ON THE GEOMETRY AND TOPOLOGY OF THE ANGULAR MOMENTUM OF LIGHT

by

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This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citation, bibliographic style, and consistency and is ready for submission to the Division of Graduate Education.

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May 2010

DEDICATION

This work is dedicated to the four most formidable women I have ever known: my mother, Elizabeth M. Gelbord, her mother, Madeline Ahern, my father's mother, Esther Gelbord, and my advisor, Sachiko Tsuruta. What you see before you would never have come to pass without their unending support.

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ABSTRACT

The classical field theory approach to the angular momentum of light, specifically how it represents the spin angular momentum of light, has been a matter of controversy for some time. This thesis analyses the aforementioned approach from the point of view of the Exterior Calculus and de Rham Cohomology. It is found purely mathematically that the spin angular momentum of a circularly polarized plane wave of light must be identically zero. It is concluded that the classical formulation of the angular momentum of a plane wave of light is, on some level, incomplete.

CHAPTER 1
INTRODUCTION

Wherein the virtues and successes of a topological approach to Electromagnetism are illuminated.

1.1 Why Geometry and Topology?

Over the past several decades the ideas and methods of geometry and topology have been appearing in multifarious ways throughout the majority of Physics. All are familiar of course with General Relativity, wherein Albert Einstein utilized the tools of Differential Geometry to great effect. Looking at the universe through the lens of geometry revolutionized our understanding not only of gravity but of the large-scale structure of the universe. While General Relativity certainly was not the first application of geometric insight into Physics, it was, at the time, the grandest.

In time, solid-state physicists would find the concept of symmetry transformations, vis-a-vis group theory, of fundamental importance. In fact, the fundamentals of Gauge Theory, the foundation of the Standard Model, are inherently geometric as they are based on symmetry groups in Representation Theory. Eventually Quantum Field Theory would give birth to what is known as Topological Quantum Field Theory. String Theory, whether true or not, has given rise to myriad advances in Physics, Topology, and Geometry. Modern Physics is now, and indeed always has been, inexorably linked in one form or another with Topology and Geometry. However, there is a specific field of Physics that, time and again, has shown itself to be rooted in the deepest concepts of geometry and topology despite not being explicitly formulated in such terms.

James Clerk Maxwell's theory of Electromagnetism, considered one of the greatest triumphs of Theoretical Physics, is almost universally expressed and taught with the tools of Vector Calculus. Observe the fundamental equations of Electromagnetism:

$$\nabla \cdot \mathbf{B} = 0 \tag{1}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3}$$

$$\frac{\partial \mathbf{E}}{\partial t} - c^2(\nabla \times \mathbf{B}) = -\frac{\mathbf{j}}{\epsilon_0}. \tag{4}$$

While the above formulae are represented here in a concise manner, their application to and subsequent solution of various physical problem can be extraordinarily complicated and computationally involved. This is a result of the fact that the above equations are vector-valued differential equations and thus contain more individual differential equations (one for each vector component) than just the four displayed above. However, such a computationally demanding formulation can lead to a loss of physical perception and insight about a given situation. The physically relevant facts can be lost or forgotten amid pages of vector calculus computations. Nevertheless, it is a necessary way of doing electromagnetism for the physicist since the current formulation of Electromagnetism allows for explicit numerical predictions, a requirement for all physicists. Yet for all its apparent computational overhead vector calculus, indeed calculus in general, can be cast in a most geometric and topological light via the fields of Differential and Algebraic Topology. Specifically, we shall show the Exterior Calculus, in conjunction with differential forms, Homology and De Rham Cohomology, and the ideas of Homotopy can be used to re-express Electromagnetism in a manifestly geometric way and apply the resulting viewpoint to the concept of the angular momentum of light.

However, as motivation for why one should do such a recasting and as a form of introduction to the methods and modes of thought of geometry and topology applied to electromagnetism, we now provide an overview of three great successes of geometric and topological considerations of electromagnetic problems.

1.2 The Aharonov-Bohm Effect

Consider a double-slit electron diffraction setup with one profound difference. A solenoid of infinite length is placed behind the screen separating the two slits as seen in figure 1 below. The upper path is p_1, the lower path is p_2, and a closed clock-wise circuit starting and ending at the electron emitter is P_c and can be expressed as $P_c = p_1 - p_2$.

Since the magnetic field of the solenoid is wholly confined to *within* the solenoid and there is no electric field of which to speak there can be no classical force acting on the electron. This is demonstrated by the Lorentz Force Law $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ and the fact that $\mathbf{B}$ and $\mathbf{E}$ are zero at any point the electron can inhabit. As a result, one would expect the resulting diffraction

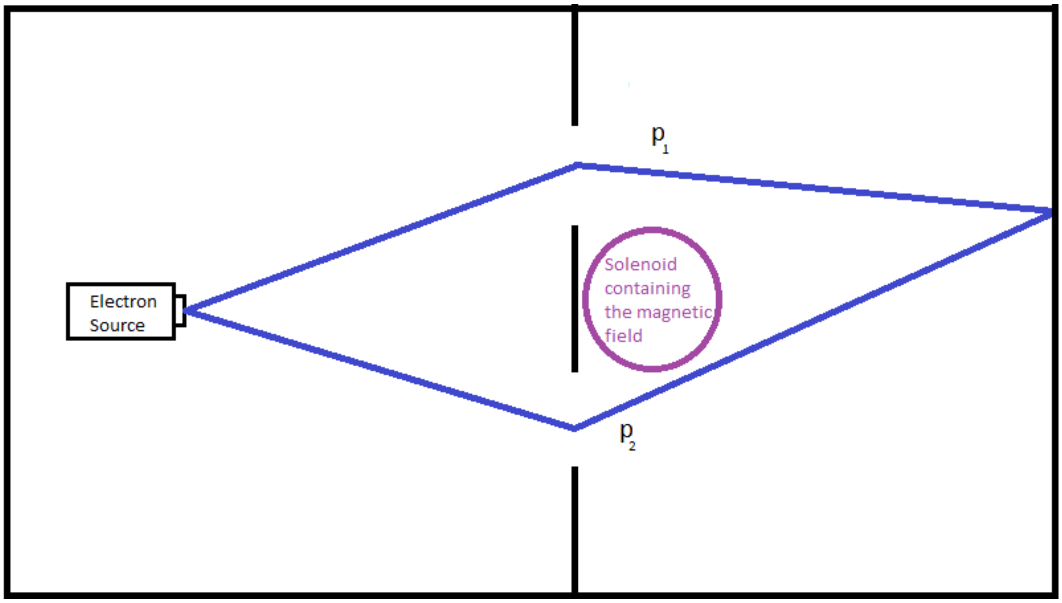

Figure 1: Aharonov-Bohm Effect Experiment Schematic

pattern to be the usual double-slit diffraction pattern of a wave incident on the two slits. That is to say, since ostensibly both paths are equivalent, then both slits, acting as apertures, should yield waves of identical phase. However, when one does the experiment one finds that the resulting diffraction pattern is dependent on the magnitude of the magnetic flux through the solenoid. It is seen that for a given magnetic flux, the phase of the wave along p_1 is potentially different from the wave along p_2, resulting in shifted fringes in the diffraction pattern. Interestingly enough, there are certain values of the magnetic flux which will yield the traditional double-slit diffraction pattern.

There are two ways of investigating this experiment. The first is by a standard quantum mechanical approach utilizing path integrals and following closely the work of M. Nakahara[1]. Using coordinate-free notation and allowing

[1]Nakahara, M.,*Geometry, Topology, and Physics*, 2nd edition, Institute of Physics Publishing, Philadelphia, PA (2003)

for no extraneous electrostatic potential in the experiment the Hamiltonian of our electron is simply

$$\boldsymbol{H} = -\frac{1}{2m}\left(\frac{\partial}{\partial x^{\mu}} - iqA_{\mu}\right)^{2} \tag{5}$$

where $\mu = \{0, 1, 2, 3\}$, q is the electric charge, and

$$A = \left(0, \frac{-y\Phi}{2\pi r^2}, \frac{x\Phi}{2\pi r^2}, 0\right), \tag{6}$$

i.e. the usual magnetic vector potential around an axial magnetic field oriented along the positive z-axis. In addition, $\frac{\partial}{\partial x^{\mu}}$ is the coordinate free representation of a first-order differential operator acting on temporal and spatial coordinates.

From here we take the solution of the Schrodinger Equation with $\mathbf{A}$ so defined to be

$$\Psi_1 = e^{iq\int \mathbf{A}(\mathbf{r}')\cdot d\mathbf{r}'}\psi_1(\mathbf{r}) \tag{7}$$

where $\Psi_1(\mathbf{r})$ is the electron's wave function along p_1 when there is a non-zero magnetic vector potential $\mathbf{A}$ in the region outside the solenoid and $\psi_1(\mathbf{r})$ is the wave function along p_1 when $\mathbf{A} = 0$. The solution is identical for $\Psi_2(\mathbf{r})$. Without going too far afield explaining how one arrives at the solution, we see that the form of $\Psi_1(\mathbf{r})$ makes intuitive sense quantum mechanically, or at the least is not unpalatable to out intuition. As it stands, $\Psi_1(\mathbf{r})$ is just the original wave function multiplied by a phase factor dependent upon the magnetic vector potential. Naively one can assume that that is all the effect $\mathbf{A}$ will have since such phase factors vanish when taking the complex-magnitude of the wave function. Nevertheless, the magnetic vector potential still has a physically demonstrable effect.

