TESTING GENERAL RELATIVITY THROUGH THE COMPUTATION
OF RADIATIVE TERMS AND WITHIN THE NEUTRON
STAR STRONG-FIELD REGIME

by

Alexander Saffer

A dissertation submitted in partial fulfillment
of the requirements for the degree

of

Doctor of Philosophy

in

Physics

MONTANA STATE UNIVERSITY
Bozeman, Montana

April 2019
DEDICATION

Dedicated to my family, who provided me with their unconditional love and support.
ACKNOWLEDGEMENTS

I would like to acknowledge the hard work of my advisor Dr. Nicolás Yunes, whose dedication and compassion for physics and those in this field has taught me more than classes ever could.

I would also like to thank the Dr. Kent Yagi and Dr. Hector Okada da Silva for their willingness to always help me no matter what I asked and for having the patience to put up with me whenever I struggled. In addition, I would like to thank Dr. Neil Cornish and the rest of the eXtreme Gravity Institute for providing an environment conducive to success.

Lastly, I would like to thank my family and friends both within and outside of the MSU physics department who provided their support and guidance over these past 6 years, especially Meg.
1. INTRODUCTION ................................................................................................ 1

2. THE GRAVITATIONAL WAVE STRESS-ENERGY (PSEUDO)-TENSOR
   IN MODIFIED GRAVITY ...................................................................................12
   Contribution of Authors and Co-Authors ...............................................................12
   Manuscript Information Page................................................................................13
   Abstract ..............................................................................................................14
   Introduction ........................................................................................................15
   General Relativity ................................................................................................18
   Perturbed Action Method .............................................................................18
   Perturbed Field Equation Method ........................................................................22
   Landau-Lifshitz Method ...............................................................................24
   Noether Current Method ..............................................................................26
   Derivation of Physical Quantities: $\dot{E}$ and $\dot{P}$ ...................................................29
   Jordan-Fierz-Brans-Dicke Theory .........................................................................30
   Perturbed Action Method .............................................................................31
   Perturbed Field Equation Method ........................................................................34
   Landau-Lifshitz Method ...............................................................................35
   Noether Current Method ..............................................................................36
   Derivation of Physical Quantities: $\dot{E}$ and $\dot{P}$ ...................................................37
   Einstein-Æther Theory .........................................................................................38
   Perturbed Action Method .............................................................................41
   Perturbed Field Equation Method ........................................................................44
   Landau-Lifshitz Method ...............................................................................47
   Noether Current Method ..............................................................................49
   Derivation of Physical Quantities: $\dot{E}$, $\dot{P}$, and $\dot{L}$ ........................................52
   Conclusion ..........................................................................................................53

3. ANGULAR MOMENTUM LOSS FOR A BINARY SYSTEM IN
   EINSTEIN-ÆTHER THEORY .............................................................................56
   Contribution of Authors and Co-Authors ...............................................................56
   Manuscript Information Page................................................................................57
   Abstract ..............................................................................................................58
   Introduction ........................................................................................................58
   Einstein-Æther Theory .........................................................................................60
   Gravitational Action .......................................................................................62
   Matter Action .................................................................................................63
   Field Equations ...............................................................................................64
TABLE OF CONTENTS – CONTINUED

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Field Decomposition</td>
<td>65</td>
</tr>
<tr>
<td>Angular Momentum Loss Rate</td>
<td>68</td>
</tr>
<tr>
<td>General Calculation</td>
<td>68</td>
</tr>
<tr>
<td>Specialization to an N-Body System</td>
<td>71</td>
</tr>
<tr>
<td>Binary Dynamics in ( \mathcal{A} )-theory</td>
<td>74</td>
</tr>
<tr>
<td>Conclusion and Future</td>
<td>78</td>
</tr>
<tr>
<td>Acknowledgments</td>
<td>79</td>
</tr>
</tbody>
</table>

4. **THE EXTERIOR SPACETIME OF RELATIVISTIC STARS IN SCALAR-GAUSS-BONNET GRAVITY** ........................................... 80

| Contribution of Authors and Co-Authors                                  | 80   |
| Manuscript Information Page                                             | 81   |
| Abstract                                                               | 82   |
| Introduction                                                           | 82   |
| Scalar Gauss-Bonnet gravity                                             | 85   |
| Action                                                                 | 85   |
| Field equations                                                        | 88   |
| Perturbative expansion for the metric and fluid variables               | 89   |
| Perturbative expansion of the field equations                           | 90   |
| Solutions of the field equations outside the star                      | 92   |
| \( O \left( \bar{\alpha}^0 \right) \) equations                        | 92   |
| \( O \left( \bar{\alpha}^1 \right) \) equations                        | 93   |
| \( O \left( \bar{\alpha}^2 \right) \) equations                        | 94   |
| Comparison with black holes spacetimes                                  | 96   |
| Solutions of the field equations inside the star                       | 97   |
| \( O \left( \bar{\alpha}^0 \right) \) equations                        | 97   |
| \( O \left( \bar{\alpha}^1 \right) \) equations                        | 98   |
| \( O \left( \bar{\alpha}^2 \right) \) equations                        | 100  |
| Astrophysical applications                                              | 103  |
| Circular orbits around the star                                        | 104  |
| Modified Kepler’s Third Law                                             | 106  |
| Quasiperiodic oscillations                                              | 106  |
| Light bending                                                          | 109  |
| Conclusions and outlook                                                | 114  |
| Addendum to sGB Work                                                   | 116  |
| On the maximum mass of the neutron star                                | 116  |
| On the study of quasi-periodic orbits                                  | 118  |
# TABLE OF CONTENTS – CONTINUED

5. CONCLUSION .................................................................................................................. 121

REFERENCES CITED ....................................................................................................... 124

APPENDICES ................................................................................................................... 138

| APPENDIX A : Brill-Hartle Averaging Scheme | 139 |
| APPENDIX B : Electromagnetic Canonical SET | 142 |
| APPENDIX C : Derivation of JFBD Reduced Field | 144 |
| APPENDIX D : Expanded Action for Einstein-Æther | 147 |
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Temporal evolution of the semi-major axis (a) and the orbital eccentricity (e) of a $1.4M_\odot$-$2M_\odot$ neutron star binary due to the modified decay rates of the binary’s orbital energy and angular momentum. To construct this figure, we modeled the sensitivities with the weak-field expansion of [1, 2], and we used an initial orbital frequency of 10 Hz and an initial orbital eccentricity of 0.9. Observe that the semi-major axis and the orbital eccentricity decay faster than in GR.</td>
</tr>
<tr>
<td>3.2</td>
<td>Example of Binary System Used</td>
</tr>
<tr>
<td>4.1</td>
<td>Mass-radius curves for various equations of state. The solid lines represent the GR solution, while the dashed lines correspond to our sGB solutions for $\alpha = 15\kappa M_\odot^2$. The central densities of the stars shown here range between $0.5-2.3 \times 10^{15} \text{g/cm}^3$.</td>
</tr>
<tr>
<td>4.2</td>
<td>Radial profiles of the Gauss-Bonnet invariant $\mathcal{G}<em>0$ (top) and the scalar field $\varphi_1$ (bottom) at $\mathcal{O}\left(\bar{\alpha}^{-1}\right)$ for and FPS EoS. In both panels, the different colors correspond to different central energy densities $\varepsilon</em>{0c}$. The vertical dashed lines correspond to the radius for each star. All scalar field solutions were calculated at a fixed $\bar{\alpha} = 15$ coupling constant strength.</td>
</tr>
<tr>
<td>4.3</td>
<td>Central values of the scalar field $\varphi_1$ for various values of $\bar{\alpha}$ as a function of the central densities of the star with an FPS EoS. Observe how the central value of the scalar field converges toward zero at small central densities irrespective of the coupling constant.</td>
</tr>
<tr>
<td>4.4</td>
<td>Mass-radius curves with a SLy EoS for varying couplings $\bar{\alpha}$. Observe that greater couplings lead to a decrease in the maximum mass of NSs, which can aid in constraining the theory with observations of massive pulsars.</td>
</tr>
<tr>
<td>4.5</td>
<td>Orbital frequencies $\Omega_r$ versus $\Omega_{\text{GB}}$ for a NS of mass $1.4 M_\odot$ to leading order in $\zeta$.</td>
</tr>
</tbody>
</table>
LIST OF FIGURES – CONTINUED

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.6</td>
<td>Diagram of emitted photon trajectory. A photon emitted in the direction $\vec{k}$ from the surface of the star orthogonal to $\vec{n}$ will have its trajectory bent by an angle $\iota = \psi - \gamma$ to an impact parameter of $b$.</td>
<td>109</td>
</tr>
<tr>
<td>4.7</td>
<td>Critical angle as a function of compactness for several coupling values $\zeta$. If we were to fix $\alpha$ as done in previous figures, we would also need to specify the NS mass. Since we can only measure $M/R$ directly (and not the NS mass) with light bending tests, it makes more sense here to fix $\zeta$ instead.</td>
<td>111</td>
</tr>
<tr>
<td>4.8</td>
<td>Light bending in sGB gravity. The solid lines represent the GR solution, while the dashed (dotted) lines correspond to $\zeta = 2.5$ ($\zeta = 5$). Deviations from GR are more noticeable when the compactnesses is large and $\zeta$ increases light bending, at fixed emission angle $\gamma$.</td>
<td>112</td>
</tr>
<tr>
<td>4.9</td>
<td>Visible fraction of a star as a function of compactness for various coupling strengths. The lines terminate at the value of compactness for which the whole surface of the star becomes visible. Large values of $\zeta$ require smaller values of compactness for this to happen.</td>
<td>113</td>
</tr>
<tr>
<td>4.10</td>
<td>Power spectrum of Sco X-1 showing the two QPO peaks over a 10 ksec observation. The best fit has been superimposed onto the graph [3].</td>
<td>119</td>
</tr>
<tr>
<td>4.11</td>
<td>$\Delta \nu$ vs. $\nu_2$ ($\Omega_r$ vs. $\Omega_{sGB}$ in Fig. 4.5) for 10 LMXBs. The numbered lines correspond to the mass of a non-rotating NS in units of $M_\odot$ [4].</td>
<td>120</td>
</tr>
</tbody>
</table>
The recent detection of coalescing black holes by the Laser Interferometer Gravitational-wave Observatory has brought forth the era of gravitational wave astronomy. Physicists are only now beginning to probe the mergers of compact objects that send ripples through space and time. These distortions carry with them the information from the system where they originated. The dynamics of black hole collisions and neutron star mergers are new and exciting events which were undetectable just a few years ago. Einstein’s theory of General Relativity has done an excellent job of describing gravity and the information that can be extracted from gravitational systems. However, his theory contains several anomalies such as the inability to explain the inflation of the universe, the effects of dark matter and energy, the presence of singularities, as well as a failure to reconcile with quantum mechanics. Modified theories of gravity have been proposed to answer any remaining questions about gravitation while prescribing solutions to the problems General Relativity still has. The work within this thesis describes how we may study modified theories of gravity in the strong field regime through two different means. The first, is through the calculation of the rate of gravitational radiation from binary systems. This rate varies depending on the theory of gravity being studied. Comparing the theoretical predictions of these rates from alternative theories to astronomical observation will allow us to place better constraints on modified gravity and test General Relativity like never before. The second way is through the investigation of the spacetime surrounding a neutron star. Unlike black holes which emit no light, we are able to see neutron stars (more specifically pulsars) through their light curve as they rotate. The shape of the light curve is dictated by the theory of gravitation used to describe the spacetime around the neutron star. My goal of constructing such a spacetime for neutron stars in modified gravity allows for future scientists to study the light curves to be detected and place constraints on the particular theory.
CHAPTER ONE

INTRODUCTION

Motivation

Developing a cohesive theory of gravity has been a constant study in the field of physics since Isaac Newton first published his *Principia* in the seventeenth century. While Newton’s ideas of gravity proved groundbreaking for the study of motion of objects on Earth as well as the orbits of (most) planets, there were some problems. To name one, the orbit of Mercury did not agree with Newton’s ideas of orbital motion. Astronomers noticed that it precessed in a way not governed by Newton’s Laws. A hypothetical series of “corpuscles,” as well as a new planet named Vulcan, were proposed as possible explanations for this anomaly [5]. However, efforts to detect these objects proved unfruitful.

It was not until 1915 when Albert Einstein published his General Theory of Relativity (GR) [6,7] that a drastic shift in how scientists viewed gravitational physics occurred. Rather than gravity being a force of attraction between two objects determined by their masses, Einstein predicted that matter and spacetime were interconnected through what he termed the field equations of gravitation. A massive object warps the spacetime around itself, which in turn dictates how other particles will move along nearby trajectories. This explanation for gravity predicted the precession of Mercury exactly, which led to both credibility and praise for the new theory.

Following the initial publication of Einstein’s theory, the understanding of gravity was further advanced by a number of predictions. Light bending around massive objects due to the curvature of spacetime was observed during an eclipse in 1919 [8]. Gravitational
redshift was seen in the spectral lines of Sirius B as predicted by GR [9–11]. The first evidence of gravitational time delay was predicted by Irwin Shapiro in 1964 [12], and then experimentally verified several years later [13]. The effect of frame dragging, or the result of a massive objects’ rotation causing the precession of orbiting bodies around it, was theorized in 1918 [14], and confirmed by the LAGEOS experiments in 1997 [15, 16] and the Gravity Probe B experiments in 2011 [17]. More recently, the detection of gravitational waves, both indirect [18, 19] and direct [20–25], further affirmed GR as the most reputable model of gravity.

So how can it be, that following the aforementioned predictions and observations, GR is anything but the correct theory of gravity? It turns out that in addition to the successful predictions, there exist a number of anomalies which still plague Einstein’s theory. The lack of compatibility with quantum mechanics (QM) is often the most discussed issue with GR. General relativity is a classical field theory and does not allow for quantization. Unlike electromagnetism, which can also be constructed as a classical field theory, GR is not renormalizable in a perturbative scheme [26]. Because of this, one cannot construct a quantum field theory to explain the effects of gravity as one can with electromagnetism. Attempts to reconcile GR with QM have led to theories such as string theory [27] and loop quantum gravity [28]. However, these theories are incomplete and do not have a complete action which can be utilized to make predictions and compare them with observations. One way to maneuver around this issue has to deal with the concept of effective field theories (EFTs) [29, 30]. One may understand these as low-energy expansions of an unknown action to a more fundamental theory. For example, since the complete action of string theory is not known, we may write the action as GR plus some low energy contribution which we state is representative of the more fundamental theory. These EFTs lead to the construction of theories which modify GR in the particular regime of interest (such as high curvature or energy), while reducing to GR in other well tested regimes. EFTs of gravity remain an
active area of research and will be discussed further below.

Another area of research which is inconsistent with our current understanding of GR is that of “dark” physics; that is, the existence of dark matter and dark energy. Dark matter has been proposed as a way of explaining the discrepancy in the galaxy rotation curve between the observed values in the orbital velocity of stars and those values predicted by GR [31]. This matter is believed to be non-baryonic, since it remains undetectable by current observation methods [32]. Dark energy, on the other hand, has been introduced as a way to explain the accelerated expansion of the universe. This can be viewed in terms of a cosmological constant, which may represent a constant energy density which permeates space [33, 34].

