy of the universe today, and this share decreases as the universe expands. Thus

$$\Omega_m + \Omega_\phi = 1 \ . \tag{4.2}$$

The evolution of dark energy can be computed by replacing the value of $f(\rho)$ appropriate for phantom energy

$$f_\phi(\rho_\phi) = -(1 + w) \rho_\phi$$

in Eq. (2.7) and integrating to find

$$\rho_\phi = \left(\frac{a_o}{a}\right)^{3+3w} (1-\Omega_m)\,\rho_c \ . \tag{4.3}$$

The time evolution of the scale factor is given by Eq. (2.8'):

$$\frac{1}{H_0}\int_0^{a(t)} \frac{\xi^{1/2}}{\left(\Omega_m + (1-\Omega_m)\xi^{-3w}\right)^{1/2}}\,\mathrm{d}\xi = t \ , \tag{4.4}$$

where the lower boundary in the integral has been chosen so that the Big Bang occurred when t was 0. The fact that the integral is convergent for any value $w < -1$ with $a \to \infty$ demonstrates that the Big Rip occurs at finite time for those values of w.

Quantum effects will become significant near the singularities. Hence for our purposes, we can approximate Eq. (4.4) near the Big Rip singularity as

$$T - t = \frac{1}{H_0\,(1-\Omega_m)^{1/2}} \int_{a(t)}^{\infty} \xi^{(1+3w)/2}\,\mathrm{d}\xi \ . \tag{4.5}$$

The limits of integration have been changed to facilitate consideration of the region near the singularity. Here, T is the time of the Big Rip, *i.e.* $a(T) \to \infty$, and the contribution from dark matter to the denominator of the integrand has been dropped on the basis that it has diluted and it is much smaller than the contribution from dark energy. The former behaves like a^{-3} and the latter as $a^{-3(1+w)}$ with $w \lesssim -1$.

Integrating Eq. (4.5), the scale factor near the singularity is then given approximately by

$$a(t) \approx \left(\left(\frac{3}{2}\right) H_0\,(1-\Omega_m)^{1/2}\,|1+w|\,(T-t)\right)^{\frac{2}{3(1+w)}} \ . \tag{4.6}$$

Using Eqs. (4.3) and (4.6), we find the behavior of the energy density of the phantom energy near the Big Rip is given by

$$\rho_\phi \sim (T-t)^{-2} \, . \tag{4.7}$$

The phantom pressure diverges at the same rate as its energy density because w is a constant throughout time in these models.

Quantum Effects

Evaluating $\langle T_{\mu\nu}\rangle$ for conformally invariant fields can now be done by replacing the scale factor of Eq. (4.6) in Eq. (2.2) and using the resulting metric to compute the curvature tensors in Eq. (2.24). We find then that the vacuum energy densities can be cast in the form

$$\rho_a := T_{00}\big|_{\mathrm{spin}=a} = \frac{P_a}{19440\,\pi^2\,(T-t)^4\,(1+w)^4} \, , \tag{4.8}$$

where a is a placeholder to denote scalar, spinor, or vector fields and P_a is a second degree polynomial in w (see Table (4.1)). Of these three polynomials, P_0 and $P_{1/2}$ are strictly positive for all $w < -1$; P_1 is positive for $w_0 < w < -1$ and negative for $w < w_0$, where $w_0 := -\frac{1}{3} - \frac{2}{27}\sqrt{174} \approx -1.31$. Because all of the other quantities in Eq. (4.8) are positive definite, the signs of these polynomials and the signs of the energy densities agree.

The pressures p_a are computed in a similar fashion:

$$p_a := T_{11}\big|_{\mathrm{spin}=a} = \frac{P_a\,(1+2\,w)}{19440\,\pi^2\,(T-t)^4\,(1+w)^4} \, . \tag{4.9}$$

Table 4.1. Spin-dependent polynomials in Eq. (4.8) and Eq. (4.9).

spin	P_a
0	$-5+18\,w+27\,w^2$
$\frac{1}{2}$	$-5+54\,w+81\,w^2$
1	$205-162\,w-243\,w^2$

The ratio of the expected value of the pressure to the expected value of the energy density is a constant independent of the spin of the field

$$\langle w\rangle = \left.\frac{T_{11}}{T_{00}}\right|_{\text{spin}=a} = 1+2\,w\ . \tag{4.10}$$