Adding $\Psi_1(\mathbf{r})$ and $\Psi_2(\mathbf{r})$ at the imaging screen yields

$$\Psi_1(\mathbf{r}) + \Psi_2(\mathbf{r}) = e^{iq\int_{p_1}\mathbf{A}(\mathbf{r}')\cdot d\mathbf{r}'}\psi_1(\mathbf{r}) + e^{iq\int_{p_2}\mathbf{A}(\mathbf{r}')\cdot d\mathbf{r}'}\psi_2(\mathbf{r}). \tag{8}$$

Now by virtue of the fact that $P_c = p_1 - p_2$ we can rewrite equation 8 in the following manner,

$$\Psi_1(\mathbf{r}) + \Psi_2(\mathbf{r}) = e^{iq\int_{p_2}\mathbf{A}(\mathbf{r}')\cdot d\mathbf{r}'}\left(e^{iq\oint_{P_c}\mathbf{A}(\mathbf{r}')\cdot d\mathbf{r}'}\psi_1(\mathbf{r}) + \psi_2(\mathbf{r})\right). \tag{9}$$

With a clever application of Stokes' Theorem to $\oint_{P_c}\mathbf{A}\left(\mathbf{r}'\right)\cdot d\mathbf{r}'$, and recalling that $\mathbf{B} = \nabla\times\mathbf{A}$, it is clear that

$$\oint_{P_c}\mathbf{A}\left(\mathbf{r}'\right)\cdot d\mathbf{r}' = \int \nabla\times\mathbf{A}\cdot d\mathbf{S} = \int \mathbf{B}\cdot d\mathbf{S} := \Phi \tag{10}$$

where Φ is the magnetic flux through the surface bounded by path 3, which clearly is just the magnetic flux through the solenoid. Before we simplify anything with equation 10 we take a moment to point out that the complex-magnitude of $\Psi_1 + \Psi_2$ will no longer eliminate the vector potential from our expression, rather it will yield

$$|\Psi_1 + \Psi_2|^2 = |\psi_1|^2 + |\psi_2|^2 + e^{iq\oint\mathbf{A}(\mathbf{r}')\cdot d\ \mathbf{r}'}|\psi_1||\psi_2| + e^{-iq\oint\mathbf{A}(\mathbf{r}')\cdot d\mathbf{r}'}|\psi_2||\psi_1| \tag{11}$$

$$= |\psi_1|^2 + |\psi_2|^2 + 2\cos\left[q\oint\mathbf{A}(\mathbf{r}')\cdot d\mathbf{r}'\right] \tag{12}$$

which clearly has a dependence on $\mathbf{A}$. Utilizing equation 10 we find that

$$|\Psi_1 + \Psi_2|^2 = |\psi_1|^2 + |\psi_2|^2 + 2\cos\left(q\Phi\right). \tag{13}$$

It is now clear that due to the periodic nature of the cosine, as the magnetic flux changes the diffraction pattern oscillates between the expected double-slit diffraction pattern and a shifted diffraction pattern caused by the two incident waves having different phases.

Inserting equation 10 into equation 9 yields

$$\Psi_1 + \Psi_2 = e^{iq\int_{p_2}\mathbf{A}(\mathbf{r}')\cdot d\mathbf{r}'}\left(e^{iq\Phi}\psi_1 + \psi_2\right). \tag{14}$$

Now let $\Psi_{1+2}^{\Phi_I}$ be the added wave functions at the screen for a given magnetic flux Φ_I and let $\Psi_{1+2}^{\Phi_{II}}$ be the same but for flux Φ_{II}. Requiring that their difference be equal to zero, i.e. there is no change in the diffraction pattern between the two fluxes causes the following:

$$\Psi_{1+2}^{\Phi_I} - \Psi_{1+2}^{\Phi_{II}} = e^{iq\int_2\mathbf{A}(\mathbf{r}')\cdot d\mathbf{r}'}\left[\left(e^{iq\Phi_I}\psi_1 + \psi_2\right) - \left(e^{iq\Phi_{II}}\psi_1 + \psi_2\right)\right] = 0 \tag{15}$$

$$\left[\left(e^{iq\Phi_I}\psi_1 + \psi_2\right) - \left(e^{iq\Phi_{II}}\psi_1 + \psi_2\right)\right] = 0 \tag{16}$$

$$e^{iq\Phi_I}\psi_1 - e^{iq\Phi_{II}}\psi_1 = 0 \tag{17}$$

$$\therefore q(\Phi_I - \Phi_{II}) = 2n\pi \tag{18}$$

Equation 18 is the condition on the difference between two magnetic fluxes such that we recover the usual double-slit diffraction patter and the crux of the Aharonov-Bohm Effect. The magnetic vector potential, via the magnetic flux, does in fact have a physically observable effect where it was once assumed to be nothing more than a mathematical convenience. As we have just seen and will see again, the magnetic vector potential is anything but a simple mathematical artifact.

Now consider the Aharonov-Bohm Effect from a purely topological (specifically homotopical) point of view. Instead of a double-slit diffraction setup we now simply have the punctured Euclidean plane $M = \mathbb{R}^2 - \{0\}$, where the absence of the origin represents the impenetrable interior of the solenoid. It is easy to see that the fundamental group $\pi_1(M) \cong \mathbb{Z}$ since M is contractible to a circle S^1. The question now is what, if any, physical insight can be gleaned from this fact.

To answer that question we first must consider what exactly the fundamental group measures. By determining whether or not two paths with fixed points can be continuously deformed into one another, the fundamental group π_1 of some space M detects holes within a given space and assigns a technical group to that space based on how many holes it has. That group can be interpreted as the condition incumbent upon two fixed end-points such that two paths between those end points are deformable into one another. In this case, since $M \cong S^2$ and S^2 can be expressed as $e^{i\theta}$, where θ is the usual polar angle, such that $e^{i\theta} = e^{i(2n\pi+\theta)}$, for $n \in \mathbb{Z}$ begin to see just what π_1 is telling us. Given a function which we shall suggestively call ψ and another we shall call ψ', both defined on $M = \mathbb{R} - \{0\}$, we claim that the only way to deform ψ into ψ' is through the identification $\psi = e^{i2n\pi}\psi'$. What we are saying is that the only way two functions that enclose the deleted origin can be deformed into one another is if they are the same function, or more accurately, if their endpoints coincide. We are aware this seems trivial but topologically this is a fairly trivial case. Nevertheless, we have still acquired enough information to deduce the content of equation 18.

Comparing the identification $\psi = e^{i2n\pi}\psi'$ to equation 14 we can immediately conclude that $i2n\pi = iq\Phi$. Furthermore, for the difference of two different fluxes Φ_I and Φ_{II} and two different integers n and m we see the following:

$$q\Phi_I - q\Phi_{II} = 2n\pi - 2m\pi \qquad (n, m \in \mathbb{Z}) \tag{19}$$

$$q(\Phi_I - \Phi_{II}) = 2\pi(n - m) \tag{20}$$

$$q(\Phi_I - \Phi_{II}) = 2l\pi \tag{21}$$

which is exactly the flux-difference condition arrived at above and where $l = n - m$ and we used the fact that the difference of integers is still an integer.

It is fortunate in this case that the corresponding topology of the physical setup is relatively simple. Regardless of the computational complexities that may arise, we have just seen how the concept of the fundamental group can be used to find physically non-trivial situations that may appear physically trivial. In this case, the electron it is detecting the "hole" in the setup by virtue of the fact that the electron cannot penetrate into the solenoid. It is doing precisely the job of the fundamental group. Now since the fundamental group in this case is non-trivial, that should immediately raise questions as to whether or not the electron's wave-function is non-trivial as well since the wave function is bound by the same deformation restrictions as the sample functions analyzed by π_1.

1.3 Magnetic Monopoles

The issue of magnetic monopoles is an intrinsically topological matter. On a superficial level, one can see this simply by observing the rules for magnetic field lines. The statement that all magnetic field lines must have endpoints is both a statement that there are no magnetic monopoles (i.e. no sources or sinks of magnetic field lines, as such objects would involve force lines radiating unbounded to infinity) and a topological restriction (i.e. magnetic field lines are necessarily closed cycles). In fact, the very existence of a smooth vector potential $\mathbf{A}$ such that $\mathbf{B} = \nabla \times \mathbf{A}$ as is used in the explanation of the Aharonov-Bohm Effect and is crucially used throughout all electrodynamic calculations is predicated on the existence of very specific topological properties of the space in which the magnetic field exists. Indeed, these conditions will form an integral aspect of the proceeding work.