While GR is unable to explain the anomalies listed above within the realms of QM and dark physics, numerous theories have been introduced in an attempt to rectify these shortcomings [35]. These theories must be proposed in such a way as to resolve one of GR’s problems, while still satisfying all of the predictions and observations which do agree with Einstein’s theory. In this sense we may argue that GR is not wrong, simply incomplete in its present form.

For the purposes of this thesis, I will introduce three such theories which are of interest to my work. The first, Jordan-Fierz-Brans-Dicke [36–38] (JFBD), is an example of a scalar-tensor theory. These theories are well motivated [39, 40] and provide a means by which gravity is mediated by not only the tensorial spin-2 spacetime field, but also a scalar spin-0 field. JFBD is one of the most well studied scalar-tensor theories and stringent constraints have already been placed on it by solar system experiments [35, 41–43].

The second theory I would like to introduce is a particular vector-tensor theory known as Einstein-Æther theory (Æ). This theory was proposed as a way to spontaneously break Lorentz symmetry by including a timelike æther field which would couple to the metric [44–46]. These theories seek to explain the consequences of Lorentz violations, which appear
in several quantum gravity theories such as string theory, non-commutative field theory, and super symmetry [47–50]. My work has been to investigate the consequences of this Lorentz violation in strongly gravitating systems, namely the gravitational radiation which stems from binary systems. Certain constraints have already been placed on \( \mathcal{E} \)-theory [2, 35, 46, 51, 52], but I aim to lay the foundation for more stringent tests to take place.

The final theory which I worked on is known as scalar-Gauss-Bonnet (sGB) gravity. This theory is a special extension of a particular group of theories known as quadratic gravity [53]. These theories modify GR by the addition of a dynamical scalar field similar to JFBD. However, this additional field couples to curvature squared terms, hence the name “quadratic” gravity. For sGB the field couples to a special curvature term known as the Gauss-Bonnet invariant, which is so called because it does not change the manifolds topology under a homeomorphic transformation. This theory is motivated by particular low-energy extensions of string theory [27, 54], and because of this, is an example of an EFT.

The consequences of these additional fields lead to solutions which differ from GR for black holes, neutron stars, and in the effects of gravitational radiation. All of these deviations can be theorized, studied, measured, and compared with observations in an attempt to place stringent constraints on the modified theories and the effects of the extra fields on physical systems.

Radiative Work

The first person to identify the SET for GR was Richard Isaacson in 1967-68 [55, 56]. In his seminal works he showed that by applying a linearized approximation to the metric one can prove that the curvature of the background geometry is directly affected by terms quadratic in the first order perturbation quantities. This term he referred to as an “effective
stress tensor for gravitational radiation," which I am calling the SET. He further goes on to show how the integration of this term over the surface of an enclosed volume is equivalent to finding the flux of a particular quantity (energy or momentum) through that surface.

The purpose of my initial work was to study a series of methods for deriving the SET in arbitrary modified theories of gravity. The motivation for this was to establish a cohesive outline to find the physical quantities which are results from the back reaction to the source of the gravitational radiation. The work of gravitational wave detectors such as the Laser Interferometer Gravitational-wave Observatory (LIGO) [57, 58], Virgo [59], and future detectors such as the Kamioka Gravitational Wave Detector (KAGRA) [60] and the Laser Interferometer Space Antenna (LISA) [61, 62], are proving that it is more important than ever to develop models which can be used to accurately describe the dynamics of massive bodies under the influence of gravitational radiation.

By formulating a methodology to calculate a SET, we can make it easier for future scientists to develop testable models in modified gravity theories for radiative systems. These models represent theoretical interpretations of the physics occurring within the systems, and can be created by a combination of analytic and numerical techniques. For example, during the inspiral phase of a binary system, a post-Newtonian (PN) expansion can be used. The PN approximation allows the Einstein equations to be solved as a series in the PN parameter $v^2/c^2$, the ratio between the characteristic velocity of the system and the speed of light, where it is assumed that $v \ll c$. This generally takes place in regions where the separation of the binary is large and orbital velocities are small. From the results of the PN calculations, the orbital evolution of the binary can be found as a series in the PN parameter, and may be used in the construction of waveforms. As the orbit decays and the binary coalesces, the PN approximation breaks down and numerical relativistic techniques need to be used in order to describe the rest of the merger. The combination of the analytical inspiral coupled to the numerical merger simulation can give physicists a template with which to compare
to observations in an attempt to characterize a GW signal buried in noise. This technique is known as matched filtering, and it is commonly employed in gravitational wave data analysis. The principle of this technique relies on correlating the constructed template with the signal, in an attempt to find any areas of high signal-to-noise.

However, before any of these calculations can be made, it must be determined exactly how a binary system will evolve under the effects of radiation reaction. This can be found through a balance law which relates the rate of change of the binding energy of a binary system to the flux of gravitational radiation which is emitted. In order to calculate this flux term, it is necessary to find the SET, which has a direct link to the energy and momentum radiated away from a system. It is crucial that these calculations be completed carefully, as the detectors are very sensitive and the models must be accurate in order to properly characterize the signal without biases.

I was able to show that there are multiple ways of deriving the SET in various gravitational theories. These different methods will, for the most part, generate identical results. However, there is no requirement that the SET be the same via all methods of derivation. The only important quantity is the physical observable derived from the results of the SET, such as the change in orbital period of a binary.

There were four mathematical schemes I made use of for deriving the SET. These methods included the Isaacson method which has been mentioned above, the variation of the gravitational action with respect to a generic background [63], the development of a gravitational stress-energy pseudo-tensor which inherently contains conserved quantities due to its construction [64,65], and finally the canonical method used in calculus of variations and attributed to the mathematician Emmy Noether [66] (See Chap. 2 for details on each method). The results for the radiative quantities for all theories of gravity studied can be shown to be identical respective to the theory in question, though the Lorentz violations in Æ-theory do cause problems in the canonical derivation. Depending on particular areas
of interest for the physicist researching in this field, certain methods may be preferred over others.

Following my initial work on finding the SET, I studied the effects of gravitational radiation for both energy and angular momentum on a binary system in $\mathcal{E}$-theory. Our goal was to determine how this theory differs from GR, and if these deviations can be measured by experiments. We find that indeed, $\mathcal{E}$-theory leads to additional modes of polarization from gravitational waves which: 1) do not propagate at the speed of light as predicted by GR and 2) exhibit dipole radiative terms, which will cause a binary to inspiral faster than predicted by GR. We also showed, for the first time, the rate at which angular momentum was carried away by gravitational waves. We applied this calculation to a binary system, and derived the rate at which the eccentricity changes due to the radiation reaction. The work here is the first step towards developing gravitational waveform models for eccentric compact binaries and comparing them to the results predicted by GR.

In summary, my work with gravitational radiation dealt with deriving a SET which could be used to derive the rates at which energy and momentum are emitted by physical systems. I was then able to apply this knowledge to a binary system in $\mathcal{E}$-theory. Along with my colleagues, I showed that the methods presented allow for results which may deviate from the values derived from GR. This helps in working towards constraining modified gravity theories, particularly that of $\mathcal{E}$-theory, which will aid in our understanding of how gravity acts under the assumption of spontaneous Lorentz violation.

**Neutron Star Work**

Neutron stars (NS) are extremely compact objects which form during the collapse of stars with masses between roughly $10 - 30 \, M_\odot$. These NSs possess masses on the order of 1.5 solar masses, but are contained within a radius 10-14 km [67]. This makes NSs the most dense stellar objects in the universe. With densities that are a few times the density
of the nucleus of an atom [68], NSs provide an excellent testbed for the study of nuclear physics. Exact numbers for the mass and radius of a NS vary depending on the equation of state (EoS). The EoS is the relationship between the pressure and density of the NS, and can be used to describe the state of matter within its interior. The trick is now to find a way to study NSs in an effort to discover what this EoS is.

In addition to high densities, NSs also possess incredibly strong magnetic fields. For NSs with a companion star, there is a chance that these fields could act on the accretion of the companion and direct charged particles to the magnetic poles of the NS, similar to how the solar winds are directed along the Earth’s magnetosphere. The infalling gas from the companion heats up the surface of the NS causing hot-spots, regions that can be $10^4$ times more luminous than the Sun [69]. These hot-spots produce X-rays which are beamed out in a cone centered around a hot spot located on the NS surface. For rotating NS this beaming effect of electromagnetic radiation can produce regular intervals of light if an observer is along the line of sight to the X-ray cone. The pulses of X-rays which can be detected follow null geodesics and are affected by the spacetime surrounding the NS [70]. This will shape the pulse profile detected by an observer. If one were to measure the shape and intensity of these light curves, it is possible to place constraints on the spacetime on which the null geodesics propagated.

The question is now, how does one go about detecting these pulse profiles? The Neutron star Interior Composition ExploreR (NICER) was designed to provide a better understanding of the conditions surrounding the structure, dynamics, and energies associated with NSs [71–73]. It does this with an array of 56 X-ray detectors which are capable of recording X-rays in the range of 0.2-12 keV. NICER’s detections will allow for measurements of a NS mass and radius via its light curve, and allow for constraints to be placed on the EoS of NSs. In addition, the data obtained from these observations can also be compared with models of the light cone constructed from null geodesics propagating on
non-GR spacetimes.

The key point of my work was to find out what the spacetime surrounding a NS was in a modified theory of gravity. If we are able to develop a metric around a NS dictated by a particular modified theory, we may then test the theory with observations since various physical observables (e.g. the light curve of a pulsar) will differ from those predicted by GR.

In the work discussed within this thesis, my colleagues and I develop an analytic exterior spacetime metric for a NS in sGB gravity. We find the metric to depend only on the observed gravitational mass of the NS and the strength of the coupling constant within the theory. This is surprising, since other theories describing the spacetime around a NS contain a scalar charge which depends on the interior of the NS, and thus the EoS being used [74,75]. The question of why our solution is independent of the scalar charge remains an open problem which we hope to investigate in the near future.

Once our spacetime was calculated, we worked on a number of physical applications to see what deviations exist between our results in sGB and the known GR solution. The innermost stable circular orbit (ISCO) is a quantity that represents the closest approach of a test particle to any object where a stable circular orbit can exist. We find that our solution differs from GR slightly, leading to a smaller ISCO that what one would expect. Accretion disk models [76–78] often use the ISCO as the inner edge of the disk when calculating the time averaged energy flux emitted [79]. Therefore, we may assume that the predictions of any such model will differ from those of GR. We also investigated the orbital frequency and the changes this makes to Kepler’s law. This frequency can be used in conjunction with the epicyclic frequency to study quasi-periodic oscillations (QPOs) which are observable in some low mass X-ray binaries (LMXB) [4,80].

In addition to our study of how matter moves in our new metric, we also investigated how photon motion is affected. We find that as the compactness of the NS is increased, the
light bending effects are more pronounced than they are in GR. Therefore, we expect that systems which contain a NS of a sufficiently high compactness may be able to constrain sGB through electromagnetic observation. This brings back the discussion of how making use of NS light curves will aid in studying the NS strong-field regime. By applying ray tracing methods to the NS hot spots for the spacetime we developed, we will be able to draw a light curve which can be compared with both the theoretical results of GR and NICER observations to allow us to further constrain sGB.

By performing the analysis we did here, we have allowed a door to be propped open for the continued study of this theory. Since our solution was found to be analytic, future investigations involving this theory will not require the use of numerical techniques to solve for the interior NS solution based on the EoS, saving time and computational costs.

**Summing it all up**

The first aspect of my thesis work dealt with the derivation of radiative terms for theories containing arbitrary fields. This work was performed and published in two papers. The first dealt with the process of developing a SET which is used in the calculation of energy and momentum flux terms. The second paper applied the results of the first to a binary system in Einstein-Æther theory.

The second aspect of my thesis work dealt with the derivation of an external metric in sGB theory. As mentioned, this theory is a particular instance of quadratic gravity which is an example of an EFT attempting to simulate string-theory in the low-energy limit. The coupling of a scalar field to higher order curvature terms, as exhibited by this theory, has the potential to lead to a number of effects which cause deviations from GR in the strong gravity regime. This was the purpose of my study. By developing a spacetime for a NS in sGB gravity, we can investigate a number of physical applications which may be tested with observations in an effort to place constraints on the theory.
The outline of this thesis is as follows: Chapter 2 detail my work in the derivation of a SET which can be used in the determination of radiative quantities. Chapter 3 calculates the angular momentum loss rate for a binary system in $\mathcal{AE}$-theory and compares the results to GR. In Chap. 4, I derive an external metric for a NS in sGB gravity, as well as perform several applications of the new spacetime to compare the results with those of GR. Lastly, in Chap. 5, I conclude the thesis by discussing the overall impact of my work and examine any future directions which can be taken.
CHAPTER TWO

THE GRAVITATIONAL WAVE STRESS-ENERGY (PSEUDO)-TENSOR IN MODIFIED GRAVITY

Contribution of Authors and Co-Authors

Manuscript in Chapter 2

Author: Alexander Saffer
Contributions: Performed mathematical analysis of problem, compared results, and wrote the manuscript.

Co-Author: Kent Yagi
Contributions: Thought of the problem, discussed implications, and edited manuscripts.

Co-Author: Nicolás Yunes
Contributions: Thought of the problem, discussed implications, and edited manuscripts.
Abstract

The recent detections of gravitational waves by the advanced LIGO and Virgo detectors open up new tests of modified gravity theories in the strong-field and dynamical, extreme gravity regime. Such tests rely sensitively on the phase evolution of the gravitational waves, which is controlled by the energy-momentum carried by such waves out of the system. We here study four different methods for finding the gravitational wave stress-energy pseudo-tensor in gravity theories with any combination of scalar, vector, or tensor degrees of freedom. These methods rely on the second variation of the action under short-wavelength averaging, the second perturbation of the field equations in the short-wavelength approximation, the construction of an energy complex leading to a Landau-Lifshitz tensor, and the use of Noether’s theorem in field theories about a flat background. We apply these methods in General Relativity, Jordan-Fierz-Brans-Dicky theory, and Einstein-Æther theory to find the gravitational wave stress-energy pseudo-tensor and calculate the rate at which energy and linear momentum is carried away from the system. The stress-energy tensor and the rate of linear momentum loss in Einstein-Æther theory are presented here for the first time. We find that all methods yield the same rate of energy loss, although the stress-energy pseudo-tensor can be functionally different. We also find that the Noether method yields a stress-energy tensor that is not symmetric or gauge-invariant, and symmetrization via the Belinfante procedure does not fix these problems because this procedure relies on Lorentz invariance, which is spontaneously broken in Einstein-Æther theory. The methods and results found here will be useful for the calculation of predictions in modified gravity theories that can then be contrasted with observations.
Introduction

With the recent announcements of the discovery of gravitational waves [20–25], we are on the brink of a new era in astrophysical science. We now have evidence that there exist events where two black holes or neutron stars collide and emit, as a consequence of this merger, powerful bursts of gravitational radiation. The information contained in these gravitational waves provide information about the compact bodies that formed the waves during the inspiral and merger. In addition, we can learn about how gravitational physics behaves in the extreme gravity regime, an area where velocities and gravitational effects are large compared to the surrounding spacetime. This regime provides an excellent test bed for investigating and constraining the predictions of gravitational theories.