Since $\langle w\rangle$ is smaller than -1 for all phantom-energy values of w, then the vacuum states behave as additional phantom energy as long as their energy density is positive. Moreover, since the pressure and energy density of the vacuum states diverge faster than those of the original phantom energy fluid, then the quantum corrections overcome the classical phantom energy at some time close to the singularity. However, we cannot feed the vacuum contributions back to the right hand side of Einstein equations (*i.e* complete the semiclassical backreaction calculation), because this would amount to complete the calculations up to first order in the Planck constant and we do not know the contributions from the phantom energy fluid up to first order in $\hbar$. We only have its classical value, this is up to zero-th order in $\hbar$. Nevertheless, it is possible to examine in which direction the inclusion of quantum effects perturbatively change the classical solution by analyzing the changes in the total stress energy tensor.

In de Sitter spacetime, the expectation value of the vacuum stress energy tensor and the cosmological constant have the same form and they are usually dealt with as one single term, the first renormalizing the second. But in the case at hand, the equation of state parameter of the vacuum states $\langle w \rangle$ is not equal to the equation of state parameter of the background phantom energy w. For the system including vacuum state and phantom energy, we define an effective equation of state parameter by

$$w_{\text{eff}} := \frac{p_{\text{total}}}{\rho_{\text{total}}} = \frac{p_\phi + p_a}{\rho_\phi + \rho_a} \,. \tag{4.11}$$

Using Eq. (4.10), this can be rewritten as

$$w_{\text{eff}} = w + (1+w)\frac{\rho_a}{\rho_\phi + \rho_a} \,. \tag{4.12}$$

The above equation shows that, because $w < -1$, $w_{\text{eff}} < w$ as long as ρ_a is positive. This is, the effect of the vacuum energy density of the quantized conformally invariant fields is to strengthen the accelerated expansion that leads to the Big Rip singularity for the values of w that yield a positive P_a (see Eq. (4.8)).

Writing it out explicitly: in this set of Big Rip cosmological models, the vacuum stress-energy of conformally invariant scalar and spinor fields *always* strengthen the classical Big Rip singularity. The vacuum stress-energy of a conformally invariant vector field strengthen the classical Big Rip singularity in cosmological models with $w \geq w_0$, whereas the singularity is weakened by the vacuum stress-energy of a conformally invariant vector field in cosmological models with $w < w_0$.

The main results of this chapter have been published in [**CH05**].

CHAPTER 5

QUANTIZED FIELDS AND SUDDEN SINGULARITIES

In this chapter, the expression for the stress energy tensor of conformal fields in conformally flat spacetimes will be applied in order to compute the contributions from the vacuum state of fields near a sudden singularity in flat FLRW cosmology [**Bar04**]. The resulting stress energy tensor will be used to qualitatively analyze the softening or enhancing of the singularity by those quantum vacua. The question of whether the singularity could be avoided as result of the quantum contributions has been studied in similar cosmological models [**NO04**] using different techniques for the calculation of the quantum perturbations.

In the second section, general thermodynamical arguments show that, in this type of singularity, the heat exchanged by the fields cannot be neglected. Thus, this chapter provides an example where gravitation is involved and thermodynamics plays a role as big as dynamics.

The fields considered in this chapter are conformally invariant and the vacuum state is chosen so that it can be conformally transformed from the Minkowski vacuum. For brevity, we will refer to these fields by their spin (*e.g.* scalar fields); but the mentioned conditions are implied.

Vacuum State Contributions

Utilizing the results of chapter 3, we begin by writing down the scale factor (which in turn determines the background metric) appropriate for a flat RW cosmology with a sudden singularity:

$$a(t) \approx a_s \left(1 - \tau + \frac{\left(\frac{3\,C\,(1+\delta)}{\kappa^2}\right)^{\frac{1}{1+\delta}} (1+\delta)}{2\,(2+\delta)\,\rho_s} \tau^{1+\frac{1}{1+\delta}} + \mathcal{O}\left(\tau^{1+\frac{1}{1+\delta}+\epsilon}\right) \right) \tag{5.1}$$

where

$$\tau := \kappa \sqrt{\frac{\rho_s}{3}} (t_s - t) \; . \tag{5.2}$$

This form of the scale parameter, using C and ρ_s (*c.f.* Eq. (3.5)), is preferable to Eq. (3.1), where A and B are used, because of two reasons. First, the resulting expressions will be shorter; and second, it incorporates naturally the physically relevant signs $\delta > 0$, $C > 0$, and $\rho_s > 0$. In addition, the energy density at the singularity determines a scale that can be used to measure time: the dimensionless quantity τ counts back the time until the singularity using such scale.

