GENERIC PROPERTIES OF THE INFINITE POPULATION GENETIC ALGORITHM

by

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Voor mijn Opa, Marinus van Zanten.

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ABSTRACT

The infinite population model for the genetic algorithm, where the iteration of the genetic algorithm corresponds to an iteration of a map G, is a discrete dynamical system. The map G is a composition of a selection operator and a mixing operator, where the latter models the effects of both mutation and crossover. This dissertation examines the finiteness and hyperbolicity of fixed points of this model. For a typical mixing operator, the fixed point set of G is finite and all fixed points are hyperbolic.

CHAPTER 1

INTRODUCTION

Theory of Genetic Algorithms

Introduction to Genetic Algorithms

Genetic algorithms are an evolution inspired class of algorithms used to solve difficult optimization problems in many areas of science and engineering. A few examples of fields in which genetic algorithms have been used include agriculture [19], bio-informatics [9], data mining [11], economics [6], [28], engineering [10], [20], finance [5], [26], fingerprint recognition [4], geography [24], molecular biology [7], pattern recognition [25], physics, chemistry and biology [13], and scheduling problems [3], [16].

Genetic algorithms are an example of a wider class of evolutionary algorithms. These include evolution strategies, evolutionary programming, artificial life, classifier systems, genetic programming, and evolvable hardware.

The algorithm is applied to a population of individuals, which differ in their genetic make-up, as well an in their phenotype. The phenotype is evaluated using a fitness function. The genetic algorithm produces a new generation of individuals by using selection, mutation and crossover. To do so, a sub-population of individuals is chosen (with replacement) based on fitness. The selected individuals are subject to mutation and crossover. Using these genetic operations, a new generation of individuals is produced. These individuals inherit genetic traits from the preceding generation.

The selection is performed by selection operator which is a function of fitness values. Examples of commonly used selection operators range from fitness proportional selection, in which an individual is chosen in proportion to it's fitness, to tournament selection, in which a collection of individuals will "fight it out" over who is most fit. Other forms of selection include, but are not limited to, roulette wheel selection and rank selection.

Crossover is an artificial implementation of mating in which two individuals are selected from a population, and offspring are created using characteristics of both parents. If individuals are binary strings, crossover may be defined by randomly choosing a crossover point (location in the bit string), and then swapping tails of the parents to the right of this point to create two offspring. With such crossover, cloning, or direct copying of individuals into the next generation, may occur when the crossover point is chosen at the end or beginning of the string, or the parents are identical.

The mutation operator is defined to imitate mutation of an individual in nature. One way to construct a mutation operator is to associate a particular mutation rate with the algorithm; these rates are generally low and are used to preserve diversity of a population. If individuals are represented as bit strings, the mutation operator may be defined by choosing a random position in the bit string and flipping bits at that position. For a more detailed example of a genetic algorithm (GA) implementation, see [22].

The next generation of individuals produced generally has a higher average fitness than the generation that preceded it because the selection operators selects preferably individuals with higher fitness values for reproduction.

The genetic algorithm runs until a termination criteria has been satisfied. This criteria may be that an individual with a sufficiently high fitness value appears in the population, a maximum number of generations has been reached (run time has been exceeded), or the algorithm has stalled in a region of suboptimal solutions.

The Dynamical Systems Model

There are many models that have been developed in the study of genetic algorithms. Nix and Vose introduced a model of the GA as Markov Chain [23], which has further been studied and developed by others including [31], [32]. A few other approaches to studying the behavior of the GA include statistical mechanics approximations, spectral analysis [32], and group theory [30].

In this thesis we consider a dynamical systems model of the genetic algorithm. This model was introduced by Vose (see [36]) by replacing finite populations with population densities modelling an infinite population. The model was further extended in [27] and [33]. Although the precise correspondence between behavior of such infinite population genetic algorithm and the behavior of the GA for finite population has not been established in detail, the infinite population model has the advantage of being a well defined dynamical system. For further discussion analyzing the relevance of both the finite and infinite population models, see [18].

The dynamical systems model of the genetic algorithm provides an attractive mathematical framework for investigating the properties of GAs. In this thesis we study the model introduced by Vose [33].

The genetic algorithm searches for solutions in the search space $\Omega = \{1, 2, \dots n\}$; each element of Ω can be thought of as a type of individual or a "species." We consider a total population of size r with $r << n$. We represent such a population as an *incidence vector*:

$$v = (v_1, v_2, ..., v_n)^T$$

where v_i is the number of times the species i appears in the population. It follows that $\sum_i v_i = r$. We also identify a population with the *population incidence vector*

$$p = (p_1, p_2, ..., p_n)^T$$

where $p_i = \frac{v_i}{r}$ is the proportion of the i-th species in the population. The vector p can be viewed as a probability distribution over Ω. In this representation, the iterations of the genetic algorithm yield a sequence of vectors $p \in \Lambda^r$ where

$$\Lambda^r := \{(p_1, p_2, ..., p_n)^T \in \mathbb{R}^n \mid p_i = \frac{v_i}{r} \text{ and } v_i \in \{0, \dots, r\} \text{ for all } i \in \{1, \dots, n\} \text{ with } \sum_i v_i = r\}. \quad (1.1)$$

We define

$$\Lambda := \{x \in \mathbb{R}^n | \sum x_i = 1 \text{ and } x_i \geq 0 \text{ for all } i \in \{1, \ldots, n\}\}.$$

Note that $\Lambda^r \subset \Lambda \subset \mathbb{R}^n$, where Λ is the unit simplex in $\mathbb{R}^n$. Not every point $x \in \Lambda$ corresponds to a population incidence vector $p \in \Lambda^r$, with fixed population size r, since p has non-negative rational entries with denominator r. However, as the population size r gets arbitrarily large, Λ^r becomes dense in Λ, that is, $\cup_{r \geq N} \Lambda^r$ is dense in Λ for all N. Thus Λ may be viewed as a set of admissible states for infinite populations. We will use p to denote an arbitrary point in Λ^r and x to denote an arbitrary point in Λ. Unless otherwise indicated, $x \in \Lambda$ is a column vector.

Let $G(x)$ represent the action of the genetic algorithm on $x \in \Lambda$, and assume that $G : \Lambda \rightarrow \Lambda$ is a differentiable map ([33]). The map G is a composition of three maps: selection, mutation, and crossover. We will now describe each of these in turn and provide some examples at the end of the section for reference.

We let $F : \Lambda \rightarrow \Lambda$ represent the selection operator. The i-th component, $F_i(x)$ represents the probability that an individual of type i will result if selection is applied to $x \in \Lambda$. As an example, consider proportional selection where the probability of an individual $k \in \Omega$ being selected is

$$Pr[k|x] = \frac{x_k f_k}{\sum_{j \in \Omega} x_j f_j},$$

where $x \in \Lambda$ is the population incidence vector, and f_k, the k-th entry of the vector f, is the fitness of $k \in \Omega$. Define $diag(f)$ as the diagonal matrix with entries from f

along the diagonal and zeros elsewhere. Then, for $F : \Lambda \to \Lambda$, proportional selection is defined as

$$F(x) = \frac{diag(f)x}{f^T x}.$$

We restrict our choice of selection operators to those which are $\mathcal{C}^1$, that is, selection operators with continuous derivatives.

We let $U : \Lambda \to \Lambda$ represent mutation. Here U is an $n \times n$ real valued matrix with ij-th entry $u_{ij} > 0$ for all i, j, and where U_{ij} represents the probability that item $j \in \Omega$ mutates into $i \in \Omega$. That is, $(Ux)_k := \sum_i u_{ki} x_i$ is the probability an individual of type k will result after applying mutation to population x.

Let crossover, $C : \Lambda \to \Lambda$, be defined by

$$C(x) = (x^T C_1 x, \ldots, x^T C_n x)$$

for $x \in \Lambda$, where $C_1, \ldots, C_n$ is a sequence of symmetric non-negative $n \times n$ real valued matrices. Here $C_k(x)$ represents the probability that an individual k is created by applying crossover to population x.

Definition 1.1. An operator $A : \mathbb{R}^n \to \mathbb{R}^n$ is **quadratic** if there exist matrices $A_1, A_2, \ldots, A_n \in Mat_n(\mathbb{R})$ such that $A(x) = (x^T A_1 x, \ldots, x^T A_n x)$. We denote a quadratic operator with its corresponding matrices as $A = (A_1, \ldots, A_n)$.

Thus, the crossover operator, $C = (C_1, \ldots, C_n)$, is a quadratic operator ([30]).

We combine mutation and crossover to obtain the mixing operator $M := C \circ U$. The k-th component of the mixing operator

$$M_k(x) = x^T(U^T C_k U)x$$

represents the probability that an individual of type k will result after applying mutation and crossover to population x. Since C_k is symmetric, M_k is symmetric. Further, since C_k is non-negative and U is positive for all k, M_k is also positive for all k. Additionally, it is easy to see check that since $\sum_{k=1}^{n}[M_k]_{ij} = 1$, $M : \Lambda \to \Lambda$, and mixing is also a quadratic operator ([30]). Here $[M_k]_{ij}$ denotes the ij-th entry of the matrix M_k. This motivates the following general definition of a mixing operator.

Definition 1.2. Let $Mat_n(\mathbb{R})$ represent the set of $n \times n$ matrices with real valued entries. We call a quadratic operator, $M = (M_1, \ldots, M_n)$, a **mixing operator** if the following properties hold:

1. $M_k \in Mat_n(\mathbb{R})$ is symmetric for all $k = 1, \ldots, n$;
2. $(M_k)_{ij} > 0$ for all $i, j \in \{1, \ldots, n\}$, and for all $k = 1, \ldots, n$;
3. $\sum_{k=1}^{n}[M_k]_{ij} = 1$ for all $j = 1, \ldots, n$ and $i = 1, \ldots, n$.

Let $\mathcal{M}$ be the set of quadratic operators M satisfying (1)-(3). Observe that (3) implies that $M \in \mathcal{M}$ maps Λ to Λ. This is easily seen since, for $x \in \Lambda$, $M(x) = (x^T M_1 x, \ldots, x^T M_n x)$, and

$$\sum_k [M(x)]_k = x^T \left(\sum_k M_k\right) x = x^T \cdot \left(\sum_i x_i, \sum_i x_i, \ldots, \sum_i x_i\right)$$

$$= x^T \cdot (1, \ldots, 1) = 1.$$

We define a norm, $||\cdot||$, on $\mathcal{M}$ by considering for $M \in \mathcal{M}$, $M \in \mathbb{R}^{n^3}$, and using the Euclidean norm.

We define

$$G := M \circ F, \text{ for } M \in \mathcal{M} \tag{1.2}$$

to be the complete operator for the genetic algorithm, or a ***GA map***.

We extend the definition of F to the positive cone in $\mathbb{R}^n$, denoted $\mathbb{R}^{n+}$. The extension of F is also denoted F and for $u \in \mathbb{R}^{n+}\backslash\Lambda$ is defined by

$$F(u) := F\left(\frac{u}{\sum_i u_i}\right).$$

Thus $F|_\Lambda = F$, and for $x \in \Lambda$, $DF(x)|_\Lambda = DF(x)$, the Jacobian of F. Since $F(\Lambda) \subset \Lambda$, it is clear that if $T_x\Lambda$ represents the tangent space to Λ at x, then $DF(x)(T_x\Lambda) \subseteq T_x\Lambda$. Finally, since

$$T_x\Lambda = \left\{x \in \mathbb{R}^n | \sum_i x_i = 0\} = \mathbb{R}^n_0\right\},$$

$DF(\mathbb{R}^n_0) \subseteq \mathbb{R}^n_0$. Because $F : \mathbb{R}^{n+} \to \Lambda$, it is also clear that $G : \mathbb{R}^{n+} \to \Lambda$ and the preceding remarks apply to G as well.

Examples

The following examples are given for easy reference and are taken from [27].

EXAMPLE 1.1 (SELECTION). Let $\Omega = \{0, 1, 2, 3\}$ denote the search space. Define the fitness function $f(0, 1, 2, 3) = (2, 1, 3, 2)$.

Let $x \in \Lambda$. The selection operator, F, corresponding to fitness proportional selection is given by

$$F(x) = \frac{1}{f^T x} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} x = \left(\frac{1}{2x_1 + x_2 + 3x_3 + 2x_4} \right) (2x_1, x_2, 3x_3, 2x_4)$$

where $f^T x = 2x_1 + x_2 + 3x_3 + 2x_4$ is the average fitness of the population $x = (x_1, x_2, x_3, x_4)$.

EXAMPLE 1.2 (MUTATION). Let $\Omega = \{0, 1, 2, 3\}$ denote the search space. Assume that there is probability $\mu > 0$ that an individual mutates. Assume also that if it mutates that the probability that one individual will mutate into another is equal for all individuals. Then, the corresponding mutation matrix is given by

$$U = \begin{pmatrix} 1-\mu & \mu/3 & \mu/3 & \mu/3 \\ \mu/3 & 1-\mu & \mu/3 & \mu/3 \\ \mu/3 & \mu/3 & 1-\mu & \mu/3 \\ \mu/3 & \mu/3 & \mu/3 & 1-\mu \end{pmatrix},$$

and for $x \in \Lambda$, the population resulting after mutation is given by Ux.

EXAMPLE 1.3 (CROSSOVER). Let $\Omega = \{0, 1, 2, 3\}$ be the search space corresponding to the integers in binary representation. We wish to create a quadratic map

$C = (C_0, \ldots, C_3)$ with the property that for $x \in \Lambda$, $k \in \{0, \ldots, 3\}$, $(Cx)_k = x^T C_k x$ is the probability that an individual of type k will result from crossover.

For $i, j, k \in \Omega$, let $r(i, j, k)$ denote the probability of obtaining k by applying crossover to i and j. Let $B_0, \ldots, B_3 \in Mat_n(\mathbb{R})$ be a set of matrices defined by $(B_k)_{ij} = r(i, j, k)$ for $i, j, k \in \{0, \ldots, 3\}$. Since it is not necessarily true that $r(i, j, k) = r(j, i, k)$, the B_k are not necessarily symmetric. For $k = 1, \ldots, 3$, let $\mathbf{B}_k = \frac{B_k + B_k^T}{2}$. Now, $\mathbf{B}_k$ is symmetric, and $C : \Lambda \to \Lambda$ is defined by $C(x)_k = x^T \mathbf{B}_k x$.

If the values of $r(i, j, 0)$ are given by

	00	01	10	11
00	1	1/3	2/3	1/3
01	2/3	0	1/3	0
10	1/3	0	0	0
11	1/3	0	0	0

,

the matrix B_0 is

$$B_0 = \begin{pmatrix} 1 & 1/3 & 2/3 & 1/3 \\ 2/3 & 0 & 1/3 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \end{pmatrix},$$

while

$$\mathbf{B_0} = \frac{B_0 + B_0^T}{2} = \begin{pmatrix} 1 & 1/2 & 1/2 & 1/3 \\ 1/2 & 0 & 1/6 & 0 \\ 1/2 & 1/6 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \end{pmatrix}.$$

Finally, for $x = \left(\frac{3}{6}, \frac{2}{6}, \frac{0}{6}, \frac{1}{6}\right)^T$ and $x^T = \left(\frac{3}{6}, \frac{2}{6}, \frac{0}{6}, \frac{1}{6}\right)$, note that $C(x)_0 = x^T \mathbb{B}_0 x = \frac{17}{36}$.

Convergence of GAs

There are many open questions in the theory of genetic algorithms, many of which center around the behavior of the map G. Therefore, the techniques of dynamical systems theory can be used to formulate and hopefully answer some fundamental questions about the GA. One such question is the question of convergence. For plausible crossover, mutation and selection, does the algorithm always converge to a unique solution for all initial states? The answer is negative in general, since Wright and Bidwell found counterexamples for mutation and selection operators that are not used in practice [34]. The question of characterization of a class of GA maps for which the algorithms converges to a unique fixed point remains open. In the infinite population model, the iterations of the GA are represented as iterations of a fixed map G on a space of admissible population densities x. Thus, the question of convergence can be reformulated in this setting as existence of a globally attracting stable fixed point, that is, a population x such that $G(x) = x$.

The fixed points, that is x such that $G(x) = x$, are fundamental objects of interest in our study. The behavior of the map G in the neighborhood of x is determined by the eigenvalues of the linearization $DG(x)$. If all the eigenvalues have absolute value less than one, then all iterates starting near x converge to x. If there is at least one eigenvalue with absolute value greater than one, then almost all iterates will diverge from x [29]. Such classification is, however, possible only if no eigenvalues lie on the unit circle in the complex plain. Fixed points x, for which $DG(x)$ has this property, are called *hyperbolic*. If at least one eigenvalue of $DG(x)$ has absolute value 1, the fixed point is *non-hyperbolic*.

It is easy to see that hyperbolicity is an open condition, i.e. if a fixed point is hyperbolic, then all small $\mathcal{C}^1$ perturbations of the map G will still admit a fixed point with eigenvalues off the unit circle. It should follow that for sufficiently large finite population, the GA will also admit a fixed point. Thus, hyperbolic fixed points under G predict behavior for finite population GA. On the other hand, non-hyperbolic fixed points can disappear under arbitrarily small perturbations. If the infinite population model wants to be a viable model of the behavior of the finite population GA, non-hyperbolic fixed points should be rare. It is clear that they must be present for some admissible maps G, since they occur when a fixed point bifurcates.

Vose and Eberlein [33] considered a class of mappings G that were a composition of a mutation and crossover map, with proportional selection scheme. The set of fitness functions was parameterized by the positive orthant. They have shown that for an open and dense set of such fitness functions, the corresponding map, G, has hyperbolic fixed points.

In this contribution we consider a class of mappings $G = M \circ F$ where F is arbitrary, but fixed, selection map and M is a mixing map from the class $\mathcal{M}$. The class of mixing maps we consider include all mixing maps that are a composition of the mutation and selection maps as described in Reeves and Rowe [27] and Vose [33]. We show that for an open and dense set of mixing maps, the corresponding map G has only hyperbolic fixed points.

To prove our result, we will also need to show that for an open and dense set of mixing maps, the corresponding map G has finitely many fixed points. We note that in contrast, Wright and Vose in [35] showed that an open and dense set of fitness operators, G under proportional selection has a finite fixed point set.

Definitions

Before we present the main result, we introduce key definitions.

Recall that if $f(x) = x$, a point x is called a *fixed point* of f.

DEFINITION 1.3. A fixed point x for $f : \mathbb{R}^n \to \mathbb{R}^n$ is called *hyperbolic* if the Jacobian, $Df(x)$, has no eigenvalues on the unit circle $\mathcal{S}^1$.

DEFINITION 1.4. A fixed point x is *non-hyperbolic* if $\text{spec}(Df(x)) \cap \mathcal{S}^1 \neq \emptyset$.

DEFINITION 1.5. A map G is *hyperbolic* if all fixed points are hyperbolic.

DEFINITION 1.6. A property is *typical*, or *generic*, in a set S, if it holds for an open and dense set of parameter values in S.

Finally, we define the following sets:

DEFINITION 1.7. For a map $f : X \to X$,

$$\text{Fix}(f) := \{x \in X | f(x) = x\}.$$

That is, $\text{Fix}(f)$ denotes the set of all fixed points of f.

DEFINITION 1.8. For a map $f : X \to X$,

$$\text{Hyp}(f) := \{x \in \text{Fix}(f) | Df(x) \cap \mathcal{S}^1 = \emptyset\}.$$

That is, Hyp(f) denotes the set of hyperbolic fixed points of f.

DEFINITION 1.9. For a map $f : X \to X$,

$$\text{NonHyp}(f) := \text{Fix}(f) \backslash \text{Hyp}(f).$$

That is, NonHyp(f) denotes the set of non-hyperbolic fixed points of f.

Main Results

We now present our main results.

THEOREM 1.10. *Let $G = M \circ F$ be a GA map (1.2). For a typical mixing operator $M \in \mathcal{M}$, G has finitely many fixed points.*

We will use Theorem 1.10 to prove Theorem 1.11 given below.

THEOREM 1.11. *Let $G = M \circ F$ be a GA map (1.2) and assume that* $\text{rank}(DF) = n - 1$ *for all $x \in \Lambda$. For a typical mixing operator, $M \in \mathcal{M}$, G is hyperbolic.*

The assumption that $\text{rank} DF(p) = n - 1$ is not very restrictive since the range of F is the $n - 1$ dimensional space Λ. In any case, this assumption is valid generically for proportional selection [33].

To prove the above theorem, we will need the following two propositions.

Proposition 1.12. *Let $G = M \circ F$ be a GA map (1.2). The set of mixing operators M, for which the fixed points of G are hyperbolic, forms an open set in $\mathcal{M}$.*

Proposition 1.13. *Let $G = M \circ F$ be a GA map (1.2) and assume* $\text{rank}(DF) = n - 1$. *The set of mixing operators for which the fixed points of G are hyperbolic, forms a dense set in $\mathcal{M}$.*

An outline of the proofs of these theorems, as well as the basic structure of this dissertation will be provided in the next section.

Outline of Proof

We start by proving that for generic mixing operator, a GA map $G = M \circ F$ has finitely many fixed points. That is, we first prove Theorem 1.10.

We use $M \in \mathcal{M}$, such that $G = M \circ F$ has finitely many fixed points to argue that G is generically hyperbolic. The proof of Theorem 1.11 is obtained by proving Propositions 1.12 and 1.13.

Proving Proposition 1.12, that the set of M for which $G = M \circ F$ is hyperbolic, is open, falls out naturally. The proof of this proposition is based on the fact that

$$\det(DG(x) - \lambda I) = \det([DM \circ F(x)]DF(x) - \lambda I)$$

is a continuous function of M and x and, therefore, if a root of this polynomial, λ_i, has $\lambda_i \notin S^1$, then small perturbations do not change this fact.

Proving Proposition 1.13, that the set of mixing operators for which $G = M \circ F$ is hyperbolic, is dense, requires greater effort. To prove this proposition, we will assume we have a fixed point x of G with one or more eigenvalues on the unit circle, $\mathcal{S}^1$. We first characterize perturbations, $M \in \mathcal{M}$, that preserve the fixed point. If the eigenvalue in question is a simple eigenvalue, we will find a perturbation in the set such that the new operator no longer has an eigenvalue on the unit circle. If the eigenvalue is a repeated eigenvalue, we perturb to the single eigenvalue case first. Finally, we find the appropriate combination of such perturbations so that no fixed points remain with eigenvalues on $\mathcal{S}^1$. Here we use strongly generic finiteness of fixed points as is guaranteed by Theorem 1.10.

Structure of this Dissertation

For the reader's convenience, we have provided three appendices for reference. In Appendix A, a notational reference is provided. In Appendix B, we provide a collection of general results that will be referenced throughout much of the dissertation. Finally, in Appendix C we provide a collection of established results in the field of differential topology.

In Chapters 2 and 3, we provide results that will provide the basic components used in the arguments to prove Theorem 1.10 and Proposition 1.13. Chapter 2 highlights results regarding the basic structure and properties of the GA map, as well as the relevant perturbations we will use. Chapter 3 determines which perturbations are appropriate for perturbation of the characteristic polynomial in the desired direction.

Finally, in Chapter 4 we prove Theorem 1.10, and in Chapters 5 and 6 we prove Propositions 1.12 and 1.13, respectively.

Notation

In addition to the above model, the following notation and terminology will be used.

- Let $Mat_n(F)$ be the set of $n \times n$ matrices with entries in the field F, usually $F = \mathbb{R}, \mathbb{C}$.
- For a matrix A, let a_{ij} denote the ij-th entry of A.
- For a matrix A, let a_i denote the i-th row of the matrix A and let a^j denote the j-th column of the matrix A.
- For $A \in Mat_n(F)$, let $\det(A)$ denote the determinant of the matrix A. Note that we also use $\det(a_1, \ldots, a_n)$ and $\det(a^1, \ldots, a^n)$ to denote $\det(A)$.
- For an $n \times n$ matrix A, let $A(i,j)$ represent the $(n-1) \times (n-1)$ matrix that results from deleting row i and column j from the matrix A. That is, $A(i,j)$ denotes the ij-th minor of A.
- For a matrix A, let A_{ij} denote the determinant of the minor $A(i,j)$. That is, $A_{ij} = \det(A(i,j))$.