The detection of gravitational waves (GWs) and the science we can extract from them depends sensitively on the models used to extract these waves from the noise. Because of the way GW detectors work, the extraction relies on an accurate modeling of the rate of change of the GW phase and frequency. In binary systems, this is calculated from the balance law between the rate of change of the binary’s binding energy and the energy and momentum extracted by all propagating degrees of freedom. In turn, the latter is obtained from the GW stress-energy (pseudo) tensor (GW SET), which can be calculated in a variety of ways.

The GW SET was first found in general relativity (GR) by Isaacson in the late 1960’s [55,56] using what we will call the perturbed field equations method. This method consists of perturbing the Einstein field equations to second order in the metric perturbation about a generic background. The first-order field equations that result describe the evolution of the gravitational radiation. The second-order field equations yield the GW SET as the source of curvature for the background spacetime.

Since Isaacson’s work, other methods for finding the GW SET have been developed.
Stein and Yunes described what we will call the perturbed action method, which consists of varying the gravitational action to second order with respect to a generic background [63]. Once the variation has been taken, they use short-wavelength averaging to isolate the leading-order contribution to the GW SET. Landau and Lifshitz developed a method that consists of constructing a pseudo-tensor from tensor densities with certain symmetries such that its partial divergence vanishes, leading to a conservation law [64]. The final method we investigate is one that makes use of Noether’s theorem, which asserts that the diffeomorphism invariance of a theory automatically leads to the conservation of a tensor [66]. This canonical energy-momentum tensor, however, is not guaranteed to be symmetric in its indices or gauge invariant, problems that can be resolved through a symmetrization procedure proposed by Belinfante [81–83].

The four methods described in this paper all give the same GW SET in GR, but this needs not be so in other gravity theories. Scalar-tensor theories, originally proposed by Jordan [36], Fierz [37], and Brans and Dicke [38], and certain vector-tensor theories, such as Einstein-Æther theory [45, 46], are two examples where a priori it is not obvious that all methods will yield the same GW SET. In scalar-tensor theories, the GW SET was first computed by Nutku [84] using the Landau-Lifshitz method. In Einstein-Æther theory, the symmetric GW SET has not yet been calculated with any of the methods discussed above; Eling [85] derived a non-symmetric GW SET using the Noether current method without applying perturbations. Foster [1] perturbed the field equations a la Isaacson and then used the Noether charge method of [86, 87] to find the rate of change of energy carried by all propagating degrees of freedom.

We here use all four methods discussed above to calculate the GW SET in GR, Jordan-Fierz-Brans-Dicke (JFBD), and Einstein-Æther theory. In the GR and JFBD cases, we find that all methods yield exactly the same GW SET. In the Einstein-Æther case, however, the perturbed field equations method, the perturbed action method and the Landau-Lifshitz
method yield slightly different GW SETs; however, the observable quantities computed from them (such as the rate at which energy is removed from the system by all propagating degrees of freedom) are exactly the same. On the other hand, the Noether method applied to Einstein-Æther theory does not yield a symmetric or gauge invariant GW SET. Symmetrization of the Noether GW SET through the Belinfante procedure fails to fix these problems. This is because this procedure relies on the action being Lorentz invariant at the level of the perturbations, while Einstein-Æther theory is constructed so as to spontaneously break this symmetry.

The results obtained here are relevant for a variety of reasons. First, we show for the first time how different methods for the calculation of the GW SET yield the same observables in three gravity theories in almost all cases. Second, we clarify why the Noether method and its Belinfante improvement can fail in Lorentz-violating theories. Third, we provide expressions for the GW SET in Einstein-Æther theory for the first time, and from this, we compute the rate of energy and linear momentum carried away by all propagating degrees of freedom. Fourth, the methods presented here can be used to compute the rate of change of angular momentum due to the emission of gravitational, vector and scalar waves in Einstein-Æther theory; in turn, this can be used to place constraints on this theory with eccentric binary pulsar observations. Fifth, the methods described here in detail can be straightforwardly used in other gravity theories that activate scalar and vector propagating modes.

Throughout this chapter, we choose the common metric signature \((-, +, +, +)\) with units in which \(c = 1\) and the conventions of [88] in which Greek letters in tensor indices stand for spacetime quantities.
General Relativity

In this section, we review how to construct the GW SET in GR and establish notation. We begin by describing the perturbed action method, following mostly [63]. We then describe the perturbed field equation method, following mostly [88] and the perturbation treatment of [55,56,89]. We then discuss the Landau-Lifshitz method, following mostly [64, 65] and conclude with a discussion of the Noether method, following mostly [89].

In all calculations in this section, we work with the Einstein-Hilbert action

\[ S_{\text{GR}} = \frac{1}{16\pi G} \int d^4 x \sqrt{-g} \ R \]  

(2.1)

in the absence of matter, where \( G \) is Newton’s constant, \( g \) is the determinant of the four-dimensional spacetime metric \( g_{\alpha\beta} \) and \( R \) is the trace of the Ricci tensor \( R_{\alpha\beta} \). By varying the action with respect to the metric, we find the field equations for GR to be

\[ G_{\alpha\beta} = 0, \]  

(2.2)

where \( G_{\alpha\beta} \) is the Einstein tensor associated with \( g_{\alpha\beta} \).

Perturbed Action Method

Let us begin by using the metric decomposition

\[ g_{\alpha\beta} = \tilde{g}_{\alpha\beta} + \epsilon \, h_{\alpha\beta} + O(\epsilon^2), \]  

(2.3)

where \( \epsilon \ll 1 \) is an order counting parameter. One should think of the metric perturbation \( h_{\alpha\beta} \) as high frequency ripples on the background \( \tilde{g}_{\alpha\beta} \), which varies slowly over spacetime; henceforth, all quantities with an overhead tilde represent background quantities. Both of
these fields will be treated as independent, and thus, one could vary the expanded action with respect to each of them separately. For our purposes, however, it will suffice to consider the variation of the expanded action with respect to the background metric order by order in \( h_{\alpha \beta} \).

The action expanded to second order in the metric perturbation is

\[
S_{\text{GR}} = S_{\text{GR}}^{(0)} + \epsilon S_{\text{GR}}^{(1)} + \epsilon^2 S_{\text{GR}}^{(2)} + O \left( \epsilon^3 \right),
\]

(2.4)

where

\[
S_{\text{GR}}^{(0)} = \frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} \tilde{R},
\]

(2.5a)

\[
S_{\text{GR}}^{(1)} = -\frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} \left[ \tilde{h}^{\alpha \beta} \tilde{R}_{\alpha \beta} - \frac{1}{2} \tilde{\nabla} \tilde{h} - \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \tilde{h}^{\alpha \beta} \right],
\]

(2.5b)

\[
S_{\text{GR}}^{(2)} = \frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} \left[ \frac{1}{8} \tilde{\nabla}_\alpha \tilde{h} \tilde{\nabla}^\alpha \tilde{h} + \frac{1}{2} \tilde{\nabla}_\beta \tilde{h}_{\alpha \gamma} \tilde{\nabla}^\gamma \tilde{h}^{\alpha \beta} - \frac{1}{4} \tilde{\nabla}_\gamma \tilde{h}_{\alpha \beta} \tilde{\nabla}^\gamma \tilde{h}^{\alpha \beta} \right],
\]

(2.5c)

and where we have introduced the trace-reversed metric perturbation

\[
\tilde{h}^{\alpha \beta} = h^{\alpha \beta} - \frac{1}{2} \tilde{g}^{\alpha \beta} h.
\]

(2.6)

Before proceeding, let us simplify the expanded action further by noticing that certain terms do not contribute to observable quantities computed from the GW SET measured at spatial infinity or under short-wavelength averaging. Equations (2.5a) and (2.5b) are exact, while we have adjusted (2.5c) for the reasons that follow. First, there is no term explicitly dependent on the background curvature. This is because the GW SET will be later used to calculate the rate at which energy and linear momentum are carried away by all propagating degrees of freedom at spatial infinity. These rates will not depend on the background curvature tensor, as the latter vanishes at spatial infinity. Second, total
divergences generated by integration by parts also become boundary terms that will not contribute to observables at spatial infinity. Third, terms which are odd in the number of perturbation quantities will not contribute to the GW SET after short-wavelength averaging (see App. A). From these considerations, we note that $S_{\text{GR}}^{(1)}$ will not contribute at all to the GW SET, and can therefore be neglected.

We are now left with a simplified effective action

$$S_{\text{GR}}^{\text{eff}} = S_{\text{GR}}^{(0) \text{eff}} + \epsilon^2 S_{\text{GR}}^{(2) \text{eff}} + O \left( \epsilon^3 \right). \quad (2.7)$$

Variation of Eq. (2.7) with respect to the background metric will yield the field equations. The variation of $S_{\text{GR}}^{(0) \text{eff}}$ gives the Einstein tensor for the background metric, and thus, the variation of $S_{\text{GR}}^{(2) \text{eff}}$ acts as a source, which we identify with the GW SET. The variation of the second-order piece of the action can be written as

$$\delta S_{\text{GR}}^{(2) \text{eff}} = \int d^4x \sqrt{-\tilde{g}} \, t_{\alpha \beta}^{\text{GR}} \delta \tilde{g}_{\alpha \beta}, \quad (2.8)$$

with the GW SET

$$\Theta_{\alpha \beta}^{\text{GR}} = -2 \left< t_{\alpha \beta}^{\text{GR}} \right>, \quad (2.9)$$

and the angle-brackets stand for short-wavelength averaging. This averaging is necessary because the background geometry contains many wavelengths of oscillation of the metric perturbation, and the details of the latter only induce higher order corrections to the metric in multiple-scale analysis [55, 56].

Before we can compute the GW SET, we first need to derive the first-order equations of motion for the field $\tilde{h}_{\alpha \beta}$. This is accomplished by varying Eq. (2.7) with respect to $\tilde{h}_{\alpha \beta}$. The result is

$$\tilde{\Box} \tilde{h}_{\alpha \beta} - 2 \tilde{\nabla}^\gamma \tilde{\nabla}_{(\alpha} \tilde{h}_{\beta)\gamma} - \frac{1}{2} \tilde{g}_{\alpha \beta} \tilde{\Box} \tilde{h} = 0. \quad (2.10)$$
Using the Lorenz gauge condition,

\[ \tilde{\nabla}_a h^{\alpha\beta} = 0, \] (2.11)

and taking the trace of Eq. (2.10), we find

\[ \Box \bar{h} = 0. \] (2.12)

If we impose the traceless condition \( \bar{h} = 0 \) on some initial hypersurface, then Eq. (2.12) forces the metric to remain traceless on all subsequent hypersurfaces [90]. The combination of the Lorentz gauge condition and the trace free condition yields the transverse-traceless (TT) gauge. Applying this condition to simplify the field equations in Eq. (2.10) gives

\[ \Box h^{\alpha\beta}_{TT} = 0. \] (2.13)

We are now able to compute the GW SET using Eq. (2.9). Performing the necessary calculations yields

\[ \Theta_{\alpha\beta} = \frac{1}{32\pi G} \left( \tilde{\nabla}_a h^{\mu\nu}_{TT} \tilde{\nabla}_b h_{\mu\nu} - \frac{1}{2} \tilde{\nabla}_a \tilde{\nabla}_b \bar{h} + \tilde{\nabla}_a \tilde{\nabla}_b \bar{h} \bar{h}_{\alpha\beta} + 2 \tilde{\nabla}_a \tilde{\nabla}_{(\delta} \bar{h}_{\beta)_{\alpha\beta}} \right). \] (2.14)

We are free to apply the TT gauge at this point, which we make use of. Since the result of the GW SET is averaged through the brackets, we are also able to integrate by parts and eliminate boundary terms. These two effects give the familiar result,

\[ \Theta_{\alpha\beta} = \frac{1}{32\pi G} \left( \tilde{\nabla}_a h^{\mu\nu}_{TT} \tilde{\nabla}_b h_{TT}^{\mu\nu} \right). \] (2.15)
Perturbed Field Equation Method

Let us begin by decomposing the metric $g_{\alpha\beta}$ as

$$ g_{\alpha\beta} = \eta_{\alpha\beta} + h^{(1)}_{\alpha\beta} + h^{(2)}_{\alpha\beta}, \quad (2.16) $$

where $\eta_{\alpha\beta}$ is the Minkowski metric and $(h^{(1)}_{\alpha\beta}, h^{(2)}_{\alpha\beta})$ are first and second order metric perturbations respectively. Unlike in the previous section, the background is here not arbitrary but chosen to be the Minkowski spacetime. We assume that $h^{(n)}_{\alpha\beta} = O(\epsilon^n)$, but we no longer keep track explicitly of the order counting parameter $\epsilon$.

With this decomposition, we now expand the Einstein field equations (Eq. (2.2)) to second order in the metric perturbation. We obtain the field equations for $h^{(1)}_{\alpha\beta}$ by truncating the expansion of the field equations to linear order in the perturbation:

$$ G_{\alpha\beta}[h^{(1)}] = 0. \quad (2.17) $$

We obtain the GW SET by truncating the expansion to second order in the perturbation:

$$ G_{\alpha\beta}[h^{(2)}] = 8\pi\Theta_{\alpha\beta} \equiv - \left\langle G_{\alpha\beta} \left[ (h^{(1)})^2 \right] \right\rangle, \quad (2.18) $$

where again the angled-brackets stand for short-wavelength averaging\(^1\).