A number of simplifications in Eq. (2.24) arise from the fact that the series in Eq. (5.1) is truncated, *e.g.* terms with $\dot{a}$ need to be computed only up to $\mathcal{O}\left(\tau^{\frac{1}{1+\delta}+\epsilon}\right)$. Also, near the singularity the metric and its derivative remain finite and the Ricci tensor and Ricci scalar diverge as $\ddot{a}$. The derivatives of the Ricci scalar $R^{;\mu}{}_{;\nu}$ diverge even faster: as the fourth derivative of the scale factor $a^{(iv)}$. Hence, the first two terms of the coefficient of α are the only ones that need to be calculated.

Plugging Eq. (5.1) into Eq. (2.24) and using the mentioned simplifications, one obtains:

$$\rho_a := T_{00}|_{\text{spin}=a} = \alpha \frac{\kappa^4 \rho_s}{3} (3\,C)^{\frac{1}{1+\delta}}\, \delta\, ((1+\delta)\,\tau)^{-2+\frac{1}{1+\delta}} \,, \tag{5.3a}$$

$$p_a := T_{11}|_{\text{spin}=a} = -\alpha \frac{\kappa^4 \rho_s}{9} (3\,C)^{\frac{1}{1+\delta}}\, \delta\, (1+2\,\delta)\, {a_s}^2\, ((1+\delta)\,\tau)^{-3+\frac{1}{1+\delta}} \,. \tag{5.3b}$$

As implied above, the spin dependence of the right hand side is coded only through the dependence of α on the spin, see Table (2.3). Both expectation values diverge as the singularity approaches, the pressure faster than the energy density. These divergences are slower than the τ^{-4} divergences in a Big Rip singularity (Eqs. (4.8) and (4.9)).

We can now compute the effective equation of state parameter of the quantum contributions of the vacuum states:

$$w = -{a_s}^2 \frac{1+2\,\delta}{3\,(1+\delta)} \tau^{-1} \,. \tag{5.4}$$

It is clearly negative and diverges at the singularity faster than the EOS parameter of the phantom fluid, which diverges as $\tau^{-1+\frac{1}{1+\delta}}$ (see Eq. (3.6)).

All these divergences mean that the approximation breaks down before the singularity occurs. As mentioned in the previous chapter, this contributions cannot be used in a back-reaction calculation because, among other reasons, we know the phantom contributions only to zero-th order in $\hbar$. Nevertheless, a qualitative analysis is in order. The behavior of the perturbation is determined by the sign of α, which is the only quantity not defined positive. Because α is positive for both scalar and spinor fields, then ρ_0 and $\rho_{1/2}$ are positive, and p_0 and $p_{1/2}$ are negative; and therefore, scalar and spinor fields enhance the singularity. This happens because adding these vacuum perturbations is equivalent to adding more phantom energy to the right hand side of Eq. (2.3). Vector fields, having negative α which yields $\rho_1 < 0$ and $p_1 > 0$, counter the contributions from dark energy and therefore soften the singularity.

Thermodynamical Considerations

It is interesting to note that phantom energy confers some of its properties to the vacuum state of scalar and spinor fields. These vacuum states represent the vacuum states of ordinary matter which would otherwise have a "normal" equation of state parameter not smaller that -1. While it is not common to study the thermodynamical properties of vacua (usually they are boring), it is not impossible, *e.g.* [**MR00**]. In a closer example [**HW86**], it is argued that the process leading to the thermalization of vacua are weak interactions that "occur through the global transfer of conformal energy".