A simple calculation can show that the magnetic vector potential $\mathbf{A}$ can go from being a computational convenience to a very demanding *prima donna*. The calculation, done after the fashion of G. L. Naber[2], shows the problems that arise when one assumes the existence of a magnetic monopole. Those problems are connected to the magnetic vector potential $\mathbf{A}$ (as all things seem to be in Topological Electromagnetism) but originate from topological concerns of the space in question. To begin, assume the existence of a magnetic monopole located at the origin. The Maxwell equation of importance is clearly $\nabla \cdot \mathbf{B} = 0$. However, as we have assumed the existence of a magnetic charge

[2]Naber, G. L., *Topology, Geometry, and Gauge Fields: Foundations*, Springer Publishing, New York, NY (1997)

then surely $\nabla \cdot \mathbf{B} \neq 0$. Naber claims that

$$\mathbf{B} = \frac{m}{r^2}\hat{\mathbf{r}} \tag{22}$$

is a satisfactory expression for the the magnetic field $\mathbf{B}$ of a point magnetic charge, where m plays the role of the hypothetical magnetic charge. However, placing that charge at the origin demonstrably adjusts the topology of the monopole's domain. It is no longer in the familiar $\mathbb{R}^3$ but now in the *seemingly* familiar but topologically distinct domain of $\mathbb{R}^3 - \{0\}$. It can be shown that since the first homology group of the space $H_1(\mathbb{R}^3 - \{0\}) = 0$, i.e. the space is simply connected, then the existence of some scalar potential for $\mathbf{B}$ is assured, however one is traditionally concerned only with the vector potential $\mathbf{A}$. Let us now perform the aforementioned simple calculation. Take M to be the surface of a sphere enclosing the magnetic charge and let E, M_{top}, M_{bottom} be the sphere's equator, upper hemisphere, and lower hemisphere respectively. Consider the following:

$$\oint_M \nabla \times \mathbf{A} \cdot d\mathbf{S} = \oint_M \mathbf{B} \cdot d\mathbf{S} \tag{23}$$

$$= \oint_M \frac{m}{r^2}\hat{\mathbf{r}} \cdot d\mathbf{S} \tag{24}$$

$$= 4\pi m. \tag{25}$$

Now consider the same integral of the curl of $\mathbf{A}$ but utilize Stokes' Theorem instead:

$$\oint_M \nabla \times \mathbf{A} \cdot d\mathbf{S} = \int_{M_{top}} \nabla \times \mathbf{A} \cdot d\mathbf{S} + \int_{M_{bottom}} \nabla \times \mathbf{A} \cdot d\mathbf{S} \tag{26}$$

$$= \oint_E \mathbf{A} \cdot d\mathbf{r} + \oint_{-E} \mathbf{A} \cdot d\mathbf{r} \tag{27}$$

$$= \oint_E \mathbf{A} \cdot d\mathbf{r} - \oint_E \mathbf{A} \cdot d\mathbf{r} \tag{28}$$

$$= 0. \tag{29}$$

Obviously there is problem with assuming the existence of a well behaved vector potential $\mathbf{A}$.

The real problem behind all of this is the fact that the second homology group of the space $H_2(\mathbb{R}^3 - \{0\}) \neq 0$. More intuitively, there are spheres in $\mathbb{R}^3$ that cannot be collapsed to points (namely, sphere enclosing the deleted origin). It can be shown that for some vector field $\mathbf{V}$ on a domain D that is simply-connected *and* for which $H_2(D) = 0$, then there exists a smooth vector field $\mathbf{S}$ such that $\mathbf{V} = \nabla \times \mathbf{S}$. Without this condition, a *globally* smooth vector potential does not exist. Such a deficiency cuts right to the heart of almost all electromagnetic considerations.

1.4 Conformal Mapping

As a final and very brief comment on the geometry and topology underlying all of electromagnetism we point out that every electro- and magnetostatic problem, that is to say, problems involving static configurations of charges, magnets, conductors, etc. can be conformally mapped to computationally and geometrically simpler setups. The fact that Electromagnetism can be expressed so well and succinctly in terms of Complex Analysis means that it can take advantage of the marvelous geometric and topological properties of said field. The fact that it is rarely presented in such form is unfortunate.

CHAPTER 2

CLASSICAL ELECTROMAGNETIC TREATMENT OF THE ANGULAR MOMENTUM OF LIGHT

Wherein the classical electromagnetic theory of, and experiments regarding, the angular momentum of light are discussed.

2.1 Classical Derivations

To describe the classical mathematics of the angular momentum of an electromagnetic plane wave we begin with the classical momentum-density of an electromagnetic field in a vacuum

$$\frac{d\mathbf{P}}{dV} = \epsilon_0 \left(\mathbf{E} \times \mathbf{B} \right), \tag{30}$$

where dV represents an infinitesimal 3-volume, and apply the definition of angular momentum, or rather, angular momentum density,

$$\frac{d\mathbf{L}}{dV} = \mathbf{x} \times \frac{d\mathbf{P}}{dV}, \tag{31}$$

where $\mathbf{x} = \{x, y, z\}$ is the usual position vector, to yield

$$\frac{d\mathbf{L}}{dV} = \mathbf{x} \times \frac{d\mathbf{P}}{dV} \tag{32}$$

$$= \mathbf{x} \times \left[\epsilon_0 \left(\mathbf{E} \times \mathbf{B} \right) \right]. \tag{33}$$

Integrating over all space and recalling that the speed of light $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ the canonical expression of the angular momentum of a source-free electromagnetic field is seen to be

$$\mathbf{L} = \frac{1}{\mu_0 c^2} \int \mathbf{x} \times (\mathbf{E} \times \mathbf{B})\, d^3x. \tag{34}$$

It should be noticed that the Poynting vector $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$ makes an appearance from the very beginning. Such an appearance intuitively makes sense when discussing momentum given the Poynting vector's description of electromagnetic energy flux and the close association between energy and momentum, particularly in electromagnetism.

It is intriguing that, as it is expressed, the angular momentum vector $\mathbf{L}$ is manifestly orthogonal to the plane wave's direction of propagation, meaning that it should not be able to impart any angular momentum to object directly in its path of propagation. Consider the following plane wave solutions to Maxwell's equations,

$$\mathbf{E} = \mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{x} - i\omega t} \tag{35}$$

$$\mathbf{B} = \mathbf{B}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{x} - i\omega t}, \tag{36}$$

where $\hat{\mathbf{k}}$ is a unit vector pointing in the plane wave's direction of propagation. Since the plane wave is in a source-free regime, $\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0$ and thus

$$\nabla \cdot \mathbf{E} = i\hat{\mathbf{k}} \cdot \left\{ \mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{x} - i\omega t} \right\} = 0 \tag{37}$$

$$= \hat{\mathbf{k}} \cdot \mathbf{E} = 0 \tag{38}$$

$$\therefore \hat{\mathbf{k}} \perp \mathbf{E}. \tag{39}$$

Clearly the same calculation holds for $\mathbf{B}$ since we have no magnetic monopoles present and thus $\hat{\mathbf{k}}$ as well. Now since the Poynting vector $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$ is orthogonal to both $\mathbf{E}$ and $\mathbf{B}$ as well, $\mathbf{S}$ is clearly the direction of propagation $\hat{\mathbf{k}}$. Within the above definition of the angular momentum vector $\mathbf{L}$, the Poynting vector $\mathbf{S}$ is crossed with the position vector $\mathbf{x}$ and so $\mathbf{L}$ *must* be in a direction orthogonal to the plane wave's direction of propagation.

It is possible, through further manipulations of equation 34, to expand the integrand of the angular momentum $\mathbf{L}$ into two terms as follows,

$$\mathbf{x} \times (\mathbf{E} \times \mathbf{B}) \rightarrow \left[\mathbf{E} \times \mathbf{A} + \sum_{j=1}^{3} E_j \left(\mathbf{x} \times \nabla\right) A_j\right]. \tag{40}$$

The first and second terms on the right side of the arrow above have been interpreted[3] as expressions for the spin angular momentum and the orbital angular momentum, respectively. Specifically, the second term has been viewed as a form of orbital angular momentum due to its similarity (sans a factor of i) the orbital angular momentum operator

$$\mathbf{L}_O = -i\left(\mathbf{x} \times \nabla\right). \tag{41}$$

At this point, it would appear as if such manipulations are ex post facto considerations. Unless one is specifically trying to match the initial form of $\mathbf{L}$ to what one knows from the quantum mechanics of the angular momentum of a photon, there is no driving reason, other than sheer curiosity and free time, to go through the effort of converting equation 34.

Why go to the trouble then? Quite simply, Nature forced us. That is to

[3]Jackson, J. D., *Classical Electrodynamics*, 3rd edition, John Wiley and Sons, Inc., Danvers, MA (1999), p. 350

say, even though equation 34 has no components in the plane wave's direction of propagation, experiments have been done which result in a transfer of angular momentum to a thin disk upon which the plane wave is incident. An explanation had to be found. Indeed this sounds very much like the mystery that surrounded Aharonov-Bohm Effect and thus ripe for topological analysis. The Lorentz force law

$$\mathbf{F} = q\left(\mathbf{E} + [\mathbf{v} \times \mathbf{B}]\right) \tag{42}$$

categorically denied the possibility that the electron could be demonstrably affected. That assumption lasted until the fundamental nature of the vector potential $\mathbf{A}$ became clear in Quantum Mechanics. Oddly enough, it takes far less work to rewrite equation 42 in a more suggestive form, that is to say, a form involving the vector potential $\mathbf{A}$, than it does for to do the same for equation 34. A trivial implementation of $\mathbf{B} = \nabla \times \mathbf{A}$ shows that

$$\mathbf{F} = q\left(\mathbf{E} + [\mathbf{v} \times \{\nabla \times \mathbf{A}\}]\right). \tag{43}$$

Of course that is only useful due to the guaranteed of the global existence of smooth vector potential $\mathbf{A}$ since the first homotopy group and thus the first homology group are both zero.