We begin by expanding the field equations to first order in the metric perturbation. The Ricci tensor takes the form

$$ R^{(1)}_{\alpha\beta} = \frac{1}{2} \left[ 2\partial_\gamma \partial_\alpha h^{(1)\gamma}_{\beta} - \partial_\alpha \partial_\gamma h^{(1)} + \partial_\gamma \partial^{\gamma} h^{(1)}_{\alpha\beta} \right]. \quad (2.19) $$

\(^1\)Strictly speaking, this method is not identical to Isaacson’s original work [55,56] because here we assume a priori that the background is Minkowski.
and plugging this and its trace into the Einstein tensor of Eq. (2.17) gives

$$\partial_\gamma \partial_\alpha h(1)_{\alpha \beta}^{(1)} - \frac{1}{2} \partial_\alpha \partial_\beta h(1) - \frac{1}{2} \partial_\gamma \partial_\alpha h(1)_{\alpha \beta}^{(1)} + \frac{1}{2} \partial_\alpha \partial_\gamma h(1)_{\alpha \beta}^{(1)} + \frac{1}{2} \eta_{\alpha \beta} \left( \partial_\gamma \partial_\alpha h(1)_{\alpha \beta}^{(1)} - \partial_\gamma \partial_\beta h(1)_{\alpha \beta}^{(1)} \right) = 0. \quad (2.20)$$

As done before, we impose the Lorenz gauge $\partial^\alpha \tilde{h}^{(1)}_{\alpha \beta} = 0$ on the trace-reversed metric perturbation $\tilde{h}^{(1)}_{\alpha \beta} = h^{(1)}_{\alpha \beta} - (1/2) \eta_{\alpha \beta} \eta^{(1)}$, and by taking the Minkowski trace in Eq. (2.20) we force $h^{(1)}_{\alpha \beta} = \eta^{\alpha \beta} h^{(1)}_{\alpha \beta}$ to satisfy a wave equation in flat spacetime. This, in turn, allows us to refine the Lorenz gauge into the TT gauge in flat spacetime to further simplify the field equations to

$$\partial_\gamma \partial_\alpha h^{(1)T T}_{\alpha \beta} = 0. \quad (2.21)$$

The next step is to investigate the field equations to $O(h^2)$. The expansion of the Ricci tensor is

$$R^{(2)}_{\alpha \beta} = \frac{1}{2} \left[ \frac{1}{2} \partial_\gamma \partial_\alpha h^{(2)\gamma}_{\beta} - \partial_\alpha \partial_\beta h^{(2)} - \partial_\gamma \partial_\alpha h^{(2)}_{\alpha \beta} \right] + \frac{1}{2} \partial_\beta \left[ h^{(1)\gamma \delta} \left( 2 \partial_\alpha h^{(1)\gamma}_\delta - \partial_\delta h^{(1)}_{\alpha \gamma} \right) \right]$$

$$- \frac{1}{2} \partial_\gamma \left[ h^{(1)\gamma \delta} \left( 2 \partial_\alpha h^{(1)\gamma}_\delta - \partial_\delta h^{(1)}_{\alpha \gamma} \right) \right] + \frac{1}{4} \partial_\gamma h^{(1)}_{\alpha \beta} \left[ 2 \partial_\alpha h^{(1)\gamma}_\beta - \partial_\beta h^{(1)}_{\alpha \gamma} \right]$$

$$- \frac{1}{4} \left[ \partial_\alpha h^{(1)\gamma \delta} \partial_\beta h^{(1)}_{\gamma \delta} + 2 \partial_\gamma h^{(1)\delta \gamma} \partial_\beta h^{(1)}_{\alpha \beta} - 2 \partial_\gamma h^{(1)\delta \gamma} \partial_\beta h^{(1)}_{\alpha \beta} \right]. \quad (2.22)$$

This equation can be simplified by imposing the Lorenz gauge on $h^{(2)}_{\alpha \beta}$, namely $\partial^\alpha \tilde{h}^{(2)}_{\alpha \beta} = 0$ with $\tilde{h}^{(2)}_{\alpha \beta} \equiv h^{(2)}_{\alpha \beta} - (1/2) \eta_{\alpha \beta} \eta^{(2)}$ and $h^{(2)} = \eta^{\alpha \beta} h^{(2)}_{\alpha \beta}$. As before, we can refine this gauge into the TT gauge by setting $h^{(2)} = 0$, which is compatible with $R^{(2)} = 0$. With this at hand, the Ricci tensor in the TT gauge reduces to

$$R^{(2)}_{\alpha \beta} = - \frac{1}{2} \partial_\gamma \partial_\alpha h^{(2)T T}_{\alpha \beta} + \frac{1}{4} \partial_\alpha h^{(1)\gamma \delta}_{T T} \partial_\beta h^{(1)T T}_{\gamma \delta} + \frac{1}{2} h^{(1)\gamma \delta}_{T T} \partial_\alpha \partial_\beta h^{(1)T T}_{\gamma \delta} + \frac{1}{2} h^{(1)\gamma \delta}_{T T} \partial_\gamma \partial_\beta h^{(1)T T}_{\alpha \beta}$$

$$- h^{(1)\gamma \delta}_{T T} \partial_\delta \partial_\alpha h^{(1)T T}_{\beta \gamma} + \partial_\alpha h^{(1)T T}_{\gamma \beta T T} \partial_\delta h^{(1)T T}_{\gamma \delta}. \quad (2.23)$$

The first line of Eq. (2.22) (and the first term of Eq. (2.23)) is nothing but Eq. (2.19) with
\( h_{\alpha \beta}^{(1)} \rightarrow h_{\alpha \beta}^{(2)} \), which will contribute to the left-hand side of Eq. (2.18). Using Eq. (2.18), the GW SET is then (suppressing superscripts for neatness)

\[
\Theta_{\alpha \beta} = \frac{1}{32\pi G} \left\langle \partial_\alpha h_{\gamma \delta}^{TT} \partial_\beta h_{\gamma \delta}^{TT} \right\rangle, \tag{2.24}
\]

after integrating by parts inside of the averaging scheme. This expression is identical to Eq. (2.15), albeit around a Minkowski background.

**Landau-Lifshitz Method**

The Landau-Lifshitz method makes use of a formulation of GR in terms of the “gothic g metric,” a tensor density defined as

\[
\eta^{\alpha \beta} \equiv \sqrt{-g} g^{\alpha \beta}. \tag{2.25}
\]

With this density, one can construct the 4-tensor density

\[
\mathcal{H}^{\alpha \mu \beta \nu} \equiv \eta^{\alpha \beta} \eta^{\mu \nu} - \eta^{\alpha \nu} \eta^{\beta \mu}, \tag{2.26}
\]

which has the remarkable property that

\[
\partial_\mu \partial_\nu \mathcal{H}^{\alpha \mu \beta \nu} = 2 (-g) G^{\alpha \beta} + 16\pi G (-g) t_{\alpha \beta}^{LL}, \tag{2.27}
\]

where \( G^{\alpha \beta} \) is exactly the Einstein tensor and \( t_{\alpha \beta}^{LL} \) is known as the Landau-Lifshitz pseudotensor [64],

\[
(-g)t_{\alpha \beta}^{LL} = \frac{1}{16\pi G} \left[ \partial_\gamma \eta^{\alpha \beta} \partial_\delta \eta^{\gamma \delta} - \partial_\gamma \eta^{\alpha \gamma} \partial_\delta \eta^{\beta \delta} + \frac{1}{2} \eta^{\alpha \beta} \eta^{\gamma \delta} \partial_\epsilon \eta^{\gamma \delta} \partial_\kappa \eta^{\delta \epsilon} - 2 \eta^{\gamma \delta} \eta^{\alpha \beta} \partial_\epsilon \eta^{\gamma \delta} \partial_\kappa \eta^{\delta \epsilon} \right].
\]
Substituting the GR field equations into Eq. (2.27) gives

$$
\partial_{\mu} \partial_{\nu} H^{\alpha \mu \beta \nu} = 16\pi G \left(-g\right) \left(T_{\text{mat}}^{\alpha \beta} + t_{\text{ll}}^{\alpha \beta}\right),
$$

(2.29)

where \(T_{\text{mat}}^{\alpha \beta}\) is the matter SET. We have kept this term here for clarity, but in this paper \(T_{\text{mat}}^{\alpha \beta} = 0\).

We can now use the symmetries of \(H^{\alpha \mu \beta \nu}\) to derive some conservation laws. First, we notice that by construction \(H^{\alpha \mu \beta \nu}\) has the same symmetries as the Riemann tensor. Since \(H^{\alpha \mu \beta \nu}\) is antisymmetric in \(\alpha\) and \(\mu\), we find

$$
\partial_{\alpha} \partial_{\mu} \partial_{\nu} H^{\alpha \mu \beta \nu} = 0 .
$$

(2.30)

Equation (2.30) implies there exists a conserved quantity, which we will define as

$$
T^{\alpha \beta} \equiv \frac{1}{16\pi G} \left(\partial_{\mu} \partial_{\nu} H^{\alpha \mu \beta \nu}\right).
$$

(2.31)

When this quantity is short-wavelength averaged, one recovers the GW SET

$$
\Theta^{\alpha \beta} = \left<T^{\alpha \beta}\right> = \left<(-g) t_{\text{ll}}^{\alpha \beta}\right> .
$$

(2.32)

What we described above is fairly general, so now we evaluate the GW SET in terms of a metric perturbation from a Minkowski background. Using the expansion

$$
g_{\alpha \beta} = \eta_{\alpha \beta} + h_{\alpha \beta} ,
$$

(2.33)
with \( h_{\alpha\beta} \sim O(\varepsilon) \), the gothic metric becomes

\[
g^{\alpha\beta} = \eta^{\alpha\beta} - \bar{h}^{\alpha\beta} + O(h^2), \tag{2.34}
\]

where \( \bar{h}^{\alpha\beta} \) is the trace reversed metric perturbation. Equation (2.28) can now be simplified and written to second order in the metric perturbation as

\[
(-g) t_{\alpha\beta}^{\alpha\beta} = \frac{1}{16\pi G} \left[ \partial_\gamma \bar{h}^{\alpha\beta} \partial_\delta h^{\gamma\delta} - \partial_\gamma \bar{h}^{\alpha\gamma} \partial_\delta \bar{h}^{\beta\delta} + \frac{1}{2} \eta^{\alpha\beta} \partial_\gamma \bar{h}^{\delta\epsilon} \partial_\delta h^{\gamma\epsilon} - 2 \partial_\gamma \bar{h}^{(\alpha} \partial^{\beta)} \bar{h}^\gamma - 2 \partial_\gamma \bar{h}^{(\alpha} \partial_\epsilon \bar{h}^\gamma_{\beta)\epsilon} \right].
\tag{2.35}
\]

Inserting this expression into Eq. (2.31) we find \( T_{\alpha\beta} \), and after short-wavelength averaging, exploiting the gauge freedom to use the TT gauge, integrating by parts, and using the first-order field equations, we obtain

\[
\Theta^{\alpha\beta} = \frac{1}{32\pi G} \left\langle \partial_\alpha h^{\gamma\delta}_{\gamma\delta} \partial_\beta h^{\gamma\delta}_{\gamma\delta} \right\rangle.
\tag{2.36}
\]

Notice that this method does not allow us to find the first-order field equations, which are typically obtained by the perturbed field equations method of the previous subsection to first order in the metric perturbation. The final result matches those found previously using the other methods.

Noether Current Method

Noether showed that symmetries of the action lead to conserved quantities [66], which have become known as Noether currents \( j^\alpha \). Taking a field theoretical approach (for fields
propagating on a Minkowski background), consider the action

$$S = \int d^4x \, \mathcal{L}(\phi_L, \partial_\alpha \phi_L),$$

(2.37)

where $\phi_L = (\phi_1, \phi_2, \ldots, \phi_L)$ are the fields of the spacetime and $\mathcal{L}$ is a Lagrangian density. From Hamilton’s principle, the variation of the action in Eq. (2.37) must vanish. This leads to

$$\left( \frac{\partial \mathcal{L}}{\partial \phi_L} - \partial_{\alpha} \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \phi_L)} \right) \delta \phi_L + \partial_{\alpha} \left( \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \phi_L)} \delta \phi_L \right) = 0.$$  

(2.38)

We choose our fields to remain constant on the boundary, which leads to the second term in Eq. (2.38) to vanish. This is because the variation still takes place within an integral and total derivatives become boundary terms. The term remaining constitutes the Euler-Lagrange equations. These must be satisfied, and give the equations of motion for the fields.

To find the conserved energy-momentum tensor, we vary the Lagrangian density with respect to the coordinates,

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = \frac{\partial \mathcal{L}}{\partial \phi_L} \partial_{\alpha} \phi_L + \frac{\partial \mathcal{L}}{\partial (\partial_\beta \phi_L)} \partial_{\beta} \partial_{\alpha} \phi_L.$$  

(2.39)

The first term on the right-hand side can be replaced with the Euler-Lagrange equations. The result, after basic analysis, is

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = \frac{\partial \mathcal{L}}{\partial x^\alpha} \left( \frac{\partial \mathcal{L}}{\partial (\partial_\beta \phi_L)} \partial_{\alpha} \phi_L \right) = 0,$$  

(2.40)

which then leads to the conservation law $\partial_{\alpha} j^\alpha_{\beta} = 0$ with the current

$$j_{\alpha \beta} = \mathcal{L} \delta_{\alpha \beta} - \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \phi_L)} \partial_{\beta} \phi_L.$$  

(2.41)
which is known as the canonical SET. The GW SET is typically assumed to be its short-wavelength average:

$$\Theta^{\alpha}_{\beta} = \langle j^{\alpha}_{\beta} \rangle .$$  \hspace{1cm} (2.42)

Unlike the previous methods, which made use of the symmetries of the variations and expansions, there is no mandate that the SET derived here is symmetric. In fact, even in Maxwell’s electrodynamics, the canonical SET is not symmetric (see Appendix B).

Equation (2.41) can now be applied to GR to find the canonical SET. In this setting, the fields are the metric perturbation themselves, which propagate on a flat background, and thus

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta},$$ \hspace{1cm} (2.43)

with $h_{\alpha\beta} \sim O(\epsilon)$. The Lagrangian must thus be expanded in the fields to the appropriate order. Any quantity that depends on the curvature of the background metric vanishes since the latter is the Minkowski metric. Expanding the Lagrangian to first order leaves odd terms in Eq. (2.41), which vanish upon short-wavelength averaging. Expanding the Lagrangian in Eq. (2.1) to second order about a Minkowski background yields

$$\mathcal{L} = \frac{1}{64\pi G} \left[ \partial_{\alpha} h \partial^{\alpha} h + 2\partial_{\alpha} h_{\beta\gamma} \partial^{\beta} h^{\alpha\gamma} - 2\partial^{\alpha} h \partial^{\beta} h_{\alpha\beta} - \partial_{\gamma} h_{\alpha\beta} \partial^{\gamma} h^{\alpha\beta} \right].$$ \hspace{1cm} (2.44)

With this Lagrangian density, we can now find the equations of motion for the metric perturbation and the GW SET. The former can be computed from the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial h_{\alpha\beta}} - \partial_{\gamma} \frac{\partial \mathcal{L}}{\partial (\partial_{\gamma} h_{\alpha\beta})} = \frac{1}{2} \partial_{\gamma} \partial^{\gamma} h_{\alpha\beta}^{TT} = 0,$$ \hspace{1cm} (2.45)

where we have imposed the TT gauge after variation of the Lagrangian. Using Eq. (2.44) in Eq. (2.41), short-wavelength averaging, integrating by parts and using the TT gauge
condition and Eq. (2.45), one finds

$$\Theta^\alpha_\beta = \frac{1}{32\pi G} \left\langle \partial^\alpha h^{\gamma\delta}_{\gamma\delta} \partial_\beta h^{\gamma\delta}_{\gamma\delta} \right\rangle.$$  \hspace{1cm} (2.46)

This expression is identical to all others found in this section.

**Derivation of Physical Quantities: $\dot{E}$ and $\dot{P}$**

Ultimately, one is interested in calculating physical, observable quantities from the GW SET that can be measured at spatial infinity, $i^0$. Two examples are the rate of energy and linear momentum transported by GWs away from any system per unit time

$$\dot{E} = -\int_{\infty} \Theta^{0i} d^2 S_i, \hspace{1cm} (2.47a)$$

$$\dot{P}^i = -\int_{\infty} \Theta^{ij} d^2 S_j, \hspace{1cm} (2.47b)$$

where $\Theta^{\alpha i}$ is the $(\alpha, i)$ component of the GW SET.