- The characteristic polynomial for the matrix $A \in Mat_n(F)$ is denoted by $\det(A - \lambda I)$, where I is the $n \times n$ identity matrix.
- Let $\mathrm{spec}(A)$ denote the set of eigenvalues of A.
- The transpose of a matrix A is denoted A^T.
- The inverse of an invertible matrix A is denoted A^{-1}.
- We use the notation $A > 0$ to indicate that $a_{ij} > 0$ for all i, j.
- For a matrix A, $\mathrm{rank}(A)$ denotes the dimension of the range of A, or the number of linearly independent columns of the matrix.
- Let $\alpha = a + bi$ be a complex number with $a, b \in \mathbb{R}$. We denote the real part of α with $\mathrm{Re}(\alpha)$ and the imaginary part is denoted $\mathrm{Im}(\alpha)$.
- Let V, W be vector spaces, and let $T : V \to W$ be an operator. We denote the null space of T with $\mathrm{null}(T)$. The image, or range, of T is denoted $\mathrm{Image}(T)$.
- Let $N_r(x)$ denote a ball of radius r about a point x.
- Given a set S, let $\mathrm{int}(S)$ denote the interior of the set S.
- We let $\mathbb{R}^{n+}$ denote the set of $x \in \mathbb{R}^n$ such that $x_k > 0$ for all $k = 1, \ldots, n$.
- Let $\mathbb{R}^n_0 := \{x \in \mathbb{R}^n | \sum_i x_i = 0\}$.

For the following items, let X, Y be n and m dimensional smooth manifolds, respectively.

- For $x \in X$, we let T_xX denote the tangent space to X at x.
- For a differentiable map $f : X \to Y$, we let $df_x : T_xX \to T_{f(x)}Y$ denote the differential of the map f at the point x.

- In the special case that $T_xX = \mathbb{R}^n$ and $T_{f(x)}Y = \mathbb{R}^m$, we note that $df_x : \mathbb{R}^n \to \mathbb{R}^m$ is given by the Jacobian matrix $Df(x)$ ([14]).
- For $r \geq 1$, let $\mathcal{C}^r(X, Y)$ be the set of all r-times continuously differentiable functions from X to Y. We let $\mathcal{C}^0(X, Y)$ then denote continuous functions from X to Y. For $r \geq 0$, if $f \in \mathcal{C}^r(X, Y)$, we call f a $\mathcal{C}^r$ function or map.

CHAPTER 2

STRUCTURE OF THE GA MAP

Domain and Range

We start by presenting a collection of results intended to describe relevant properties of the GA map $G : \mathbb{R}^{n+} \to \Lambda$.

LEMMA 2.1. *Let $G = M \circ F$ be a GA map. If $x \in \Lambda$ is a fixed point of G, with $x = (x_1, \ldots, x_n)$, then $x \in \text{int}(\Lambda)$.*

PROOF. Assume $x \in \Lambda$ with $G(x) = x$. To show $x \in \text{int}(\Lambda)$ it suffices to show that for all $k \in \{1, \ldots, n\}$, $x_k \neq 0$. By assumption, since $x \in \Lambda$ with $G(x) = x$, for all $k \in \{1, \ldots, n\}$,

$$x_k = G_k(x) = M_k \circ F(x). \tag{2.1}$$

Recall that $F : \mathbb{R}^{n+} \to \Lambda$. Thus, $F(x) \neq (0, \ldots, 0)$ for all $x \in \Lambda$. And, by definition of a GA map, $M_k > 0$ for all $k = 1, \ldots, n$. Thus, for all $k = 1, \ldots, n$, $M_k \circ F(x) \neq 0$, proving the desired result. □

LEMMA 2.2. *Let $H : \mathbb{R}^{n+} \to \Lambda$ be a differentiable map. Then for $x \in \text{int}(\Lambda)$, $DH(x)(\mathbb{R}^n) \subseteq \mathbb{R}^n_0$.*

PROOF. We consider arbitrary $x \in \text{int}(\Lambda)$ and $v \in \mathbb{R}^n$ and show $DH(x)v \in \mathbb{R}^n_0$. To do so, we compute and show $\sum_j [DH(x)v]_j = 0$. By definition,

$$\sum_j [DH(x)v]_j = \sum_j \lim_{\alpha \to 0} \frac{1}{\alpha} \left[H_j(x + \alpha v) - H_j(x) \right].$$

Since $x \in \text{int}(\Lambda)$, for α sufficiently small, $x + \alpha v$ is in the first orthant, $\mathbb{R}^{n+}$. Thus, and $x + \alpha v \notin \mathbb{R}^n_0$, and,

$$\begin{aligned}
\sum_j [DH(x)v]_j =& \lim_{\alpha \to 0} \frac{1}{\alpha} \left[\sum_j H_j \left(\frac{x + \alpha v}{\sum_i (x + \alpha v)_i} \right) - \sum_j H_j(x) \right] \\
=& \lim_{\alpha \to 0} \frac{1}{\alpha} [1 - 1] \\
=& \lim_{\alpha \to 0} \frac{1}{\alpha} [0] \\
=& 0.
\end{aligned}$$

Finally, because for all $v \in \mathbb{R}^n$, $DH(x)v \in \mathbb{R}^n_0$, it is clear that $DH(x)v \cdot (1, \ldots, 1) = 0$. □

LEMMA 2.3. *Let* $H : \mathbb{R}^{n+} \to \Lambda$ *be a differentiable map. Assume for all* $x \in \Lambda$, *that* $\text{rank}(DH(x)) = n - 1$. *Then, for* $x \in \text{int}(\Lambda)$, $(\text{Image}(DH(x))) = (1, \ldots, 1)^\perp$.

PROOF. By assumption, $\text{rank}(DH(x)) = n - 1$. Note also that $\dim(1, \ldots, 1)^\perp = n-1$. By Lemma 2.2, for all $v \in \text{Image}(DH(x))$, $v \in (1, \ldots, 1)^\perp$. Thus, $\text{Image}(DH(x)) = (1, \ldots, 1)^\perp$. □

COROLLARY 2.4. *Let* $G = M \circ F$ *be a GA map. For all* $v \in \mathbb{R}^n$, $(DG(x))v \cdot (1, \ldots, 1) = 0$, *where* $DG(x)$ *is the Jacobian of* G *at an arbitrary fixed point* $x \in \Lambda$.

Proof. Since $F : \mathbb{R}^{n+} \to \Lambda$ and $M : \Lambda \to \Lambda$, the map G maps $\mathbb{R}^{n+} \to \Lambda$. By Lemma 2.1, $x \in \mathrm{int}(\Lambda)$. The result now follows from Lemma 2.2. □

Corollary 2.5. *Let $G = M \circ F$ be a GA map with a fixed point x. Let $\lambda = \alpha + i\beta \in \mathbb{C}\backslash\mathbb{R}$ be an eigenvalue of $DG(x)$ with corresponding eigenvector v, where $DG(x)$ is the Jacobian of G at a fixed point $x \in \Lambda$. Let $\mathrm{Re}(v), \mathrm{Im}(v) \in \mathbb{R}^n$ denote the real and imaginary parts of v. Then,*

$$[DG(x)](\mathrm{Re}(v)) \cdot (1, \dots, 1) = 0 = (\alpha \mathrm{Re}(v) - \beta \mathrm{Im}(v)) \cdot (1, \dots, 1)$$

and

$$[DG(x)](\mathrm{Im}(v)) \cdot (1, \dots, 1) = 0 = (\beta \mathrm{Re}(v) + \alpha \mathrm{Im}(v)) \cdot (1, \dots, 1)$$

Proof. The result follows directly from Lemma 7.12 and Corollary 2.4. □

Lemma 2.6. *Let $H : \mathbb{R}^{n+} \to \Lambda$ be a differentiable map. Let $DH(x)$ be the Jacobian of H at the point $x \in \Lambda$ and assume $\mathrm{rank}(DH(x)) = n - 1$. Then, $\mathrm{span}(H(x)) \nsubseteq \mathrm{Image}(DH(x))$.*

Proof. Take $v \neq 0$ and $v \in \mathrm{span}(H(x))$. Then there exists $a \neq 0$ such that $v = aH(x)$. To show $\mathrm{span}(H(x)) \nsubseteq \mathrm{Image}(DH(x))$ we assume to the contrary that $\mathrm{span} H(x) \nsubseteq \mathrm{Image}(DH(x))$. Therefore, there exists $w \in \mathbb{R}^{n+}$ such that $v = DH(x)w$. Then,

$$DH(x)w = aH(x),$$

and,

$$DH(x)w \cdot (1, \ldots, 1) = aH(x) \cdot (1, \ldots, 1). \qquad (2.2)$$

But, by Lemma 2.3, $DH(x)w \cdot (1, \ldots, 1) = 0$. And, because $H : \mathbb{R}^{n+} \to \Lambda$,

$$aH(x) \cdot (1, \ldots, 1) = a \sum_k H_k(x) = a1 = a. \qquad (2.3)$$

Thus, equation (2.2) gives the contradiction that $0 = a$. □

COROLLARY 2.7. *Let $G = M \circ F$ be a GA map with a fixed point x. Let $\lambda \in \mathbb{C} \backslash \mathbb{R}$ be an eigenvalue of $DG(x)$ with corresponding eigenvector v. Let* $\mathrm{Re}(v), \mathrm{Im}(v) \in \mathbb{R}^n$ *denote the real and imaginary parts of v. Then,* $\mathrm{Re}(v), \mathrm{Im}(v) \in (1, \ldots, 1)^{\perp}$.

PROOF. Note that by Corollary 2.4, it suffices to show that for $\lambda = \alpha + i\beta$, with $\alpha, \beta \in \mathbb{R}$, not both zero, that $\mathrm{Re}(v), \mathrm{Im}(v) \in \mathrm{Image}(DG(x))$. By Lemma 7.12,

$$[DG(x)](\mathrm{Re}(v)) = (\alpha\mathrm{Re}(v) - \beta\mathrm{Im}(v)) \qquad (2.4)$$

and

$$[DG(x)](\mathrm{Im}(v)) = (\beta\mathrm{Re}(v) + \alpha\mathrm{Im}(v)). \qquad (2.5)$$

First consider the case $\alpha = 0$, then by equations (2.4) and (2.5),

$$[DG(x)](\mathrm{Re}(v)) = -\beta\mathrm{Im}(v), \qquad (2.6)$$

and

$$[DG(x)](\mathrm{Im}(v)) = \beta\mathrm{Re}(v). \qquad (2.7)$$

Thus, by equation (2.6)

$$[DG(x)]\left(\frac{\mathrm{Re}(v)}{-\beta}\right) = \mathrm{Im}(v). \tag{2.8}$$

That is, $\mathrm{Im}(v) \in \mathrm{Image}(DG(x))$ and by Corollary 2.4, $\mathrm{Im}(v) \perp (1,\ldots,1)$. Similarly, we see that for $\alpha = 0$, $\mathrm{Re}(v) \perp (1,\ldots,1)$. By assumption, $\beta \neq 0$, since $\lambda \in \mathbb{C}\backslash\mathbb{R}$.

For the case $\alpha \neq 0$, we solve the linear system of equations (2.4) and (2.5), to show that

$$[DG(x)]\left(\frac{\mathrm{Re}(v)}{\beta} + \frac{\mathrm{Im}(v)}{\alpha}\right) = \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)(\mathrm{Re}(v)), \tag{2.9}$$

which show that $\mathrm{Re}(v) \in \mathrm{Image}(DG(x))$ if $\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) \neq 0$. Note that if $\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) = 0$, then $\alpha^2 = -\beta^2$ for non-zero $\alpha, \beta \in \mathbb{R}$. Thus, $\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) \neq 0$, and $\mathrm{Re}(v) \in \mathrm{Image}(DG(x))$. Finally, by Corollary 2.4, $\mathrm{Re}(v) \perp (1,\ldots,1)$. To show that $\mathrm{Im}(v) \perp (1,\ldots,1)$, for $\alpha \neq 0$, we once again compute using equations (2.4) and (2.5) that

$$[DG(x)]\left(\frac{\mathrm{Re}(v)}{\alpha} - \frac{\mathrm{Im}(v)}{\beta}\right) = \left(\frac{-\beta}{\alpha} + \frac{-\alpha}{\beta}\right)(\mathrm{Im}(v)). \tag{2.10}$$

Thus, if $(\frac{-\beta}{\alpha} + \frac{-\alpha}{\beta}) \neq 0$, then $\mathrm{Im}(v) \in \mathrm{Image}(DG(x))$. If $(\frac{-\beta}{\alpha} + \frac{-\alpha}{\beta}) = 0$, then $-\beta^2 = \alpha^2$ for non-zero $\alpha, \beta \in \mathbb{R}$. Since this is a contradiction, we see that $(\frac{-\beta}{\alpha} + \frac{-\alpha}{\beta}) \neq 0$ and therefore $\mathrm{Im}(v) \in \mathrm{Image}(DG(x))$ and once again by Corollary 2.4, $\mathrm{Im}(v) \perp (1,\ldots,1)$. $\square$

The Class of Perturbations of G

We now describe the set of perturbations of $M \in \mathcal{M}$. In particular, we are interested in perturbations of M that are still elements of the set $\mathcal{M}$, and additionally have the property that they preserve the fixed point of interest.

Let $G = M \circ F$ be a GA map (1.2) with fixed point x. Let $\mathcal{Q}(x)$ represent a class of quadratic operators $Q = (Q_1, \ldots, Q_n)$ for which the following properties hold:

1. $Q_k \in Mat_n(\mathbb{R})$ is symmetric for all $k = 1, \ldots, n$;
2. $\sum_k Q_k = 0$;
3. $[F(x)]^T Q_k F(x) = 0$ for all $k = 1, \ldots, n$ where x is the fixed point.

It is worth noting that elements of $\mathcal{Q}(x)$ may be viewed as elements of the tangent space to $\mathcal{M}$. Given a fixed mixing operator $M \in \mathcal{M}$, let $\mathcal{P}(x, M) \subset \mathcal{Q}(x)$ be defined as follows:

$$\mathcal{P}(x, M) := \{Q \in \mathcal{Q}(x) | M_k \pm Q_k > 0 \text{ for all } k = 1, \ldots, n\}.$$

Lemma 2.8. *Let $G = M \circ F$ be a GA map (1.2) with fixed point x. Given $Q \in \mathcal{Q}(x)$, there exists $\bar{\epsilon}$ such that for all $0 \leq \epsilon \leq \bar{\epsilon}$, $\epsilon Q \in \mathcal{P}(x, M)$.*

Proof. Let $Q \in \mathcal{Q}(x)$. By definition of $\mathcal{Q}(x)$, it follows that for any $t \in \mathbb{R}$, $tQ \in \mathcal{Q}(x)$. We now show that for $Q \in \mathcal{Q}(x)$ there exists $\bar{\epsilon}$ such that $M_k \pm (\epsilon Q)_k > 0$ for all $k = 1, \ldots, n$, that is, such that $\epsilon Q \in \mathcal{P}(x, M)$ for all $0 \leq \epsilon \leq \bar{\epsilon}$.

Consider first that for $Q \in \mathcal{Q}(x)$, $(\epsilon Q)_k = \epsilon Q_k$. Thus, to show $M_k \pm (\epsilon Q)_k > 0$, it suffices to show that $M_k \pm \epsilon Q_k > 0$ for all $k = 1, \ldots, n$. For $k \in \{1, \ldots, n\}$, we let $(Q_k)_{ij}$ denote the ij-th entry of the matrix Q_k. Similarly, $(M_k)_{ij}$ denotes the ij-th entry of the matrix M_k. The requirement for $\epsilon Q \in \mathcal{P}(x, M)$ that $M_k \pm P_k > 0$ is equivalent to the requirement that $|\epsilon(Q_k)_{ij}| < (M_k)_{ij}$ for all i, j, k. Thus, we will show that there exists $\bar{\epsilon}$ such that for all $0 \leq \epsilon \leq \bar{\epsilon}$,

$$\epsilon|(Q_k)_{ij}| \leq \bar{\epsilon}|(Q_k)_{ij}| < (M_k)_{ij}.$$

We assume $Q \neq 0$ since the case $Q = 0$ is trivial. For $Q \neq 0$, let $\alpha = \max\{|(Q_k)_{ij}| \text{ for all i,j,k }\}$. Similarly, let $\beta = \min\{(M_k)_{ij} \text{ for all i,j,k }\}$. Now, take $\bar{\epsilon} \in \mathbb{R}^+$ such that $\frac{\beta}{\alpha} > \bar{\epsilon}$. Thus,

$$\beta > \alpha\bar{\epsilon} > \alpha\epsilon \text{ for all } \bar{\epsilon} > \epsilon \geq 0.$$

Since for all i, j, k,

$$(M_k)_{ij} > \min\{(M_k)_{ij} \text{ for all i,j,k }\} \geq \beta$$

and

$$\epsilon\alpha > \epsilon(\max\{|(Q_k)_{ij}| \text{ for all i,j,k }\}) \geq \epsilon|(Q_k)_{ij}|$$

we have for all i, j, k and $0 \leq \epsilon \leq \bar{\epsilon}$ that

$$(M_k)_{ij} > \epsilon|(Q_k)_{ij}|.$$

Thus, for all i, j, k and $0 \leq \epsilon \leq \bar{\epsilon}$,

$$(M_k)_{ij} \pm \epsilon(Q_k)_{ij} > 0,$$

and $\epsilon Q \in \mathcal{P}(x, M)$ for $0 \leq \epsilon \leq \bar{\epsilon}$. $\square$

COROLLARY 2.9. *If $P \in \mathcal{P}(x, M)$, then $tP \in \mathcal{P}(x, M)$ for $0 \leq t \leq 1$.*

PROOF. Let $P \in \mathcal{P}(x, M)$ and let $t \in [0, 1]$. Clearly $tP \in \mathcal{Q}(x)$. By definition of $\mathcal{P}(x, M)$, we know that for all k, $M_k \pm P_k > 0$. For $t \in [0, 1]$, for all i, j, k, $t|(P_k)_{ij}| \leq |(P_k)_{ij}|$. Thus, $M_k \pm tP_k > 0$ naturally follows, and $tP \in \mathcal{P}(x, M)$. $\square$

We now form the set $\overline{\mathcal{Q}(x)} \subset \mathcal{Q}(x)$ which we will use to prove various lemmas. To form this subset, first consider the $n-1$ dimensional space $(F(x))^{\perp} := [span\{F(x)\}]^{\perp}$.

Let $v \in (F(x))^{\perp} := \{u \in \mathbb{R}^n \mid u \cdot F(x) = 0\}$ with $v \neq 0$. Assume without loss of generality that $F_i(x) = 0$ for $i < k$ and $F_i(x) \neq 0$ for $i \geq k$. Note that $k \leq n$ since $F(x) \in \Lambda$. Select arbitrary integers i, j, $i \neq j$, with $1 \leq i < j \leq n$. We create a quadratic operator $Q = Q(i, j, v) := (Q_1, \ldots, Q_n)$. For $l \neq i, j$, define $Q_l = 0$, the zero matrix and let $Q_j = -Q_i$ with entries

$$(Q_j)_{rs} := \begin{cases} \frac{v_s}{F_k(x)} & r = k \text{ and } s \leq k-1 \\ \frac{v_r}{F_k(x)} & s = k \text{ and } r \leq k-1 \\ \frac{v_r}{F_r(x)} & r = s \text{ and } r > k-1 \\ 0 & \text{elsewhere} \end{cases}.$$

One can verify by a direct computation that $Q_j F(x) = v$ for $j = 1, \ldots, n$. We define $\overline{\mathcal{Q}(x)}$ as the set of quadratic maps created as above for $v \in (F(x))^{\perp}$:

$$\overline{\mathcal{Q}(x)} := \{Q(i, j, v) | v \in (F(x))^{\perp}, i, j \in \{1, \ldots, n\}\}.$$

LEMMA 2.10. $\{0\} \subset \overline{\mathcal{Q}(x)} \subset \mathcal{Q}(x)$.

Proof. By definition, since $0 \in (F(x))^{\perp}$, $Q = (0, \ldots, 0) \in \overline{\mathcal{Q}(x)}$. Now, let $Q \in \overline{\mathcal{Q}(x)}$ for some $v \in (F(x))^{\perp}$ with $v \neq 0$. By construction, Q is symmetric and $\sum Q_i = 0$. Similarly, it is clear that for $i = 1, \ldots, n$, $Q_i F(x) = v$. Thus, $[F(x)]^T Q_i F(x) = [F(x)]^T v = 0$ since $v \in (F(x))^{\perp}$. Thus, we have shown that for $Q \in \overline{\mathcal{Q}(x)}$, $Q \in \mathcal{Q}(x)$, finally proving $\{0\} \subset \overline{\mathcal{Q}(x)} \subset \mathcal{Q}(x)$. □

Note that Lemma 2.10 in conjunction with Lemma 2.8, show that for a GA map $G = M \circ F$ with fixed point $x \in \Lambda$, the corresponding set of quadratic operators $\mathcal{P}(x, M) \neq \{0\}$.

We now show that the above constructed set $\mathcal{P}(x, M)$ defines a collection of perturbations of M with the desired fixed point preserving property for the GA map (1.2). For $P \in \mathcal{P}(x, M)$, let $M_P := M + P$ and $G_P = M_P \circ F$.

Lemma 2.11. *Let $G = M \circ F$ be a GA map (1.2). Assume $x \in \Lambda$ is a fixed point of G. If $P \in \mathcal{P}(x, M)$, then $M_P = M + P$ satisfies*

1. *$M_P \in \mathcal{M}$*

2. *$G_P(x) = M_P \circ F(x) = x$.*

That is, G_P has the same fixed point x as G.

Proof. Let $P \in \mathcal{P}(x, M)$. Consider then a quadratic operator, $M_P = ([M_1 + P_1], \ldots, [M_n + P_n])$. To show $M_P \in \mathcal{M}$, we need the following:

1. $(M_P)_k \in Mat_n(\mathbb{R})$ and is symmetric. Since $(M_P)_k = M_k + P_k$ where $M_k \in Mat_n(\mathbb{R})$ and $P_k \in Mat_n(\mathbb{R})$ are each symmetric by definitions of $M \in \mathcal{M}$

and $P \in \mathcal{P}(p, M)$. Clearly, $(M_P)_k = M_k + P_k \in Mat_n(\mathbb{R})$ is symmetric for $k = 1, \ldots, n$.

2. $(M_P)_k > 0$ for all $k = 1, \ldots, n$. This follows from directly from the definition of $\mathcal{P}(x, M)$.

3. For all $i, j, k \in \{1, \ldots, n\}$ let $(M_k)_{ij}$ and $(P_k)_{ij}$ denote the ij-th entries of M_k and P_k respectively. We show $\sum_{k=1}^{n} [M_k]_{ij} = 1$ for all $j = 1, \ldots, n$ and $i = 1, \ldots, n$. Since $\mathcal{P}(x, M) \subset \mathcal{Q}(x)$, by (2) of the definition of $\mathcal{Q}(x)$, and (3) of the definition of $\mathcal{M}$,

$$\begin{aligned}\sum_{k=1}^{n} [[M_P]_k]_{ij} =& \sum_{k=1}^{n} [M_k + P_k]_{ij} \\ =& \sum_{k=1}^{n} (M_k)_{ij} + \sum_{k=1}^{n} (P_k)_{ij} \\ =& 1 + 0 \\ =& 1.\end{aligned}$$

Thus we have shown part 1: $M_P \in \mathcal{M}$.