In general, small unknown terms in the Hamiltonian lead to the thermalization of the system. Basically, even if the system begins its evolution as a pure state, it will become a mixed state. Let us elaborate. Assume that the system was a pure state with density matrix D_0 corresponding to the eigenstate of some well known Hamiltonian H_0. This is, ${D_0}^2 = D_0$ and $H_0 D_0 = \epsilon D_0$. Since the true Hamiltonian $H = H_0 + \Delta H$ contains small unknown terms ΔH, then D_0 is only an approximate eigenstate of H; but most importantly, the true solution D satisfies $D^2 = D$ only to the same order as $\Delta H \approx 0$ and the system is not in a pure state. For a canonical exposition, see [**Kat67**]; in particular, notice how the Hamiltonian (equation (12.2) in the reference) with small unknown terms yields a thermal state (equation (13.13)).

In this section, we will consider the individual vacua as subsystems. We admit that we know their evolution only up to order $\hbar$ and we reckon that Lagrangian higher order terms in $\hbar$, possible self-interactions, and interactions with the phantom are among the mechanisms for thermalization. We don't need to assume that the process is quasi-static because we will compute neither the temperature nor the entropy; we shall be satisfied with the sign of the change of the enthalpy of formation. The expected value of the stress energy tensor computed in the previous section gives the energy and the pressure. They depend on the spin of the field. The fact that the pressures are not equal should not be interpreted as the vacua not being in hydrostatic equilibrium with each other, but those pressures should be thought of as partial pressures in a mixture.

Consider the first law of thermodynamics $\delta U = \delta Q - \delta W$ where

$$U \propto \rho_a a^3 \tag{5.5a}$$

and

$$W \propto p_a a^3 \tag{5.5b}$$

This last expression, for the work done by the system, doesn't contain interaction terms because they would arise from terms in the Lagrangian other than the kinetic energy. But the approximations leading to Eq. (2.24) are valid only for free fields (see [**BD84**] page 5).

Let us investigate whether the expansion is exothermic or endothermic. Given that, as shown above, the pressure diverges faster that the energy then $\delta Q \approx \delta W$ and

$$\mathrm{sign}\,(\delta Q) = \mathrm{sign}\left(3\, p_a\, a^2\, \delta a + a^3\, \delta p_a\right) \tag{5.6}$$

The first term on the right hand side can be dismissed because it diverges as $\tau^{-3+\frac{1}{1+\delta}}$ which is slower than the $\tau^{-4+\frac{1}{1+\delta}}$ divergence of δp_a. Therefore, the sign of the heat flowing into the system is the same as the sign of pressure change. The latter is determined by $-\mathrm{sign}\,(\alpha)$ because the other quantities in Eq. (5.3b) are defined positive. The expansion is then exothermic for both scalar and spinor fields and endothermic for vector fields. Exothermic reactions are spontaneous and thus they enhance the singularity. We conclude again that scalar and spinor vacua enhance the singularity and vector vacua soften it. This is in agreement with the dynamical results of the previous section.

CHAPTER 6

QUANTIZED MASSIVE SCALAR FIELDS IN SCHWARZSCHILD - DE SITTER BLACK HOLES

The fields considered in previous chapters have been massless and they have coupled conformally to the gravitational field. The fields considered in this chapter are no longer massless. The mass introduces a characteristic length to the system and destroys conformal invariance. The background metric is that of a Schwarzschild-(anti) de Sitter black hole of mass M, with a cosmological constant Λ. We solve the semiclassical Einstein equations to find the first-order in $\hbar/M^2$ semiclassical corrections to the classical Schwarzschild-deSitter metric, treating the vacuum stress-energy of the quantized massive scalar field as a perturbation on the “bare" classical background spacetime:

$$G^{\mu}{}_{\nu} + \Lambda\,\delta^{\mu}{}_{\nu} = 8\pi\,T^{\mu}{}_{\nu}\ , \tag{6.1}$$

where $T^{\mu}{}_{\nu}$ is the renormalized value of the energy tensor. The backreaction will be computed by writing the perturbations in the following form

$$\mathrm{d}s^2 = -(1+2\epsilon\,\rho(r))\left(1-\frac{2m(r)}{r}-\Lambda\frac{r^2}{3}\right)\mathrm{d}t^2+\left(1-\frac{2m(r)}{r}-\Lambda\frac{r^2}{3}\right)^{-1}\mathrm{d}r^2+r^2\,\mathrm{d}\Omega\ , \tag{6.2}$$

where

$$m(r) := M(1+\epsilon\mu(r))\ . \tag{6.3}$$

This metric can be feed into Eq. (6.1). The resulting equations are:

$$\frac{\mathrm{d}\mu}{\mathrm{d}r} = -\frac{4\,\pi\,r^2}{M\,\epsilon} T^t{}_t\,, \tag{6.4a}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{4\,\pi\,r}{\epsilon} \frac{T^r{}_r - T^t{}_t}{1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}}\,. \tag{6.4b}$$