The first and principal experiment that was sensitive enough to detect the energy transfer is now detailed.

2.2 The Experiment of R. A. Beth

The angular deflection of a wave due to an impinging beam of light was first considered by John H. Poynting[4] in 1909. At the time however he dismissed the deflection as being too small to ever be able to measure. It was not until the 1930s that Richard A. Beth[5] actually designed and executed an experiment which could detect the deflection predicted by Poynting. We now briefly describe his experiment in order to get a physical sense of what is actually happening before we apply topological considerations.

Figure 2 below reproduces the schematic diagram of R. A. Beth's experimental setup, from his paper cited above, designed to detect the angular momentum of of light by observing the angular deflection of a quartz wave plate suspended by a fine quartz filament. Light from a tungsten filament F (all italicized capital letters refer to identification in the two figures 2 and 3 below) was passed through a quartz lens L that focused it through a Nicol prism N which further focused the light onto the fine wave plate setup shown in figure 3 (also reproduced from Beth's paper).

J. Poynting was not exaggerating when he felt that the angular deflection caused by light's angular momentum was too small to ever measure. While his statement was ultimately proven to be incorrect, it was only done so through considerable ingenuity on Beth's part. His wave plate design quadrupled the possible total angular momentum incident upon the wave plate M shown in figure 3.

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[4]Poynting, J. H., 1909, *Proc. Roy. Soc.* A **82** 560

[5]Beth, R. A., 1936, *Phys. Rev.* **50** 115

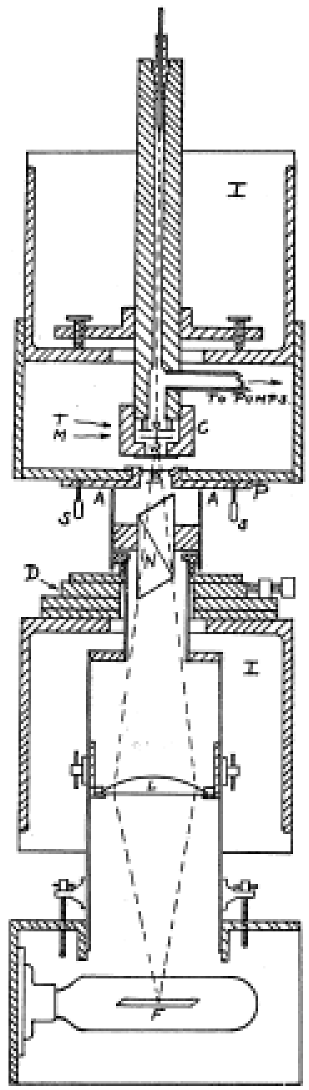

Figure 2: R. A. Beth's Experiment Schematic

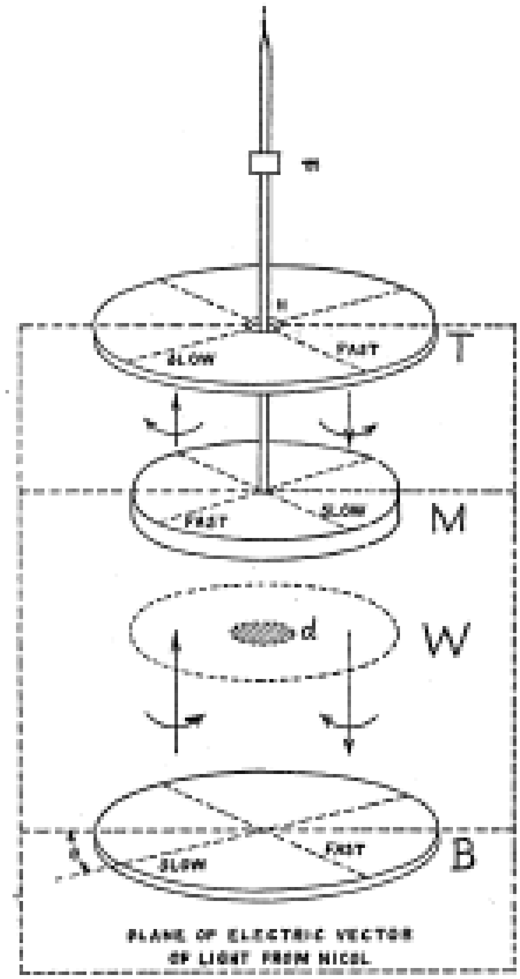

Figure 3: wave plate setup from R. A. Beth's Experiment

Light entering from the bottom of figure 3 encounters the quarter wave plate B, causing the resultant light to become right-circularly polarized. Passing through wave plate M, the resulting angular deflection of which is to be measured, it is then incident upon wave plate T. The light is then reflected back by an aluminum coating on the backside of T to once again be incident upon M, thereby delivering further angular momentum. Wave plate M is itself another quarter plate so that only right-circularly polarized light is ever incident upon it (since one must take into account the reflection at T) and the fact that only left-circularly polarized light is ever emitted from wave plate M. The result of Beth's setup is that the angular momentum delivered to the plate per second is four times what it could be if circularly polarized light was merely incident upon plate M once.

2.3 Poynting's Considerations

A surprisingly elementary calculation done by Beth (*loc. cit.*) shows the expected angular momentum delivered per second per area by right-circularly polarized light. From Beth, the number of right-circularly polarized photons per second per area, called the Poynting energy flow P_E is given by

$$P_E = \frac{cR^2}{4\pi}, \tag{44}$$

where R is the amplitude of the right-circularly polarized component of a polarized plane wave. To get a pure energy flux per time we then divide P_E by the energy of a photon, $2\pi h\frac{c}{\lambda}$, yielding

$$\frac{P_E}{2\pi h\frac{c}{\lambda}} = \frac{\frac{cR^2}{4\pi}}{2\pi h\frac{c}{\lambda}}. \tag{45}$$

Multiplying by $\hbar$ results in the angular momentum delivered per second per area by right-circularly polarized light to be

$$\frac{dL_R}{dAdt} = \frac{\lambda R^2}{8\pi^2}. \tag{46}$$

For an average wavelength of monochromatic red light $\lambda_{red} = 685\text{nm}$,

$$\frac{dL_R}{dAdt} = \frac{\lambda R^2}{8\pi^2} = (867\text{nm})R^2. \tag{47}$$

It is thus no small wonder that Poynting did not believe such a value could ever be measured experimentally. Yet regardless of how straightforward the above calculation appears, the canonical field definition of the angular momentum of light suggests that because $\mathbf{L} \cdot \hat{\mathbf{k}} = 0$ (cf. section 1.0) no angular momentum can be imparted to an object directly in front of a propagating plane wave.

CHAPTER 3

THE APPLICATION OF DIFFERENTIAL FORMS

Wherein the canonical expression for the angular momentum of light is converted from its standard Vector Calculus notation into the language of Differential Forms.

3.1 Definitions

We shall begin with the standard definition of a p-form defined on a manifold M. The following definitions will closely follow the presentations of M. Spivak[6] and G. Bredon[7].

Consider the space of alternating multilinear p-forms of an n-dimensional vector space V over $\mathbb{R}$. Call that space $\Lambda^p(\mathrm{M})$ and its elements ω. The fact that the elements ω are multilinear suggests that there is a connection to tensors, which are themselves multilinear functions. Moving on for the moment, define a p-form ω to take elements of the vector space V, specifically p-tuples of vectors $\langle v_1, v_2, ..., v_p\rangle$ with $v_i \in V$ and for $i \in \{1, ..., p\}$, and return a real number, i.e.

$$\omega\left(v_1, v_2, ..., v_p\right) \in \mathbb{R}. \tag{48}$$

By alternating it is meant that $\omega\left(v_1, v_2, ..., v_k, ..., v_l, ...v_p\right) = 0$ if $v_k = v_l$ for $l \neq k$. It can be shown that for ω alternating, ω is skew-symmetric. That is

[6] Spivak, M., *A Comprehensive Introduction to Differential Geometry, Volume 1*, 3rd edition, Publish or Perish, Inc., Houston, TX (2005)

[7] Bredon, G. E., *Topology and Geometry*, Springer Publishing, New York, NY (1993)

to say

$$\omega\left(v_{\sigma_1}, v_{\sigma_2}, ..., v_{\sigma_p}\right) = \operatorname{sgn}(\sigma)\,\omega\left(v_1, v_2, ..., v_p\right) \tag{49}$$

for σ-permutations of i. With what is known about the general definition of tensors, a most intriguing connection can be made, as is made by B. Felsager[8], namely that p-forms are in fact the skew-symmetric duals of tensors, or rather are skew-symmetric *cotensors.*

It is often the case that a dual form of a theory carries with it many benefits that the non-dual form does not. In this case, the dual presentation (i.e. the differential form presentation) of electromagnetism is computationally and intuitively simpler and more clear. Before this can be seen however, it is required to define the binary operation between forms of different orders p and q called the *exterior* or *wedge* product.