Now, we prove part (2): $G_P(x) = M_P \circ F(x) = x$. Clearly,

$$\begin{aligned}G_P(x) =& M_P \circ F(x) \\ =& (M + P) \circ F(x) \\ =& M \circ F(x) + P \circ F(x) \\ =& G(x) + (F^T(x) P_1 F(x), \ldots, F^T(x) P_n F(x))^T.\end{aligned}$$

And by definition of $P \in \mathcal{P}(x, M)$, we know that $(F^T(x)P_1F(x), \ldots, F^T(x)P_nF(x)) = (0, \ldots, 0)$, thus

$$\begin{aligned} G_P(x) =& G(x) + (F^T(x)P_1F(x), \ldots, F^T(x)P_nF(x))^T \\ =& x + (0, \ldots, 0)^T \\ =& x. \end{aligned}$$

□

Corollary 2.12. *Let $G = M \circ F$ be a GA map (1.2). Assume $x \in \Lambda$ is a fixed point of G and $P^1, \ldots, P^k \in \mathcal{P}(x, M)$. There exists $\epsilon > 0$ such that*

1. *$\epsilon \sum_{i=1}^{n} P^i \in \mathcal{P}(x, M)$;*
2. *$M + \epsilon \sum_{i=1}^{k} P^i \in \mathcal{M}$;*
3. *$G_P := (M + \epsilon \sum_{i=1}^{k} P^i) \circ F$ admits the same fixed point x as the map $G = M \circ F$.*

Proof. For part (1), it suffices to show that if $P^1, \ldots, P^k \in \mathcal{P}(x, M)$, then $\sum_{i=1}^{k} P^i \in \mathcal{Q}(x)$, since by Lemma 2.8, it follows that there exists $\epsilon > 0$ such that $\epsilon \sum_{i=1}^{k} P^i \in \mathcal{P}(x, M)$. Once we have shown part (1), part (2) follows automatically. Further, to show part (3), if $\epsilon \sum_{i=1}^{k} P^i \in \mathcal{P}(x, M)$, by Lemma 2.11, $G_P := (M + \epsilon \sum_{i=1}^{k} P^i) \circ F$ admits the same fixed point x as the map $G = M \circ F$.

To show that if $P^1, \ldots, P^k \in \mathcal{P}(x, M)$, then $\sum_{i=1}^{k} P^i \in \mathcal{Q}(x)$, we show that parts (1)-(3) of the definition of $\mathcal{Q}(x)$ are satisfied.

1. For $i = 1, \ldots, k$, because $P^i \in \mathcal{P}(x, M) \subset \mathcal{Q}(x)$, it is clear that $P^i \in Mat_n(\mathbb{R})$, and is symmetric. Thus, $\sum_{i=1}^k P^i \in Mat_n(\mathbb{R})$ and is symmetric.

2. Similarly, since for each $i = 1, \ldots, k$; $P^i = (P_1^i, \ldots, P_n^i)$, and $\sum_{j=1}^n P_j^i = 0$, it follows that $\sum_{j=1}^n \sum_{i=1}^k P_j^i = 0$.

3. By definition of $\mathcal{P}(x, M)$, for $i = 1, \ldots, k$, and $j = 1, \ldots, n$, $[F(x)]^T P_j^i F(x) = 0$. Thus, for $i = 1, \ldots, k$,

$$[F(x)]^T (\sum_j P^i)_j F(x) = \sum_j [F(x)]^T (P^i)_j F(x) = 0.$$

So we have shown $\sum_{i=1}^k P^i \in \mathcal{P}(x, M)$, which leads to the desired result. $\square$

We observe that

$$\begin{aligned} G_P =& M_P \circ F \\ =& (M + P) \circ F \\ =& (M \circ F) + (P \circ F) \\ =& G + (P \circ F). \end{aligned}$$

Thus,

$$\begin{aligned} DG_P(x) =& D[G + (P \circ F)](x) \\ =& DG(x) + H \end{aligned} \tag{2.11}$$

where $H \in Mat_n(\mathbb{R})$. In order to trace the effects of perturbations of M on the derivative DG_P, we define

$$\mathcal{H} = \{H \in Mat_n(\mathbb{R}) | H = D(P \circ F)(x) \text{ for } P \in \mathcal{P}(x, M)\}.$$

We examine properties of this set in the section that follows.

Perturbations of DG

LEMMA 2.13. *Let* $G = M \circ F$ *be a GA map (1.2) with fixed point* $x \in \Lambda$ *and* $\mathrm{rank}(DF(x)) = n-1$. *There exists a perturbation* $M_P \in \mathcal{M}$ *with* $G_P = M_P \circ F$ *and* $DG_P(x) = DG(x) + H$ *such that* H *is of rank* $n-1$.

PROOF. That such an H exists can be shown by explicitly forming an operator $P \in \mathcal{P}(x, M)$ so that the corresponding $H \in \mathcal{H}$ has $\mathrm{rank}(H) = n-1$. To show there exists such a $P \in \mathcal{P}(x, M)$, we will find a corresponding $Q \in \mathcal{Q}(x)$ and then apply Lemma 2.8 to show there exists an ϵ such that $\epsilon Q \in \mathcal{P}(x, M)$.

Let $\{v_1, \ldots, v_{n-1}\}$ be a basis for the $n-1$ dimensional space $(F(x))^{\perp}$. Thus, for all $j \in \{1, \ldots, n-1\}$, $v_j \perp F(x)$ or equivalently, $v_j \circ F(x) = 0$.

Assume without loss of generality that $F_i(x) = 0$ for $i < k$ and $F_i(x) \neq 0$ for $i \geq k$. Note that $k \leq n$ since $F(x) \in \Lambda$. For $v_j \in (F(x))^{\perp}$, for $j = 1, \ldots, n-1$, we define the following matrices associated with the quadratic map Q. We create a quadratic operator $Q = (Q_1, \ldots, Q_n)$: for $j = 1, \ldots, n-1$ and corresponding vector

$v_j = ((v_j)_1, \ldots, (v_j)_n)$, define Q_j with entries

$$(Q_j)_{rs} := \begin{cases} \frac{(v_j)_s}{F_k(x)} & r = k \text{ and } s \leq k-1 \\ \frac{(v_j)_r}{F_k(x)} & s = k \text{ and } r \leq k-1 \\ \frac{(v_j)_r}{F_r(x)} & r = s \text{ and } r > k-1 \\ 0 & \text{elsewhere} \end{cases}.$$

Finally, define $Q_n = -\sum_{j=1}^{n-1} Q_j$. By computation it is clear that for $j = 1, \ldots, n-1$, $Q_j F(x) = v_j$. Thus,

$$Q_n F(x) = -\sum_{j=1}^{n-1} Q_j F(x) = -\sum_{j=1}^{n-1} v_j.$$

Of course, since $v_j \in (F(x))^{\perp}$, for $j = 1, \ldots, n-1$, $-\sum_{j=1}^{n-1} v_j \in (F(x))^{\perp}$. Finally, because $Q_i F(x) = v_i \in (F(x))^{\perp}$ for $i = 1, \ldots, n$, we see that property (3) of the definition of $\mathcal{Q}(x)$ is upheld as $F^T(x) Q_i F(x) = 0$ for all i.

By definition, each matrix $Q_i \in Mat_n(\mathbb{R})$ and is symmetric for $i = 1, \ldots, n$, thus showing part (1) of the definition of $\mathcal{Q}(x)$. By construction, $\sum_{i=1}^{n} Q_i = 0$, thus showing part (2) of the definition of $\mathcal{Q}(x)$. Finally, we have shown for the Q constructed above that properties (1)-(3) of the definition of $\mathcal{Q}(x)$ hold. Thus, $Q \in \mathcal{Q}(x)$. By Lemma 2.8, since $Q \in \mathcal{Q}(x)$, there exists a corresponding ϵ such that $x := \epsilon Q \in \mathcal{P}(x, M)$.

Finally, we must show that $\text{rank}(H) = n-1$ where H corresponds to $P = \epsilon Q$. Recall that by equation (2.11),

$$H = D[(P \circ F)](x)$$

$$=2\begin{pmatrix}(P_1F(x))^T\\(P_2F(x))^T\\\vdots\\(P_nF(x))^T\end{pmatrix}DF(x). \tag{2.12}$$

Observe for $k < n$,

$$P_kF(x) = \epsilon Q_kF(x) = \epsilon v_k.$$

And, since $Q_n = -\sum_{k=1}^{n-1} Q_k$,

$$\begin{aligned}P_nF(x) =&\epsilon Q_nF(x)\\ =&\epsilon\left(-\sum_{k=1}^{n-1} Q_k\right)F(x)\\ =&\epsilon\left(-\sum_{k=1}^{n-1} v_k\right)\\ =&-\epsilon\sum_{k=1}^{n-1} v_k.\end{aligned}$$

Thus, from equation (2.12), we see that

$$H = 2\epsilon\begin{pmatrix}v_1\\v_2\\\vdots\\v_{n-1}\\-\sum_{k=1}^{n-1} v_k\end{pmatrix}DF(x). \tag{2.13}$$

Since $v_1, \ldots, v_{n-1}$ form a basis of $(F(x))^\perp$, $\text{rank}(v_1, \ldots, v_{n-1}, -\sum_{k=1}^{n-1} v_k) = n-1$.

Furthermore, by construction of $v_1, \ldots, v_{n-1}$,

$$\text{null}\begin{pmatrix}v_1\\v_2\\\vdots\\v_{n-1}\\-\sum_{k=1}^{n-1} v_k\end{pmatrix} = \text{span}[(F(x))^T]. \tag{2.14}$$

By Lemma 2.6, $\text{span}(F(x)) \nsubseteq \text{Image}(DF(x))$. Thus, because H is the product of the two matrices given in equation (2.13), the rank of H is also $n-1$. $\square$

Lemma 2.14. *Let $G = M \circ F$ be a GA map. Assume $G(x) = x$ for some $x \in \Lambda$ and that* $\operatorname{rank}(\mathrm{DF(x)}) = n-1$. *Let $\mathcal{R} := \{h | h = v^T DF(x),\ \text{and } v \in F(x)^\perp\}$. Then*

1. *$\mathcal{R}$ has dimension $n-1$;*
2. *$\mathcal{R}^\perp = \operatorname{span}(1, \ldots, 1)$.*

Proof. We first prove part (1). By assumption, $\operatorname{rank}(DF(x)) = n-1$, so if

$$\operatorname{null}(DF(x))^T \cap (F(x))^\perp = \emptyset$$

then $\dim(\mathrm{Image}(\mathrm{DF(x)})) = n-1$. We now show

$$\operatorname{null}(DF(x))^T \cap (F(x))^\perp = \emptyset.$$

By Lemma 2.3, we know that $(\mathrm{Image}(DF(x)))^\perp = \operatorname{span}(1, \ldots, 1)$, and by the Fredholm alternative, $\operatorname{null}(DF(x))^T = \mathrm{Image}(DF(x))^\perp = \operatorname{span}(1, \ldots, 1)$. That is, $\operatorname{null}(DF(x))^T = \operatorname{span}(1, \ldots, 1)$. Take $v \in \operatorname{null}(DF(x))^T \cap (F(x))^\perp$, then $v = \alpha(1, \ldots, 1)$ and $\alpha \neq 0$. Now $v \perp F(x)$. Then

$$v \cdot F(x) = \alpha(1, \ldots, 1) \cdot F(x) = \alpha \sum_i F_i(x) = \alpha \neq 0,$$

which is a contradiction, thus $\operatorname{null}(DF(x))^T \cap F(x)^\perp = \emptyset$. This proves part (1).

The proof of part (2) follows directly from Lemma 2.3. □

Lemma 2.15. *Let $G = M \circ F$ be a GA map with fixed point x. For all $x \in \Lambda$ assume* $\operatorname{rank}(DF(x)) = n-1$. *Given $h \in \mathcal{R}$ with $h \neq 0$, for all $1 \leq i < j \leq n$ there exists $H \in \mathcal{H}$ such that*

1. $h = h_j = -h_i \neq 0$;

2. $h_k = 0$ *for* $k \neq i, j$.

PROOF. That such an H exists can be shown by explicitly forming an operator $P \in \mathcal{P}(x, M)$ so that the corresponding $H \in \mathcal{H}$ with $H = PF(x)DF(x)$ has the desired properties.

Let $v \in (F(x))^\perp$ with $v \neq 0$. Define $Q \in \overline{\mathcal{Q}}(x)$ as in equation (2.11). By Lemma 2.8, there exists $\epsilon > 0$ such that $P := \epsilon Q \in \mathcal{P}(x, M)$.

Finally, we show that $H = DP \circ F(x) = DP(F(x))DF(x)$ has the advertised properties. Since

$$h_l = (P_l F(x))^T DF(x)$$

we have $h_l = 0$ for $l \neq i, j$. For $l = i$,

$$h_i = (P_i F(x))^T DF(x) = v^T DF(x).$$

Clearly, $h_j = -h_i$.

$\square$

LEMMA 2.16. *Let* $G = M \circ F$ *be a GA map (1.2). If* $H \in \mathcal{H}$, *then for all* $t \in [0, 1]$, $tH \in \mathcal{H}$.

PROOF. Let $H \in \mathcal{H}$, then by definition there exists $P \in \mathcal{P}(x, M)$ such that $G_P = (M + P) \circ F$ and $DG_P(x) = DG(x) + H$ where

$$H = D(P \circ F(x)) = P(F(x))DF(x).$$

Now, by Corollary 2.9, for all $t \in [0,1]$, $tP \in \mathcal{P}(x, M)$, and

$$DG_{tP}(x) = DG(x) + tP(F(x))DF(x) = DG(x) + tH.$$

Thus, by definition of $\mathcal{H}$, there exists $tP \in \mathcal{P}(x, M)$ such that $DG_{tP}(x) = DG(x)+tH$. That is, $tH \in \mathcal{H}$. □

COROLLARY 2.17. *Let $G = M \circ F$ be a GA map (1.2). Assume $G(x) = x$ for $x \in \Lambda$. Let $H \in \mathcal{H}$, $B(\lambda) := DG(x) - \lambda I$ and $\epsilon \in (0,1)$. Then*

$$\begin{aligned}\det(B(\lambda) + \epsilon H) = \det B(\lambda) &+ \epsilon h_1 \cdot (B_{11}(\lambda), -B_{12}(\lambda), B_{13}(\lambda), \ldots, \pm B_{1n}(\lambda)) \\ &+ \epsilon h_2 \cdot (-B_{21}(\lambda), B_{22}(\lambda), -B_{23}(\lambda), \ldots, \pm B_{2n}(\lambda)) \\ &+ \cdots + \epsilon h_n \cdot (\pm B_{n1}(\lambda), \ldots, \pm B_{nn}(\lambda)) + \mathcal{O}(\epsilon^2),\end{aligned}$$

where $B_{ij}(\lambda)$ denotes the determinant of the ij-th minor for the matrix $B(\lambda)$.

PROOF. The result follows directly from Lemma 7.2. □

Perturbations of DG in Jordan Normal Form

We will often use a change of basis to obtain the Jordan normal form for the matrix $DG(x)$. Let $C \in Mat_n(\mathbb{R})$ be the change of basis matrix so that $C^{-1}[DG(x)]C$ is in Jordan normal form. We observe that

$$\begin{aligned}C^{-1}[DG(x) - \lambda I + \epsilon H]C =& C^{-1}[DG(x)]C - \lambda I + \epsilon C^{-1}HC \\ =& \widetilde{DG}(x) - \lambda I + \epsilon K\end{aligned}$$

where $\widetilde{DG}(x)$ denotes $DG(x)$ in the Jordan normal form. We define the set $\mathcal{K}$,

$$\mathcal{K} := \{K | \text{ there exists } H \in \mathcal{H} \text{ such that } K = C^{-1}HC\}.$$

In general, we will use an ordering of rows such that the eigenvalue of interest appears in the top left corner of the matrix $\widetilde{DG}(x)$. Note that c_i^{-1} denotes the i-th row of the matrix C^{-1} and c_{ij}^{-1} denotes the ij-th entry of the matrix C^{-1}. Recall that $B(\lambda) = DG(x) - \lambda I$ and we define $\widetilde{B}(\lambda) := \widetilde{DG}(x) - \lambda I$.

LEMMA 2.18. *For $K \in \mathcal{K}$ with $K = C^{-1}HC$ for $H \in \mathcal{H}$ with rows $h_1, \ldots, h_n$,*

$$k_{ij} = c_{i1}^{-1}(h_1 \cdot c^j) + c_{i2}^{-1}(h_2 \cdot c^j) + \cdots + c_{in}^{-1}(h_n \cdot c^j).$$

Here c^j denotes j-th column of C and c_{ij}^{-1} denotes the ij-th entry of C^{-1}.

PROOF. Let h_i denote i-th row of the matrix H, c^j denote j-th column of C, and c_k^{-1} denote the k-th row of the matrix C^{-1}. Then,

$$\begin{aligned}
k_{ij} &= [C^{-1}HC]_{ij} \\
&= [C^1(HC)]_{ij} \\
&= \left[C^{-1} \begin{pmatrix} h_1 \cdot c^1 & h_1 \cdot c^2 & \ldots & h_1 \cdot c^n \\ h_2 \cdot c^1 & h_2 \cdot c^2 & \ldots & h_2 \cdot c^n \\ \vdots & \vdots & \ddots & \vdots \\ h_n \cdot c^1 & h_n \cdot c^2 & \ldots & h_n \cdot c^n \end{pmatrix} \right]_{ij} \\
&= c_i^{-1} \cdot (h_1 \cdot c^j, h_2 \cdot c^j, \ldots, h_n \cdot c^j) \\
&= c_{i1}^{-1}(h_1 \cdot c^j) + c_{i2}^{-1}(h_2 \cdot c^j) + \cdots + c_{in}^{-1}(h_n \cdot c^j).
\end{aligned}$$

□

Corollary 2.19. *Let $K \in \mathcal{K}$ with $K = C^{-1}HC$ where H is defined as in Lemma 2.15, that is, for all $1 \le i < j \le n$,*

1. $h_j = -h_i \neq 0$;

2. $h_k = 0$ *for* $k \neq i, j$.

Then, $k_{11} = (-c_{1j}^{-1} + c_{1i}^{-1})(h \cdot c^1)$, $k_{12} = (-c_{1j}^{-1} + c_{1i}^{-1})(h \cdot c^2)$, $k_{21} = (-c_{2j}^{-1} + c_{2i}^{-1})(h \cdot c^1)$, *and* $k_{22} = -(-c_{2j}^{-1} + c_{2i}^{-1})(h \cdot c^2)$. *Here* c^j *denotes* j*-th column of* C *and* c_{ij}^{-1} *denotes the* ij*-th entry of* C^{-1}.

Proof. $H \in \mathcal{H}$ is of the form:

$$H = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -h \\ 0 \\ \vdots \\ 0 \\ h \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{2.15}$$

where $h \neq 0$. By Lemma 2.18,

$$\begin{aligned} k_{11} =& c_1^{-1} \cdot (h_1 \cdot c^1, h_2 \cdot c^1, \dots, h_n \cdot c^1) \\ =& c_1^{-1} \cdot (0, \dots, 0, -h \cdot c^1, 0, \dots, 0, h \cdot c^1, 0, \dots, 0) \\ =& (-c_{1j}^{-1} + c_{1i}^{-1})(h \cdot c^1) \end{aligned}$$

and

$$k_{22} = c_2^{-1} \cdot (h_1 \cdot c^2, h_2 \cdot c^2, \dots, h_n \cdot c^2)$$

$$= \left[c_2^{-1} \cdot (0, \dots, 0, -h \cdot c^2, 0, \dots, 0, h \cdot c^2, 0, \dots, 0)\right]$$

$$= k_{22} = -(-c_{2j}^{-1} + c_{2i}^{-1})(h \cdot c^2).$$

Once again, using Lemma 2.18, we arrive at the remaining desired results: $k_{12} = (-c_{1j}^{-1} + c_{1i}^{-1})(h \cdot c^2)$ and $k_{21} = (-c_{2j}^{-1} + c_{2i}^{-1})(h \cdot c^1)$. □

LEMMA 2.20. *Assume that for all $K \in \mathcal{K}$,*

$$(k_{11} + k_{22}) = 0 \text{ and } (k_{21} - k_{12}) = 0.$$

Then for all $h \in \mathcal{R}$,

$$(-c_{1j}^{-1} + c_{1k}^{-1})(h \cdot c^1) = -(-c_{2j}^{-1} + c_{2k}^{-1})(h \cdot c^2) \tag{2.16}$$

and

$$(-c_{1j}^{-1} + c_{1k}^{-1})(h \cdot c^2) = (-c_{2j}^{-1} + c_{2k}^{-1})(h \cdot c^1). \tag{2.17}$$

Here c^j denotes j-th column of C and c_{ij}^{-1} denotes the ij-th entry of C^{-1}.

PROOF. By Lemma 2.18,

$$k_{11} = c_1^{-1} \cdot (h_1 \cdot c^1, h_2 \cdot c^1, \dots, h_n \cdot c^1)$$

and

$$k_{22} = c_2^{-1} \cdot (h_1 \cdot c^2, h_2 \cdot c^2, \dots, h_n \cdot c^2).$$

Since $k_{11} + k_{22} = 0$, this implies $k_{11} = -k_{22}$, or equivalently,

$$c_1^{-1} \cdot (h_1 \cdot c^1, h_2 \cdot c^1, \dots, h_n \cdot c^1) = -\left[c_2^{-1} \cdot (h_1 \cdot c^2, h_2 \cdot c^2, \dots, h_n \cdot c^2)\right].$$

Similarly, by Lemma 2.18, and the assumption that $k_{12} - k_{21} = 0$,

$$k_{12} = c_1^{-1} \cdot (h_1 \cdot c^2, h_2 \cdot c^2, \ldots, h_n \cdot c^2) = \left[c_2^{-1} \cdot (h_1 \cdot c^1, h_2 \cdot c^1, \ldots, h_n \cdot c^1)\right] = k_{21}.$$

Since

$$(k_{11} + k_{22}) = 0 \text{ and } (k_{21} - k_{12}) = 0$$

for all $K \in \mathcal{K}$, then the result follows from Corollary 2.19. $\square$

CHAPTER 3

PERTURBING THE CHARACTERISTIC POLYNOMIAL

Simple Eigenvalue Case

For $G = M \circ F$ with fixed point $x \in \Lambda$, we first consider the case where $DG(x)$ has a simple eigenvalue λ_0 of absolute value one. Clearly, either $\lambda_0 \in \mathbb{C} \backslash \mathbb{R}$ or $\lambda_0 \in \mathbb{R}$. In each case, there exists a change of basis matrix, C, such that $\widetilde{DG}(x) := C^{-1}DG(x)C$ is a Jordan normal form of the matrix $DG(x)$ ([29]):

$$\widetilde{DG}(x) = \begin{pmatrix} \lambda_0 & 0 & \dots & 0 \\ 0 & * & * & * \\ \vdots & * & * & * \\ 0 & * & * & *, \end{pmatrix}$$

or

$$\widetilde{DG(x)} = \begin{pmatrix} \alpha & \beta & 0 & \dots & 0 \\ -\beta & \alpha & 0 & \dots & 0 \\ 0 & 0 & * & * & * \\ \vdots & \vdots & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix}. \tag{3.1}$$

We note that for $P \in \mathcal{P}(x, M)$,

$$DG_P(x) = DG(x) + H \text{ implies } \widetilde{DG}_P(x) = \widetilde{DG}(x) + K$$

and

$$\{\lambda | \det(DG_P(x) - \lambda I) = 0\} = \{\lambda | \det(\widetilde{DG}_P(x) - \lambda I) = 0\},$$

thus

$$\{\lambda | \det(DG(x) + H - \lambda I) = 0\} = \{\lambda | \det(\widetilde{DG}(x) + K - \lambda I) = 0\}.$$

We want to find $H \in \mathcal{H}$ such that $\det(DG(x) - \lambda I + H) \neq 0$ for all $\lambda \in \mathcal{S}^1$. To do so, we find $K = C^{-1}HC$ for $K \in \mathcal{K}$ such that $\det(\widetilde{DG}(x) - \lambda I + K) \neq 0$ for all $\lambda \in \mathcal{S}^1$.