Backreaction

Using the results of [**AHS95**] for the case of a quantized massive scalar field in the Schwarzschild-deSitter spacetime, the following approximate values for the expectation value of the stress-energy tensor are obtained:

$$\begin{aligned} \langle T^r{}_r \rangle = \frac{\epsilon\,M^2}{272160\,\pi^2 m^2 r^9} \Big(& 54\,M^3\,(1237 - 5544\,\xi) + 27\,M^2\,r^3\,\Lambda\,(263 - 1176\,\xi) \\ & - 1215\,M^2\,r\,(25 - 112\,\xi) \\ & - r^9\,\Lambda^3\,\left(185 - 3654\,\xi + 22680\,\xi^2 - 45360\,\xi^3\right)\Big)\,, \end{aligned} \tag{6.5a}$$

$$\begin{aligned} \langle T^t{}_t \rangle = \frac{\epsilon\,M^2}{272160\,\pi^2 m^2 r^9} \Big(& 1701\,M^2\,r\,(7 - 32\,\xi) - 189\,M^2\,r^3\,\Lambda\,(37 - 168\,\xi) \\ & - 378\,M^3\,(47 - 216\,\xi) \\ & - r^9\,\Lambda^3\,\left(185 - 3654\,\xi + 22680\,\xi^2 - 45360\,\xi^3\right)\Big)\,. \end{aligned} \tag{6.5b}$$

where $\epsilon = 1/M^2$ is our expansion parameter for the perturbation (in conventional units, $\epsilon = M^2_{\text{Planck}}/M^2$) and ξ is the curvature coupling for the field. We do not explicitly display the value of $\langle T^{\theta}{}_{\theta}\rangle$, as it carries no new information – it can be computed from Bianchi identities. Knowledge of the two components shown above is sufficient to find the perturbing μ and ρ by Eq. (6.4).

The overall factor ϵ in the expressions for $\langle T^{t}{}_{t}\rangle$ and $\langle T^{r}{}_{r}\rangle$ will cancel the ϵ factors in the denominator of the leading terms in both Eq. (6.4a) and Eq. (6.4b). These differential equations can be integrated to find the general solutions for μ and ρ. They are

$$\mu = \frac{M\,r^3\,\Lambda^3\,\left(185 - 3654\,\xi + 22680\,\xi^2 - 45360\,\xi^3\right)}{204120\,m^2\,\pi} - \frac{M^3\,(25 - 112\,\xi)}{280\,m^2\,\pi\,r^5} + \frac{M^3\,\Lambda\,(263 - 1176\,\xi)}{7560\,m^2\,\pi\,r^3} + \frac{M^4\,(1237 - 5544\,\xi)}{7560\,m^2\,\pi\,r^6} + C_1\ , \tag{6.6a}$$

and

$$\rho = -\frac{M^4\,(87 - 392\,\xi)}{840\,m^2\,\pi\,r^6} + C_2\ . \tag{6.6b}$$

Renormalization of the Cosmological Constant

The term in $\mu(r)$ that is proportional to r^3 enters the perturbed metric with the same algebraic form as the background cosmological constant; hence it may be absorbed into a renormalization of the cosmological constant,

$$\Lambda_R = \Lambda + \frac{\epsilon\,M^2\,\Lambda^3\,\left(185 - 3654\,\xi + 22680\,\xi^2 - 45360\,\xi^3\right)}{68040\,m^2\,\pi}\ . \tag{6.7}$$

The apparent dependence of the renormalization upon the black hole mass M is not real, since $\epsilon = \hbar/M^2$; the actual renormalization depends only on the field mass m, the cosmological constant itself, Λ, and the scalar field's curvature coupling, ξ.