The wedge product $\wedge : \Lambda^p \times \Lambda^q \to \Lambda^{p+q}$ of a p-form ω^p and a q-form η^q is canonically defined as follows.

$$\omega^p\left(v_1, ..., v_p\right) \wedge \eta^q\left(v_1, ..., v_q\right) = \omega \wedge \eta\left(v_1, ..., v_p, ..., v_q, ...v_{p+q}\right) \tag{50}$$

$$= \sum_\sigma \operatorname{sgn}(\sigma)\,\omega\left(v_1, ..., v_p\right)\eta\left(v_{p+1}, ..., v_{p+q}\right). \tag{51}$$

The above definition is admittedly somewhat abstract, however a most illustrative demonstration of the inherent meaning of equation 51 can be seen in the wedge product of a k-number of 1-forms

$$\omega_1\wedge, ..., \wedge\omega_k\left(v_1, ..., v_k\right) = \det\left[\omega_i\left(v_j\right)\right], \tag{52}$$

[8]Felsager, B., *Geometry, Particles, and Fields*, Springer Publishing, New York, NY (1998)

which is manifestly the definition of the determinant. The following is a list of the algebraic properties of the wedge product:

- Distributivity: $\omega \wedge (\eta + \alpha) = \omega \wedge \eta + \omega \wedge \alpha$
- Associativity: $(\omega \wedge \eta) \wedge \alpha = \omega \wedge (\eta \wedge \alpha)$
- Commutativity: $\omega \wedge \eta = (-1)^{pq} (\eta \wedge \omega)$, where p and q are the order of the forms ω and η, respectively.

We now introduce the concept of the *differential* or *exterior derivative* of a p-form, or more precisely, a differential p+1-form. First off, we have the *exterior derivative map d*

$$d : \Lambda^p \to \Lambda^{p+1}.$$

Next, given a basis of differentials $dx_1, ..., dx_n$, such as is commonly chosen in General Relativity, a p-form ω can be expressed as follows.

$$\omega = \sum_i f_{i_1,\ldots,i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}, \tag{53}$$

where $f_{i_1,\ldots,i_p}$ are smooth functions. From the above we now define $d\omega$.

$$d\omega = \sum_i df_{i_1,\ldots,i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}, \tag{54}$$

where

$$df_{i_1,\ldots,i_p} = \sum_i \left(\frac{\partial f}{\partial x_i} \right) dx_i. \tag{55}$$

By definition it is now clear that $dd\omega = 0$ for any p-form ω. In addition, the exterior derivative is distributive and obeys the chain rule, after a fashion:

- Distributivity: $d(\omega + \eta) = d\omega + d\eta$

- Chain Rule: $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p (\omega \wedge d\eta)$, where p is again the order of ω.

Furthermore, the notions of *exact* forms and *closed* forms can now be defined:

- A p-form ω is said to be closed if its exterior derivative is zero, i.e. $d\omega = 0$.

- A p-form ω is said to be exact if it can be expressed as the exterior derivative of some (p-1)-form η. That is, $\omega = d\eta$ for some (p-1)-form η.

Clearly all exact forms are closed since given an exact p-form ω the following is self-evident:

$$d\omega = d(d\eta) = 0. \tag{56}$$

Finally, we introduce the Hodge star operator $*$.

$$* : \Lambda^p \to \Lambda^{n-p}$$

where n is the dimension of our manifold M. Given a p-form $\omega = \sum_i f_{i_1,\ldots,i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ we define $*$ in the following manner.

$$* \, \omega = \operatorname{sgn}(i) \sum_i f_{i_1,\ldots,i_p} dx_{i_{p+1}} \wedge \cdots \wedge dx_{i_n} \tag{57}$$

for cyclic permutations of i_j with $j \in \{1, ..., p, ..., n\}$. The Hodge star operator allows for a proper analogous definition of the cross product operation from linear algebra. While the wedge product can mimic the cross product, it can only do so in special cases. That is to say, in $\mathbb{R}^3$ the cross product of any number of vectors is again a vector. Such is not the case with the wedge product of any number of 1- or 2-forms. This is due to the fact that the wedge product always results in a new form of order equivalent to the sum of the orders of the forms involved. Consequently, one cannot indefinitely wedge 1-forms or 2-forms and always yield a 1- or 2-form. However, with the combined use of Hodge star and the wedge product, the effect of the cross product can be realized in the language of exterior forms by the following formula.

$$* (\mu \wedge \nu) = \mathbf{u} \times \mathbf{v}, \tag{58}$$

where μ and ν are either 1- or 2-forms and $\mathbf{u}$ and $\mathbf{v}$ are their corresponding vectors, respectively.

The space of 1-forms Λ^1 and the space of 2-forms Λ^3 are both isomorphic to $\mathbb{R}^3$ and so it is most convenient that elements of those spaces can be used to represent vector fields. In addition, elements of the space of 0-forms Λ^0 and the space of 3-forms Λ^3 are just smooth functions. Observe the representative elements of the aforementioned spaces of forms below.

- $\Lambda^0 : f(x, y, z)$
- $\Lambda^1 : f(x, y, z)\, dx + g(x, y, z)\, dy + h(x, y, z)\, dz$
- $\Lambda^2 : p(x, y, z)\, dy \wedge dz + q(x, y, z)\, dz \wedge dx + r(x, y, z)\, dx \wedge dy$
- $\Lambda^3 : t(x, y, z)\, dx \wedge dy \wedge dz$

The language of Exterior Calculus can now replicate the operations of linear algebra and vector calculus. This is thanks to the above definitions of the exterior derivative d, the wedge product $\wedge$, the Hodge star $*$, and the fact that elements of Λ^1 and Λ^2 mimic vector fields while Λ^0 and Λ^3 are just smooth functions.

3.2 Application

In order to convert equation 34, the expression for the angular momentum $\mathbf{L}$, into the language of differential forms, we must first consider each part of the integrand and analyze them with the tools and definitions that have been developed regarding forms. To see how electromagnetism provides a natural environment for differential forms and to discover the appropriate p-forms that represent the electric field $\mathbf{E} = \langle E_i\,(x, y, z, t)\,, E_j\,(x, y, z, t)\,, E_k\,(x, y, z, t)\rangle$ and the magnetic field $\mathbf{B} = \langle B_i\,(x, y, z, t)\,, B_j\,(x, y, z, t)\,, B_k\,(x, y, z, t)\rangle$, consider the integral forms of two of Maxwell's equations, all the while keeping the forms of elements of the four spaces of forms itemized above in mind. Beginning with Faraday's law of induction, where S is a surface bounded by ∂S, Φ_B and Φ_E are the magnetic and electric fluxes through S, respectively, and $d\mathbf{l} =$

$dx\hat{i} + dy\hat{j} + dz\hat{k}$ is the canonical Euclidean line element.

$$-\frac{\partial \Phi_B}{\partial t} = \oint_{\partial S} \mathbf{E} \cdot d\mathbf{l} \tag{59}$$

$$= \oint_{\partial S} \langle E_i, E_j, E_k \rangle \cdot \langle dx, dy, dz \rangle \tag{60}$$

$$= \oint_{\partial S} E_i dx + E_j dy + E_k dz \tag{61}$$

$$= \oint_{\partial S} \Sigma. \tag{62}$$

E_i, E_j, E_k are all functions of the coordinates x, y, and z and time t and where $\Sigma = E_i dx + E_j dy + E_k dz$. Clearly $\Sigma \in \Lambda^1$ and thus the Electric field 1-form Σ is found right under out noses.

An identical analysis to find the magnetic field 1-forms can be done with the integral expression of Ampere's Law in free space, with the same definitions for S, d$\mathbf{l}$,Φ_B, and Φ_E as above.

$$\mu_0 \epsilon_0 \frac{\partial \Phi_E}{\partial t} = \oint_{\partial S} \mathbf{B} \cdot d\mathbf{l} \tag{63}$$

$$= \oint_{\partial S} \langle B_i, B_j, B_k \rangle \cdot \langle dx, dy, dz \rangle \tag{64}$$

$$= \oint_{\partial S} B_i dx + B_j dy + B_k dz \tag{65}$$

$$= \oint_{\partial S} \beta, \tag{66}$$

with B_i, B_j, B_j all functions of space and time too, and where $\beta = B_i dx + B_j dy + B_k dz$ is also clearly a member of Λ^1. The magnetic field 1-form has now been determined.

Finally, the 1-form corresponding to the position vector

$\mathbf{x} = \left\langle x\hat{i}, y\hat{j}, z\hat{k} \right\rangle$ is needed and by analogy to Σ and β we claim that

$$\rho = xdx + ydy + zdz \tag{67}$$

is appropriate.