Recall that $B(\lambda) = DG(x) - \lambda I$ and $\widetilde{B}(\lambda) = \widetilde{DG}(x) - \lambda I$. Note that for $\lambda_0 = \alpha + \beta i \in \mathbb{C}\backslash\mathbb{R}$, by equation (3.1),

$$\widetilde{B}(\lambda_0) = \widetilde{B}(\alpha+\beta i) = \begin{pmatrix} \alpha - (\alpha+\beta i) & \beta & 0 & \dots & 0 \\ -\beta & \alpha - (\alpha+\beta i) & 0 & \dots & 0 \\ 0 & 0 & * & * & * \\ \vdots & \vdots & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix}.$$

That is,

$$\widetilde{B}(\lambda_0) = \begin{pmatrix} -\beta i & \beta & 0 & \dots & 0 \\ -\beta & -\beta i & 0 & \dots & 0 \\ 0 & 0 & * & * & * \\ \vdots & \vdots & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix}. \tag{3.2}$$

LEMMA 3.1. *Let $G = M \circ F$ be a GA map (1.2). Assume $G(x) = x$ for $x \in \Lambda$. Let $\epsilon \in (0,1)$, $K \in \mathcal{K}$ and $B(\lambda) := DG(x) - \lambda I$ where λ_0 is a simple eigenvalue of $DG(x)$. Then*

1. *If $\lambda_0 \in \mathbb{R}$,*

$$\det(\widetilde{B}(\lambda_0) + \epsilon K) = \epsilon k_{11}\widetilde{B}_{11}(\lambda_0) + \mathcal{O}(\epsilon^2);$$

2. *If $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$,*

$$\det(\widetilde{B}(\lambda_0) + \epsilon K) = \epsilon[(k_{11} + k_{22})\widetilde{B}_{11}(\lambda_0) + (k_{21} - k_{12})\widetilde{B}_{12}(\lambda_0)] + \mathcal{O}(\epsilon^2).$$

PROOF. We first prove part (1). By Corollary 2.17 and Lemma 7.5,

$$\det(\widetilde{B}(\lambda_0) + \epsilon K) = \det\widetilde{B}(\lambda_0) + \epsilon k_1 \cdot (\widetilde{B}_{11}(\lambda_0), -\widetilde{B}_{12}(\lambda_0), \widetilde{B}_{13}(\lambda_0), \dots, \pm\widetilde{B}_{1n}(\lambda_0))$$

$$+ \epsilon k_2 \cdot (-\widetilde{B}_{21}(\lambda_0), \widetilde{B}_{22}(\lambda_0), -\widetilde{B}_{23}(\lambda_0), \ldots, \pm\widetilde{B}_{2n}(\lambda_0))$$

$$+ \cdots + \epsilon k_n \cdot (\pm\widetilde{B}_{n1}(\lambda_0), \ldots, \pm\widetilde{B}_{nn}(\lambda_0)) + \mathcal{O}(\epsilon^2)$$

$$= 0 + \epsilon k_1 \cdot (\widetilde{B}_{11}(\lambda_0), 0, \ldots, 0)$$

$$+ \epsilon k_2 \cdot (0, 0, \ldots, 0)$$

$$+ \cdots + \epsilon k_n \cdot (0, \ldots, 0) + \mathcal{O}(\epsilon^2)$$

$$= \epsilon k_{11} \widetilde{B}_{11}(\lambda_0) + \mathcal{O}(\epsilon^2).$$

proving the result.

We now prove part (2). Recall that according to Corollary 2.17,

$$\begin{aligned}\det(\widetilde{B}(\lambda_0) + \epsilon K) =& \det\widetilde{B}(\lambda_0) + \epsilon k_1 \cdot (\widetilde{B}_{11}(\lambda_0), -\widetilde{B}_{12}(\lambda_0), \widetilde{B}_{13}(\lambda_0), \ldots, \pm\widetilde{B}_{1n}(\lambda_0)) \\ &+ \epsilon k_2 \cdot (-\widetilde{B}_{21}(\lambda_0), \widetilde{B}_{22}(\lambda_0), -\widetilde{B}_{23}(\lambda_0), \ldots, \pm\widetilde{B}_{2n}(\lambda_0)) \\ &+ \cdots + \epsilon k_n \cdot (\pm\widetilde{B}_{n1}(\lambda_0), \ldots, \pm\widetilde{B}_{nn}(\lambda_0)) + \mathcal{O}(\epsilon^2).\end{aligned}$$

Because λ_0 is an eigenvalue of $DG(x)$, $\det(\widetilde{B}(\lambda_0)) = 0$, and by equation (3.2) and Lemma 7.9,

$$\widetilde{B}_{11}(\lambda_0) = \widetilde{B}_{22}(\lambda_0) \text{ and } \widetilde{B}_{12}(\lambda_0) = -\widetilde{B}_{12}(\lambda_0).$$

Thus,

$$\begin{aligned}\det(\widetilde{B}(\lambda_0) + \epsilon K) &= \det(\widetilde{B}(\lambda_0)) + \epsilon k_1 \cdot (\widetilde{B}_{11}(\lambda_0), -\widetilde{B}_{12}(\lambda_0), 0, \ldots, 0) \\ &\quad + \epsilon k_2 \cdot (-\widetilde{B}_{21}(\lambda_0), \widetilde{B}_{22}(\lambda_0), 0, \ldots, 0) + \epsilon k_3 \cdot 0 + \cdots + k_n \cdot 0 + \mathcal{O}(\epsilon^2) \\ &= \epsilon k_1 \cdot (\widetilde{B}_{11}(\lambda_0), -\widetilde{B}_{12}(\lambda_0), 0, \ldots, 0) + \epsilon k_2 \cdot (-\widetilde{B}_{21}(\lambda_0), \widetilde{B}_{22}(\lambda_0), 0, \ldots, 0) \\ &\quad + \mathcal{O}(\epsilon^2).\end{aligned}$$

By a short computation, we arrive at the desired result,

$$\begin{aligned}\det(\widetilde{B}(\lambda_0)+\epsilon K) =&\epsilon[k_{11}\widetilde{B}_{11}(\lambda_0)-k_{12}\widetilde{B}_{12}(\lambda_0)+k_{21}\widetilde{B}_{21}(\lambda_0)+k_{22}\widetilde{B}_{11}(\lambda_0)]+\mathcal{O}(\epsilon^2)\\ =&\epsilon[(k_{11}+k_{22})\widetilde{B}_{11}(\lambda_0)+(k_{21}-k_{12})\widetilde{B}_{12}(\lambda_0)]+\mathcal{O}(\epsilon^2).\end{aligned}$$

□

Lemma 3.1 motivates us to find $H \in \mathcal{H}$ or corresponding $K \in \mathcal{K}$ so that $\det(\widetilde{DG}(x)+\epsilon K-\lambda_0 I) = \det(\widetilde{B}(\lambda_0+\epsilon K)) \neq 0$. But because Lemma 3.1 is split into two parts based on whether λ_0 is real or complex, we split our exploration into two parts:

- If $\lambda_0 \in \mathbb{R}$, we search for $K \in \mathcal{K}$ such that $k_{11} \neq 0$. We prove that such a $K \in \mathcal{K}$ exists in Lemma 3.4. To prove Lemma 3.4 we use Lemmas 3.2 and 3.3.
- If $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$, we search for $K \in \mathcal{K}$ such that $(k_{11}+k_{22})$ and $(k_{21}-k_{12})$ are not both zero. We prove that such a $K \in \mathcal{K}$ exists in Lemma 3.8. To prove Lemma 3.8, we use Corollary 3.5, and Lemmas 3.6 and 3.7.

Lemma 3.2. *Let $G = M \circ F$ be a GA map (1.2). Assume $G(x) = x$ for $x \in \Lambda$, with $\lambda_0 \in \mathbb{R}$ a simple eigenvalue of $DG(x)$ corresponding to F with* $\operatorname{rank}(DF(x)) = n-1$. *There exists $h \in \mathcal{R}$ with $h \cdot c^1 \neq 0$, where c^1 denotes the first column of the change of basis matrix C.*

PROOF. Proof by contradiction. Assume for all $h \in \mathcal{R}$, that $h \perp c^1$. By definition, since $h \in \mathcal{R}$,

$$h = v^T DF(x) = (v \cdot (DF(x))^1, v \cdot (DF(x))^2, \ldots, v \cdot (DF(x))^n)$$

for some v with $v \perp F(x)$. Note that λ_0 is simple, and C is the change of basis matrix such that $[C^{-1}DG(x)C]_{11} = \lambda_0$. Thus, c^1 is the eigenvector of $DG(x)$ corresponding to λ_0.

Because, $h = v^T DF(x)$ for some $v \perp F(x)$,

$$h \perp c^1 \Longleftrightarrow v^T DF(x) \perp c^1 \text{ for all } v \perp F(x).$$

By definition,

$$h \perp c^1 \iff (v \cdot (DF(x))^1, v \cdot (DF(x))^2, \ldots, v \cdot (DF(x))^n) \cdot c^1 = 0 \text{ for all } v \perp F(x).$$

Finally,

$$h \perp c^1 \iff v \cdot (c_1^1 (DF(x))^1, c_2^1 (DF(x))^2, \ldots, c_n^1 (DF(x))^n) = 0 \text{ for all } v \perp F(x),$$

and by definition of $[DF(x)]c^1$,

$$h \perp c^1 \iff [DF(x)]c^1 \cdot v = 0 \text{ for all } v \perp F(x).$$

Since $[[DF(x)]c^1] \cdot v = 0$ for all $v \perp F(x)$ if and only if $DF(x)c^1 = \alpha F(x)$ for some $\alpha \in \mathbb{R}$,

$$h \perp c^1 \iff DF(x)c^1 = \alpha F(x) \text{ for some } \alpha \in \mathbb{R}.$$

If $\alpha = 0$, then $DF(x)u = 0$ which cannot correspond to an eigenvector c^1 of $DG(x) = 2M(F(x))DF(x)$ of a nonzero eigenvalue λ_0. Thus, $\alpha \neq 0$. Note further that by Lemma 2.2, $DF(x)u \cdot (1, \ldots, 1) = 0$. But, if $DF(x)u = \alpha F(x)$ for $\alpha \neq 0$, then since $F(x) \in \Lambda$,

$$0 = DF(x)u \cdot (1, \ldots, 1) = \alpha F(x) \cdot (1, \ldots, 1) = \alpha \sum_{i=1}^{n} F_i(x) = \alpha,$$

a contradiction. □

Lemma 3.3. *Let $G = M \circ F$ be a GA map (1.2). Assume $G(x) = x$ for $x \in \Lambda$, with $\lambda_0 \in \mathbb{R}$ a simple eigenvalue of $DG(x)$ corresponding to F with* $\text{rank}(DF(x)) = n - 1$. *If $k_{11} = 0$ for all $K \in \mathcal{K}$, then c_1^{-1} is a non-zero multiple of the vector* $(1, \ldots, 1)$. *That is, $c_1^{-1} = a(1, 1, \ldots, 1)$, $a \in \mathbb{R} \backslash \{0\}$.*

Proof. Assume $k_{11} = 0$ for all $K \in \mathcal{K}$, that is, for all $H \in \mathcal{H}$, $k_{11} = [C^{-1}HC]_{11} = 0$. By Lemma 3.2, there exists h such that $h \cdot c^1 \neq 0$, where c^1 is the eigenvector associated with λ_0. We construct H as in Lemma 2.15 with $i = 1$ and $j = 2$:

$$H = \begin{pmatrix} h \\ -h \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

By Lemma 2.15, $H \in \mathcal{H}$, and, by assumption $k_{11} = 0$. Thus, by Lemma 2.18,

$$0 = k_{11} = ((c^{-1})_{11}h + (c^{-1})_{12}(-h)) \cdot c^1$$

$$= ((c^{-1})_{11} - (c^{-1})_{12})(h \cdot c^1)$$

$$= 0.$$

Thus $(c^{-1})_{11} = (c^{-1})_{12}$, since $h \cdot c^1 \neq 0$. Similarly, construct $H \in \mathcal{H}$ as in Lemma 2.15 with $i = 1$ and $j = 3$,

$$H = \begin{pmatrix} h \\ 0 \\ -h \\ \vdots \\ 0 \end{pmatrix},$$

with corresponding k_{11}:

$$\begin{aligned} k_{11} &= ((c^{-1})_{11}h + (c^{-1})_{13}(-h)) \cdot c^1 \\ &= ((c^{-1})_{11} - (c^{-1})_{13})(h \cdot c^1) \\ &= 0. \end{aligned}$$

Thus $(c^{-1})_{11} = (c^{-1})_{13}$. Continuing in this manner, we get

$$(c^{-1})_{11} = (c^{-1})_{12} = (c^{-1})_{13} = \cdots = (c^{-1})_{1n}.$$

Thus, we have

$$\begin{aligned} c^{-1} =& ((c^{-1})_{11}, (c^{-1})_{12}, (c^{-1})_{13}, \ldots, (c^{-1})_{1n}) \\ =& (a, a, a, \ldots, a) \\ =& a(1, \ldots, 1), \end{aligned}$$

where $a = (c^{-1})_{11}$. Finally, note that $a \neq 0$ by Lemma 7.6. Thus, we have the desired result that $c_1^{-1} = a(1, 1, \ldots, 1)$. □

Lemma 3.4. *Let* $G = M \circ F$ *be a GA map. Assume* $G(x) = x$ *for* $x \in \Lambda$, *with* $\lambda_0 \in \mathbb{R}\backslash\{0\}$ *a simple eigenvalue of* $DG(x)$ *corresponding to* F *with* $\operatorname{rank}(DF(x)) = n-1$. *There exists* $H \in \mathcal{H}$ *such that* $K = C^{-1}HC$ *has* $k_{11} \neq 0$. *That is, there exists* $K \in \mathcal{K}$ *such that* $\det(\widetilde{DG}(x) + \epsilon K - \lambda_0 I) \neq 0$.

Proof. Proof by contradiction. Assume for all $K \in \mathcal{K}$, $k_{11} = 0$. Recall that C is the change of basis matrix corresponding to $DG(x)$ in Jordan normal form, with c^1 is the eigenvector corresponding to λ_0. By Lemma 3.3, $c_1^{-1} = a(1, \ldots, 1)$ for some non-zero $a \in \mathbb{R}$. Because $C^{-1}C = I$,

$$1 = c_1^{-1} \cdot c^1 = a(1, \ldots, 1) \cdot c^1. \tag{3.3}$$

Because λ_0 is a simple eigenvalue of $DG(x)$,

$$DG(x)c^1 = \lambda_0 c^1. \tag{3.4}$$

Thus, by equations (3.3) and (3.4),

$$\begin{aligned}(DG(x)c^1) \cdot (1, \ldots, 1) =& \lambda_0 c^1 \cdot (1, \ldots, 1) \\ =& \frac{\lambda_0}{a}.\end{aligned}$$

Thus $(DG(x)c^1) \cdot (1, \ldots, 1) \neq 0$. This contradicts Corollary 2.4 which states that for all $v \in \mathbb{R}^n$, $(DG(x)v) \cdot (1, \ldots, 1) = 0$. Finally, by Lemma 3.1 we see there exists $K \in \mathcal{K}$ such that $\det(\widetilde{DG}(x) + \epsilon K - \lambda_0 I) \neq 0$. □

Thus, we have shown that if $\lambda_0 \in \mathbb{R}$, that there exists $K \in \mathcal{K}$ such that $k_{11} \neq 0$ and therefore $\det(\widetilde{DG}(x) + \epsilon K - \lambda_0 I) \neq 0$ for this particular K.

We now proceed to the complex eigenvalue case. We prove that if $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$, there exists $K \in \mathcal{K}$ such that $(k_{11}+k_{22})$ and $(k_{21}-k_{12})$ are not both zero, and therefore $\det(\widetilde{DG}(x)+\epsilon K-\lambda_0 I)\neq 0$ for this particular K.

Corollary 3.5. *Let $G = M\circ F$ be a GA map. Assume $G(x)=x$ for some $x\in\Lambda$. Let $B(\lambda):=DG(x)-\lambda I$ where $\lambda_0\in\mathbb{C}\backslash\mathbb{R}$ is a simple eigenvalue of $DG(x)$. Then $\widetilde{B}_{11}(\lambda_0)=i\widetilde{B}_{12}(\lambda_0)$.*

Proof. Since $B(\lambda_0)$ is of the form given in equation (3.2), we apply Lemma 7.8 to obtain

$$\widetilde{B}_{11}(\lambda_0) =(-\beta i)\det(\widetilde{B}(\lambda_0)(1,1)(1,1)) \tag{3.5}$$

and

$$\widetilde{B}_{12}(\lambda_0) =(-\beta)\det(\widetilde{B}(\lambda_0)(1,2)(1,1)). \tag{3.6}$$

Recall that $\widetilde{B}(i,j)(k,l)$ denotes the matrix obtained by deleting row k and column l from the matrix $\widetilde{B}(i,j)$. $\widetilde{B}(i,j)$, in turn, was first obtained by deleting row i and column j from matrix $\widetilde{B}$. Because λ_0 is a simple eigenvalue,

$$\det(\widetilde{B}(\lambda_0)(1,1)(1,1))\neq 0\neq \det(\widetilde{B}(\lambda_0)(1,2)(1,1)).$$

Since $\widetilde{B}(\lambda_0)$ is of the form given in equation (3.2),

$$\det(\widetilde{B}(\lambda_0)(1,1)(1,1)) = \det(\widetilde{B}(\lambda_0)(1,2)(1,1)). \tag{3.7}$$

Thus, by equations (3.5) (3.6), and (3.7),

$$\widetilde{B}_{11}(\lambda_0)=i\widetilde{B}_{12}(\lambda_0).$$

□

LEMMA 3.6. *Let $G = M \circ F$ be a GA map (1.2). Assume $G(x) = x$ for some $x \in \Lambda$, with $\lambda_0 \in (\mathbb{C}\backslash\mathbb{R})\backslash\{0\}$ a simple eigenvalue of $DG(x)$ corresponding to F with* $\text{rank}(DF(x)) = n-1$. *There exists $h \in \mathcal{R}$ with $h \cdot c^2 \neq 0$, where c^2 denotes the second column of the change of basis matrix C corresponding to Jordan normal form of $DG(x)$.*

PROOF. Assume, to the contrary, that for all $h \in \mathcal{R}$ that h is perpendicular to c^2. Note that c^2 is the imaginary part of the complex eigenvector corresponding to $\lambda_0 = \alpha + \beta i$ with $\beta \neq 0$. By definition, for any $h \in \mathcal{R}$, there exists $v \in (F(x))^{\perp}$, such that

$$\begin{aligned} h =& v^T DF(x) \\ =& (v \cdot (DF(x))^1, v \cdot (DF(x))^2, \ldots, v \cdot (DF(x))^n). \end{aligned}$$

That is, $h = v^T DF(x)$ for some $v \perp F(x)$. Since $h = v^T DF(x)$ for some $v \perp F(x)$,

$$h \perp c^2 \iff v^T DF(x) \perp c^2 \text{ for all } v \perp F(x).$$

By definition,

$$h \perp c^2 \iff (v \cdot (DF(x))^1, v \cdot (DF(x))^2, \ldots, v \cdot (DF(x))^n) \cdot c^2 = 0 \text{ for all } v \perp F(x).$$

By computation,

$$h \perp c^2 \iff v \cdot (c_1^2 (DF(x))^1, c_2^2 (DF(x))^2, \ldots, c_n^2 (DF(x))^n) = 0 \text{ for all } v \perp F(x),$$

and by definition of $[DF(x)]c^2$,

$$h \perp c^2 \iff [DF(x)]c^2 \cdot v = 0 \text{ for all } v \perp F(x).$$

Since $[[DF(x)]c^2] \cdot v = 0$ for all $v \perp F(x)$ if and only if $DF(x)c^2 = \gamma F(x)$ for some $\gamma \in \mathbb{R}$,

$$h \perp c^2 \iff DF(x)c^2 = \gamma F(x) \text{ for some } \gamma \in \mathbb{R}.$$

If $\gamma = 0$, then $DF(x)u = 0$ which does not correspond to the imaginary part of a complex eigenvector u for an eigenvalue $\lambda_0 = \alpha + \beta i$. Thus, $\gamma \neq 0$. By Lemma 2.2, $DF(x)u \cdot (1, \ldots, 1) = 0$. But, if $DF(x)u = \gamma F(x)$ for $\gamma \neq 0$, then since $F(x) \in \Lambda$,

$$\begin{aligned} 0 = DF(x)u \cdot (1, \ldots, 1) &= \gamma F(x) \cdot (1, \ldots, 1) \\ &= \gamma 1 = \gamma \end{aligned}$$

thus implying a contradiction. □

LEMMA 3.7. *Let $G = M \circ F$ be a GA map (1.2). Assume $G(x) = x$ for $x \in \Lambda$, with $\lambda_0 = \alpha + \beta i$, $\beta \neq 0$ a simple eigenvalue of $DG(x)$. Let C represent the change of basis matrix corresponding to Jordan normal form of $DG(x)$, with c^1 the real part of the complex eigenvector corresponding to λ_0 and c^2 the complex part of the eigenvector corresponding to λ_0.*

1. *If $\mathrm{Re}(\lambda_0) = \alpha \neq 0$, then there exist $i, j \in \{1, \ldots, n\}$, $i \neq j$ such that $-c^{-1}_{1i} + c^{-1}_{1j} \neq 0$.*

2. *There exist $i, j \in \{1, \ldots, n\}$, $i \neq j$ such that $-c^{-1}_{2i} + c^{-1}_{2j} \neq 0$.*

Proof. Let c^1 be the real part of the eigenvector corresponding to λ_0, and let c^2 be the imaginary part of the eigenvector corresponding to λ_0. By Corollary 2.5,

$$\begin{aligned} 0 =&(\alpha c^1 - \beta c^2) \cdot (1, \ldots, 1) \\ 0 =&(\beta c^1 + \alpha c^2) \cdot (1, \ldots, 1). \end{aligned} \tag{3.8}$$

We first prove (1) by contradiction. Assume $\mathrm{Re}(\lambda_0) \neq 0$ and that for all $i, j \in \{1, \ldots, n\}$,

$$-c^{-1}_{1i} + c^{-1}_{1j} = 0. \tag{3.9}$$

By Lemma 7.6, there exists $i \in \{1, \ldots, n\}$ such that $c^{-1}_{1i} \neq 0$. If equation (3.9) holds for all $i, j \in \{1, \ldots, n\}$, there exists $a \neq 0$ such that $c^{-1}_1 = a(1, \ldots, 1)$. Because $C^{-1}C = I$,

$$1 = c^{-1}_1 \cdot c^1 = a(1, \ldots, 1) \cdot c^1. \tag{3.10}$$

That is,

$$c^1 \cdot (1, \ldots, 1) = 1/a. \tag{3.11}$$

Because c^1 and c^2 are columns of the invertible matrix C, c^1 and c^2 are linearly independent, thus $(\alpha c^1 - c^2\beta) \neq 0$ and $(\beta c^1 + \alpha c^2) \neq 0$. Thus the hypothesis condition that $\alpha \neq 0$ and equations (3.8) and (3.11) imply that

$$0 = (\alpha c^1 - \beta c^2) \cdot (1, \ldots, 1) = \frac{\alpha}{a} - \beta(c^2 \cdot (1, \ldots, 1)). \tag{3.12}$$

Note also that

$$0 = (\beta c^1 + \alpha c^2) \cdot (1, \ldots, 1) = \frac{\beta}{a} + \alpha(c^2 \cdot (1, \ldots, 1)). \tag{3.13}$$

Finally, by equations (3.12) and (3.13), we compute that

$$\frac{\alpha}{\beta a} = c^2 \cdot (1, \ldots, 1) = \frac{-\beta}{a\alpha}. \tag{3.14}$$

Recall that $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq 0 \neq \beta$, thus equation (3.14) leads to the contradiction $\alpha^2 + \beta^2 = 0$, which implies $\lambda_0 = 0$.