These observables can be simplified through the shortwave approximation, which assumes the characteristic wavelength of radiation $\lambda_c$ is much shorter than the observer’s distance to the center of mass $r$, i.e. the observer is in the so-called far-away wave zone so that $r \gg \lambda_c$. When this is true, the propagating fields can be expanded as [65]

$$\phi_L = \frac{\lambda_c}{r} f_{1L}(\tau) + \left( \frac{\lambda_c}{r} \right)^2 f_{2L}(\tau) + O \left( \left[ \frac{\lambda_c}{r} \right]^3 \right), \hspace{1cm} (2.48)$$

where $\tau = t - r/v$ is retarded time and $v$ is the speed of propagation of the field. Moreover, the spacetime (partial) derivative of the field then satisfies

$$\partial_\alpha \phi_L = -\frac{1}{v} k_\alpha \partial_\tau \phi_L + O \left( \frac{\lambda_c^2}{r^2} \right), \hspace{1cm} (2.49)$$
where \( k^\alpha \) is a unit normal 4-vector normal to the \( r = \text{const} \) surface \((k_\alpha \equiv (-1, N_i))\) with \( N_i = x_i/r \) and \( x^i \) are spatial coordinates on the 2-sphere.

With this at hand, we can now simplify the observable quantities \( \dot{E} \) and \( \dot{P}^i \) in GR. The rate at which energy and linear momentum are removed from the system is

\[
\dot{E}_{GR} = -\frac{R^2}{32\pi G} \int \left\langle \frac{\partial h^\gamma_\delta}{\partial \tau} \partial^j h^\gamma^\delta_{TT} \right\rangle d^2 S_i, \\
\dot{P}^i_{GR} = -\frac{R^2}{32\pi G} \int \left\langle \partial^i h^\gamma_\delta \partial^j h^\gamma^\delta_{TT} \right\rangle d^2 S_j,
\]

where \( \dot{h}_{\alpha\beta} \) is the partial derivative of \( h_{\alpha\beta} \) with respect to coordinate time \( \tau \) and \( d^2 S_i = R^2 N_i d\Omega \). Incorporating the shortwave approximation into Eq. (2.50) gives

\[
\dot{E}_{GR} = -\frac{R^2}{32\pi G} \int \left\langle \partial_\tau h^\gamma_\delta \partial_\tau h^\gamma^\delta_{TT} \right\rangle d\Omega, \\
\dot{P}^i = -\frac{R^2}{32\pi G} \int \left\langle N^j \partial_\tau h^\gamma_\delta \partial_\tau h^\gamma^\delta_{TT} \right\rangle d\Omega,
\]

which are the final expressions we seek for the GW SET-related observables. To proceed further, one would have to specify a particular physical system, solve the field equations for the metric perturbation for that given system, and then insert the solution in the above equations to carry out the integral; all of this is system specific and outside the scope of this paper.

Jordan-Fierz-Brans-Dicke Theory

Following the procedure presented in the previous section, we now derive the GW SET in JFBD theory \([36–38]\). This theory was developed in an attempt to satisfy Mach’s
principle and its action is

\[ S_{\text{JFBD}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( \Phi R - \omega \frac{\nabla_\alpha \Phi \nabla^\alpha \Phi}{\Phi} \right), \]  

(2.52)

where \( R \) is the Ricci scalar, \( \Phi \) is the scalar field, and \( \omega \) is a coupling constant. In the GR limit, the scalar field \( \Phi \) becomes unity. The field equations for JFBD are

\[ X_{\alpha\beta} = G_{\alpha\beta} + \frac{1}{\Phi} \left( \nabla_\alpha \nabla_\beta \Phi - g_{\alpha\beta} \nabla_\gamma \nabla^\gamma \Phi \right) \frac{\omega}{\Phi} \left( \nabla_\alpha \Phi \nabla_\beta \Phi - \frac{1}{2} g_{\alpha\beta} \nabla_\gamma \Phi \nabla^\gamma \Phi \right) = 0, \]  

(2.53)

and

\[ Y_{\alpha\beta} = \nabla_\gamma \nabla^\gamma \Phi = 0, \]  

(2.54)

where we vary the action with respect to the metric and scalar field respectively. Lastly, we will make use of the “reduced field” [40] (see Appendix C for derivation of this quantity),

\[ \theta_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} \tilde{g}_{\alpha\beta} h - \frac{1}{\Phi} \tilde{g}_{\alpha\beta} \varphi, \]  

(2.55)

where \( \tilde{g}_{\alpha\beta} \) is the background metric, \( \Phi \) is a background scalar field, and \( h_{\alpha\beta} \) and \( \varphi \) are the perturbations for each field respectively. As we will show, the field equation for the reduced field is simply

\[ \Box \theta_{\alpha\beta}^{\text{RT}} = 0 \]  

(2.56)

in vacuum.

**Perturbed Action Method**

Let us begin by decomposing the metric as in Eq. (2.3) and the scalar field via

\[ \Phi = \tilde{\Phi} + \epsilon \varphi. \]  

(2.57)
The action will decompose in a similar manner to that of Eq. (2.4), where the expanded action terms are

\begin{align*}
S_{(0)}^{(0)} & = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \Phi \left( \tilde{R} - \frac{\omega}{\Phi} \tilde{\nabla}_a \tilde{\Phi} \tilde{\nabla}^a \tilde{\Phi} \right), \\
S_{(1)}^{(1)} & = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} \Phi \left[ \tilde{R}^{\alpha\beta} \theta_{\alpha\beta} - \frac{2}{\Phi} (6 + 2\omega) \Box \varphi - (1 + \omega) \tilde{\nabla}_a \tilde{\nabla}_\beta \theta^{\alpha\beta} - \frac{1}{2} \Box \theta \right], \\
S_{(2)}^{(2)} & = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} \Phi \left[ \frac{1}{\Phi^2} \tilde{R} \varphi^2 + \frac{1}{\Phi} \tilde{R}^{\alpha\beta} \theta_{\alpha\beta} \varphi + \frac{1}{4} \tilde{R} \theta_{\alpha\beta} \theta_{\alpha\gamma} \theta_{\beta\gamma} \\
& - \frac{1}{8} \tilde{R} \theta^2 + 2 \tilde{R} \theta \theta_{\alpha\beta} + \frac{1}{\Phi_0} \left( \frac{3}{2} + \omega \right) \tilde{\nabla}_a \varphi \tilde{\nabla}^a \varphi + \frac{1}{4} \tilde{\nabla}_\gamma \theta_{\alpha\beta} \tilde{\nabla}^\gamma \theta^{\alpha\beta} \\
& - \frac{1}{8} \tilde{\nabla}_a \theta \tilde{\nabla}^a \theta - \frac{1}{2} \tilde{\nabla}_\beta \theta_{\alpha\gamma} \tilde{\nabla}^\gamma \theta^{\alpha\beta} \right].
\end{align*}

(2.58a)

(2.58b)

(2.58c)

All trace terms have been taken with respect to the background metric. As argued before, the $S_{(1)}^{(1)}$ term will not contribute to the GW SET upon variation, leaving us with the effective action

\begin{equation}
S_{\text{eff}}^{(0)} = S_{(0)}^{(0)} \text{eff} + \varepsilon^2 S_{(2)}^{(2) \text{eff}} + \mathcal{O} \left( \varepsilon^3 \right).
\end{equation}

(2.59)

The independent variation of the effective action with respect to $\theta^{\alpha\beta}$ and $\varphi$ gives the first order field equations. In particular, variation with respect to $\theta^{\alpha\beta}$ gives

\begin{equation}
\Box \theta_{\alpha\beta} - 2 \tilde{\nabla}^\gamma \tilde{\nabla}_{(\alpha} \theta_{\beta)} \gamma - \frac{1}{2} \tilde{g}_{\alpha\beta} \theta = 0,
\end{equation}

(2.60)

while variation with respect to $\varphi$ yields

\begin{equation}
\Box \varphi = 0.
\end{equation}

(2.61)

Notice that Eq. (2.60) is identical to the first-order Einstein equations in Eq. (2.10) with the
replacement \( h_{\alpha\beta} \rightarrow \theta_{\alpha\beta} \). Therefore, by utilizing the TT gauge condition on \( \theta_{\alpha\beta} \), namely

\[
\tilde{\nabla}^\alpha \theta_{\alpha\beta}^{\text{TT}} = 0,
\]

\[
\theta^{\text{TT}} = 0,
\]

the equation of motion for the reduced field \( \theta_{\alpha\beta}^{\text{TT}} \) is

\[
\tilde{\Box} \theta_{\alpha\beta}^{\text{TT}} = 0.
\]

Variation of the effective action with respect to the background metric

\[
\delta S_{\text{JFBD}}^{(2)\text{eff}} = \int d^4x \sqrt{-\tilde{g}} \tilde{t}_{\alpha\beta}^{\text{JFBD}} \delta \tilde{g}^{\alpha\beta}
\]

gives the GW SET

\[
\Theta_{\alpha\beta}^{\text{JFBD}} = -2 \langle \tilde{t}_{\alpha\beta}^{\text{JFBD}} \rangle.
\]

Carrying out the variation, we find

\[
\Theta_{\alpha\beta}^{\text{JFBD}} = \frac{1}{32\pi G} \hat{\Phi} \left\{ \tilde{\nabla}_\alpha \theta^{\mu\nu} \tilde{\nabla}_\beta \theta_{\mu\nu} + \frac{6 + 4\omega}{\Phi^2} \tilde{\nabla}_\alpha \varphi \tilde{\nabla}_\beta \varphi
\right. \\
- \frac{1}{2} \tilde{\nabla}_\alpha \theta \tilde{\nabla}_\beta \theta + \tilde{\nabla}_\gamma \theta \tilde{\nabla}_\alpha \theta_{\alpha\beta} + 4 \tilde{\nabla}_{[\gamma} \theta_{\delta]} \tilde{\nabla}^\delta \theta_{\alpha\gamma}
\left. + \tilde{g}_{\alpha\beta} \left( \tilde{\nabla}_\gamma \theta_{\delta\sigma} \tilde{\nabla}^\sigma \theta^{\gamma\delta} + \frac{1}{4} \tilde{\nabla}_\gamma \theta \tilde{\nabla}_\gamma \theta - \frac{1}{2} \tilde{\nabla}_\gamma \theta_{\delta\sigma} \tilde{\nabla}_\delta \theta^{\gamma\sigma} - (3 + 2\omega) \tilde{\nabla}_\gamma \varphi \tilde{\nabla}^\gamma \varphi \right) \right\}.
\]

At this point we may impose the TT gauge condition and use integration by parts to simplify Eq. (2.66) with the use of Eqs. (2.61) and (2.63). The resulting GW SET is

\[
\Theta_{\alpha\beta}^{\text{JFBD}} = \frac{\hat{\Phi}}{32\pi G} \left\{ \tilde{\nabla}_\alpha \theta^{\gamma\epsilon} \tilde{\nabla}_\beta \theta_{\gamma\epsilon}^{\text{TT}} + \frac{1}{\Phi^2} (6 + 4\omega) \tilde{\nabla}_\alpha \varphi \tilde{\nabla}_\beta \varphi \right\}.
\]
This result is consistent with that found in [40] for $\tilde{g}_{\alpha\beta} \rightarrow \eta_{\alpha\beta}$.

**Perturbed Field Equation Method**

Let us begin by expanding the metric as in Eq. (2.16) and similarly expand the scalar field as

$$\Phi = \phi_0 + \varphi^{(1)} + \varphi^{(2)},$$

(2.68)

where $\phi_0$ is now the constant background field.

The expansion of the field equations to first-order yields the equations of motion for the fields $h^{(1)}_{\alpha\beta}$ and $\varphi^{(1)}$

$$\frac{\phi_0}{2} \partial_\gamma \partial^{\gamma} \varphi^{(1)}_{\alpha\beta} = \phi_0 \partial_\gamma \partial_{(\alpha} \varphi^{(1)\gamma}_{\beta)} - \frac{\phi_0}{2} \eta_{\alpha\beta} \partial_\gamma \partial_\delta \varphi^{(1)\gamma\delta},$$

(2.69)

$$\partial_\gamma \partial^{\gamma} \varphi^{(1)} = 0.$$  

(2.70)

Working in the TT gauge as in the previous subsection (see Eq. (2.62)) confirms our vacuum field equation (Eq. (2.56)) in a Minkowski background.

As in Eq. (2.18), the GW SET will be defined as the average of the expansion of the field equations to quadratic order in the linear perturbation terms:

$$8\pi \Theta^{\text{FBD}}_{\alpha\beta} \equiv - \left\langle X_{\alpha\beta} \left[ \left( h^{(1)} \right)^2, \left( \varphi^{(1)} \right)^2, h^{(1)} \varphi^{(1)} \right] \right\rangle,$$

(2.71)

where we recall $X_{\alpha\beta}$ is given by Eq. (2.53). The details of the expansion of the field equations to second order in the perturbations will be omitted for brevity, but the procedure follows that in the perturbed field equation method for GR. Once this is done, we use Eq. (2.71) to find the GW SET

$$\Theta^{\text{FBD}}_{\alpha\beta} = \frac{\phi_0}{32\pi G} \left\langle \partial_\alpha \theta^{\gamma\epsilon}_{\text{TT}} \partial_\beta \theta_{\gamma\epsilon}^{\text{TT}} \frac{1}{\phi_0^2} (6 + 4\omega) \partial_\alpha \varphi \partial_\beta \varphi \right\rangle.$$  

(2.72)
We arrive at this equation using integration by parts, imposing the TT gauge condition, Eqs. (2.54) and (2.56) for a Minkowski background. This GW SET agrees with that found previously for a flat background.

Landau-Lifshitz Method

The Landau-Lifshitz method requires the use of the $\mathcal{H}^{\alpha\mu\beta\nu}$ tensor density or a variation of it to obtain a conservation law of the form of Eq. (2.30). In principle, one could use the same $\mathcal{H}^{\alpha\mu\beta\nu}$ tensor density as that used in GR (see Eq. (2.26)), and one would obtain the same rate of energy and linear momentum loss in a binary system [91]. But in practice, it is easier to use an improved $\mathcal{H}^{\alpha\mu\beta\nu}$ tensor density that simplifies the calculations of the modified Landau-Lifshitz pseudo-tensor $t_{\text{LL,JFBD}}^{\alpha\beta}$. Following the work of [84], we choose

$$\mathcal{H}^{\alpha\mu\beta\nu} = \Phi^2 \left( g^{\alpha\beta} g^{\mu\nu} - g^{\alpha\mu} g^{\beta\nu} \right).$$

(2.73)

This satisfies the relation

$$\partial_\mu \partial_\nu \mathcal{H}^{\alpha\mu\beta\nu} = 2 \left(-g\right) \Phi^2 \left(X^{\alpha\beta} + \frac{8\pi}{\Phi} t_{\text{LL,JFBD}}^{\alpha\beta} \right),$$

(2.74)

where $X^{\alpha\beta} = 0$ are the field equations (Eq. (2.53)). When the GR limit is taken, this agrees exactly with the results obtainin in the GR section. The final term in Eq. (2.74) will be called the JFBD pseudo-tensor and is given by [84]

$$t_{\text{LL,JFBD}}^{\alpha\beta} = \Phi_{\text{LL}}^{\mu\nu} + \frac{1}{8\pi} \left[ 2\Phi^{(\alpha\Gamma_{\gamma\delta})\gamma\delta} g^{\gamma\delta} - 2g^{\gamma(\alpha\Gamma_{\gamma\delta})\delta} \Phi_{\gamma\delta} + g^{\mu(\alpha\Gamma_{\gamma\delta})\mu} g^{\nu(\beta\Gamma_{\delta\gamma})\nu} + g^{\alpha\beta} (2\Phi^{(\gamma(\alpha\Gamma_{\gamma\delta})\delta)} - \Phi_{(\gamma\delta)\gamma\delta}) - 2\Phi^{(\gamma(\alpha\Gamma_{\gamma\delta})\delta)} \right] + \frac{1}{16\pi\Phi} \left[ 2(\omega - 1) \Phi^{\alpha\beta} \Phi_{\gamma\delta} (2 - \omega) g^{\alpha\beta} \Phi_{\gamma\delta} \Phi \right].$$

(2.75)
One could write this expression entirely in terms of the gothic metric, but this does not simplify the resulting expression.