The Poynting vector $\mu_0 \mathbf{S} = \mathbf{E} \times \mathbf{B}$ is now explicitly computed, via the definitions in the previous section, by taking the Hodge star of the wedge product of Σ and β,

$$
\begin{aligned}
*(\Sigma \wedge \beta) &= *(E_i dx + E_j dy + E_k dz) \wedge (B_i dx + B_j dy + B_k dz) && (68)\\
&= *(E_i B_i \underbrace{dx \wedge dx}_{=0} + E_i B_j dx \wedge dy + E_i B_k dx \wedge dz + E_j B_i dy \wedge dx + \ldots\\
&\ldots + E_j B_j \underbrace{dy \wedge dy}_{=0} + E_j B_k dy \wedge dz + E_k B_i dz \wedge dx + \ldots\\
&\ldots + E_k B_j dz \wedge dy + E_k B_k \underbrace{dz \wedge dz}_{=0}) && (69)\\
&= *(E_i B_j dx \wedge dy + E_i B_k \underbrace{dx \wedge dz}_{-dz \wedge dx} + E_j B_i \underbrace{dy \wedge dx}_{=-dx \wedge dy} + \ldots\\
&\ldots + E_j B_k dy \wedge dz + E_k B_i dz \wedge dx + E_k B_j \underbrace{dz \wedge dy}_{=-dy \wedge dz}) && (70)\\
&= *((E_i B_j - E_j B_i)\, dx \wedge dy + (E_k B_i - E_i B_k)\, dz \wedge dx + \ldots\\
&\ldots + (E_j B_k - E_k B_j)\, dy \wedge dz) && (71)\\
&= *(\tau_1 dx \wedge dy + \tau_2 dz \wedge dx + \tau_3 dy \wedge dz) && (72)\\
&= (\tau_1 dz + \tau_2 dy + \tau_3 dx) = \Xi. && (73)
\end{aligned}
$$

Note the use of the fact that $d\mu \wedge d\nu = 0$ for $\mu = \nu$ (i.e. the *alternating* property of forms) and that $d\mu \wedge d\nu = -d\nu \wedge d\nu$ for anti-cyclic permutations of μ and ν (i.e. the *skew-symmetric* property of forms) to reduce the above equations. In addition, observe the following identifications

$$*(\Sigma \wedge \beta) \rightarrow \Xi$$

$$(E_i B_j - E_j B_i) \rightarrow \tau_1$$

$$(E_k B_i - E_i B_k) \rightarrow \tau_2$$

$$(E_j B_k - E_k B_j) \rightarrow \tau_3$$

and thus created the Poynting energy-flux 1-form Γ.

Next wedge Ξ with ρ and take the Hodge star of that, since the cross product between $\mathbf{x}$ and $\mathbf{S}$ is being replicated.

$$*(\rho \wedge \Xi) = *((xdx + ydy + zdz) \wedge (\tau_1 dz + \tau_2 dy + \tau_3 dx)) \tag{74}$$

$$= *((y\tau_1 - z\tau_2)dy \wedge dz + (z\tau_3 - x\tau_1)dz \wedge dx + (z\tau_2 - y\tau_3)dx \wedge dy) \tag{75}$$

$$= (y\tau_1 - z\tau_2)dx + (z\tau_3 - x\tau_1)dy + (z\tau_2 - y\tau_3)dz = \lambda. \tag{76}$$

We have now created the angular momentum-density 1-form $\lambda = *(\rho \wedge * (\Sigma \wedge \beta))$. As is desired, with the use of the Hodge star and the wedge product, the integrand is still a 1-form, i.e. a vector field. However, integration of such an object will result in an element of $\mathbb{R}$ and not another 1- or 2- form. It is at this point that we must diverge from a strict analogy to the physical definition of the angular momentum of light $\mathbf{L}$. As it stands, $\mathbf{L}$ is a vector, but by converting to the underlying language of exterior forms, a "dimensional" problem occurs. Having started from the ground up by performing the double cross product

$\mathbf{x} \times (\mathbf{E} \times \mathbf{B})$ we shall take our analysis thus far as valid and the analogous physical definition of $\mathbf{L}$ as flawed on some level. The resulting *exterior form* definition of $\mathbf{L}$ has now been computed.

$$\Gamma = \frac{1}{c^2\mu_0}\int_{\partial\gamma} *(\rho \wedge *(\Sigma \wedge \beta)) \tag{77}$$

$$= \frac{1}{c^2\mu_0}\int_{\partial\gamma} (\tau_1 dz + \tau_2 dy + \tau_3 dx) \tag{78}$$

$$= \frac{1}{c^2\mu_0}\int_{\partial\gamma} \lambda, \tag{79}$$

where $\partial\gamma$ is the boundary of some surface γ through which the plane wave passes. We are now in a prime position to analyze the angular momentum of light from a geometric and topological point of view.

CHAPTER 4

ANALYSES

Wherein the geometric and topological considerations of the angular momentum of light, as it has been reformulated in the geometric language of differential p-forms in the previous chapter, are analyzed.

4.1 Geometric Analysis

To begin, the problem is that the angular momentum vector $\mathbf{L}$ has no vector components in the direction of propagation and so is manifestly incapable of imparting any angular momentum unto an object that lies directly in (i.e. transverse to) a plane wave's direction of propagation. However, as is stated in relation 2.11, it is possible, through manipulation of vector identities and the fact that the wave propagates in free space devoid of charges (either electric or magnetic), to break the expression for $\mathbf{L}$ into a spin angular momentum part, $\mathbf{L}_{spin} = \frac{1}{c^2\mu_0}\int \mathbf{E}\times\mathbf{A}d^3x$, and an orbital angular momentum part, $\mathbf{L}_{orbital} = \frac{1}{c^2\mu_0}\int\sum_{j=1}^{3} E_j\left(\mathbf{x}\times\nabla\right)A_j d^3x$, so called because of the appearance of a term which bears a close resemblance to the orbital angular momentum operator $\hat{\mathbf{L}}_O = -i\left(\mathbf{x}\times\nabla\right)$, resulting in the following.

$$\begin{aligned}\mathbf{L} &= \frac{1}{c^2\mu_0}\int \mathbf{x}\times\left[\mathbf{E}\times\mathbf{B}\right]d^3x \\ &= \frac{1}{c^2\mu_0}\int \mathbf{x}\times\left[\mathbf{E}\times\left\{\nabla\times\mathbf{A}\right\}\right]d^3x \\ &= \frac{1}{c^2\mu_0}\int \mathbf{E}\times\mathbf{A}d^3x + \frac{1}{c^2\mu_0}\int\sum_{j=1}^{3}E_j\left(\mathbf{x}\times\nabla\right)A_j d^3x. \end{aligned} \tag{80}$$

At a glance, this would appear to provide a quasi-quantum mechanical solution to the problem by acknowledging the fact that a photon, which is what the plane wave represents, has an intrinsic spin and that it is this spin which is the source of the imparted angular momentum. An attempt to expand equation 79 to find the differential form analog of equation 80 will yield some startling results.

The first step in finding a comparison with equation 80 is claiming that the magnetic field 1-form β is in fact an exact form, i.e. $\beta = d\alpha$ for some form α. But of what order form should α be? In order for α to be representative of the magnetic vector potential $\mathbf{A}$ it must be at least a 1-form initially since only 1- and 2-forms can represent vector fields. In addition, after applying the exterior derivative to α we must recover either a 1-from or a 2-form since the curl of $\mathbf{A}$ is the magnetic field *vector*. As defined earlier, the exterior derivative results in a form of order $p + 1$ compared to the form upon which it acted. Consequently, the magnetic field 1-form β must be promoted to a 2-form β' such that $\beta' = d\alpha$ works within the rules of the exterior calculus and matches the physical significance from the vector calculus representation of electromagnetism and where $\alpha = A_i dx + A_j dy + A_k dz$ is the magnetic vector potential 1-form. Thus

$$\beta' = B_i dy \wedge dz + B_j dz \wedge dx + B_k dx \wedge dy \tag{81}$$

and

$$d\alpha = \left(\frac{\partial A_j}{\partial x} - \frac{\partial A_i}{\partial y}\right) dx \wedge dy + \dots$$
$$\dots + \left(\frac{\partial A_i}{\partial z} - \frac{\partial A_k}{\partial x}\right) dz \wedge dx + \left(\frac{\partial A_k}{\partial y} - \frac{\partial A_j}{\partial z}\right) dy \wedge dz. \tag{82}$$

With $d\alpha$ defined we can now supplant β with it in equation 79 to get

$$\Gamma = \frac{1}{c^2\mu_0} \int_{\partial\Omega} *(\rho \wedge *\,(\Sigma \wedge d\alpha)), \tag{83}$$

where $\partial\Omega$ is some closed surface through which the plane wave passes. In particular, we consider a cylindrical pill-box intersecting a wave plate absorber, like the ones in figure 3, in a transverse manner such that Ω is the three-dimensional space contained therein, $\partial\Omega$ is the bounding surface of that space, γ is a face of the pill-box and $\partial\gamma$ is the boundary of such a face. Expanding the above integrand we see the following.