We now prove (2) by contradiction. Assume that for all $i, j \in \{1, \ldots, n\}$,

$$-c_{2i}^{-1} + c_{2j}^{-1} = 0. \tag{3.15}$$

By Lemma 7.6, there exists $i \in \{1, \ldots, n\}$ such that $c_{2i}^{-1} \neq 0$. Thus, if equation (2) holds for all $i, j \in \{1, \ldots, n\}$, there exists $a \neq 0$ such that $c_2^{-1} = a(1, \ldots, 1)$. Since $C^{-1}C = I$, we see that

$$1 = c_2^{-1} \cdot c^2 = a(1, \ldots, 1) \cdot c^2. \tag{3.16}$$

Recall that c^1 and c^2 are linearly independent columns of the invertible matrix C, thus $(\alpha c^1 - \beta c^2) \neq 0$ and $(\beta c^1 + \alpha c^2) \neq 0$. Using equations (3.8) and (3.16) we compute

$$0 = (\alpha c^1 - \beta c^2) \cdot (1, \ldots, 1) = \alpha(c^1 \cdot (1, \ldots, 1)) - \frac{\beta}{a}. \tag{3.17}$$

Note also that

$$0 = (\beta c^1 + \alpha c^2) \cdot (1, \ldots, 1) = \beta c^1 \cdot (1, \ldots, 1) + \frac{\alpha}{a}. \tag{3.18}$$

Note first that if $\alpha = 0$, then by equation (3.17), $\frac{-\beta}{a} = 0$, which is a contradiction. Finally, for $\alpha \neq 0$, by equations (3.17) and (3.18), we compute that

$$\frac{-\alpha}{\beta a} = c^1 \cdot (1, \ldots, 1) = \frac{\beta}{a\alpha}, \tag{3.19}$$

once again leading to the contradiction $\alpha^2+\beta^2=0$, which implies $\lambda_0=0$. □

LEMMA 3.8. *Let $G=M\circ F$ be a GA map with fixed point x. Let $\lambda\in\mathbb{C}\backslash\mathbb{R}$ be a simple eigenvalue of $DG(x)$ corresponding to F with* $\operatorname{rank}(DF(x))=n-1$. *Then for all $\epsilon>0$, there exists $K\in\mathcal{K}$ such that* $\det(\widetilde{DG}(x)+\epsilon K-\lambda_0 I)\neq 0$.

PROOF. Proof by contradiction. Let $\epsilon>0$ and assume $\det(\widetilde{DG}(x)+\epsilon K-\lambda_0 I)\neq 0$ for all $K\in\mathcal{K}$. Let $B(\lambda):=DG(x)-\lambda I$, and $\widetilde{B}(\lambda):=\widetilde{\mathcal{DG}}(x)-\lambda I$. Recall by equation (3.2), that

$$\widetilde{B}(\lambda_0)=\begin{pmatrix} -\beta i & \beta & 0 & \dots & 0 \\ -\beta & -\beta i & 0 & \dots & 0 \\ 0 & 0 & * & * & * \\ \vdots & \vdots & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix}. \tag{3.20}$$

By Lemma 3.1, for all $K\in\mathcal{K}$,

$$\det(\widetilde{B}(\lambda_0)+\epsilon K)=\epsilon[(k_{11}+k_{22})\widetilde{B}_{11}(\lambda_0)+(k_{21}-k_{12})\widetilde{B}_{12}(\lambda_0)]+\mathcal{O}(\epsilon^2). \tag{3.21}$$

Thus, our assumption that $\det(\widetilde{B}(\lambda_0)+\epsilon K)=0$ for all $K\in\mathcal{K}$ is equivalent to the assumption that

$$(k_{11}+k_{22})\widetilde{B}_{11}(\lambda_0)+(k_{21}-k_{12})\widetilde{B}_{12}(\lambda_0)=0. \tag{3.22}$$

Recall that by Corollary 3.5, $\widetilde{B}_{11}(\lambda_0)=i\widetilde{B}_{12}(\lambda_0)$. Thus, equation (3.22) is equivalent to

$$i(k_{11}+k_{22})\widetilde{B}_{12}(\lambda_0)+(k_{21}-k_{12})\widetilde{B}_{12}(\lambda_0)=0, \tag{3.23}$$

which, in turn, is equivalent to

$$(k_{11}+k_{22})=0 \text{ and } (k_{21}-k_{12})=0. \tag{3.24}$$

Furthermore, by Lemma 2.20,

$$(k_{11} + k_{22}) = 0 \text{ and } (k_{21} - k_{12}) = 0,$$

for all $K \in \mathcal{K}$ if and only if for all $h \in \mathcal{R}$,

$$(-c_{1j}^{-1} + c_{1k}^{-1})(h \cdot c^1) = -(-c_{2j}^{-1} + c_{2k}^{-1})(h \cdot c^2) \tag{3.25}$$

and

$$(-c_{1j}^{-1} + c_{1k}^{-1})(h \cdot c^2) = (-c_{2j}^{-1} + c_{2k}^{-1})(h \cdot c^1). \tag{3.26}$$

By Lemma 3.6, we know there exists $h \in \mathcal{R}$ such that $h \cdot c^2 \neq 0$. Similarly, by Lemma 3.7, there exist i, j such that $(-c_{2i}^{-1} + c_{2j}^{-1}) \neq 0$. If we assume equation (3.25) holds, then $h \cdot c^1 \neq 0$, and $(-c_{1i}^{-1} + c_{1j}^{-1}) \neq 0$.

Now, let

$$(-c_{1i}^{-1} + c_{1j}^{-1}) = a \neq 0 \text{ and } (-c_{2i}^{-1} + c_{2j}^{-1}) = b \neq 0. \tag{3.27}$$

Then, equations (3.25),(3.26) and (3.27), imply

$$\begin{aligned} a(h \cdot c^1) &= -\, b(h \cdot c^2) \\ a(h \cdot c^2) &= b(h \cdot c^1). \end{aligned} \tag{3.28}$$

Letting $z = (h \cdot c^1)$ and $y = (h \cdot c^2)$, we see that the system of equations (3.28) is equivalent to

$$\begin{aligned} az &= -\, by \\ ay &= bz, \end{aligned} \tag{3.29}$$

which in turn is equivalent to

$$a\begin{pmatrix} z \\ y \end{pmatrix} = b\begin{pmatrix} -y \\ z \end{pmatrix}. \tag{3.30}$$

Since $b \neq 0$ by equation (3.27), $c = \frac{a}{b} \neq 0$, and equation (3.30) is

$$c\begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} -y \\ z \end{pmatrix}. \tag{3.31}$$

Finally, we arrive at a contradiction, since now (3.31) implies that

$$c^2 y = cx = -y \Longrightarrow c^2 = -1, \tag{3.32}$$

but, $c = \frac{a}{b} \in \mathbb{R}$. The contradiction implies that there exists a $K \in \mathcal{K}$ such that $k_{11} + k_{22} \neq 0$ or $k_{12} - k_{21} \neq 0$. Now equations (3.23) and (3.24) prove the lemma. $\square$

Thus, we have shown that if $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$, that there exists $K \in \mathcal{K}$ such that $\det(\widetilde{DG}(x) + \epsilon K - \lambda_0 I) \neq 0$ for this particular K.

We have not however, shown that we have perturbed the eigenvalue λ_0 off of the unit circle. To show this, we consider the direction of movement this particular perturbation forces on the old eigenvalue λ_0. These ideas are presented in the next section.

Direction of Movement Off of the Unit Circle

LEMMA 3.9. *Let $G = M \circ F$ with fixed point x. Let $\lambda_0 = \alpha + i\beta$ be a simple eigenvalue of $DG(x)$ with $\lambda_0 \in \mathcal{S}^1 \cap \mathbb{C}$. For $K \in Mat_n(F)$, define*

$$g(K) = \frac{(k_{11} + k_{22})}{2} + \frac{(k_{12} - k_{21})i}{2} + \mathcal{O}(\epsilon)^2.$$

Then there exists $K \in \mathcal{K}$ such that, $g(K) \neq a(\beta - i\alpha)$, $a \in \mathbb{R}$.

PROOF. Proof by contradiction. Assume that for all $K \in \mathcal{K}$, $g(K) = a(K)\theta$ where $\theta = \beta - i\alpha$. Because $K \in \mathcal{K}$, there exists $H \in \mathcal{H}$ such that $K = C^{-1}HC$, where C is the change of basis matrix. Note first that by Lemma 2.18,

$$k_{ij} = (c^{-1})_{i1}(h_1 \cdot c^j) + (c^{-1})_{i2}(h_2 \cdot c^j) + \cdots + (c^{-1})_{in}(h_n \cdot c^j), \tag{3.33}$$

where c^1 is the first column of the change of basis matrix C, and also the real part of the complex eigenvector v_{λ_0} corresponding to λ_0. Similarly, c^2 is the second column of the change of basis matrix C, and also the imaginary part of the complex eigenvector v_{λ_0} corresponding to λ_0. We let $h \in \mathcal{R}$, and define $H \in \mathcal{H}$ according to Lemma 2.15. Thus, we can consider $g = g(h)$, where $h \in \mathcal{R}$ and g is expressed as a function of h rather than K. That is, $g : \mathbb{R}^n \to \mathbb{C}$,

$$g(h) := (A(h \cdot c^1) + B(h \cdot c^2)) + i(A(h \cdot c^2)) - B(h \cdot c^1)). \tag{3.34}$$

Here, by Corollary 2.19, $A = (-c^{-1}_{1j} + c^{-1}_{1i}) \in \mathbb{R}$ and $B = -(-c^{-1}_{2j} + c^{-1}_{2i}) \in \mathbb{R}$ are constant for all $h \in \mathcal{R}$.

Recall that the eigenvector corresponding to λ_0, $v_{\lambda_0} = c^1 + ic^2$ where $c^1, c^2 \in \mathbb{R}^n$. Let $v := (Ac^1 + Bc^2)$ and let $w := (Ac^2 - Bc^1)$, then $g(h) = (h \cdot v) + i(h \cdot w)$. Now assume that for all $h \in \mathcal{R}$, $g(h) = a(h)(\beta - i\alpha)$. Since $g : \mathbb{R}^n \to \mathbb{C}$, there exists $h_1 \in \mathbb{R}^n$ such that $h_1 \perp v$ and $h_1 \cdot w \neq 0$. Then,

$$\begin{aligned} g(h_1) =& (h_1 \cdot v) + i(h_1 \cdot w) \\ =& 0 + i(h_1 \cdot w) \end{aligned} \tag{3.35}$$

where $(h_1 \cdot w) \neq 0$. Similarly, there exists $h_2 \perp w$, with $h_2 \cdot v \neq 0$ such that

$$\begin{aligned} g(h_2) =& (h_2 \cdot v) + i(h_2 \cdot w) \\ =& (h_2 \cdot v) + i0. \end{aligned} \tag{3.36}$$

But there is no $a \in \mathbb{R}$ such that $g(h_1) = g(h_2)$, thus the assumption that for all h, $g(h) = a(h)(\beta - i\alpha)$ must be false. □

LEMMA 3.10. *Let $G = M \circ F$ be a GA map with fixed point x and assume F has* rank$(DF(x)) = n - 1$. *If $DG(x)$ has a simple eigenvalue $\lambda_0 \in \mathcal{S}^1$, then there exists $P \in \mathcal{P}(x, M)$ such that for all $0 < \epsilon < 1$, λ_0 is not an eigenvalue of $D(G_{\epsilon P}(x))$, and the perturbed eigenvalue $\lambda_{\epsilon P}$ is not on $\mathcal{S}^1$.*

PROOF. Let spec$(DG(x)) = \{\lambda_0, \mu_2, \ldots, \mu_n\}$. Let $\widetilde{B}(\lambda) = \widetilde{DG}(x) - \lambda I$, and for $K \in \mathcal{K}$, let

$$q_i(\lambda_0) := k_i \cdot (\widetilde{B}_{i1}(\lambda_0), -\widetilde{B}_{i2}(\lambda_0), \widetilde{B}_{i3}(\lambda_0), \ldots, \pm\widetilde{B}_{in}(\lambda_0)). \tag{3.37}$$

By Lemma 7.10 and Corollary 7.11, for any $K \in Mat_n(\mathbb{R})$ and $\epsilon \in [0,1)$, there exist polynomials $r(\lambda), q(\lambda)$, and $s(\lambda)$, each of degree less than n, such that

$$\det(\widetilde{B}(\lambda) + \epsilon K) = r(\lambda)\left[(\lambda - \lambda_0) + \frac{\epsilon q(\lambda) + \epsilon(\lambda - \lambda_0)s(\lambda) + \mathcal{O}(\epsilon^2)}{r(\lambda)}\right]. \tag{3.38}$$

Here

$$r(\lambda) := \prod_{i=2}^{n}(\mu_i - \lambda) \tag{3.39}$$

and if $\lambda_0 \in \mathbb{R}$, then $q(\lambda) := k_{11}\det B_{11}(\lambda_0)$, and $s(\lambda) := \frac{\sum_{i=2}^{n} q_i(\lambda)}{\lambda - \lambda_0}$. Similarly, if $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$, then $q(\lambda) := [(k_{11} + k_{22}) + i(k_{12} - k_{21})]\ \det B_{11}(\lambda_0)$, and $s(\lambda) := \frac{\sum_{i=3}^{n} q_i(\lambda)}{\lambda - \lambda_0}$. By equation (3.38), we see that for $\epsilon = 0$, $\det(\widetilde{B}(\lambda) + \epsilon K) = r(\lambda)(\lambda - \lambda_0)$. To determine the direction of motion of an eigenvalue off of the unit circle as we perturb $DG(x)$ using ϵK, we analyze how

$$(\lambda - \lambda_0) + \frac{\epsilon q_1(\lambda) + \epsilon(\lambda - \lambda_0)s(\lambda) + \mathcal{O}(\epsilon^2)}{r(\lambda)} \tag{3.40}$$

changes as $\epsilon \to 0$. We start with the Taylor series representations of $r(\lambda)$, $s(\lambda)$, and $q(\lambda)$ about the point λ_0. Because r, s and q are each polynomials of finite degree, we know that these expansions exist, and certainly converge for $\lambda \in N_\delta(\lambda_0)$. That is,

$$r(\lambda) = r(\lambda_0) + \frac{r'(\lambda_0)}{1!}(\lambda - \lambda_0) + \cdots =: R_0 + R_1(\lambda - \lambda_0) + \dots \tag{3.41}$$

$$s(\lambda) = s(\lambda_0) + \frac{s'(\lambda_0)}{1!}(\lambda - \lambda_0) + \cdots =: S_0 + S_1(\lambda - \lambda_0) + \dots \tag{3.42}$$

$$q(\lambda) = q(\lambda_0) + \frac{q'(\lambda_0)}{1!}(\lambda - \lambda_0) + \cdots =: Q_0 + Q_1(\lambda - \lambda_0) + \dots. \tag{3.43}$$

By equation (3.39), we see that $r(\lambda_0) = R_0 \neq 0$. For the cases $\lambda_0 \in \mathbb{R}$ and $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$, by Lemmas 3.4 and 3.8, respectively, there exist $K \in \mathcal{K}$ such that $q(\lambda_0) = Q_0 \neq 0$.

Let $\lambda_\epsilon = f(\epsilon)$ with $\lambda_0 = f(0)$ be the root of the equation (3.38). By equation (3.39), λ_ϵ satisfies

$$0 = (\lambda_\epsilon - \lambda_0) + \frac{\epsilon q(\lambda_\epsilon) + \epsilon(\lambda_\epsilon - \lambda_0)s(\lambda_\epsilon) + \mathcal{O}(\epsilon^2)}{r(\lambda_\epsilon)}. \tag{3.44}$$

Consider the Taylor series expansion of $\lambda_\epsilon = f(\epsilon)$ about the point $\epsilon = 0$ given by

$$\lambda_\epsilon = f(\epsilon) = f(0) + \epsilon f'(0) + \epsilon^2 \frac{f''(0)}{2!} + \dots\ . \tag{3.45}$$

By definition of λ_ϵ, $f(0) = \lambda_0$, thus

$$\lambda_\epsilon - \lambda_0 = \epsilon f'(0) + \mathcal{O}(\epsilon^2). \tag{3.46}$$

The term $f'(0)$ describes the first order direction of movement of λ_0 as we perturb by ϵK. To find this direction, expand all factors in equation (3.44) using equations (3.41)-(3.44), and (3.46)

$$\begin{aligned} 0 =& \epsilon f'(0) + \mathcal{O}(\epsilon^2) + \epsilon \frac{Q_0 + Q_1(\epsilon f'(0) + \mathcal{O}(\epsilon^2)) + \mathcal{O}((\epsilon)^2)}{R_0 + R_1(\epsilon f'(0) + \mathcal{O}(\epsilon^2)) + \mathcal{O}((\epsilon)^2)} \\ -& \ \epsilon(\epsilon f'(0) + \mathcal{O}(\epsilon^2)) \frac{S_0 + S_1(\epsilon f'(0) + \mathcal{O}(\epsilon^2)) + \mathcal{O}((\epsilon)^2)}{R_0 + R_1(\epsilon f'(0) + \mathcal{O}(\epsilon^2)) + \mathcal{O}((\epsilon)^2)} + \mathcal{O}(\epsilon^2). \end{aligned}$$

Taking the common denominator, we get

$$0 = \epsilon f'(0) R_0 + \epsilon Q_0 + \mathcal{O}(\epsilon)^2.$$

That is, the first order movement is in the direction

$$f'(0) = -\frac{Q_0}{R_0} + \mathcal{O}(\epsilon)^2. \tag{3.47}$$

Thus we must calculate Q_0 and R_0. We consider the cases $\lambda_0 \in \mathbb{R}$ and $\lambda_0 \in \mathbb{C} \backslash \mathbb{R}$ separately.

First, consider $\lambda_0 \in \mathbb{R}$. Recall that by $q(\lambda) = k_{11}\widetilde{B}_{11}(\lambda)$. Since λ_0 is real and the matrix $DG(x)$ is real valued, the minor $B_{11}(\lambda_0) \in \mathbb{R}$. Thus, $Q_0 = q(\lambda_0) = k_{11}\widetilde{B}_{11}(\lambda_0) \in \mathbb{R}$, since $k_{11} \in \mathbb{R}$. Similarly, $R_0 = r(\lambda_0) \in \mathbb{R}$ with $R_0 \neq 0$. Since both Q_0 and R_0 are nonzero and real, $f'(0) = -\frac{Q_0}{R_0} \neq 0$ and is real valued. Thus, for the case that $\lambda_0 \in \mathbb{R}$, we know that $f'(0) \in \mathbb{R}$ and since $Q_0 \neq 0$, we move off the unit circle in a real direction. Therefore the direction in which λ_0 is moving off the unit circle is perpendicular to the unit circle.

Now we consider $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$. Once again, we note that $R_0 = r(\lambda_0) \neq 0$ and because $\lambda_0 \in \mathbb{C}$,

$$R_0 = r(\lambda_0) = \prod_{i=2}^{n}(\lambda - \lambda_0) = (\lambda_0 - \overline{\lambda_0})\prod_{i=3}^{n}(\lambda_0 - \lambda_i) = 2i(\text{Im}(\lambda_0))\prod_{i=3}^{n}(\lambda_0 - \lambda_i). \quad (3.48)$$

Similarly,

$$Q_0 = q(\lambda_0) = k_{11}B_{11}(\lambda_0) - k_{21}B_{21}(\lambda_0) - k_{12}B_{12}(\lambda_0) + k_{22}B_{22}(\lambda_0).$$

Note also that by Lemma 7.9, for $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$,

$$B_{11}(\lambda_0) = -\text{Im}(\lambda_0)i\prod_{i=3}^{n}(\lambda_0 - \lambda_i) = B_{22}(\lambda_0)$$

and

$$B_{21}(\lambda_0) = -\text{Im}(\lambda_0)\prod_{i=3}^{n}(\lambda_0 - \lambda_i) = -B_{21}(\lambda_0).$$

Thus,

$$Q_0 = k_{11}B_{11}(\lambda_0) - k_{21}B_{21}(\lambda_0) - k_{12}B_{12}(\lambda_0) + k_{22}B_{22}(\lambda_0)$$

which implies

$$Q_0 = (k_{11} + k_{22})(-\mathrm{Im}(\lambda_0))i \det \prod_{i=3}^{n}(\lambda_0 - \lambda_i) + (k_{12} - k_{21})\mathrm{Im}(\lambda_0) \prod_{i=3}^{n}(\lambda_0 - \lambda_i). \quad (3.49)$$

Finally, by equations (3.47), (3.48), and (3.49),

$$\begin{aligned} f'(0) &= -\frac{Q_0}{R_0} + \mathcal{O}(\epsilon)^2 \\ &= -\frac{(k_{11} + k_{22})(-\mathrm{Im}(\lambda_0))i \prod_{i=3}^{n}(\lambda_0 - \lambda_i) + (k_{12} - k_{21})\mathrm{Im}(\lambda_0) \prod_{i=3}^{n}(\lambda_0 - \lambda_i)}{2i(\mathrm{Im}(\lambda_0)) \prod_{i=3}^{n}(\lambda_0 - \lambda_i)} + \mathcal{O}(\epsilon)^2 \\ &= \frac{(k_{11} + k_{22})i - (k_{12} - k_{21})}{2i} + \mathcal{O}(\epsilon)^2 \\ &= \frac{(k_{11} + k_{22})}{2} + \frac{(k_{12} - k_{21})i}{2} + \mathcal{O}(\epsilon)^2. \end{aligned} \quad (3.50)$$

Note that by equation (3.50), $f'(0)$ is a function of k. Define $g(K) := f'(0)$, then g represents the first order movement off the unit circle. By Lemma 3.9, there exists $K \in \mathcal{K}$ such that the direction of movement off of the unit circle is not tangent to the unit circle. □

Repeated Eigenvalues

LEMMA 3.11. *Let $G = M \circ F$ be a GA map (1.2) with fixed point x. If $DG(x)$ has eigenvalue $\lambda_0 \in \mathcal{S}^1$ and multiplicity $k > 1$, then there exists $P \in \mathcal{P}(x, M)$ such that for $0 < t \leq 1$, $DG_{tP}(x)$ has eigenvalue $\lambda_0 \in \mathcal{S}^1$ with multiplicity at most 1.*

PROOF. Assume $\lambda_0 \in \mathcal{S}^1$ is the eigenvalue of $DG(x)$ with multiplicity $k > 1$. Since the polynomial $g(c) = \det(DG(x) - \lambda_0 I + cH)$, $g : \mathbb{R} \to \mathbb{C}$, defines an analytic function in c, either

1. $g \equiv 0$; or

2. $g(c)$ has isolated zeros, [17].

By Lemma 4.3, we can choose H to have rank $n-1$. Thus 0 is a simple eigenvalue of H. For large values of c, we have $0 \in \text{spec}(cH)$ but for $\mu \in [\text{spec}(cH)\backslash\{0\}]$, $|\mu| > L$ for some large $L = \mathcal{O}(c)$. If $\|DG(x)\| << L$, then we can view $DG(x)$ as a small perturbation of cH. Two possibilities arise:

(a) There exists $c \in \mathbb{R}$ such that $g(c) = \det(cH + DG(x) - \lambda_0 I) \neq 0$.

(b) For all $c \in \mathbb{R}$, $g(c) = \det(cH + DG(x) - \lambda_0 I) = 0$.

Case (a) implies (2), i.e. g has isolated zeros. Since $g(0) = 0$, there is δ arbitrarily close to 0 such that $g(\delta) \neq 0$. The proof now follows from Lemma 2.16.

In case (b), we note that since H has the simple eigenvalue 0, λ_0 must be a simple eigenvalue of $(cH+DG(x))$ for large c. Since $g \equiv 0$, λ_0 is an eigenvalue of $cH+DG(x)$ for all c. Therefore there exists a function $h(c, \lambda)$ such that $\det(cH + DG(x) - \lambda_0 I) = (\lambda - \lambda_0)h(c, \lambda)$. Observe that $h(c, \lambda)$ is a polynomial in λ and, since $g(c)$ is analytic in c, the function

$$h(c) := h(c, \lambda_0)$$

is also analytic in c. Since λ_0 is a simple eigenvalue of $(cH + DG(x))$ for large c, $h(c) \neq 0$ for large c. Therefore $h(c)$ has isolated zeros and there is $\delta_0 > 0$ such that for all $\delta < \delta_0$ we have $h(\delta) \neq 0$. Therefore, the Jacobian of the map $G_{\delta P}$ corresponding to a perturbation $\delta P \in \mathcal{P}(x, M)$ with $\delta < \delta_0$, has eigenvalue λ_0 with multiplicity 1.