The GW SET can now be obtained through Eq. (2.32) after expanding Eq. (2.74) to leading non-vanishing order. Doing so, making use of the TT gauge, integrating by parts, and using the field equations for the linear perturbations, the final GW SET is found to be

$$
\Theta_{\alpha\beta}^{WBD} = \frac{\phi_0}{32\pi G} \left( \partial_\alpha \theta^\gamma_{\TT} \partial_\beta \theta^\epsilon_{\TT} + \frac{1}{\phi_0^2} (6 + 4\omega) \partial_\alpha \varphi \partial_\beta \varphi \right).
$$

Equation (2.76) is identical to the GW SETs previously found.

**Noether Current Method**

The derivation of the canonical SET relies on Eq. (2.41), which requires we expand the Lagrangian to second order through the metric decomposition of Eq. (2.43) and the field decomposition of Eq. (2.57). Doing so, we find

$$
\mathcal{L}_{\text{WBD}} = \frac{\phi_0}{32\pi G} \left( \frac{1}{4} \partial^\alpha \theta \partial_\alpha \theta + \partial_\beta \theta_{\alpha\gamma} \partial^\gamma \theta^{\alpha\beta} - \frac{1}{2} \partial_\gamma \theta_{\alpha\beta} \partial^\gamma \theta^{\alpha\beta} - \frac{(3 + 2\omega)}{\phi_0^2} \partial_\alpha \varphi \partial^\alpha \varphi \right).
$$

(2.77)

With this second order Lagrangian at hand, we can now follow the same steps as in the GR Noether current section to calculate the canonical SET. The first step is to ensure the Euler-Lagrange equations are satisfied. For the reduced field $\theta_{\alpha\beta}$, the Euler-Lagrange equations give

$$
\partial_\gamma \partial^\gamma \theta^{\alpha\beta}_{\TT} = 0,
$$

(2.78)

where we have used the TT gauge after varying the Lagrangian. As for $\varphi$, the Euler-Lagrange equations lead to

$$
\partial_\gamma \partial^\gamma \varphi = 0.
$$

(2.79)

The next step is to compute the Noether’s current. Summing over the variation of all fields,
we find
\[ j_{\alpha \beta} = \left( -\frac{\partial L}{\partial (\partial_\alpha \theta_{\mu \nu})} \partial_\beta \theta_{\mu \nu} - \frac{\partial L}{\partial (\partial_\alpha \varphi)} \partial_\beta \varphi + \eta^\alpha_\beta \mathcal{L} \right). \] (2.80)

The current in Eq. (2.80) gives a pseudo-tensor that is not gauge invariant, but after short-wavelength averaging, these terms vanish. Using the TT gauge condition and imposing the first-order field equations, one then finds
\[ \Theta_{JFBD}^{\alpha \beta} = \phi_0^2 \frac{R^2}{32 \pi G} \int \frac{1}{\phi_0^2} (6 + 4 \omega) \partial_\alpha \varphi \partial_\beta \varphi \, d\Omega. \] (2.81)

The result here is consistent with those presented in the previous sections.

**Derivation of Physical Quantities: $\dot{E}$ and $\dot{P}$**

The four different ways of deriving the GW SET in JFBD all produced the same result, and hence, one can use any of them to derive $\dot{E}$ and $\dot{P}$. Inserting the GW SET into Eqs. (2.47a) and (2.47b), the rate of change of energy and linear momentum carried away by all propagating degrees of freedom is
\[ \dot{E}_{JFBD} = \frac{\phi_0^2 R^2}{32 \pi G} \int \left( \partial_\tau \theta_{\tau \gamma \delta} \partial_\tau \theta_{\tau \gamma \delta} + \frac{6 + 4 \omega}{\phi_0^2} \partial_\tau \varphi \partial_\tau \varphi \right) \, d\Omega, \] (2.82a)
\[ \dot{P}_i_{JFBD} = -\frac{\phi_0^2 R^2}{32 \pi G} \int N_i \left( \partial_\tau \theta_{\tau \gamma \delta} \partial_\tau \theta_{\tau \gamma \delta} + \frac{6 + 4 \omega}{\phi_0^2} \partial_\tau \varphi \partial_\tau \varphi \right) \, d\Omega. \] (2.82b)

In the GR limit, we find that the rate of energy and momentum loss is identical to that found previously in the GR section. Notice that the overall rate losses are modified by terms quadratic in the time derivatives of the scalar field perturbation. In fact, Eq. (2.82a) can be shown to generate a dipole term in energy loss, which does not exist in GR [40]. The effect of a dipole term would be to accelerate the rate at which a binary inspirals, and its associated GW frequency chirps, as is well known [40]. Binary pulsar and GW observations consistent
with GR could thus be used to test this theory, as first suggested in the 20th century [92, 93].

**Einstein-Æther Theory**

In this section, we study Einstein-Æther theory [45] by following the work of Foster [1].

We begin with the action

\[
S_{\text{Æ}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R - K_{\gamma\delta}^{\alpha}\nabla_\gamma u^\gamma \nabla_\delta u^\delta + \lambda (u^\alpha u_\alpha + 1) \right),
\]

(2.83)

where

\[
K_{\gamma\delta}^{\alpha\beta} = c_1 g_{\alpha\beta} g_{\gamma\delta} + c_2 \delta_\alpha^{\gamma} \delta_\beta^{\delta} \epsilon + c_3 \delta_\alpha^{\epsilon} \delta_\beta^{\gamma} - c_4 u^\alpha u^\beta g_{\gamma\delta}. 
\]

(2.84)

The vector \(u^\alpha\) in Eq. (2.83) is the Æther field, which is unit time-like due to the Lagrange multiplier \(\lambda\) constraint. The quantities \(c_i\) are coupling constants of the theory. Certain combinations of these constants will typically appear in the perturbed field equations, namely

\[
c_{14} = c_1 + c_4, 
\]

(2.85a)

\[
c_\pm = c_1 \pm c_3, 
\]

(2.85b)

\[
c_{123} = c_1 + c_2 + c_3. 
\]

(2.85c)

Varying the action with respect to the metric tensor and the Æther vector field yields the field equations of the theory in vacuum:

\[
G_{\alpha\beta} = S_{\alpha\beta}, 
\]

(2.86)

\[
\nabla_\beta K_{\alpha}^{\beta} = -\lambda u_\alpha - c_4 (u_\beta \nabla^\gamma u^\gamma) \nabla_\alpha u_\gamma, 
\]

(2.87)
where \( G_{\alpha\beta} \) is the Einstein tensor and \( S_{\alpha\beta} \) is the Æther contribution to the field equations given by

\[
S_{\alpha\beta} = \nabla_\gamma \left( K^\gamma_{(\alpha u}\beta) + K_{(\alpha\beta)u}\gamma - K_{(\alpha\gamma u}\beta) \right) + c_1 \left( \nabla_\alpha u_\gamma \nabla_\beta u^\gamma - \nabla_\gamma u_\alpha \nabla_\beta u^\gamma \right) + c_4 \left( u^\gamma u^\delta \nabla_\gamma u_\alpha \nabla_\delta u_\beta \right) + \lambda u_\alpha u_\beta - \frac{1}{2} g_{\alpha\beta} \left( K^\gamma_\delta \nabla_\gamma u^\delta \right),
\]

(2.88)

and where

\[
K^\alpha_{\gamma} = K^\alpha_{\beta\gamma\delta} \nabla_\beta u^\delta.
\]

(2.89)

Variation of the action with respect to the Lagrange multiplier \( \lambda \) gives the constraint,

\[
u^\alpha u_\alpha = -1.
\]

(2.90)

Contracting Eq. (2.87) with \( u^\alpha \), we can solve for \( \lambda \) to obtain

\[
\lambda = u^\alpha \nabla_\beta K^\beta_{\alpha} + c_4 \left( u^\alpha \nabla_\alpha u^\beta \right) \left( u^\gamma \nabla_\gamma u_\beta \right).
\]

(2.91)

Unlike in the previous sections for GR and JFBD, we will use an irreducible decomposition of all fields rather than a single reduced field. This will have the effect of cleanly separating the independent modes of propagation. Expanding the metric as in Eq. (2.3), the Æther field can be decomposed via

\[
u^\alpha = t^\alpha + \omega^\alpha,
\]

(2.92)

where \( |\omega^\alpha| = O(h) \) and \( t^\alpha = (1, 0, 0, 0) \) and it is a timelike unit vector with respect to the background metric \( \bar{g}_{\alpha\beta} \), i.e. \( \bar{g}_{\alpha\beta} t^\alpha t^\beta = -1 \). We next decompose the metric perturbation
into tensor, vector, and scalar modes as follows

\[ h^{\alpha\beta} = t^\alpha t^\beta h_{00} + 2 P_i^\alpha t^\beta h^{0i} + P_i^\alpha P_j^\beta h^{ij}, \]  

(2.93)

where \( P_i^\alpha \) is the background spatial projector

\[ P^\alpha \beta = \tilde{g}^{\alpha \beta} + t^\alpha t^\beta. \]  

(2.94)

The Æther, vector and tensor perturbations are further decomposed into their transverse and longitudinal parts,

\[ \omega^\alpha = t^\alpha \omega_0 + P_i^\alpha \left( \gamma^i + \partial^i \gamma \right), \]  

(2.95a)

\[ h^{0i} = \gamma^i + \partial^i \gamma, \]  

(2.95b)

\[ h^{ij} = \phi_{\text{TT}}^{ij} + \frac{1}{2} P_{ij} \left[ f \right] + 2 \partial_{(i} \phi_{j)} + \partial_i \partial_j \phi, \]  

(2.95c)

with \( P_{ij} [f] \equiv \tilde{g}_{ij} \tilde{\nabla}_k \tilde{\nabla}^k f - \tilde{\nabla}_i \tilde{\nabla}_j f \), \( \phi_{\text{TT}}^{ij} \) a TT spatial tensor and \( \gamma^i, \gamma, \phi_{\text{TT}}^{ij}, f, \phi^i, \phi \). These quantities can be further constrained. The time-like condition on the Æther field \( u^\alpha \) and the metric tensor \( g_{\alpha\beta} \), have thus been replaced by their decompositions \( (\omega_0, \gamma^i, \gamma) \) and \( (h^{00}, \gamma^i, \gamma, \phi_{\text{TT}}^{ij}, f, \phi^i, \phi) \), but these quantities can be further constrained. The time-like condition on the Æther field requires that

\[ \omega_0 = -\frac{1}{2} h_{00}, \]  

(2.96)

to leading order in the perturbations. We also choose the gauge conditions

\[ \partial_i \omega^i = 0, \]  

(2.97a)

\[ \partial_i h^{0i} = 0, \]  

(2.97b)

\[ \partial_i \partial_j [h_{kj}]^i = 0, \]  

(2.97c)
or equivalently $\nu = \gamma = \phi_i = 0$.

**Perturbed Action Method**

The metric is decomposed as in Eq. (2.3) while the Æther field takes the form of Eq. (2.92) with $\omega^\alpha$ replaced by $\epsilon \omega^\alpha$. We expand the action to second-order (see App. D) and find the effective action

$$S_{\text{eff}}^\epsilon = S_{\text{eff}}^{(0)} + \epsilon^2 S_{\text{eff}}^{(2)} + O(\epsilon^3).$$  \hspace{1cm} (2.98)

Recall that the $S_{\text{eff}}^{(1)}$ is not important for our consideration in this paper due to the angular averaging.

We start by solving for the first-order equations of motions for the fields. After expanding the action, we decompose the perturbations into the various transverse and longitudinal parts given in Eq. (2.95). Starting with the tensor mode $\phi_{TT}^{ij}$,

$$\frac{\delta S_{\text{eff}}^{(2)}}{\delta \phi_{TT}^{ij}} = (1 - c_+) \tilde{\partial}_0^2 \phi_{TT}^{ij} - \tilde{\Delta} \phi_{TT}^{ij} = 0,$$  \hspace{1cm} (2.99)

where we have focused on terms quadratic in the tensor mode from the action. The $\tilde{\Delta}$ in Eq. (2.99) is the spatial Laplacian operator associated with the background spacetime $\tilde{\Delta} = g^{ij} \tilde{\nabla}_i \tilde{\nabla}_j$, while the $\tilde{\partial}_0$ operator is the time derivative in this background $\tilde{\partial}_0 = t^\alpha \tilde{\nabla}_\alpha$. We may rewrite Eq. (2.99) in a more compact form using a modified wave equation,

$$\tilde{\Box}_2 \phi_{TT}^{ij} = 0,$$  \hspace{1cm} (2.100)

where the wave operator is defined as

$$\tilde{\Box}_2 \equiv (1 - c_+) \tilde{\partial}_0^2 - \tilde{\Delta}.$$  \hspace{1cm} (2.101)
From this, we are able to see that the wave speed for the tensor mode is

\[ v_T^2 \equiv \frac{1}{1 - c_+}. \]  

(2.102)

We perform the same procedure to the vector modes \( \gamma^i \) and \( \nu^i \). The variations become

\[
\frac{\delta S_{\text{eff}}^{(2)}}{\delta \gamma^i} = c_{14} \partial_0^2 \left( \gamma_i + v_i \right) + \frac{1}{2} \tilde{\Delta} \left[ (1 - c_-) \gamma_i - c_- v_i \right] = 0, \quad \text{(2.103a)}
\]

\[
\frac{\delta S_{\text{eff}}^{(2)}}{\delta \nu^i} = c_{14} \partial_0^2 \left( \gamma_i + v_i \right) - \frac{1}{2} \tilde{\Delta} \left[ c_- \gamma_i + 2 c_+ v_i \right] = 0. \quad \text{(2.103b)}
\]

We can combine these equations to give us a relation between the variables \( \gamma_i \) and \( v_i \),

\[ \gamma_i = -c_+ v_i. \]  

(2.104)