$$*\,(\rho \wedge *\ \underbrace{(\Sigma \wedge d\alpha)}_{[d\Sigma \wedge \alpha - d(\Sigma \wedge \alpha)]}\ ) \tag{84}$$

$$= *(\rho \wedge *\,(d\Sigma \wedge \alpha - d(\Sigma \wedge \alpha))) \tag{85}$$

$$= *(\rho \wedge (*(d\Sigma \wedge \alpha) - *(d(\Sigma \wedge \alpha))) \tag{86}$$

$$= *(\rho \wedge *(d\Sigma \wedge \alpha)) - *(\rho \wedge *d(\Sigma \wedge \alpha)). \tag{87}$$

Reinserting equation 87 into equation 83 yields

$$\Gamma = \frac{1}{c^2\mu_0}\int_{\partial\Omega} *(\rho\wedge *(\Sigma\wedge d\alpha)) \tag{88}$$
$$= \frac{1}{c^2\mu_0}\int_{\partial\Omega} [*(\rho\wedge *(d\Sigma\wedge \alpha)) - *(\rho\wedge *d(\Sigma\wedge\alpha))] \tag{89}$$
$$= \frac{1}{c^2\mu_0}\int_{\partial\Omega} *(\rho\wedge *(d\Sigma\wedge \alpha)) - \frac{1}{c^2\mu_0}\int_{\partial\Omega} *(\rho\wedge *d(\Sigma\wedge\alpha)) \tag{90}$$

and comparing this to equation 80 shows that:

$$\mathbf{E}\times\mathbf{A}d^3x \rightarrow *(\rho\wedge *d(\Sigma\wedge\alpha))$$
$$\sum_{j=1}^{3} E_j\left(\mathbf{x}\times\nabla\right)A_j d^3x \rightarrow *(\rho\wedge *(d\Sigma\wedge\alpha)).$$

With the above identifications the total spin angular momentum form, which we shall call λ_{spin}, is

$$\lambda_{spin} = -\frac{1}{c^2\mu_0}\int_{\partial\Omega} *(\rho\wedge *d(\Sigma\wedge\alpha)). \tag{91}$$

λ_{spin} is not a 1-form or a 2-form, i.e. it does not correspond to a vector-valued angular momentum $\mathbf{L}_{spin}$. Clearly, vector tricks aside, the *classical* field approach to the angular momentum of light is inherently flawed. By expressing it in terms of the underlying geometry, accomplished by translating it into the language of differential forms, this flaw is immediately recognized as a poorly defined vector-valued integral.

Further efforts to make sense of this involve taking a time-average of the Poynting vector $\langle\mathbf{S}\rangle_T = \mathrm{Re}\left\{\mathbf{E}\times\mathbf{B}^*\right\}$. Before embarking upon that analysis, a brief excursion to the realm of complex-valued differential forms is necessary.

The general rules of differential forms, including the wedge product, Hodge star operator, and exterior derivative outlined in the previous chapter are still adhered to, just with the three following addenda. We define a complex differential form as follows:

$$\omega = \alpha + i\beta \tag{92}$$

such that $\omega^* = \alpha - i\beta$ and $\omega \in \mathbb{R}$ if $\omega = \omega^*$. In addition, the wedge product and the exterior derivative act as one might expect. Let $\omega = \alpha + i\beta$ and $\eta = \mu + i\nu$ be complex differential forms. Then

$$\omega \wedge \eta = (\alpha + i\beta) \wedge (\mu + i\nu) \tag{93}$$

$$= (\alpha \wedge \mu - \beta \wedge \nu) + i\,(\alpha \wedge \nu + \beta \wedge \mu) \tag{94}$$

and

$$d\omega = d\,(\alpha + i\beta) = d\alpha + i d\beta. \tag{95}$$

All the other algebraic properties of the wedge product and the exterior derivative still apply in addition to the above.

Consider now the time-average of the Poynting vector $\langle \mathbf{S} \rangle_T = \mathrm{Re}\,\{\mathbf{E} \times \mathbf{B}^*\}$ with:

$$\mathbf{E} \to \Sigma$$

$$\mathbf{B} \to \beta' = d\alpha$$

as before except now allow for complex values of the fields and potentials.

Thus:

$$\Sigma = \sigma + i\theta \tag{96}$$

$$\alpha = \Delta + i\epsilon. \tag{97}$$

Now we compute

$$\mathbf{E} \times \mathbf{B}^* = \mathbf{E} \times (\nabla \times \mathbf{A})^* \tag{98}$$

$$\downarrow \tag{99}$$

$$*(\Sigma \wedge (\beta')^*) = *(\Sigma \wedge (d\alpha)^*) \tag{100}$$

$$= *((\sigma + i\theta) \wedge (d\Delta + id\epsilon)^*) \tag{101}$$

$$= *((\sigma + i\theta) \wedge (d\Delta - id\epsilon)) \tag{102}$$

$$= *((\sigma \wedge d\Delta + \theta \wedge d\epsilon) - (\sigma \wedge id\epsilon + i\theta \wedge d\Delta)) \tag{103}$$

$$= *((\sigma \wedge d\Delta + \theta \wedge d\epsilon) - i\,(\sigma \wedge d\epsilon + \theta \wedge d\Delta)). \tag{104}$$

Ultimately however, only the real part of equation 104 is required,

$$\mathrm{Re}\,\{*((\sigma \wedge d\Delta + \theta \wedge d\epsilon) - i\,(\sigma \wedge d\epsilon + \theta \wedge d\Delta))\} = *\,(\sigma \wedge d\Delta + \theta \wedge d\epsilon)\,. \tag{105}$$

$*(\rho \wedge *(\mathrm{Re}\{\Sigma \wedge \beta^*\}))$ can now be calculated,

$$
\begin{aligned}
*(\rho \wedge *(\mathrm{Re}\{\Sigma \wedge \beta^*\})) &= *(\rho \wedge *(\mathrm{Re}\{\Sigma \wedge (d\alpha)^*\})) && (106)\\
&= *(\rho \wedge *(\sigma \wedge d\Delta + \theta \wedge d\epsilon)) && (107)\\
&= *(\rho \wedge * \underbrace{(\sigma \wedge d\Delta)}_{[=d(\sigma\wedge\Delta)-d\sigma\wedge\Delta]}) + *(\rho \wedge *(\theta \wedge d\epsilon)) && (108)\\
&= *(\rho \wedge *((\sigma \wedge \Delta) - (\underbrace{d\sigma}_{[=0]} \wedge\Delta))) + *(\rho \wedge *(\theta \wedge d\epsilon)) && (109)\\
&= *(\rho \wedge * \underbrace{(\sigma \wedge \Delta)}_{[=\mathrm{Re}\{\Sigma\}\wedge\mathrm{Re}\{\alpha\}]}) + *(\rho \wedge *(\theta \wedge d\epsilon)) && (110)
\end{aligned}
$$

By the presence of a term which is equivalent to $\mathrm{Re}\{\Sigma\} \wedge \mathrm{Re}\{\alpha\}$, it is reasonable to assign λ_{spin} to the following:

$$\lambda_{spin} = \frac{1}{c^2\mu_0}\int_{\partial\gamma} *(\rho \wedge *(\sigma \wedge \Delta)). \tag{111}$$

The final step in the analysis is impose the condition of circularly polarized light. Specifically, we have $\mathrm{Re}\{\Sigma\} = i\,\mathrm{Im}\{\alpha\}$ and $\mathrm{Im}\{\Sigma\} = \mathrm{Re}\{\alpha\}$, resulting in the following:

$$
\begin{aligned}
*(\Sigma \wedge (d\alpha)^*) &= *((-\epsilon + i\Delta) \wedge (d\Delta - id\epsilon)) && (112)\\
&= *(-\epsilon \wedge d\Delta + \Delta \wedge d\epsilon) + *i(\Delta \wedge d\Delta + \Delta \wedge d\epsilon) && (113)\\
&= *(\underbrace{-\epsilon \wedge d\Delta + \Delta \wedge d\epsilon}_{[=d(\epsilon\wedge\Delta)]}) + *i((\epsilon \wedge d\epsilon + \Delta \wedge\ d\Delta)) && (114)\\
&= *d(\epsilon \wedge \Delta) + *i((\epsilon \wedge d\epsilon + \Delta \wedge\ d\Delta)). && (115)
\end{aligned}
$$

Inserting the term in the underbrace in equation 114 into equation 106 and taking the hodge star of its wedge product with ρ yields

$$* \left(\rho \wedge *\mathrm{Re}\left\{\Sigma \wedge (d\alpha)^*\right\} = *(\rho \wedge *d(\epsilon \wedge \Delta)). \tag{116}$$

Equation 116 above now leads to a new version of the total spin angular momentum form

$$\begin{aligned} \lambda'_{spin} &= \frac{1}{c^2\mu_0}\int_{\partial\Omega} *(\rho \wedge *d(\epsilon \wedge \Delta)) \\ &= \frac{1}{c^2\mu_0}\int_{\partial\Omega} *(\rho \wedge \zeta), \end{aligned} \tag{117}$$

where we have set the 0-form ζ equal to $*d(\epsilon \wedge \Delta)$. Now, thanks to Stokes' Theorem, the above integral can be rewritten as the following:

$$\lambda'_{spin} = \frac{1}{c^2\mu_0}\int_{\partial\Omega} *(\rho \wedge \zeta) = \frac{1}{c^2\mu_0}\int_{\Omega} d * (\rho \wedge \zeta). \tag{118}$$

4.2 Topological Analysis

The topological analysis of the situation is as profoundly deep as it is conceptually straightforward. We draw a strong analogy between this experiment, the Aharonov-Bohm Effect, and the topological considerations of magnetic monopoles, for in all cases physical objects are performing the tasks of mathematical objects. In the Aharonov-Bohm Effect, the electron is doing the job of the first fundamental group π_1 by detecting the region in the experiment where it is not permitted to go, i.e. the hole caused by deleting the origin from $\mathbb{R}^2$. Specifically, by making the inside of the solenoid impenetrable to the electron it has found that space by virtue of the fact that it is the one place it cannot go.