Set $P = \delta P$, it follows that for $0 < t \leq 1$, $tP \in \mathcal{P}(x, M)$ and the Jacobian of the map G_P has eigenvalue λ_0 with multiplicity 1.

□

Conclusion

We have dealt with the simple eigenvalue case and the repeated eigenvalue case separately. We now show that given a GA map $G = M \circ F$ with fixed point x and $\text{spec}(DG(x)) \cap \mathcal{S}^1 \neq \emptyset$, that we can perturb using $P \in \mathcal{P}(x, M)$ such that the resulting map G_P has $\text{spec}(DG_P(x)) \cap \mathcal{S}^1 = \emptyset$.

LEMMA 3.12. *Let $G = M \circ F$ with fixed point x and F with* $\text{rank}(DF(x)) = n-1$. *If x is non-hyperbolic, there exists $P \in \mathcal{P}(x, M)$ such that $G_P = (M + cP) \circ F$ has hyperbolic fixed point x for all* $0 < c \leq 1$.

PROOF. Let $\text{spec}(DG(x)) \cap \mathcal{S}^1 = \{\lambda_1, \ldots, \lambda_k\}$ with multiplicities $m_1, \ldots, m_k$, respectively and let $\text{spec}(DG(x)) \backslash \mathcal{S}^1 = \{\lambda_{k+1}, \ldots, \lambda_n\}$. We define

$$\epsilon := \min_{i \in \{k+1, \ldots, n\}}(dist(\mathcal{S}^1, \lambda_i)). \tag{3.51}$$

If $m_1 > 1$, by Lemma 3.11, there exists $P^r \in \mathcal{P}(x, M)$ such that $DG_{P^r}(x)$ has eigenvalue λ_1 with multiplicity at most 1. If, or once, this eigenvalue does have multiplicity 1, then by Lemma 3.10, there exists $P \in \mathcal{P}(x, M)$ such that the perturbed map G_{P^r} has $\lambda_0 \notin \text{spec}(DG_x(x))$. By Corollary 2.12, there exists $\delta > 0$ such that $\delta(P^r + P) \in \mathcal{P}(x, M)$. By Corollary 2.9, since $\delta(P^r + P) \in \mathcal{P}(x, M)$, for any $t \in [0, 1)$,

$t(\delta(P^r + P)) \in \mathcal{P}(x, M)$. We choose t small enough so that the perturbed eigenvalues $\lambda_{k+1}, \ldots, \lambda_n$ are still outside the unit circle.

Set $P^1 = t(\delta(P^r + P))$. Note that for $0 < c \leq 1$, $ct < t$, thus for $cP_1 = ct(\delta(P^r + P)) \in \mathcal{P}(x, M)$, perturbed eigenvalues $\lambda_{k+1}, \ldots, \lambda_n$ are still outside the unit circle. Clearly G_{cP^1} has $\lambda_1 \notin \text{spec}(DG_{cP^1}(x))$ for all c with $0 < c \leq 1$.

If there are still eigenvalues on the unit circle, we use Lemma 3.11 to reduce the multiplicity to 1, if necessary, and then use Lemma 3.10 to perturb the eigenvalue off of the unit circle. We hereby produce a perturbation and note that it is once again the a sum of perturbations. We choose factor t small enough so the remaining eigenvalues, as well as the perturbed eigenvalues are still off of the unit circle. This terminates in a finite number of steps.

$\square$

CHAPTER 4

GENERICITY OF FINITE FIXED POINT SET

Background and Terminology

Let X, Y be n and m dimensional manifolds, respectively. For $x \in X$, we let T_xX denote the tangent space to X at x. For a differentiable map $f : X \to Y$, we let $df_x : T_xX \to T_{f(x)}Y$ denote the derivative of the map f at the point x. In the special case that $T_xX = \mathbb{R}^n$ and $T_{f(x)}Y = \mathbb{R}^m$, we note that $df_x : \mathbb{R}^n \to \mathbb{R}^m$ is given by the Jacobian matrix $Df(x)$ ([14]).

The following notation is adopted from [1]. Let $\mathcal{A}, X, Y$ be $\mathcal{C}^r$ manifolds. Let $\mathcal{C}^r(X, Y)$ be the set of $\mathcal{C}^r$ maps from X to Y. Let $\rho : \mathcal{A} \to \mathcal{C}^r(X, Y)$ be a map. For $a \in \mathcal{A}$ we write ρ_a instead of $\rho(a)$. That is, $\rho_a : X \to Y$ is a $\mathcal{C}^r$ map.

Definition 4.1. The map $\mathrm{ev}_\rho : \mathcal{A} \times X \to Y$ defined by $\mathrm{ev}_\rho(a, x) := \rho_a(x)$ for $a \in \mathcal{A}$ and $x \in X$, is called *evaluation map*. We say that ρ is a $\mathcal{C}^r$ representation if $\mathrm{ev}_\rho(a, x) = \rho_a(x)$ is a $\mathcal{C}^r$ map, for every $a \in \mathcal{A}$.

Definition 4.2 (Restricted definition from [1] for finite vector spaces). Let X and Y be $\mathcal{C}^1$ manifolds, $f : X \to Y$ a $\mathcal{C}^1$ map, and $W \subset Y$ a submanifold. We say that f *is transversal to* W *at a point* $x \in X$, in symbols: $f \pitchfork_x W$, if, where $y = f(x)$, either $y \notin W$ or $y \in W$ and the image $(T_x f)(T_x X)$ contains a closed complement to $T_y W$ in $T_y Y$.

We say f *is transversal to* W, in symbols: $f \pitchfork W$, if and only $f \pitchfork_x W$ for every $x \in X$.

Genericity of Finite Fixed Point Set

Let $\mathcal{A} = \mathcal{M}$, where $\mathcal{M}$ denotes the set of mixing operators given by Definition 1.2, and let $X = \Lambda \subset \mathbb{R}^n$ and $Y = \mathbb{R}^n_0$. For $M \in \mathcal{M}$, we define $\rho : \mathcal{M} \to \mathcal{C}^r(\Lambda, \mathbb{R}^n_0)$ by $\rho_M := (M \circ F - I)$. Recall that $\mathbb{R}^n_0 := \{x \in \mathbb{R}^n | \sum_i x_i = 0\}$. Note that because we assume $F, M, \in \mathcal{C}^1(\Lambda, \Lambda)$, $\rho_M \in \mathcal{C}^1(\Lambda, \mathbb{R}^n_0)$.

Finally, we define $\text{ev}_\rho : \mathcal{M} \times \Lambda \to \mathbb{R}^n_0$ by $\text{ev}_\rho(M, x) := \rho_M(x)$ for $M \in \mathcal{M}$ and $x \in \Lambda$. That is,

$$\begin{aligned} \text{ev}_\rho(M, x) :=& \rho_M(x) \\ =& (M \circ F - I)(x) \\ =& M(F(x)) - x. \end{aligned} \tag{4.1}$$

Finally, note that since G, F are $\mathcal{C}^1$, ev_ρ is $\mathcal{C}^1$; that is, ev_ρ is a $\mathcal{C}^1$ representation.

LEMMA 4.3. *For* $\text{ev}_\rho(M, x) := M(F(x)) - x$, $\text{rank}(d(\text{ev}_\rho)) = n - 1$.

PROOF. Note first that since $\Lambda \subset \mathbb{R}^{n+}$, $d(\text{ev}_\rho) = D\ \text{ev}_\rho$, a Jacobian of ev_ρ. Similarly, we note that

$$D(\text{ev}_\rho|_{(\mathcal{M}\times\Lambda)}) = D\text{ev}_\rho|_{T_{(P,y)}(\mathcal{M}\times\Lambda)}.$$

Because $T(\mathbb{R}^n_0) = \mathbb{R}^n_0$, and

$$T(\mathcal{M} \times \Lambda) = \{(P, y) | P = (P_1, \dots, P_n) \text{ with } \sum_i P_i = 0 \text{ and } y \in \mathbb{R}^n\}, \tag{4.2}$$

it suffices to show that the Jacobian $D\ \mathrm{ev}_\rho : T(\mathcal{M} \times \Lambda) \to \mathbb{R}^n_0$ is onto. Thus, for $(P, y) \in T(\mathcal{M} \times \Lambda)$, we calculate

$$\begin{aligned} D\ \mathrm{ev}_{(M,x)}(P, y) =& [\frac{\partial \mathrm{ev}_\rho}{\partial M}, \frac{\partial \mathrm{ev}_\rho}{\partial x}] \begin{pmatrix} P \\ y \end{pmatrix} \\ =& \frac{\partial \mathrm{ev}_\rho}{\partial M} P + \frac{\partial \mathrm{ev}_\rho}{\partial x} y. \end{aligned} \tag{4.3}$$

By a short computation we get

$$\frac{\partial \mathrm{ev}_\rho}{\partial M} P = PF(x), \tag{4.4}$$

and

$$\frac{\partial \mathrm{ev}_\rho}{\partial x} y = 2(MF(x))DF(x)y - y. \tag{4.5}$$

Finally, for any $z \in \mathbb{R}^n_0$, and $(M, x) \in \mathcal{M} \times \Lambda$, we show there exists $(P, y) \in T(\mathcal{M} \times \Lambda)$ such that

$$D\mathrm{ev}_{(M,x)}(P, y) = PF(x) + 2(MF(x))DF(x)y - y = z. \tag{4.6}$$

We start by choosing $y = 0 \in \mathbb{R}^n$. Now, by equation (4.6), it suffices to find $P = (P_1, \dots, P_n)$ such that

$$D\mathrm{ev}_{(M,x)}(P, y) = PF(x) + 0 - 0 = z. \tag{4.7}$$

Because $F : \mathbb{R}^{n+} \to \Lambda$, we let $u = F(x) \in \Lambda$. By equation (4.7), we see that for fixed $u \in \Lambda$, we want $P = (P_1, \dots, P_n)$ such that $Pu = (u^T P_1 u, \dots, u^T P_n u) = z$. Clearly, for $i = 1, \dots, n-1$, we can choose P_i such that $u^T P_i u = z_i$. Finally, because $z \in \mathbb{R}^n_0$, that is $\sum_i z_i = 0$, we see that $z_n = -\sum_i z_i$. Thus, for our choice of $P_1, \dots, P_{n-1}$,

$$z_n = -\sum_i z_i = -\sum_i u^T P_i u = u^T(\sum_i P_i)u, \tag{4.8}$$

and $P_n = -\sum_{i=1}^{n-1} P_i$. Because $P_n = -\sum_{i=1}^{n-1} P_i$, it is clear that $\sum_{i=1}^{n} P_i = 0$, and for this choice of $P = (P_1, \dots, P_n)$ with $y = 0$ we have $D\mathrm{ev}_{(M,x)}(P, y) = z$. Because z, M, x were arbitrary, we have shown that $D\mathrm{ev}_\rho : T(\mathcal{M} \times \Lambda) \to \mathbb{R}^n_0$ is onto. That is, $\mathrm{rank}(D\mathrm{ev}_\rho) = n - 1$.

□

Lemma 4.4. *Let* $\mathrm{ev}_\rho : \mathcal{M} \times \Lambda \to \mathbb{R}^n_0$, $x \in \Lambda$, *and* $M \in \mathcal{M}$. *Then* $\mathrm{ev}_\rho \pitchfork \{0\}$.

Proof. Definition 4.2 requires that the image $(T_x\mathrm{ev}_\rho)(T_xX)$ contains a closed complement to T_yW in T_yY. In our case $T_yW = \{0\}$, and the only closed complement to $\{0\}$ in $T_yY = \mathbb{R}^{n-1}$ is $T_yY = \mathbb{R}^{n-1}$. Thus, the requirement that the image $(T_x\mathrm{ev}_\rho)(T_xX)$ contains a closed complement to T_yW in T_yY is that $D\mathrm{ev}_\rho(x)$ is surjective. By Lemma 4.3, $\mathrm{rank}(D\mathrm{ev}_\rho(x)) = n - 1$, and therefore $D\mathrm{ev}_\rho(x)$ is surjective and $\mathrm{ev}_\rho \pitchfork \{0\}$. □

Lemma 4.5. *Let* $\mathcal{M}_{\{0\}} := \{M \in \mathcal{M} | \rho_M \pitchfork \{0\}\}$. *Then* $\mathcal{M}_{\{0\}}$ *is dense in* $\mathcal{M}$. *That is, the set of parameter values for which* ρ_M *is transversal to* $\{0\}$ *is dense in* $\mathcal{M}$.

PROOF. We apply the Transversal Density Theorem: Theorem 7.13. We first note that by Lemma 4.4, $\mathrm{ev}_\rho \pitchfork \{0\}$, and therefore hypothesis condition (4) of Theorem 7.13 holds. We now verify the remaining hypothesis conditions (1)-(3).

1. $X = \Lambda$ has finite dimension m and $W = \{0\}$ has finite codimension q in $Y = \mathbb{R}^n_0$. Because $X = \Lambda$, $m = n - 1 < \infty$. Clearly, the codimension of $\{0\}$ in $\mathbb{R}^n_0$ is $n - 1$. That is, $q = n - 1$.

2. $\mathcal{M}$ and X are second countable. Clearly, $\mathbb{R}^{n-1}$ and $\mathcal{M} \subset \mathbb{R}^{n^2} \times \mathbb{R}^n$ are second countable, as they are the product of and therefore subsets of second countable spaces. [21].

3. $r > \max(0, m - q)$. Since $r = 1$, clearly $r > \max(0, 0) = 0$.

□

LEMMA 4.6. *Let* $\mathcal{M}_{\{0\}} := \{M \in \mathcal{M} | \rho_M \pitchfork \{0\}\}$. *Then* $\mathcal{M}_{\{0\}}$ *is open in* $\mathcal{M}$. *That is, the set of parameter values for which* ρ_M *is transversal to* $\{0\}$ *is open in* $\mathcal{M}$.

PROOF. We apply Theorem 7.15, and therefore start by verifying its hypothesis. Clearly $\mathcal{M}, X, Y$ are $\mathcal{C}^1$ manifolds. We take $K = X = \Lambda$, and thus K is a compact subset of the finite dimensional manifold X. Similarly $W = \{0\} \subset Y$ is closed. Because ρ is a representation, ρ is also a pseudorepresentation. Thus, all hypothesis requirements have been met and by Theorem 7.15,

$$\mathcal{M}_{K\{0\}} = \{M \in \mathcal{M} | \rho_M \pitchfork_x \{0\} \text{ for } x \in K = X\}$$

is open in $\mathcal{M}$. □

LEMMA 4.7. *For generic $M \in \mathcal{M}$,*

1. *$\rho_M \pitchfork \{0\}$. That is, the set of parameter values for which ρ_M is transversal to $\{0\}$ is open and dense in $\mathcal{M}$.*

2. *The set of parameter values for which $\rho_M^{-1}(\{0\})$ has finitely many solutions is open and dense in $\mathcal{M}$.*

PROOF. The proof of part (1) follows directly from Lemmas 4.5 and 4.6.

We now prove part (2). By part (1), the set of parameter values for which ρ_M is transversal to $\{0\}$ is open and dense in $\mathcal{M}$. Thus by Theorem 7.14, for this dense set, $\rho_M^{-1}(\{0\})$ has only finitely many connected components. Furthermore, since $\rho_M^{-1}(\{0\})$ has finitely many connected components for these same parameter values, we know that the set of parameter values for which $\rho_M^{-1}(\{0\})$ has finitely many connected components is open and dense. We now show by contradiction that there are finitely many solutions to $\rho_M(x) = 0$ in Λ.

For $x \in \rho_M^{-1}(\{0\})$, let $M_x \subset \rho_M^{-1}(\{0\})$ denote the connected component with $x \in M_x$. Assume x is not isolated in $\rho_M^{-1}(\{0\})$. Then, there exists a sequence $\{x_n\} \subset M_x$ such that $x_n \to x$, and by choosing a subsequence $\{x_{n_k}\}$,

$$\lim_{n_k \to \infty} \frac{x_{n_k} - x}{||x_{n_k} - x||} = v, \tag{4.9}$$

where $v \in T_x(\rho_M^{-1}(\{0\}))$. Here $v \neq 0$ because the terms in the quotient are on the unit sphere. Since

$$M \circ F(x_{n_k}) - M \circ F(x) = D(M \circ F)(x) \cdot (x_{n_k} - x) + R \tag{4.10}$$

where R is a remainder, then

$$\lim_{n_k \to \infty} \frac{M \circ F(x_{n_k}) - M \circ F(x)}{||x_{n_k} - x||} = \lim_{n_k \to \infty} \frac{D(M \circ F)(x) \cdot (x_{n_k} - x) + R}{||x_{n_k} - x||}. \tag{4.11}$$

By equations (4.9) and (4.11), and because x_{n_k}, x are fixed points,

$$v = D(M \circ F)(x) \cdot \left(\lim_{n_k \to \infty} \frac{x_{n_k} - x}{||x_{n_k} - x||} \right) + \lim_{n_k \to \infty} \frac{R}{||x_{n_k} - x||}. \tag{4.12}$$

That is,

$$v = D(M \circ F)(x) \cdot v + 0, \tag{4.13}$$

and $v \neq 0$ is an eigenvector of $D(M \circ F)(x)$ with eigenvalue 1. However, since $x \in \rho_M^{-1}(\{0\}$, and ρ_M is transversal to $\{0\}$ at x, $D(M \circ F)(x) - I$ is a linear isomorphism on a finite vector space. Thus, for all $v \neq 0$, $(D(M \circ F)(x) - I)v \neq 0$ which is a contradiction. Thus, all the components of $\rho_M^{-1}(\{0\})$ only contain isolated points and each connected component of $\rho_M^{-1}(\{0\})$ is itself an isolated point. Since there are finitely many connected components of $\rho_M^{-1}(\{0\})$, there are finitely many solutions to $\rho_M(x) = 0$ for $x \in \Lambda$. □

Corollary 4.8. *There exists an open and dense set $\mathcal{B} \subset \mathcal{M}$ such that if $M \in \mathcal{B}$, then $G = M \circ F$ has finitely many fixed points.*

By Lemma 4.7, for $M \in \mathcal{M}_{\{0\}} \subset \mathcal{M}$, $\rho_M(x) = 0$ has finitely many solutions in Λ. That is, for generic $M \in \mathcal{M}$,

$$\rho_M(x) = M(F(x)) - x \tag{4.14}$$

has finitely many solutions in Λ. Thus solutions to $\rho_M(x) = 0$ correspond to fixed points of $G = M \circ F$.

CHAPTER 5

PROOF OF OPENNESS (PROPOSITION 1.12)

If a GA map G has hyperbolic fixed points, then, since Λ is compact there can be only finitely many of them in Λ.

Consider one such fixed point x. Since

$$\det(DG(x) - \lambda I) = \det([DM \circ F(x)]DF(x) - \lambda I)$$

is a continuous function of M, if the spectrum of $DG(x)$ does not intersect the unit circle, then there is a $\delta_0 = \delta_0(x) > 0$ such that the spectrum of $DG_{M'}$ corresponding to any M' with $||M - M'|| < \delta_0(x)$ will not intersect the unit circle. Since there are finitely many fixed points, there is a minimal $\delta = min_x \delta_x$. Then all maps $G_{M'}$ corresponding to M' with $||M - M'|| \leq \delta$ are hyperbolic.

CHAPTER 6

PROOF OF DENSENESS (PROPOSITION 1.13)

We will use the following notation. For $M \in \mathcal{M}$, define $G^M := M \circ F$.

LEMMA 6.1. *Let $M \in \mathcal{M}_{\{0\}}$ and x be a fixed point of $G^M := M \circ F$. Then $DG^M(x)$ does not have eigenvalue* 1.

PROOF. By definition, since $\rho_M \pitchfork \{0\}$, we know that $DG^M(x) - I : \mathbb{R}^n_0 \to \mathbb{R}^n_0$ is surjective. Thus $DG^M(x) - I$ is a linear isomorphism on a finite dimensional vector space [2]. Thus, for all $v \neq 0$, $(DG^M(x) - I)v \neq 0$ and 1 cannot be an eigenvalue of $DG^M(x)$. □

Note that by Proposition 1.1.4 in [15], fixed points that do not have eigenvalue 1 for the Jacobian are isolated and remain isolated under small perturbations. That is, when perturbing the map, new fixed points are not introduced in the neighborhood(s) of fixed points that do not have eigenvalue 1.

Recall that $\mathrm{Fix}(f)$ denotes the set of fixed points of f, while $\mathrm{NonHyp}(f) \subset \mathrm{Fix}(f)$ and $\mathrm{Hyp}(f) \subset \mathrm{Fix}(f)$ denote the sets of non-hyperbolic and hyperbolic fixed points of f, respectively.

LEMMA 6.2. *Let $M \in \mathcal{M}_{\{0\}}$, then there exists $\epsilon > 0$ such that if $||M' - M|| < \epsilon$, then $\#\{\mathrm{Fix}(M \circ F)\} = \#\{\mathrm{Fix}(M' \circ F)\} < \infty$.*

PROOF. Since $M \in \mathcal{M}_{\{0\}}$, by Lemma 4.7, there exists $j < \infty$ such that $\#\{\mathrm{Fix}(M \circ F)\} = j$. Thus it suffices to show that $\#\{\mathrm{Fix}(M \circ F)\} = \#\{\mathrm{Fix}(M' \circ F)\}$. By Lemma 6.1, for all $i = 1, \ldots, j$ and $x_i \in \mathrm{Fix}(M \circ F)$, x_i is non-degenerate, that is, for all $i = 1, \ldots, j$; $D(M \circ F)(x_i)$ does not have eigenvalue 1. By part (1) of Lemma 4.7, $M_{\{0\}}$ is open and dense in $\mathcal{M}$, thus there exists $\delta > 0$ such that if $||M - M'|| < \delta$, then $M' \in M_{\{0\}}$. By part (2) of Lemma 4.7, since $M' \in M_{\{0\}}$ there exists $k < \infty$ such that $\#\{\mathrm{Fix}(M' \circ F)\} = k$. We now show that for δ above chosen small enough, $k = j$.

By Proposition 1.1.4 of [15], because x_i is non-degenerate, for all $i = 1, \ldots, j$, there exist $\eta_i > 0$, such that for all $\gamma_i < \eta_i$, there exist $\epsilon_i > 0$ such that for all $\epsilon < \epsilon_i$, if $||(M \circ F) - (M' \circ F)||_1 < \epsilon$, then there is a unique fixed point $x' \in \mathrm{Fix}(M' \circ F)$ with $||x_i - x_i'|| < \gamma_i$. Pick $\gamma = \min_i\{\gamma_i/2\}$. Now choose $\epsilon = \mathrm{min_i}\{\epsilon_i(\gamma)\}$. By this choice, if $||M \circ F - M' \circ F|| < \epsilon$, then for $i = 1, \ldots, j$ there exist unique fixed points x_i' with $||x_i - x_i'|| < \gamma$.

Note that for all $\epsilon > 0$, there exists $\delta_1 = \delta_1(\epsilon) > 0$ such that if $||M - M'|| < \delta_1$ then $||(M \circ F) - (M' \circ F)|| < \epsilon$ on Λ in the $\mathcal{C}^1$ topology.