We will now assume that Λ has been renormalized according to Eq. (6.7); we will however continue to simply label the (now renormalized) cosmological constant as Λ for notational simplicity. We can then eliminate the terms involved in renormalizing Λ from the expression for $\mu(r)$. We may also replace Λ by its renormalized value in the metric perturbations, since the changes in the metric caused by this replacement will be of second order in ϵ. Let us then define a new metric perturbation function

$$\tilde{\mu} = -\frac{M^3\,(25-112\,\xi)}{280\,m^2\,\pi\,r^5} + \frac{M^3\,\Lambda\,(263-1176\,\xi)}{7560\,m^2\,\pi\,r^3} + \frac{M^4\,(1237-5544\,\xi)}{7560\,m^2\,\pi\,r^6} + C_1\ , \qquad (6.8)$$

which will hereafter replace μ.

The perturbed spacetime is now defined to first order in ϵ to within the two integration constants, C_1 and C_2. Also note that, after renormalizing the cosmological constant, the perturbation becomes linear in both the curvature coupling and the cosmological constant.

The only zero of $\Lambda_R - \Lambda$ is located at $\xi_0 = 0.102268$. Below that value, the perturbation $\Lambda_R - \Lambda$ has the same sign as Λ; and above ξ_0, it has the opposite sign to Λ. Therefore, for scalar fields with $\xi \lesssim 0.1023$ (including the minimally coupled field, $\xi = 0$), the effect of the renormalization is to *increase* the effective value of the cosmological constant, while for fields with larger curvature couplings, $\xi \gtrsim 0.1023$ (including the conformally invariant field, for which $\xi = 1/6$), the effect of the renormalization is to *decrease* the value of the cosmological constant.

Location of the Horizons

The horizon radii are no longer located at r_+ or r_{++} due to the metric perturbation caused by the vacuum state of the quantized field. The radii, r_h, of the horizons in the perturbed spacetime are defined as the solutions to the equation $g^{rr} = 0$, this is:

$$\Lambda {r_h}^3 + 6\, m(r_h) - 3\, r_h = 0\,. \tag{6.9}$$

Since the above is a ninth degree polynomial, we might have up to nine horizons. For example, using Descartes Rule of Signs, it can be proven that Eq. (6.9) has at least three positive zeros for $\lambda > 0$ and $\xi > \frac{263}{1173} \simeq .224$. However, by taking Mm big enough, those extra zeros will be outside the region between the black hole and cosmological horizons, for Schwarzschild - de Sitter, and inside the black hole for Schwarzschild - anti de Sitter. And, our calculations are not valid in those regions anyway because the lack of a timelike Killing vector; i.e. quantum perturbations, in the way computed here, cannot be used to create more horizons than the ones already existent in the classical case.

Let $r_h = r_x + \epsilon\, \delta r_x$ where r_x stands for either r_+ or r_{++}. Replace this in equation Eq. (6.9) to obtain

$$\frac{\delta r_x}{r_x} = \frac{\mu(r_x)}{3 - r_x/M}\quad . \tag{6.10}$$

Now we impose the requirement that the quantum perturbations shall displace the horizons by a small amount. This demand can only be fulfilled by choosing the integration constant C_1 in such way that $\mu(r_h) = 0$ for a degenerate black hole. Therefore,

$$C_1 = -\frac{1}{5511240 m^2 M^2 \pi} \,. \tag{6.11}$$

Note that C_1 does not depend on the coupling ξ. Also, it can be proven that the quantum perturbation is of the order of the radius of the horizon, $\left|\delta r_+\right| \sim \left|r_+\right|$, only for very small r_+, which corresponds to very negative values of λ. Again, this is outside the limits of validity of our calculations; and so, the quantum perturbations computed here cannot push a zero of g^{rr} from a very small positive value (existence of a black hole horizon) to a negative value (no black hole horizon).

A degenerate black hole might occur not only for $(\lambda = \frac{1}{9}, \xi \in \mathbb{R})$ but also at the pair (λ, ξ) that satisfies the equation

$$\epsilon\,(\delta r_{++} - \delta r_+) = -(r_{++} - r_+) \tag{6.12}$$

It can be proven that there are no other solutions, within the limits of validity of our calculations, than the classical one.