Equipped with this relation, we are able to solve for another wave equation, this time for the vector mode,

\[
\tilde{\square}_1 v_i \equiv \frac{2 (1 - c_+) c_{14} \partial_0^2 \gamma_i}{2c_1 - c_+ c_-} v_i - \tilde{\Delta} v_i
\]

\[ = 0. \]  

(2.105)

The wave speed for the vector mode can be read off as

\[ v_V^2 \equiv \frac{2c_1 - c_+ c_-}{2(1 - c_+)c_{14}}. \]  

(2.106)

Lastly, we investigate the scalar modes \( h_{00} \), \( \phi \), and \( F \). The relevant variations are

\[
\frac{\delta S_{\text{eff}}^{(2)}}{\delta h_{00}} = \frac{1}{32\pi G} \tilde{\Delta} \left( c_{14} h_{00} - F \right), \quad \text{(2.107a)}
\]
\[
\frac{\delta S^{(2)\text{eff}}}{\delta \phi} = \frac{1}{32\pi G} \partial_0 \partial_0 \tilde{\Delta} \left[ (1 + c_2) F + c_{123} \tilde{\Delta} \phi \right], \quad (2.107b)
\]
\[
\frac{\delta S^{(2)\text{eff}}}{\delta F} = \frac{1}{32\pi G} \left[ \tilde{\Delta} h_{00} + \frac{1}{2} (1 + c_+ + 2c_2) \tilde{\Delta}^2 F + (1 + c_2) \tilde{\Delta} \tilde{\Delta}^2 \phi - \frac{1}{2} \tilde{\Delta} F \right], \quad (2.107c)
\]

with \( F \equiv \tilde{\nabla}_i \tilde{\nabla}^i f \). From the vanishing of the first two scalar mode variations, we are able to find the relations
\[
\begin{align*}
    h_{00} &= \frac{1}{c_{14}} F, \quad \text{(2.108a)} \\
    \tilde{\Delta} \phi &= -\frac{1 + c_2}{c_{123}} F. \quad \text{(2.108b)}
\end{align*}
\]

Using Eq. (2.108) in the final variation, Eq. (2.107c), we are able to find the modified wave equation for the scalar mode
\[
\tilde{\Box}_0 F = \frac{(1 - c_+)(2 + 2c_2 + c_{123})c_{14}}{(2 - c_{14})c_{123}} \tilde{\Delta}^2 F - \tilde{\Delta} F = 0, \quad \text{(2.109)}
\]

with a propagation speed of
\[
\nu_s^2 \equiv \frac{(2 - c_{14})c_{123}}{(1 - c_+)(2 + 2c_2 + c_{123})c_{14}}. \quad \text{(2.110)}
\]

We know from the previous sections that we can find the GW SET by using
\[
\delta S^{(2)\text{eff}}_E = \int d^4x \sqrt{-\tilde{g}} t^\mu_{\alpha \beta} \delta \tilde{g}^{\alpha \beta}, \quad \text{(2.111)}
\]

where we define the GW SET to be
\[
\Theta^\mu_{\alpha \beta} = -2 \left< t^\mu_{\alpha \beta} \right>. \quad \text{(2.112)}
\]
To simplify the result, we decompose the resulting GW SET into its tensor \((T)\), vector \((V)\), and scalar \((S)\) pieces. In addition to this, we make use of the equations of motion given in Eqs. (2.100), (2.105) and (2.109). The GW SET is

\[
(T) : \quad \Theta^E_{\alpha\beta} = \frac{1}{32\pi G} \left\langle \tilde{\nabla}_\alpha \phi^T_{ij} \tilde{\nabla}_\beta \phi^T_{ij} \right\rangle, \tag{2.113a}
\]

\[
(V) : \quad \Theta^E_{\alpha\beta} = \frac{1 - c_+}{16\pi G} \left(2c_1 - c_+ c_- \right) \tilde{\nabla}_\alpha v^j_i \tilde{\nabla}_\beta v_i + c_+ \left(\frac{c_+(2c_1 - c_+ c_-) - 2c_{14}}{2c_1 - c_+ c_-} \right) t_\alpha t_\beta \left(\partial^2_{0} v^j_i \right) v_i, \tag{2.113b}
\]

\[
(S) : \quad \Theta^E_{\alpha\beta} = \frac{1}{64\pi G} \left\langle \left(\frac{2 - c_{14}}{c_{14}} \right) \tilde{\nabla}_\alpha F \tilde{\nabla}_\beta F - \frac{2}{c_{14}} t_\alpha t_\beta \left(\frac{2c_{14} - 3c_2 + 2c_{14}c_2 - c_+ + 2c_2c_+}{c_{123}} \right) \left(\partial^2_{0} F \right) - \frac{3c_1 - 2(c_+ + c_{14})}{c_{14}} \tilde{\Delta} F \right\rangle F. \tag{2.113c}
\]

Note that in this situation, the GW SET has terms explicitly dependent on the \(t^\alpha\), the Lorentz-violating background Æther field. However, we will see that these terms do not affect physical observables.

**Perturbed Field Equation Method**

We begin by expanding the metric as in Eq. (2.16) and expanding the Æther field as

\[
u^\alpha = t^\alpha + \omega^{(1)}\alpha + \omega^{(2)}\alpha. \tag{2.114}
\]

We next expand the field equations in Eqs. (2.86) and (2.87) to \(O(h)\) in order to find the equations of motion for the perturbed fields

\[
G^{(1)}_{\alpha\beta} - S^{(1)}_{\alpha\beta} = \frac{1}{2} \left[ 2\partial_\gamma \partial_{(\alpha} h^{(1)}_{\beta)} - \partial_{\alpha} \partial_\beta h^{(1)} - \partial_\gamma \partial^\gamma h^{(1)}_{\alpha\beta} + g_{\alpha\beta} \left( \partial_\gamma \partial^\gamma h^{(1)} - \partial^\gamma \partial^\delta h^{(1)}_{\gamma\delta} \right) \right] + c_1 \left[ t^{\gamma} t_{(\alpha} \partial_{\beta)} \partial_\delta h^{(1)}_{\gamma\delta} - \partial_\gamma \partial^\gamma \left( t^\delta t_{(\alpha} h^{(1)}_{\beta)} \right) - \frac{1}{2} h^{(1)}_{\alpha\beta} \right]
\]
The vector modes are found by contracting Eqs. (2.115a) and (2.115b) with the projector which matches what we found in Eq. (2.100) with the background taken to be Minkowski. For the tensor mode,

\[ \nabla^\beta K^\alpha_a \] \hspace{1cm} (1)

We are now able to decompose the first-order field equations into the modes of propagation. For the tensor mode,

\[ \partial_\gamma \partial_\gamma \varphi_{ij}^{TT} + c_+ \varphi_{ij}^{TT} = 0. \]  

(2.116)

We notice that Eq. (2.116) may be written to leading order as a modified wave equation,

\[ \Box_2 \varphi_{ij}^{TT} = 0, \]  

(2.117)

which matches what we found in Eq. (2.100) with the background taken to be Minkowski. The vector modes are found by contracting Eqs. (2.115a) and (2.115b) with the projector \( \mathcal{P}^\alpha_i \). The results are,

\[ c_{14} (\ddot{\gamma}_i + \dot{v}_i) + \frac{1}{2} \partial_j \partial^j \left[ (1 - c_+) \gamma_i - c_- v_i \right] = 0, \]  

(2.118a)

\[ c_{14} (\ddot{\gamma}_i + \dot{v}_i) - \frac{1}{2} \partial_j \partial^j \left[ c_- \gamma_i + 2c_1 v_i \right] = 0. \]  

(2.118b)
Equations (2.118a) and (2.118b) are then solved simultaneously to generate the relation
\[ \gamma_i = -c_+ \nu_i, \]
leading to the modified wave equation,
\[ \square_1 \nu_i \equiv \frac{2 (1 - c_+) c_{14}}{2 c_1 - c_+ c_-} \dot{\nu}_i - \partial_j \partial^j \nu_i = 0, \tag{2.119} \]
which matches Eq. (2.105) for a Minkowski background. The scalar mode equation of motion is found by looking at the scalar field equations:
\[ G_{ii}^{(1)} - S_{ii}^{(1)} = 0, \quad G_{00}^{(1)} - S_{00}^{(1)} = 0, \]
and \( (\nabla_\beta K^\beta_1)^{(1)} = 0 \), namely
\[ -\partial_j \partial^j h_{00} - \frac{1}{2} (2 + 2 c_2 + c_{123}) \ddot{F} + \frac{1}{2} \partial_j \partial^j F - \frac{1}{2} (2 + 2 c_2 + c_{123}) \partial_j \partial^j \phi = 0, \tag{2.120a} \]
\[ \frac{1}{2} (c_{14} h_{00} - F) = 0, \tag{2.120b} \]
\[ \frac{1}{2} \partial_0 \partial_i \left( c_{123} \partial_j \partial^j \phi + c_{14} h_{00} + c_2 F \right) = 0. \tag{2.120c} \]
From Eqs. (2.120b) and (2.120c) we find the relations
\[ h_{00} = \frac{1}{c_{14}} F, \tag{2.121a} \]
\[ \partial_i \partial^i \phi = -\frac{1 + c_2}{c_{123}} F. \tag{2.121b} \]
These combined with Eq. (2.120a) give the modified wave equations for the scalar mode
\[ \square_0 F \equiv \frac{(1 - c_+) (2 + 2 c_2 + c_{123}) c_{14}}{(2 - c_{14}) c_{123}} \ddot{F} - \partial_j \partial^j F = 0, \tag{2.122} \]
which agrees with Eq. (2.109) for a Minkowski background.

We now expand the field equations to \( O(h^2) \) to find the GW SET. Retaining only the
tensor mode \((T)\) terms gives

\[
\Theta^{(T)}_{\alpha \beta} = \frac{1}{32 \pi G} \left\{ \partial_\alpha \phi^{TT}_{ij} \partial_\beta \phi^{ji}_{TT} - 2 \Box_2 \phi^{TT}_{\alpha i} \phi^{TT}_{\beta j} - \frac{1}{2} \eta_{\alpha \beta} \Box_2 \phi^{TT}_{ij} \phi^{TT}_{ji} + t_\alpha t_\beta c_+ \phi^{TT}_{ij} \phi^{TT}_{ji} \right\} .
\]  \tag{2.123}

We use Eq. (2.117) to simplify this expression to

\[
\Theta^{(T)}_{\alpha \beta} = \frac{1}{32 \pi G} \left\{ \partial_\alpha \phi^{TT}_{ij} \partial_\beta \phi^{ji}_{TT} + t_\alpha t_\beta c_+ \phi^{TT}_{ij} \phi^{TT}_{ji} \right\} .
\]  \tag{2.124}

This procedure is again repeated for the vector \((V)\) and scalar \((S)\) modes to give additional terms for the GW SET,

\[
\Theta^{(V)}_{\alpha \beta} = \frac{1 - c_+}{16 \pi G} \left\{ (2 c_1 - c_+) \partial_\alpha v^i \partial_\beta v^i + t_\alpha t_\beta \left( c_+^2 - 2 c_4 (1 - c_+) \right) \ddot{v}^i \nu^i \right\} , \tag{2.125}
\]

\[
\Theta^{(S)}_{\alpha \beta} = \frac{1}{64 \pi G c_{14}} \left\{ (2 - c_{14}) \partial_\alpha F \partial_\beta F - \left( \frac{c_{14} (1 - c_+) (2 + 2 c_2 + c_{123})}{c_{123}} \right) t_\alpha t_\beta \ddot{F} \right\} .
\]  \tag{2.126}

Note that all of the terms proportional to \(t^\alpha\) here differ from those found in the previous section. This is okay, since there is no way to define a unique GW SET. However, as we will see below, the resulting physical observables will be the same.

**Landau-Lifshitz Method**

We now construct a tensor density \(H^{\alpha \mu \beta \nu}\) to obtain a conservation law of the form of Eq. (2.30). By keeping \(H^{\alpha \mu \beta \nu}\) the same as that found in Eq. (2.26) (as also done in [94] to derive the GW SET for other vector-tensor theories), we obtain Eq. (2.27) where \(G^{\alpha \beta}\) is again the Einstein tensor and \(t^{\alpha \beta}_{\alpha \beta}\) is the Landau-Lifshitz pseudotensor found in Eq. (2.28).

We now substitute the field equations found in Eq. (2.86) which gives us (including \(T^{\alpha \beta}_{\text{mat}}\) for
completeness)

\[ \partial_\mu \partial_\nu H^{\alpha \mu \beta \nu} = 2 \left( -g \right) \left( 8 \pi G T_{\text{mat}}^{\alpha \beta} + S^{\alpha \beta} \right) + 16 \pi G \left( -g \right) t_{\text{LL}}^{\alpha \beta} \]

\[ = 16 \pi G \left( -g \right) \left( T_{\text{mat}}^{\alpha \beta} + \frac{1}{8 \pi G} S^{\alpha \beta} + t_{\text{LL}}^{\alpha \beta} \right) \]

\[ = 16 \pi G \left( -g \right) \left( T_{\text{mat}}^{\alpha \beta} + \bar{t}_{\text{LL}}^{\alpha \beta} \right) , \quad (2.127) \]

where \( \bar{t}_{\text{LL}}^{\alpha \beta} \) is the modified Landau-Lifshitz pseudotensor

\[ \bar{t}_{\text{LL}}^{\alpha \beta} = \frac{1}{8 \pi G} S^{\alpha \beta} + t_{\text{LL}}^{\alpha \beta} . \quad (2.128) \]

Following the same procedure as in the GR section, we find the GW SET to be

\[ \Theta_{\alpha \beta}^{\text{e}} = \frac{1}{16 \pi G} \left\langle \partial_\mu \partial_\nu H^{\alpha \mu \beta \nu} \right\rangle . \quad (2.129) \]

As we have done previously in the Landau-Lifshitz Method sections, we expand the pseudotensor in Eq. (2.128) to second order in the metric and \( \mathcal{A} \)ether perturbations given in Eqs. (2.34) and (2.92) respectively. Once this is done, we impose the decomposition and solve Eq. (2.129) for each mode. Recall that this method provides no way to solve for the field equations for the first order perturbations. We already calculated these equations in the previous section, and so we can use them here to simplify the GW SET. Doing so, tensor part of the GW SET is

\[ \Theta_{\text{e}}^{(T) \alpha \beta} = \frac{1}{32 \pi G} \left\langle \partial_\alpha \phi_{ij}^{TT} \partial_\beta \phi_{ij}^{TT} + t_{\alpha \beta} c_4 \phi_{ij}^{TT} \phi_{ij}^{TT} \right\rangle . \quad (2.130) \]

We apply this analysis to the vector and scalar modes and find

\[ \Theta_{\text{e}}^{(V) \alpha \beta} = \frac{1 - c_+}{16 \pi G} \left\langle (2c_1 - c_+ c_-) \partial_\alpha \mathcal{A} \partial_\beta \mathcal{A} \right. \left. + t_{\alpha \beta} c_4 \left( \frac{c_+}{c_+ - 2c_4(1 - c_+)} \right) \mathcal{A} \mathcal{A} \right\rangle , \quad (2.131) \]
\[
\Theta_{\kappa}^{(S)}_{\alpha\beta} = \frac{1}{64\pi G_{c14}} \left( (2 - c_{14}) \partial_{a} F \partial_{\beta} F - \left( \frac{c_{14} (1 - c_{+}) (2 + 2c_{2} + c_{123})}{c_{123}} \right) t_{a} t_{\beta} \ddot{F} \right). 
\]

(2.132)

Notice that the results in this section match those given in the perturbed field equation method section. This is because in both cases, the expansion of the field equations was used to find the GW SET.