Let U_i be a neighborhood of radius γ of x_i, and let $U = \bigcup_{i=1,\ldots,j} U_i$. Having fixed neighborhood U, the previous argument shows that there exists $\delta_1 > 0$ such that if $||M - M'|| < \delta_1$ then

$$\#\{\mathrm{Fix}(M \circ F|_U)\} = \#\{\mathrm{Fix}(M' \circ F)|_U\}.$$

We now show that on the compact set $K := \Lambda \backslash U$,

$$\#\{\mathrm{Fix}(M \circ F|_K\} = \#\{\mathrm{Fix}(M' \circ F)|_K\} = 0, \tag{6.1}$$

provided M' is close enough to M. Since $d(x, M \circ F(x)) : K \to \mathbb{R}$ is continuous, and K is compact, by the Minimum Value Theorem [21], there exists $c > 0$ such that for all $x \in K$, $d(x, M \circ F(x)) > c$. Thus, there exists $\delta_2 > 0$ such that if $||M' - M|| < \delta_2$, then $d(x, M' \circ F(x)) > c/2$ for all $x \in K$. This implies that if $||M - M'|| < \delta_2$, then $M' \circ F$ has no fixed points in K.

Finally, let $\epsilon = \min\{\ \delta, \delta_1, \delta_2\}$. Then, if $||M' - M|| < \epsilon$,

$$\#\{\mathrm{Fix}(M \circ F|_K\} = \#\{\mathrm{Fix}(M' \circ F)|_K\} = j. \tag{6.2}$$

□

LEMMA 6.3. *Let* $M, M' \in \mathcal{M}$.

1. *For all* $x \in \mathrm{Hyp}(G^M)$*, there exist* $\delta > 0$ *such that if* $||M' - M|| < \delta$*, then* $d(\mathrm{spec}(DG^{M'}(x)), \mathcal{S}^1) > 0$.

2. *If there exists* $m < \infty$ *such that* $\mathrm{Hyp}(G^M) = \{x_1, \ldots, x_m\}$*, then there exists* $\delta > 0$ *such that if* $||M - M'|| < \delta$ *then* $\min_{i=1,\ldots,m} d(\mathrm{spec}(DG^{M'}(x_i)), \mathcal{S}^1) > 0$.

PROOF. We first prove part (1). Let $x \in \mathrm{Fix}(G^M)$, and note that for $\lambda \in \mathrm{spec}(DG^M(x))$, λ changes continuously with M. That is, for all i such that $\lambda_i \in \mathrm{spec}(DG^M(x))$, $\lambda_i = f_i(M)$, for some continuous f_i.

Similarly, since $x \in \mathrm{Hyp}(G^M)$, we know that $d(\mathrm{spec}(DG^M(x)), \mathcal{S}^1) = \epsilon > 0$. By definition,

$$d(\mathrm{spec}(DG^M), \mathcal{S}^1) = \min_i d(\lambda_i, \mathcal{S}^1), \tag{6.3}$$

is a continuous function of λ_i for $i = 1, \ldots, n$ and thus a continuous function of M. Thus given $\epsilon > 0$ there exists δ such that if $||M' - M|| < \delta$,

$$|d(\text{spec}(DG^M)(x), \mathcal{S}^1) - d((\text{spec}(DG^{M'})(x), \mathcal{S}^1))| < \epsilon.$$

Therefore, $d(\text{spec}(DG^{M'}(x)), \mathcal{S}^1) > 0$.

We now prove part (2). Assume there exists $m < \infty$ such that $\text{Hyp}(G^M) = \{x_1, \ldots, x_m\}$. Then by part (1), given $\epsilon > 0$, for $i = 1, \ldots, m$ there exist $\delta_1, \ldots, \delta_m$ such that if $||M' - M|| < \delta_i$, $d(\text{spec}(DG^M(x_i)), \mathcal{S}^1) = \epsilon > 0$. Let $\delta = \min_{i=1,\ldots,m}(\delta_i)$, then for all $i \in \{1, \ldots, m\}$, $\delta \leq \delta_i$ and for $M \in \mathcal{M}$ with $||M - M'|| < \delta$, $d(\text{spec}(DG^M(x_i)), \mathcal{S}^1) > 0$. Because $m < \infty$, this implies $\min_{i=1,\ldots,m} d(\text{spec}(DG^{M'}(x_i)), \mathcal{S}^1) > 0$. □

Lemma 6.4. *Assume* $M \in \mathcal{M}_{\{0\}}$ *and* $\text{Hyp}(G^M) = \{x_1, \ldots, x_m\}$ *with* $m < \infty$. *There exists* $\delta < 0$ *such that if* $||M - M'|| < \delta$, *then for* $\{x'_1, \ldots, x'_m\}$ *perturbed fixed points of* $G^{M'}$,

$$\min_{i=1,\ldots,m} d(spec(DG^{M'}(x_i), \mathcal{S}^1)) > 0.$$

Proof. First "compactify" a set of $M's$ by choosing $||M - M'|| \leq \epsilon/2$. Given $\gamma > 0$ by uniform continuity of $DG^{M'}(x)$ in M' and x, there are ϵ' and η' so that if $||M - M'|| < \epsilon'$ and $||x - x'|| < \eta'$, then

$$||DG^{M'}(x') - DG^M(x)|| < \gamma.$$

Next, if $\text{Hyp}(G^M) = \{x_1, \ldots, x_m\}$, then there is a $\gamma > 0$ such that if $||DG^{M'}(x_i') - DG^M(x_i)|| < \gamma$ then

$$d(spec(DG^{M'}(x_i')), \mathcal{S}^1) > 0$$

for $i = 1, \ldots, m$. Given this γ, choose ϵ' and η' as above. Since $x_i' = x_i'(M')$ is continuous in M', for M' near M, there exists ϵ'' such that if $||M - M'|| < \epsilon''$ then $||x_i' - x_i|| < \eta'$ for all i. Finally, let $\epsilon''' = \min(\epsilon, \epsilon', \epsilon'')$. It follows that if $||M - M'|| < \epsilon'''$, then

$$\min_{i=1,\ldots,m} d(spec(DG^{M'}(x_i), \mathcal{S}^1)) > 0. \tag{6.4}$$

□

We now present the proof of Proposition 1.13.

Proof. Let $G^M = M \circ F$ be a GA map which is not hyperbolic. We claim that for any $\epsilon > 0$, we can find $\overline{M} \in N_\epsilon(M) \subset \mathcal{M}$ such that $G^{\overline{M}} = \overline{M} \circ F$ is hyperbolic.

By Corollary 4.8 and Lemma 6.1, for any $\epsilon > 0$, there exists $M' \in \mathcal{M}_{\{0\}}$ such that if $||M' - M|| < \epsilon$, then $G^{M'}$ has finitely many fixed points each of which has a Jacobian with no eigenvalue 1.

Clearly, by choice of M', there exists $m < \infty$ such that

$$\text{Fix}(G^{M'}) = \{x_1, \ldots, x_m\}$$

and for some $l \leq m$,

$$\text{Hyp}(G^{M'}) = \{x_1, \ldots, x_l\}$$

$$\text{NonHyp}(G^{M'}) = \{x_{l+1}, \ldots, x_m\}.$$

We now construct a finite sequence of perturbations, indexed by j, which will perturb non-hyperbolic fixed points in such a way that they will become hyperbolic.

Assume that after the j-th step we have the map $G^{M_j} := M_j \circ F$ with

$$\text{Fix}(G^{M^j}) = \{y_1, \ldots, y_m\}$$

and

$$\text{Hyp}(G^{M_j}) = \{y_1, \ldots, y_k\}$$

$$\text{NonHyp}(G^{M_j}) = \{y_{k+1}, \ldots, y_m\}.$$

We construct the $j+1$-th perturbation M_{j+1} and define $G^{M_{j+1}} = M_{j+1} \circ F$. By Lemma 6.2 there exists $\epsilon_1 > 0$ such that if $||M - M_j|| < \epsilon_1$, then

$$\#\{\text{Fix}(M \circ F)\} = \#\{\text{Fix}(M_j \circ F)\}.$$

By part (2) of Lemma 6.3, there exists $\epsilon_2 > 0$ such that for $M \in \mathcal{M}$ with $||M - M_j|| < \epsilon_2$

$$\tau_j = \min_{i=1,\ldots,k} d(\text{spec}(DG^{M_j}(y_i)), \mathcal{S}^1) > 0. \tag{6.5}$$

By Lemma 3.12, there exists $P \in \mathcal{P}(y_{k+1}, M)$ such that for $(M_j + cP) \circ F$, y_{k+1} is a hyperbolic fixed point for all $0 < c \leq 1$.

Let $\eta < \min\{\epsilon_1, \epsilon_2\}$. Since $\mathcal{M}_{\{0\}} \subset \mathcal{M}$ is open and dense, there exists $0 < \mu < 1$ such that for $M_{j+1} := M_j + \mu P$, then $M_{j+1} \in \mathcal{M}_{\{0\}}$ and $||M_{j+1} - M_j|| < \eta$. By construction, $\eta < \epsilon_1$, thus

$$\#\{\text{Fix}(M_{j+1} \circ F)\} = \#\{\text{Fix}(M_j \circ F)\}.$$

Additionally, because $\mu < 1$, by Corollary 2.9, $\mu P \in \mathcal{P}(y_{k+1}, M)$ which by definition implies $G^{M_{j+1}}(y_{k+1}) = y_{k+1}$. That is, for this choice of μ, we note $M_{j+1} \in \mathcal{M}_{\{0\}}$, y_{k+1} is a fixed point of $G^{M_{j+1}}$, and y_{k+1} is hyperbolic.

Finally, because $\eta < \epsilon_2$, for $G^{M_{j+1}}$,

$$\tau_j = \min_{i=1,\dots,k} d(\text{spec}(DG^{M_{j+1}}(y_i)), \mathcal{S}^1) > 0.$$

Thus, by Lemma 6.4,

$$\text{Hyp}(G^{M_{j+1}}) \supseteq \{y_1', \dots, y_k', y_{k+1}\}$$

where $y_1', \dots, y_k'$ are perturbed fixed points $y_1, \dots, y_k$ which by our choice $\eta < \epsilon_2$ are hyperbolic. Therefore,

$$|\text{NonHyp}(G^{M_{j+1}})| < |\text{NonHyp}(G^{M_j})|.$$

This process terminates in a finite number of steps when for j large enough $\text{NonHyp}(G^{M_j}) = \emptyset$. $\square$

CHAPTER 7

PROOF OF MAIN RESULT

We set out to prove Theorem 1.11 which states that for a typical mixing operator, $M \in \mathcal{M}$, G is hyperbolic.

We split the proof up into two parts based on Proposition 1.12, which shows the set of mixing operators for which G is hyperbolic is open, and Proposition 1.13, which shows the set of mixing operators for which G is hyperbolic is dense.

Proposition 1.12 is proved in Chapter 5 and Proposition 1.13 is proved in Chapter 6 using the results of Theorem 1.10 which was proved in Chapter 4.

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APPENDICES

APPENDIX A

Notation

In addition to the above model, the following notation and terminology will be used.

- Let $Mat_n(F)$ be the set of $n \times n$ matrices with entries in the field F, usually $F = \mathbb{R}, \mathbb{C}$.
- For a matrix A, let a_{ij} denote the ij-th entry of A.
- For a matrix A, let a_i denote the i-th row of the matrix A and let a^j denote the j-th column of the matrix A.
- For $A \in Mat_n(F)$, let $\det(A)$ denote the determinant of the matrix A. Note that we also use $\det(a_1, \ldots, a_n)$ and $\det(a^1, \ldots, a^n)$ to denote $\det(A)$.
- For an $n \times n$ matrix A, let $A(i, j)$ represent the $(n-1) \times (n-1)$ matrix that results from deleting row i and column j from the matrix A. That is, $A(i, j)$ denotes the ij-th minor of A.
- For a matrix A, let A_{ij} denote the determinant of the minor $A(i, j)$. That is, $A_{ij} = \det(A(i, j))$.
- The characteristic polynomial for the matrix $A \in Mat_n(F)$ is denoted by $\det(A - \lambda I)$, where I is the $n \times n$ identity matrix.
- Let $\mathrm{spec}(A)$ denote the set of eigenvalues of A.
- The transpose of a matrix A is denoted A^T.
- The inverse of an invertible matrix A is denoted A^{-1}.
- We use the notation $A > 0$ to indicate that $a_{ij} > 0$ for all i, j.

- For a matrix A, rank(A) denotes the dimension of the range of A, or the number of linearly independent columns of the matrix.
- Let $\alpha = a + bi$ be a complex number with $a, b \in \mathbb{R}$. We denote the real part of α with $\mathrm{Re}(\alpha)$ and the imaginary part is denoted $\mathrm{Im}(\alpha)$.
- Let V, W be vector spaces, and let $T : V \to W$ be an operator. We denote the null space of T with null(T). The image, or range, of T is denoted Image(T).
- Let $N_r(x)$ denote a ball of radius r about a point x.
- Given a set S, let int(S) denote the interior of the set S.
- We let $\mathbb{R}^{n+}$ denote the set of $x \in \mathbb{R}^n$ such that $x_k > 0$ for all $k = 1, \ldots, n$.
- Let $\mathbb{R}^n_0 := \{x \in \mathbb{R}^n | \sum_i x_i = 0\}$.

For the following items, let X, Y be n and m dimensional smooth manifolds, respectively.

- For $x \in X$, we let $T_x X$ denote the tangent space to X at x.
- For a differentiable map $f : X \to Y$, we let $df_x : T_x X \to T_{f(x)} Y$ denote the differential of the map f at the point x.
- In the special case that $T_x X = \mathbb{R}^n$ and $T_{f(x)} Y = \mathbb{R}^m$, we note that $df_x : \mathbb{R}^n \to \mathbb{R}^m$ is given by the Jacobian matrix $Df(x)$ ([14]).
- For $r \geq 1$, let $\mathcal{C}^r(X, Y)$ be the set of all r-times continuously differentiable functions from X to Y. We let $\mathcal{C}^0(X, Y)$ then denote continuous functions from X to Y. For $r \geq 0$, if $f \in \mathcal{C}^r(X, Y)$, we call f a $\mathcal{C}^r$ function or map.

APPENDIX B

Matrix Properties

The determinant of an $n \times n$ matrix, A, is a multi-linear alternating n-form on the vector rows, or columns, of the matrix [8], [14]. Letting $a_1, \ldots, a_n$ denote the rows of the matrix A, we will write $\det(a_1, \ldots, a_n) = \det(A)$.

Lemma 7.1. *Let $A^1, A^2 \in Mat_n(F)$, and let $\sigma(n)$ be the set of mappings from $\{1, \ldots, n\}$ to $\{1, 2\}$. That is, $\sigma(n) := \{f : \{1, \ldots, n\} \to \{1, 2\}\}$. Then,*

$$\det(A^1 + A^2) = \sum_{f \in \sigma(n)} \det(a_1^{f(1)}, a_2^{f(2)}, \ldots, a_n^{f(n)}).$$

Here a_j^i denotes the j-th row of the matrix A^i, $i = 1, 2$.

Proof. Let a_j^i denote the j-th row of the matrix A^i, $i = 1, 2$. We define the set of mappings from $\{1, \ldots, k\}$ to $\{1, 2\}$ by $\sigma(k) := \{f : \{1, \ldots, k\} \to \{1, 2\}\}$, and show that for any $k = 1, \ldots, n$,

$$\det(A^1 + A^2) = \sum_{f \in \sigma(k)} \det(a_1^{f(1)}, \ldots, a_k^{f(k)}, a_{k+1}^1 + a_{k+1}^2, \ldots, a_n^1 + a_n^2). \tag{7.1}$$

Let $k = 1$, then $\sigma(1) = \{f : \{1\} \to \{1, 2\}\}$. That is,

$$\sigma(1) = \{f : f(1) = 1\} \cup \{f : f(1) = 2\}.$$

We show that

$$\det(A^1 + A^2) = \sum_{f \in \sigma(1)} \det(a_1^{f(1)}, a_2^1 + a_2^2, \ldots, a_n^1 + a_n^2).$$

Note that

$$\det(A^1 + A^2) = \det(a_1^1 + a_1^2, a_2^1 + a_2^2, \ldots, a_n^1 + a_n^2)$$

$$
\begin{aligned}
=&\det(a_1^1, a_2^1 + a_2^2, \ldots, a_n^1 + a_n^2) + \det(a_1^2, a_2^1 + a_2^2, \ldots, a_n^1 + a_n^2) \\
=&\det(a_1^{f(1)=1}, a_2^1 + a_2^2, \ldots, a_n^1 + a_n^2) + \det(a_1^{f(1)=2}, a_2^1 + a_2^2, \ldots, a_n^1 + a_n^2) \\
=&\sum_{f \in \sigma(1)} \det(a_1^{f(1)}, a_2^1 + a_2^2, \ldots, a_n^1 + a_n^2).
\end{aligned}
$$

Thus we have proved the case $k = 1$.

And now we arrive at the induction step. We assume that (7.1) holds for $k = j$, that is, for $\sigma(j) = \{f : \{1, \ldots, j\} \to \{1, 2\}\}$,

$$\det(A^1 + A^2) = \sum_{f \in \sigma(j)} \det(a_1^{f(1)}, \ldots, a_j^{f(j)}, a_{j+1}^1 + a_{j+1}^2, \ldots, a_n^1 + a_n^2).$$

Now we show that (7.1) holds for $k = j + 1$. We compute

$$
\begin{aligned}
\det(A^1 + A^2) =&\det(a_1^1 + a_1^2, \ldots, a_j^1 + a_j^2, a_{j+1}^1 + a_{j+1}^2, \ldots, a_n^1 + a_n^2) \\
=&\sum_{f \in \sigma(j)} \det(a_1^{f(1)}, \ldots, a_j^{f(j)}, a_{j+1}^1 + a_{j+1}^2, \ldots, a_n^1 + a_n^2) \\
=&\sum_{f \in \sigma(j)} [\det(a_1^{f(1)}, \ldots, a_j^{f(j)}, a_{j+1}^1, a_{j+2}^1 + a_{j+2}^2, \ldots, a_n^1 + a_n^2) \\
&+ \det(a_1^{f(1)}, \ldots, a_j^{f(j)}, a_{j+1}^2, a_{j+2}^1 + a_{j+2}^2, \ldots, a_n^1 + a_n^2)] \\
=&\sum_{f \in \sigma(j+1)} \det(a_1^{f(1)}, \ldots, a_{j+1}^{f(j+1)}, a_{j+2}^1 + a_{j+2}^2, \ldots, a_n^1 + a_n^2).
\end{aligned}
$$

By induction equation (7.1) is true for all $1 \le k \le n$, which shows that for $A^1, A^2 \in Mat_n(F)$,

$$\det(A^1 + A^2) = \sum_{f \in \sigma(n)} \det(a_1^{f(1)}, \ldots, a_n^{f(n)}).$$

□

Lemma 7.2. *Let $B, C \in Mat_n(F)$. Then for $0 < \epsilon << 1$,*

$$\det(B + \epsilon C) = \det(B) + \epsilon c_1 \cdot (B_{11}, -B_{12}, \ldots, \pm B_{1n})$$

$$+\epsilon c_2 \cdot (-B_{21}, B_{22}, \ldots, \pm B_{2n}) + \cdots + \epsilon c_n \cdot (\pm B_{n1}, -(\pm) B_{n2}, \ldots, \pm B_{nn})$$
$$+\mathcal{O}(\epsilon^2).$$

Here c_j denotes the j-th row of the matrix C, and B_{ij} denotes the determinant of the ij-th minor of B. That is, B_{ij} is the determinant of the matrix obtained by removing row i and column j from matrix B.

Proof. Let $B, C \in Mat_n(F)$ and let $A^1 := B$ and $A^2 := \epsilon C$. We let a_j^i denote the j-th row of the matrix A^i, $i = 1, 2$ so that by Lemma 7.1,

$$\det(B + \epsilon C) = \det(A^1 + A^2) = \sum_{f \in \sigma(n)} \det(a_1^{f(1)}, \ldots, a_n^{f(n)})$$

where $\sigma(n) = \{f : \{1, \ldots, n\} \to \{1, 2\}\}$. We define $\sigma_1(n)$ to denote the set of functions $f : \{1, \ldots, n\} \to \{1, 2\}$ with the property that there is exactly one $i \in \{1, \ldots, n\}$ for which $f(i) = 2$:

$$\sigma_{(1)}(n) := \{f \in \sigma(n) | \#\{i | f(i) = 2\} = 1\}.$$

Similarly, we denote the complementary set of functions $f : \{1, \ldots, n\} \to \{1, 2\}$ for which there exist i, j with $i \neq j$ and $f(i) = 2 = f(j)$, as follows:

$$\sigma_{(>1)}(n) := \{f \in \sigma(n) | \#\{i | f(i) = 2\} > 1\}.$$

Then,

$$\sigma(n) = \sigma_{(1)}(n) \cup \sigma_{(>1)}(n) \text{ and } \sigma_{(1)}(n) \cap \sigma_{(>1)}(n) = \emptyset.$$

Thus,

$$\det(A^1 + A^2) = \sum_{f \in \sigma(n)} \det(a_1^{f(1)}, \ldots, a_n^{f(n)})$$

$$= \sum_{f \in \sigma_{(1)}(n)} \det(a_1^{f(1)}, \ldots, a_n^{f(n)}) + \sum_{f \in \sigma_{(>1)}(n)} \det(a_1^{f(1)}, \ldots, a_n^{f(n)}).$$

Note every determinant in the second sum has $f(i) = 2$ for at least two $i = 1, \ldots, n$. That is, for at least two i, the row of the matrix is of the form $a_i^{f(i)=2} = \epsilon a_i^2$. Thus, for all $f \in \sigma_{(>1)}$, $\det(a_1^{f(1)}, a_2^{f(2)}, \ldots, a_n^{f(n)})$ is an $\mathcal{O}(\epsilon^2)$ term. Thus, we have:

$$\begin{aligned}
\det(A^1 + A^2) =& \sum_{f \in \sigma_{(1)}(n)} \det(a_1^{f(1)}, \ldots, a_n^{f(n)}) + \sum_{f \in \sigma_{(>1)}(n)} \det(a_1^{f(1)}, \ldots, a_n^{f(n)}) \\
=& \sum_{f \in \sigma_{(1)}(n)} \det(a_1^{f(1)}, \ldots, a_n^{f(n)}) + \sum_{f \in \sigma_{(>1)}(n)} \mathcal{O}(\epsilon^2) \\
=& \sum_{f \in \sigma_{(1)}(n)} \det(a_1^{f(1)}, \ldots, a_n^{f(n)}) + \mathcal{O}(\epsilon^2) \\
=& \det(a_1^1, a_2^1, \ldots, a_n^1) + \det(a_1^2, a_2^1, \ldots, a_n^1) + \det(a_1^1, a_2^2, a_3^1, \ldots, a_n^1) \\
& + \cdots + \det(a_1^1, a_2^1, \ldots, a_{n-1}^1, a^2) + \mathcal{O}(\epsilon^2) \\
=& \det(b_1, b_2, \ldots, b_n) + \det(\epsilon c_1, b_2, \ldots, b_n) + \det(b_1, \epsilon c_2, b_3, \ldots, b_n) \\
& + \cdots + \det(b_1, b_2, \ldots, b_{n-1}, \epsilon c_n) + \mathcal{O}(\epsilon^2). \qquad (7.2)
\end{aligned}$$

By properties of determinants [12], for any $C \in Mat_n(F)$, and for all $\epsilon \in \mathbb{R}$,

$$\det(c_1, \ldots, c_{i-1}, \epsilon c_i, c_{i+1}, \ldots, c_n) = \epsilon \det(c_1, \ldots, c_{i-1}, c_i, c_{i+1}, \ldots, c_n).$$

for all $i = 1, \ldots, n$. Note also that by expanding about the first row of a matrix C,

$$\det(c_1, \ldots, c_n) = c_1 \cdot (C_{11}, -C_{12}, \ldots, \pm C_{1n}). \qquad (7.3)$$

Thus, by equations (7.3) and (7.2),

$$\begin{aligned}
\det(B + \epsilon C) =& \det(B) + \epsilon c_1 \cdot (B_{11}, -B_{12}, \ldots, \pm B_{1n}) \\
& + \epsilon c_2 \cdot (-B_{21}, B_{22}, \ldots, \pm B_{2n}) + \cdots + \epsilon c_n \cdot (\pm B_{n1}, -(\pm) B_{n2}, \ldots, \pm B_{nn})
\end{aligned}$$