Similarly, an a traveling electromagnetic wave, whether purely plane or whether its intensity tapers off to zero at some point around its edges, does the job of the second fundamental group π_2 by finding deleted regions of the ambient space $\mathbb{R}^3$, just like a sphere centered around a magnetic monopole cannot be contracted to a point. Consider spherical electromagnetic wave expanding in $\mathbb{R}^3$ encountering a perfectly reflecting disk at the origin. Physically the wave will of course diffract past it, not to mention impart angular momentum, but very close to the disk we see that a hole has been torn into the surface of that spherical wave. That action is equivalent to a sphere in $\mathbb{R}^3 - \{0\}$ attempting to expand *continuously* past the deleted origin, something which it cannot do. Thus we find that π_2 is nontrivial for both the applied expanding spherical electromagnetic wave case and the abstract homotopy example.

In addition, as we saw with the magnetic monopole example, a continuous vector potential for the magnetic field cannot exist globally on $\mathbb{R}^3 - \{0\}$, due to the non-triviality of the second fundamental group π_2, and thus the second homology group H^2. Thus any computations that make intrinsic use of such a vector potential near the disk are risking considerable danger in terms of utilizing a tool that simply does not exist in the region in question.

To consider the situation in terms of de Rham Cohomology, the object of interest is the final, non-zero, expression derived for the spin angular momentum of the plane wave, $\lambda'_{spin} = \frac{1}{c^2\mu_0}\int_\Omega d*(\rho \wedge \zeta)$. The q-th de Rham

Cohomology group is defined as follows:

$$H^q_{dR} = \frac{\text{closed } q\text{-forms}}{\text{exact } q\text{-forms}}. \tag{119}$$

Now since a closed q-form ω is said to be exact if

$$\int \omega = 0 \tag{120}$$

it is clear that the integral is able to detect identity elements of the q-th de Rham Cohomology. For if the integral of a closed form is zero then it is exact and thus an identity element of H^q_{dR}. Whereas if the integral is non-zero, then the form is closed but not exact.

Consequently, the integral in λ'_{spin},

$$\lambda'_{spin} = \frac{1}{c^2\mu_0}\int_\Omega d * (\rho \wedge \zeta), \tag{121}$$

is now a point of tremendous concern. It is physically desirable, even necessary, to demand that the integral above take on some non-zero value, for it represents a physically observable effect and a physically measurable quantity. Mathematically however, we know that all closed 3-forms, such as $d * (\rho \wedge \zeta)$, are also exact forms since $H^3_{dR}(\mathbb{R}^3 - \{0\}) = 0$. Thus the above integral must be equal to zero. Like the previous consideration of magnetic monopoles, a contradiction between physical requirements and mathematical conclusions has been reached.

Ultimately, the preceding differential form considerations, paired with purely topological results, are able to specifically indicate where breakdowns in the theory of the angular momentum of light occur. The physical question of whether a plane wave can carry spin angular momentum has now been recast as the question of the exactness of the differential form $d * (\rho \wedge \zeta)$.

CHAPTER 5
CONCLUSIONS

Wherein conclusions are drawn.

5.1 The Topology of the Ambient Space

More than just an abstract notion, it has been shown how the topology of the ambient space in which a physical theory is defined plays a paramount role in the success and application of said theory. The simple removal of the origin from $\mathbb{R}^3$ for example, completely removes the ability to define a continuous magnetic vector potential quantity as thus handicaps the efforts of the physicist. In such a situation, the magnetic vector potential goes from being a handy mathematical convenience to a tremendous liability that can cause serious problems. Yet it is not completely fickle, for if the physicist is confined to $\mathbb{R}^2$ then the removal of the origin is less of a concern. In fact, in that case it is the magnetic vector potential that comes to the rescue and in fact detects that deleted point.

The foregoing analysis of this work took place in $\mathbb{R}^3 - \{0\}$ and as such the use of a vector potential 1-form α is suspect, though not entirely unacceptable. The removal of the origin can create great complications but it does not prevent an accurate analysis, for there are advanced methods involving the notion of *fibre bundles* that can be used to ameliorate the difficulties. In addition, the conclusion that equation 121 is zero because $H^3_{dR}(\mathbb{R}^3 - \{0\}) = 0$ works just as well if the analysis is allowed to be done solely in $\mathbb{R}^3$ since $H^3_{dR}(\mathbb{R}^3) = 0$, i.e. all closed 3-forms are still exact 3-forms in $\mathbb{R}^3$.

It is also extremely important to recognize the necessity of using complex-valued differential forms to perform the time average. However, while complex-valued objects are often used in physical calculations and theories, they are still just tools. Complex analysis allows for notationally simple solutions to physical differential equations. Regardless, in the end it is the real part of such solutions which obviously are the only part of concern for the physicist. It may be computationally cumbersome, but the same solutions could be achieved through purely real calculations. Not so with the time-averaging trick. While the real part of $\mathbf{E} \times \mathbf{B}^*$ is ultimately used, the work to show that it represents the time-averaging of the Poynting vector is wholly complex in nature. This was seen in the above analysis by the fact that purely real forms could not adequately express the situation and a move to complex-valued forms was required. Nevertheless, this should not be considered a problem, but rather an indication that a purely quantum mechanical picture is necessary to accurately represent the angular momentum of light. After all, quantum mechanics is itself a manifestly complex theory.

5.2 The Spin Angular Momentum of Light

It now appears that the accepted classical vector calculus expression for spin angular momentum

$$\mathbf{L}_{spin} = \frac{1}{c^2\mu_0}\int \mathbf{E}\times\mathbf{A}d^3x, \tag{122}$$

is a poorly formulated one. The exterior form expression for the spin angular momentum,

$$\lambda'_{spin} = \frac{1}{c^2\mu_0}\int_{\partial\Omega} *(\rho\wedge\zeta), \tag{123}$$

has an extremely intuitive feel. The integrand is a 2-form integrated over some two-dimensional surface and thus is clearly the integral of a some flux density through a surface, and considering that the Poynting vector is itself an energy flux-density, such a viewpoint seems most appropriate. In addition, such a representation can be though of physically as the momentum "current" entering and leaving Ω and so the above expression has parallels with Gauss' Law. Thus if the above integral is indeed zero as 3rd de Rham Cohomology claims, then there must not be any spin angular momentum within Ω. Furthermore, any physical tricks of the trade that are able to recover a non-zero answer are mathematical sublimations of the underlying and more complete quantum mechanical description.

The preceding mathematical arguments notwithstanding, the results of R. A. Beth's experiment are undeniable. There *is* angular momentum transferred to the wave plate. Yet our geometric (i.e. exterior calculus) and topological (i.e. de Rham cohomology) analyses assume no physics other than simple electric, magnetic, and potential vector fields but are able to independently make physically relevant conclusions. Thus it is clear that it is only the classical field formulation and understanding of the angular momentum of light that is in error and certainly not, of course, nature herself. The preceding analyses does not suggest to solve this problem. Rather it is only meant to demonstrate, through a contradiction in the classical theory, that geometric and topological methods are very well suited to analyzing classical electromagnetic problems for contradictions in the theory's internal consistency. Due to the strong and undeniable geometric and topological foundations of the classical theory of electromagnetism, it is without a doubt a worthwhile endeavor to look before one leaps, as it were, with geometry and topology as a guide.

While every experimentalist and theorist certainly is not expected to be as competent at all levels of geometry and topology as the professional geometer and topologist are, it is not unreasonable, and in fact it has been shown that it is quite helpful, for physicists to broaden their mathematical horizons, particularly at this point in modern physics. Geometric and topological considerations are thus most assuredly not outside of the physicist's area of concern and in fact should be welcome and well-worn tools of the trade.

5.3 Prospective Research

There are two potential avenues to explore. The first is purely topological in nature. Clearly there is some non-zero amount of angular momentum transferred and thus the integral 4.50 is not zero in some context. Of course there do exist spaces M such that $H^3_{dR}(M) \neq 0$, i.e. such that all closed 3-forms are not necessarily exact 3-forms as well. Determining whether or not any or all such spaces are physically meaningful is a potential area of exploration.

Secondly, a broader geometric investigation of the phenomenon is possible via complex-valued differential forms on complex manifolds. Only the simplest aspects of complex exterior calculus were utilized in this analysis. Wider considerations using the Dolbeault operator $\bar{\partial}$, the complex analog of the exterior derivative d, and Dolbeault Cohomology, the extension of de Rham Cohomology onto complex manifolds, could yield more and deeper insights than a mostly real analysis of the problem.

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