Renormalization of the Black Hole Mass

We could utilize the location of the black hole horizon to define the perturbed mass of the black hole,

$$M_1 = m(r_h) = M\left[1 + \epsilon\,\mu(r_+)\right] \,, \tag{6.13}$$

to first order in ϵ; r_h has been changed to r_+ in the final expression on the right, as the difference would be of order ϵ^2. However, in the Schwarzschild-deSitter case, one could also equally well define a perturbed mass using the radius of the cosmological horizon,

$$M_2 = m(r_h) = M\left[1 + \epsilon\, \mu(r_{++})\right] \; . \tag{6.14}$$

However, M_1 and M_2 are distinct, different, values for the perturbed mass; neither has any particular claim to being a more appropriate physically renormalized mass than the other (or, indeed, any other mass defined by choosing an arbitrary radius at which to evaluate $\mu(r)$ and then define the renormalized mass).

Temperature of the Field

The surface gravity at the black hole horizon, up to first order in ϵ, is $\kappa_h = \kappa_+ + \epsilon\, \delta\kappa_+$, where κ_+ is the classical surface gravity, Eq. (2.16a), and $\delta\kappa_+$ is given by

$$\delta\kappa_+ = \frac{M\left(3M - 2r_+\right)\mu\left(r_+\right) - \left(3M - r_+\right)\left(\left(r_+ - 3M\right)\rho\left(r_+\right) + M r_+ \mu'\left(r_+\right)\right)}{\left(3M - r_+\right) r_+^2} \tag{6.15}$$

A similar equation holds for the surface gravity at the cosmological horizon.

As in [**AHWY94, Yor85**], a cavity surrounding the black hole is introduced in order to prevent its evaporation. The cavity not only limits the support of the quantized fields, but it mainly helps to specify the thermodynamical state of the black hole.

The temperature will be just $\kappa_h/(2\,\pi)$ and the constant C_2 will be

$$C_2 = -\frac{M^4(392\xi - 87)}{840 m^2 \pi r_0^6} \tag{6.16}$$

where r_0 is the radius of the cavity.

Determination of both constants C_1 and C_2 and renormalization of the mass and cosmological constant complete the problem.

CHAPTER 7

SUMMARY

In this dissertation, three new applications of QFTCS have been calculated. The metrics considered (flat RW with Big Rip singularity, flat RW with sudden singularity, and Schwarzschild-de Sitter black holes) have two properties in common. First, they deal, in one or another, with fluids representing dark energy. In this sense, this dissertation studies aspects related to the most abundant form of energy in the universe. Second, there is a singularity in all three metrics. Having a singularity is the feature that creates a region of spacetime where the quantum contributions from the vacuum expectation value of the stress-energy tensor of quantized fields is significant.

In flat RW with Big Rip singularity and in flat RW with sudden singularity, the vacuum contributions considered are those of massless, conformally-coupled fields. The spins of the fields are constrained, in this study, to either scalar, spinor, or vector cases. We presented the answer to a common question: do vacuum contributions enhance of soften the singularity? The result is important in two accounts. First, it is related to questions about the fate of the universe; and second, it constitutes a limiting case that a full theory of quantum gravity should either reproduce or explain any discrepancy.

The case of flat RW with sudden singularity was also analyzed using thermodynamics. Thermodynamics of vacuum are usually uninteresting, but this case is an example where gravitation is present and thermodynamics would play a role as substantial as dynamics.

We have found that the dynamical and the thermodynamical results coincide. This coincidence could be the result of a more general theorem, but at this point the thermodynamics of gravitational systems is not well understood.

A couple of strictly classical results regarding finite-time future singularities were also presented in this dissertation. They were calculated to disprove some claims in the literature.

Back in the quantum realm, the vacuum contributions of a quantized massive scalar field with arbitrary curvature coupling were computed outside of a Schwarzschild-de Sitter black hole in the region where there are timelike Killing vectors. The field was considered enclosed in a perfectly reflecting cavity. The results were used to calculate the backreaction in the metric. We found that renormalization of the mass of the black hole and of the cosmological constant, relocation of the horizons, and the location of the cavity were enough to write the perturbed metric without unphysical constants. Further, we studied the perturbation to see whether quantum effects could produce remnants other than the classical case with degenerate horizons at zero temperature. We found that semiclassical corrections do not produce zero-temperature solutions other that the classical ones and thus there are no other candidates for "remnant" geometry.

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