$+\mathcal{O}(\epsilon^2).$

$\square$

LEMMA 7.3. *Let* $B \in Mat_n(F)$ *with* $\operatorname{spec}(B) = \{\mu_1, \ldots, \mu_n\}$*. Let* $B(\lambda) := B - \lambda I$ *and* $\epsilon \in [0, 1)$*, then for any* $K \in Mat_n(F)$*,*

$$\det(B(\lambda) + \epsilon K) = (\lambda - \mu_1) \prod_{i=2}^{n} (\lambda - \mu_i) + \epsilon \sum_{i=1}^{n} q_i(\lambda) + \mathcal{O}(\epsilon^2)$$

For all $i = 1, \ldots, n$*,*

$$q_i(\lambda) = \pm k_i \cdot (B_{i1}(\lambda), -B_{i2}(\lambda), B_{i3}(\lambda), \ldots, \pm B_{in}(\lambda)).$$

Additionally, $r(\lambda) := \prod_{i=2}^{n} (\lambda - \mu_i)$ *and for all* i*,* $q_i(\lambda)$ *are each of degree* $n - 1$*.*

PROOF. By Lemma 7.2, for $\epsilon \in [0, 1)$

$$\begin{aligned} \det(B(\lambda) + \epsilon K) = \det B(\lambda) &+ \epsilon k_1 \cdot (B_{11}(\lambda), -B_{12}(\lambda), B_{13}(\lambda), \ldots, \pm B_{1n}(\lambda)) \\ &+ \epsilon k_2 \cdot (-B_{21}(\lambda), B_{22}(\lambda), -B_{23}(\lambda), \ldots, \pm B_{2n}(\lambda)) \\ &+ \cdots + \epsilon k_n \cdot (\pm B_{n1}(\lambda), \ldots, \pm B_{nn}(\lambda)) + \mathcal{O}(\epsilon^2). \end{aligned}$$

For all i, j, $B_{ij}(\lambda)$ is the determinant of the matrix obtained by removing row i and column j from $B(\lambda) = B - \lambda I$. Thus, for all i, j, the degree of $B_{ij}(\lambda)$ is $n - 1$. Let

$$q_i(\lambda) := \pm k_i \cdot (B_{i1}(\lambda), -B_{i2}(\lambda), B_{i3}(\lambda), \ldots, \pm B_{in}(\lambda)),$$

then for $i = 1, \ldots, n$; $q_i(\lambda)$ is a polynomial of degree less than n and

$$\det(B(\lambda) + \epsilon K) = \det B(\lambda) + \epsilon \sum_i q_i(\lambda) + \mathcal{O}(\epsilon^2). \tag{7.4}$$

Furthermore, since λ_0 is an eigenvalue of B, then λ_0 is a root of the characteristic polynomial $\det(B(\lambda))$. That is,

$$\det(B(\lambda)) = (\lambda - \mu_1)\prod_{i=2}^{n}(\lambda - \mu_i). \tag{7.5}$$

Let

$$r(\lambda) := \prod_{i=2}^{n}(\lambda - \mu_i), \tag{7.6}$$

then clearly r is a polynomial of degree $n-1$. The result now follows from equations (7.4), (7.5), and (7.6). □

Corollary 7.4. *Let $B \in Mat_n(F)$ with $\operatorname{spec}(B) = \{\mu_1, \dots, \mu_n\}$ and $\mu_1 \in \operatorname{spec}(B)$ a simple eigenvalue. Let $B(\lambda) := B - \lambda I$ and $\epsilon \in [0, 1)$. Then for any $K \in Mat_n(F)$, there exists $\delta > 0$ such that*

$$\det(B(\lambda) + \epsilon K) = \prod_{i=2}^{n}(\lambda - \mu_i)\left[(\lambda - \lambda_0) + \frac{\epsilon \sum_i q_i(\lambda) + \mathcal{O}(\epsilon^2)}{\prod_{i=2}^{n}(\lambda - \mu_i)}\right] \tag{7.7}$$

is well-defined in $N_\delta(\mu_1)$. For all $i = 1, \dots, n$,

$$q_i(\lambda) = \pm k_i \cdot (B_{i1}(\lambda), -B_{i2}(\lambda), B_{i3}(\lambda), \dots, \pm B_{in}(\lambda)).$$

Proof. By Lemma 7.3,

$$\det(B(\lambda) + \epsilon K) = (\lambda - \mu_1)\prod_{i=2}^{n}(\lambda - \mu_i) + \epsilon\sum_{i=1}^{n} q_i(\lambda) + \mathcal{O}(\epsilon^2)$$

where for $i = 1, \dots, n$;

$$q_i(\lambda) := \pm k_i \cdot (B_{i1}(\lambda), -B_{i2}(\lambda), B_{i3}(\lambda), \dots, \pm B_{in}(\lambda)).$$

Furthermore,

$$\det(B(\lambda)) = (\lambda - \mu_1)\prod_{i=2}^{n}(\lambda - \mu_i). \tag{7.8}$$

Since μ_1 is a simple eigenvalue of B, for all $i > 1$, $\mu_i \neq \mu_1$, thus $\prod_{i=2}^{n}(\mu_1 - \mu_i) \neq 0$. Thus, for some $\delta > 0$, there exists $N_\delta(\mu_1)$, such that for all $\lambda \in N_\delta(\mu_1)$, $\prod_{i=2}^{n}(\lambda - \mu_i) \neq 0$. Now for $\lambda \in N_\delta(\mu_1)$,

$$\begin{aligned} \det(\widetilde{B}(\lambda) + \epsilon K) =& (\lambda - \mu_1)\prod_{i=2}^{n}(\lambda - \mu_i) + \epsilon \sum_i q_i(\lambda) + \mathcal{O}(\epsilon^2) \\ =& \prod_{i=2}^{n}(\lambda - \mu_i)\left[(\lambda - \mu_1) + \frac{\epsilon \sum_i q_i(\lambda) + \mathcal{O}(\epsilon^2)}{\prod_{i=2}^{n}(\lambda - \mu_i)}\right] \end{aligned} \tag{7.9}$$

is well-defined. □

Lemma 7.5. *Let B denote a matrix with a simple eigenvalue λ_0. Let $B(\lambda) := B - \lambda I$ and define $\widetilde{B}(\lambda) := \widetilde{B} - \lambda I$, where $\widetilde{B}$ denotes B in Jordan normal form. Then*

$$(\widetilde{B})_{ij}(\lambda_0) = \begin{cases} \neq 0 & i = j = 1 \\ 0 & otherwise \end{cases}.$$

Here $\widetilde{B}_{ij}(\lambda_0)$ denotes the determinant of the ij-th minor of the matrix $\widetilde{B}(\lambda_0)$.

Proof. After changing the order of the rows and columns if necessary, by Jordan normal form,

$$\widetilde{B}(\lambda_0) = [\widetilde{B} - \lambda_0 I] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}. \tag{7.10}$$

Consider first the case $i \neq 1$. To calculate the appropriate determinant, $\widetilde{B}_{ij}(\lambda_0)$, for $i \neq 1$ we remove the i-th row and j-th column from the above matrix shown in equation (7.10). The result is a matrix of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}.$$

Thus, it is clear that for $i \neq 1$, $\widetilde{B}_{ij}(\lambda_0) = 0$. Analogously, we can show that $\widetilde{B}_{ij}(\lambda_0) = 0$ in the case $j \neq 1$. Finally, we are left with $(\widetilde{B})_{11}(\lambda_0)$. To show $(\widetilde{B})_{11}(\lambda_0) \neq 0$, we note that the $\text{rank}(\widetilde{B}(\lambda_0)) = n-1$ since λ_0 is a simple eigenvalue of B. Thus, by Theorem 4.17 in Greub [12], there is a minor of order $n-1$. That is, there exists a $(n-1) \times (n-1)$ minor for which the determinant is non-zero. Since there exists a minor of order $n-1$, and $\widetilde{B}_{ij}(\lambda_0) = 0$ unless $i = j = 1$, this implies $\widetilde{B}_{11}(\lambda_0) \neq 0$. □

LEMMA 7.6. *Take $A \in Mat_n(\mathbb{R})$, and let C be a change of basis matrix such that $\widetilde{A} = C^{-1}AC$ represents the matrix A expressed in the Jordan normal form. Then, for all i, there exists $j \in \{1, \dots, n\}$ such that $(c^{-1})_{ij} \neq 0$.*

PROOF. Proof by contradiction. Let $i \in \{1, \dots, n\}$. Assume that for all $j = 1, \dots, n$, $(c^{-1})_{ij} = 0$. That is, the i-th row of the matrix C^{-1} is the zero vector: $(c^{-1})_i = (0, \dots, 0)$. This is a contradiction since C^{-1} is invertible by nature of its definition. □

LEMMA 7.7. *Let $B \in Mat_n(F)$. If B is of the form*

$$B := \begin{pmatrix} \alpha & \beta & 0 & \dots & 0 \\ \gamma & \delta & 0 & \dots & 0 \\ 0 & 0 & * & * & * \\ \vdots & \vdots & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix}, \tag{7.11}$$

then $B_{ij} = 0$ *for* $i = 1, 2$ *and* $j \geq 2$ *as well as* $j = 1, 2$ *and* $i \geq 2$. *Here* B_{ij} *denotes the determinant of the minor obtained by deleting row* i *and column* j *from the matrix* B.

PROOF. Let $B(i, j)$ denote the matrix obtained by deleting row i and column j from the original matrix B. Let $i \in \{1, 2\}$ and $j > 2$, then $\det(B(i, j)) = 0$ since columns one and two of $B(i, j)$ are constant multiples of each other. That is, $\text{rank}(B(i, j)) \leq n - 1$, so $B_{ij} = 0$. The rest follows by the same argument. □

LEMMA 7.8. *Let* $B \in Mat_n(F)$. *If* B *is of the form*

$$B := \begin{pmatrix} \alpha & \beta & 0 & \dots & 0 \\ \gamma & \delta & 0 & \dots & 0 \\ 0 & 0 & * & * & * \\ \vdots & \vdots & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix}, \tag{7.12}$$

then $B_{11} = \delta \det(*)$, $B_{12} = \gamma \det(*)$, $B_{21} = \beta \det(*)$ *and* $B_{22} = \alpha \det(*)$. *Here* $\det(*)$ *denotes the determinant of the matrix obtained by deleting rows* $1, 2$ *and columns* $1, 2$ *from the matrix* B.

PROOF. Let $i \in \{1, 2\}, j \in \{1, 2\}$ and let $B(i, j)$ denote the matrix obtained by deleting row i and column j from B as defined above. Then, $B(i, j)$ is an $(n - 1) \times (n - 1)$ diagonal matrix of the form

$$B(i, j) = \begin{pmatrix} \mu & 0 & \dots & 0 \\ 0 & * & * & * \\ \vdots & * & * & * \\ 0 & * & * & * \end{pmatrix}.$$

Let $[B(i,j)](k,l)$ denote the matrix obtained by deleting row k and column l from the minor $B(i,j)$. Thus,

$$\det(B_{ij}) = \mu \det([B(i,j)](1,1)), \tag{7.13}$$

where $[B(i,j)](1,1)$ matrix obtained by removing row 1, column 1 from matrix $B(i,j)$. Now note that for $i,j \in \{1,2\}$,

$$\det(*) = \det([B(i,j)](1,1)).$$

The desired result now follows. □

LEMMA 7.9. *Let $B \in Mat_n(\mathbb{C})$ with*

$$B = \begin{pmatrix} -\beta i & \beta & 0 & \dots & 0 \\ -\beta & -\beta i & 0 & \dots & 0 \\ 0 & 0 & * & * & * \\ \vdots & \vdots & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix},$$

where $\beta \in \mathbb{R}$. Then,

1. *$B_{ij} = 0$ for $i \geq 2$, $j \geq 2$;*

2. *$B_{11} = B_{22}$ and $B_{12} = -B_{21}$.*

Here B_{ij} denotes the minor obtained by deleting row i and column j from the matrix B.

PROOF. We start by proving (1). Let $i \geq 2, j \geq 2$ and let $B(i,j)$ denote the matrix obtained be deleting row i and column j from B as defined above. Then,

$B(i,j)$ is an $(n-1)\times(n-1)$ block diagonal matrix of the form

$$B(i,j) = \begin{pmatrix} -\beta i & \beta & 0 & \dots & 0 \\ -\beta & -\beta i & 0 & \dots & 0 \\ 0 & 0 & * & * & * \\ \vdots & \vdots & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix}.$$

Thus,

$$\det(B_{ij}) = \det\begin{pmatrix} -\beta i & \beta \\ -\beta & -\beta i \end{pmatrix} \cdot \det(*), \qquad (7.14)$$

where $\det(*)$ is the determinant of the lower block diagonal matrix; that is, the determinant of the matrix obtained by removing rows $1,2$ and columns $1,2$ from matrix $B(i,j)$. Since

$$\begin{aligned} \det\begin{pmatrix} -\beta i & \beta \\ -\beta & -\beta i \end{pmatrix} &= (-\beta i)^2 - (-\beta)(\beta) \\ &= -\beta^2 + \beta^2 \\ &= 0, \end{aligned}$$

we can see that by equation (7.14),

$$\det(B_{ij}) = 0 \cdot \det(*) = 0.$$

This concludes the proof of part (1). To prove part (2), consider that

$$B(1,1) = B(2,2) = \begin{pmatrix} -\beta i & 0 & \dots & 0 \\ 0 & * & * & * \\ \vdots & * & * & * \\ 0 & * & * & * \end{pmatrix}.$$

Similarly,

$$B(1,2) = \begin{pmatrix} \beta & 0 & \dots & 0 \\ 0 & * & * & * \\ \vdots & * & * & * \\ 0 & * & * & * \end{pmatrix} \text{ and } B(2,1) = \begin{pmatrix} -\beta & 0 & \dots & 0 \\ 0 & * & * & * \\ \vdots & * & * & * \\ 0 & * & * & * \end{pmatrix}.$$

Part (2) now follows. □

LEMMA 7.10. *Let $B \in Mat_n(F)$ with simple eigenvalue λ_0. Let $\widetilde{B}(\lambda) := \widetilde{B} - \lambda I$ where $\widetilde{B}$ represents B expressed in Jordan normal form. Then for any $K \in Mat_n(F)$ and*

$$q_i(\lambda) := k_i \cdot (\widetilde{B}_{i1}(\lambda), -\widetilde{B}_{i2}(\lambda), \widetilde{B}_{i3}(\lambda), \ldots, \pm\widetilde{B}_{in}(\lambda)),$$

we have the following:

1. *If $\lambda_0 \in \mathbb{R}$, for $i > 1$, $q_i(\lambda_0) = 0$. Thus, for $i > 1$ there exists $p_i(\lambda)$ a polynomial of degree at most $n-1$ such that $q_i(\lambda) = (\lambda - \lambda_0)p_i(\lambda)$;*
2. *If $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$, for $i > 2$, $q_i(\lambda_0) = 0$. Thus, for $i > 2$ there exists $p_i(\lambda)$ a polynomial of degree at most $n-1$ such that $q_i(\lambda) = (\lambda - \lambda_0)p_i(\lambda)$.*

PROOF. We first prove (1), the case for $\lambda_0 \in \mathbb{R}$. By Lemma 7.5, we know that $\widetilde{B}_{ij}(\lambda_0) = 0$ for $i \neq 1$ and $j \neq 1$. Thus,

$$q_i(\lambda_0) = \pm k_i \cdot (\widetilde{B}_{i1}(\lambda_0), -\widetilde{B}_{i2}(\lambda_0), \widetilde{B}_{i3}(\lambda_0), \ldots, \pm\widetilde{B}_{in}(\lambda_0)) = 0 \tag{7.15}$$

for all $i > 1$. That is, for all $i > 1$, the polynomial q_i has root λ_0. Thus, for $i > 1$, there exists $p_i(\lambda)$, a polynomial of degree less than $n-1$ such that

$$q_i(\lambda) = (\lambda - \lambda_0)p_i(\lambda). \tag{7.16}$$

Thus we have proved (1). Now we prove (2), for $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$. By Lemma 7.9, we know that $\widetilde{B}_{ij}(\lambda_0) = 0$ for $i \geq 2$, $j \geq 2$; and for all $i > 2$,

$$q_i(\lambda_0) = k_i \cdot (\widetilde{B}_{i1}(\lambda_0), -\widetilde{B}_{i2}(\lambda_0), \widetilde{B}_{i3}(\lambda_0), \ldots, \pm\widetilde{B}_{in}(\lambda_0)) = 0.$$

That is, for all $i > 2$, the polynomial q_i has root λ_0. Thus, for $i > 2$, there exists $p_i(\lambda)$, a polynomial of degree less than $n-1$ such that

$$q_i(\lambda) = (\lambda - \lambda_0)p_i(\lambda). \tag{7.17}$$

Thus we have proved (2). □

COROLLARY 7.11. *Let* $B \in Mat_n(F)$ *with* $\operatorname{spec}(B) = \{\lambda_1, \ldots, \lambda_n\}$ *and* $\lambda_1 \in \operatorname{spec}(B)$ *a simple eigenvalue. Let* $\widetilde{B}(\lambda) := \widetilde{B} - \lambda I$ *and* $\epsilon \in [0,1)$. *Then for any* $K \in Mat_n(F)$, *for* $i = 1, \ldots, n$ *with* $q_i(\lambda) := k_i \cdot (\widetilde{B}_{i1}(\lambda), -\widetilde{B}_{i2}(\lambda), \widetilde{B}_{i3}(\lambda), \ldots, \pm\widetilde{B}_{in}(\lambda))$.*, there exists* $\delta > 0$ *such that*

$$\det(B(\lambda) + \epsilon K) = \prod_{i=2}^{n}(\lambda - \lambda_i)\left[(\lambda - \lambda_1) + \frac{\epsilon q(\lambda) + \epsilon(\lambda - \lambda_1)s(\lambda) + \mathcal{O}(\epsilon^2)}{\prod_{i=2}^{n}(\lambda - \lambda_i)}\right] \tag{7.18}$$

is well-defined in $N_\delta(\lambda_1)$. *Additionally,* $q(\lambda)$ *is a polynomial of degree* $n-1$ *and* $s(\lambda)$ *is a polynomial of degree* $n-2$*, where*

1. *If* $\lambda_1 \in \mathbb{R}$, *then* $q(\lambda) := q_1(\lambda)$, *and* $s(\lambda) = \frac{\sum_{i=2}^{n} q_i(\lambda)}{\lambda - \lambda_1}$;
2. *If* $\lambda_1 \in \mathbb{C}\backslash\mathbb{R}$, *then* $q(\lambda) := q_1(\lambda) + q_2(\lambda)$, *and* $s(\lambda) = \frac{\sum_{i=3}^{n} q_i(\lambda)}{\lambda - \lambda_1}$.

PROOF. Let λ_1 be a simple eigenvalue of B with $\operatorname{spec}(B) = \{\lambda_1, \ldots, \lambda_n\}$. We first consider the case $\lambda_1 \in \mathbb{R}$. By Lemma 7.10, for $i > 1$, there exist polynomials $p_i(\lambda)$ such that $q_i(\lambda) = (\lambda - \lambda_1)p_i(\lambda)$. Thus, for $q(\lambda) := q_1(\lambda)$ and $s(\lambda) := \sum_{i=2}^{n} p_i(\lambda)$, the result follows by Corollary 7.4 and Lemma 7.10. Next, consider the case $\lambda_1 \in \mathbb{C}\backslash\mathbb{R}$. By Lemma 7.10, for $i > 2$ there exist polynomials $p_i(\lambda)$ such that $q_i = (\lambda - \lambda_1)p_i(\lambda)$.

Thus, for $q(\lambda) := q_1(\lambda) + q_2(\lambda)$, $s(\lambda) := \sum_{i=3}^{n} p_i(\lambda)$. The result now follows directly from Corollaries 7.4 and Lemma 7.10. $\square$

LEMMA 7.12. *Let $A \in Mat_n(R)$ with eigenvalue $\lambda \in \mathbb{C}$ for which v is the corresponding eigenvector. Let $\mathrm{Re}(v), \mathrm{Im}(v) \in \mathbb{R}^n$ denote the real and imaginary parts of v, respectively, and let $\lambda = \alpha + i\beta$. Then, $A(\mathrm{Re}(v)) = (\alpha\mathrm{Re}(v) - \beta\mathrm{Im}(v))$ and $A(\mathrm{Im}(v)) = (\beta\mathrm{Re}(v) + \alpha\mathrm{Im}(v))$.*

PROOF. By definition, since $\lambda = \alpha + i\beta$ is an eigenvalue corresponding to v, $Av = (\alpha + i\beta)v$. Since $v = \mathrm{Re}(v) + i\mathrm{Im}(v)$,

$$A(v) = A(\mathrm{Re}(v) + i\mathrm{Im}(v)) = (\alpha + i\beta)(\mathrm{Re}(v) + i\mathrm{Im}(v)).$$

By computation, we see that

$$A(\mathrm{Re}(v)+i\mathrm{Im}(v)) = A(\mathrm{Re}(v))+iA(\mathrm{Im}(v)) = (\alpha\mathrm{Re}(v)-\beta\mathrm{Im}(v))+i(\beta\mathrm{Re}(v)+\alpha\mathrm{Im}(v)), \tag{7.19}$$

that is,

$$A(\mathrm{Re}(v)) + iA(\mathrm{Im}(v)) = (\alpha\mathrm{Re}(v) - \beta\mathrm{Im}(v)) + i(\beta\mathrm{Re}(v) + \alpha\mathrm{Im}(v))$$

where $A(\mathrm{Re}(v)), A(\mathrm{Im}(v)) \in \mathbb{R}^n$. The desired result follows directly. $\square$

APPENDIX C

Transversality Theorems

THEOREM 7.13 (Transversal Density Theorem, [1]). *Let $\mathcal{A}, X, Y$ be $\mathcal{C}^r$ manifolds, $\rho : \mathcal{A} \to \mathcal{C}^r(X,Y)$ a $\mathcal{C}^r$ representation, $W \subset Y$ a submanifold (not necessarily closed) and $\mathrm{ev}_\rho : \mathcal{A} \times X \to Y$ the evaluation map. Define $\mathcal{A}_W \subset \mathcal{A}$ by*

$$\mathcal{A}_W = \{a \in \mathcal{A} | \rho_a \pitchfork W\}.$$

Assume that

1. *X has finite dimension m and W has finite codimension q in Y;*

2. *$\mathcal{A}$ and X are second countable;*

3. *$r > \max(0, m-q)$;*

4. *$\mathrm{ev}_\rho \pitchfork W$.*

Then $\mathcal{A}$ is residual (and hence dense) in $\mathcal{A}$.

THEOREM 7.14 (Corollary 17.2, [1]). *Let X, Y be $\mathcal{C}^r$ manifolds ($r \geq 1$), $f : X \to Y$ a $\mathcal{C}^r$ map, $W \subset Y$ a $\mathcal{C}^r$ submanifold. Then if $f \pitchfork W$:*

1. *W and $f^{-1}(W)$ have the same codimension;*

2. *If W is closed and X is compact, $f^{-1}(\{W\})$ has only finitely many connected components.*

THEOREM 7.15 (Openness of Transversal Intersection, [1]). *Let $\mathcal{A}, X, Y$ be $\mathcal{C}^1$ manifolds with X finite dimensional, $W \subset Y$ a closed $\mathcal{C}^1$ submanifold, $K \subset X$ a* *compact subset of* X*, and* $\rho : \mathcal{A} \to \mathcal{C}^1(X, Y)$ *a* $\mathcal{C}^1$ *representation. Then the subset* $\mathcal{A}_{KW} \subset \mathcal{A}$ *defined by* $\mathcal{A}_{KW} = \{a \in \mathcal{A} | \rho_a \pitchfork_x W \text{ for } x \in K\}$ *is open.*

Remark: The Openness of Transversal Intersection Theorem in [1] actually requires that $\rho : \mathcal{A} \to \mathcal{C}^1(X, Y)$ is a pseudorepresentation. Since every $\mathcal{C}^r$ representation is also a pseudorepresentation ([1]), we present a weaker version of the theorem.