**Noether Current Method**

The derivation of the canonical SET relies on Eq. (2.41), which requires we expand the Lagrangian density to second order (see App. D) through the metric decomposition of Eq. (2.43) and the \( \mathcal{A} \)ether decomposition of Eq. (2.92)\(^2\). We decompose the results into each tensor, vector, and scalar component before applying the Euler-Lagrange equations to each individual field contribution. For the tensor mode, we only have one field \( \phi_{ij}^{TT} \). The resulting Euler-Lagrange equation is

\[
\frac{1}{32\pi} \left[ (1 - c_{+}) \phi_{ij}^{TT} - \Delta \phi_{ij}^{TT} \right] = 0.
\]

(2.133)

This is again identical to the modified wave equation of Eq. (2.99) around a Minkowski background. The Euler-Lagrange equations for the vector modes are

\[
\frac{c_{14}}{8\pi} (\ddot{y}_{i} + \ddot{v}_{i}) + \frac{1}{16\pi} \Delta \left[ (1 - c_{-}) \gamma_{i} - c_{-} v_{i} \right] = 0, 
\]

(2.134a)

\[
\frac{c_{14}}{8\pi} (\ddot{y}_{i} + \ddot{v}_{i}) - \frac{1}{16\pi} \Delta (c_{-} \gamma_{i} + 2c_{1} v_{i}) = 0,
\]

(2.134b)

\(^2\)See [85] for related work on deriving the non-symmetric GW SET in Einstein-\( \mathcal{A} \)ether theory using the Noether current method without applying any perturbations.
for $\gamma_i$ and $\nu_i$ respectively. These reduce to the same modified field equations for the vector mode around a Minkowski background. Using the relation $\gamma_i = -c_+ \nu_i$, these equations can be combined as

$$\Box_1 \nu_i = 0.$$  (2.135)

The scalar Euler-Lagrange equations are

$$\frac{1}{32\pi} \Delta (c_{14} h_{00} - F) = 0,$$  (2.136a)

$$\frac{1}{32\pi} \partial_\alpha \partial^\alpha \left[ c_{123} \Delta \phi + (1 + c_2) F \right] = 0,$$  (2.136b)

$$\frac{1}{64\pi} \Delta \left[ \Delta F - 2 \Delta h_{00} - (1 + c_+ + 2 c_2) \ddot{F} - 2 (1 + c_2) \Delta \dot{\phi} \right] = 0,$$  (2.136c)

when we vary with respect to $h_{00}$, $\phi$, and $f$ respectively. Combining these equations gives us

$$\Box_0 F = 0,$$  (2.137)

with the relations from Eq. (2.108) found explicitly.

With the equations of motion at hand, we may use Eq. (2.41) to solve for the GW SET. We again look at each mode individually and use Eqs. (2.97), (2.133), (2.135), and (2.137) to simplify the results

$$\Theta^{(T)}_{\alpha\beta} = \frac{1}{32\pi G} \left\{ \partial_\alpha \phi^{ij}_{\parallel} \partial_\beta \phi^{ij}_{\parallel} + c_+ t_\alpha \partial_\beta \phi^{jk}_{\parallel} \phi^{jk}_{\parallel} \right\},$$  (2.138a)

$$\Theta^{(V)}_{\alpha\beta} = \frac{1 - c_+}{32\pi G} \left\{ (2c_1 - c_+ c_-) \partial_\alpha \nu^i \partial_\beta \nu_i + \left( c_+^2 - 2 c_4 (1 - c_+) \right) t_\alpha \partial_\beta \nu_i \nu^i \right\},$$  (2.138b)

$$\Theta^{(S)}_{\alpha\beta} = \frac{1}{64\pi G c_{123} c_{14}} \left\{ (2c_+ + c_{14} (4 + 3 c_2 - c_+) - 2 c_2 - 4) \partial_\alpha F \partial_\beta F 
+ \left( 2(c_+ - 2) + c_2 (3 c_{14} c_+ - 2) + c_{14} (2 + c_+^2) \right) t_\alpha \partial_\beta F \dot{F} 
+ 4(1 + c_2) (1 - c_{14}) \mathcal{P}_\alpha^j \partial_\beta F \partial_i \dot{F} \right\}.$$  (2.138c)
The above SET is indeed conserved, i.e. \( \nabla_a (\Theta^\alpha_{(T)} + \Theta^\alpha_{(V)} + \Theta^\alpha_{(S)}) = 0 \), and thus, we can calculate the rate of change of the energy and linear momentum carried away to spatial infinity by all propagating modes. If we do so, we indeed find the correct answer, i.e. the same answer as what one finds with all other methods to compute these quantities. Indeed, Foster [1] used the Noether charge method, which is related to the Noether method shown here, to find the correct \( \dot{E} \). However, the above GW SET is somewhat unsatisfactory because it is clearly not symmetric in its two indices or gauge invariant, and thus, if we tried to calculate \( \dot{E} \) by taking the covariant divergence with respect to the second index in the GW SET, we would find the wrong answer.

To fix these problems, we must apply the Belinfante procedure [81–83] (App. B). Applying this method to Einstein-Æther theory gives the GW SET

\[
\Theta^\alpha_{(T)} = \frac{1}{32\pi G} \left( \partial_a \phi_{ij} \partial_\beta \phi_{ij} + c_+ t_{(\alpha} \partial_\beta) \phi_{ij} \phi_{jk} \right),
\]

\[
\Theta^\alpha_{(V)} = \frac{1 - c_+}{16\pi G} \left( (2c_1 - c_+ c_-) \partial_\alpha v^j \partial_\beta v_j + \left( c_+^2 - 2c_4 (1 - c_+) \right) t_{(\alpha} \partial_\beta) v^j v_j \right),
\]

\[
\Theta^\alpha_{(S)} = \frac{1}{64\pi G c_{123} c_{14}} \left( (2c_+ + c_{14} (4 + 3c_2 - c_+)) - 2c_2 - 4 \right) \partial_\alpha F \partial_\beta F
\]

\[
+ \left( 2(c_+ - 2) + c_2 (3c_{14} c_+ - 2) + c_{14} (2 + c_+^2) \right) t_{(\alpha} \partial_\beta) \partial_\alpha \partial_\beta \partial_\alpha F
\]

\[
+ 4(1 + c_2)(1 - c_{14}) \nabla_{(\alpha} \partial_\beta) F \partial_\alpha F \right),
\]

Observe that this new GW SET is indeed symmetric, but it is not gauge invariant because of all the terms that are proportional to \( t_{(\alpha} \partial_\beta) \). As before, the symmetrized GW SET of Eq. (2.139) does not match that of any previous sections, but this time the differences actually do propagate into observable quantities.

Given an actual observation, however, there will be a unique measured value of, for example, the energy flux carried by GWs, so what went wrong? The answer is in the application of the Belinfante procedure [81,83]. This algorithm is derived assuming the
Lagrangian density is invariant under Lorentz transformations. Einstein-Æther is indeed diffeomorphism invariant, but the solutions of this theory do spontaneously break Lorentz-symmetry. Therefore, when the Einstein-Æther Lagrangian density is expanded about a Lorentz-violating background solution, it loses its diffeomorphism invariance, and in particular, it loses its Lorentz invariance, making the Belinfante procedure inapplicable. Indeed, we find that all of the differences in observables calculated with the above GW SET are proportional to the Lorentz-violating background Æther field $t^\alpha$. One could in principle generalize the Belinfante procedure to allow for the construction of SETs in Lorentz-violating theories, but this is beyond the scope of this paper.

**Derivation of Physical Quantities: $\dot{E}$, $\dot{P}$, and $\dot{L}$**

Now that we have acquired the GW SET, we can solve for the physical observables. The first one we look at is $\dot{E}$, which we use Eq. (2.47a) to solve for. The first three methods in the Einstein-Æther section give $\dot{E}$ as

$$
\dot{E}_\kappa = -\frac{R^2}{16\pi G} \int \left\{ \frac{1}{2} \partial_\tau \phi_{ij}^{TT} \partial_\tau \phi_{ij}^{TT} + \frac{(1-c_+)(2c_1-c_+c_-)}{v_\nu} \partial_\tau v_i \partial_\tau v^i \\
+ \frac{2-c_{14}}{4v_s c_{14}} \partial_\tau F \partial_\tau F \right\}.
$$

(2.140)

The results here agree with those previously found by Foster [1] using the Noether charge method [86,87], which is different from the Noether current method adopted in this paper. Similarly, we can solve for the loss rate of linear momentum. Using the GW SET from the first three Einstein-Æther sections we find,

$$
\dot{P}_i^\kappa = -\frac{R^2}{16\pi G} \int \left\{ \frac{1}{2} \partial_\tau \phi_{ij}^{TT} \partial_\tau \phi_{ij}^{TT} + \frac{(1-c_+)(2c_1-c_+c_-)}{v_\nu} \partial_\tau v_i \partial_\tau v^i \\
+ \frac{2-c_{14}}{4v_s c_{14}} \partial_\tau F \partial_\tau F \right\}.
$$

(2.141)
The physical quantities found in Eqs. (2.140) and (2.141) agree with GR in the proper limits. Modifications to GR, of course, do arise for generic values of the coupling constants. The first relevant effect is the existence of dipole and possible monopole radiation [51]. These radiation terms should dominate over the quadrupole term predicted in GR, thus leading to an accelerated rate of inspiral and chirping, as in the JFBD case of the previous section. Binary pulsar and GW observations consistent with GR could then be used to constrain Einstein-Æther theory, as first suggested in [95]. A second effect is in the speed of propagation of all radiation, which deviates from the speed of light. In particular, GWs in Einstein-Æther theory do not travel at the speed of light. The recent coincident electromagnetic and GW observation of the merger of a binary neutron star system has then allowed for the most stringent constraints on the speed of propagation of tensor modes in Einstein-Æther theory [96]. Note, however, that this only constrains $c_+$, without placing any constraints on the other 3 coupling constants of the theory.

**Conclusion**

In this paper we studied a wide array of methods to calculate the GW SET in theories with propagating scalar, vector and tensor fields. The methods include the variation of the action with respect to a generic background, the second-order perturbation of the field equations, the calculation of a pseudotensor from the symmetries of a tensor density, and the use of Noether’s theorem to derive a canonical GW SET. Generally, all methods yield the same results, but care should be taken when dealing with theories that break Lorentz symmetry. This is because the procedure that symmetrizes the canonical SET and makes it gauge-invariant (the so-called Belinfante procedure) fails in its standard form in theories that are not Lorentz invariant. In addition to all of this, we present here and for the first time the symmetric GW SET for Einstein-Æther theory, from which we calculated the rate of energy and linear momentum carried away by all propagating degrees of freedom; the rate
of energy loss had been calculated before \[1\] and it agrees with the results we obtained.

The work we presented here opens the door to several possible future studies. One crucial ingredient in binary pulsar and GW observations that has not been calculated here is the rate of angular momentum carried away by all propagating degrees of freedom. The best way to calculate this quantity is through the Landau-Lifshitz pseudo-tensor, but this requires expansions to one higher order in perturbation theory and the careful application of short-wavelength averaging. Once this is calculated, for example in Einstein-Æther theory, one could combine the result with the rate of energy loss computed in this paper to calculate the rate at which the orbital period and the eccentricity decay in compact binaries. These results could then be used to constrain Einstein-Æther theory with binary pulsar observations and GW observations. The latter would require the construction of models for the GWs emitted in eccentric inspirals of compact binaries, which in turn requires the energy and angular momentum loss rate.

Another possible avenue for future work is to calculate the GW SET in more complicated theories, such as TeVeS \[97\] and MOG \[98,99\]. Both of these theories modify Einstein’s through the inclusion of a scalar and vector field with non-trivial interactions with the tensor sector and the matter sector. The methods studied in this paper are well-suited for the calculation of the GW SET in these more complicated theories. Once that tensor has been calculated, one could then compute the rate of energy carried away by all propagating degrees of freedom in this theory, and from that, one could compute the rate of orbital period decay in binary systems. Binary pulsar observations and GW observations could then be used to stringently constrain these theories in a new independent way from previous constraints.
Acknowledgments

N.Y and A.S. acknowledge support through the NSF CAREER grant PHY-1250636 and NASA grants NNX16AB98G and 80NSSC17M0041. K.Y. acknowledges support from Simons Foundation and NSF grant PHY-1305682.
CHAPTER THREE

ANGULAR MOMENTUM LOSS FOR A BINARY SYSTEM IN EINSTEIN-ÆTHER THEORY

Contribution of Authors and Co-Authors

Manuscript in Chapter 3

Author: Alexander Saffer
Contributions: Thought of the problem, performed mathematical analysis of problem, compared results, and wrote the manuscript.

Co-Author: Nicolás Yunes
Contributions: Thought of the problem, discussed implications, and edited manuscripts.
Abstract

The recent gravitational wave observations provide insight into the extreme gravity regime of coalescing binaries, where gravity is strong, dynamical and non-linear. The interpretation of these observations relies on the comparison of the data to a gravitational wave model, which in turn depends on the orbital evolution of the binary, and in particular on its orbital energy and angular momentum decay. In this paper, we calculate the latter in the inspiral of a non-spinning compact binary system within Einstein-Æther theory. From the theory’s gravitational wave stress energy tensor and a balance law, we compute the angular momentum decay both as a function of the fields in the theory and as a function of the multipole moments of the binary. We then specialize to a Keplerian parameterization of the orbit to express the angular momentum decay as a function of the binary’s orbital elements. We conclude by combining this with the orbital energy decay to find expressions for the decay of the semi-major axis and the orbital eccentricity of the binary. We find that these rates of decay are typically faster in Einstein-Æther theory than in General Relativity due to the presence of dipole radiation. Such modifications will imprint onto the chirp rate of gravitational waves, leaving a signature of Einstein-Æther theory that if absent in the data could be used to stringently constrain it.

Introduction

Since the first detection of gravitational waves (GW) by the advanced Laser Interferometer Gravitational-Wave Observatory (aLIGO), gravitational wave astronomy has moved to the forefront of scientific study [20, 21, 23–25, 100]. These observations have confirmed the existence of gravitational radiation emanating from coalescing compact binaries. The GWs emitted depend sensitively on the orbital evolution of such binaries, and thus, their observation is a prime target for new tests in the extreme gravity regime, where
the gravitational interaction is simultaneously strong, dynamical and non-linear.

Einstein-Æther theory (Æ) [45] is an ideal model to test with GWs because it encompasses one of the most general ways to break one of General Relativity’s (GR) fundamental pillars: local Lorentz invariance. This symmetry is broken through a unit timelike vector field, called the æther field, which couples non-minimally to the metric tensor and induces the activation and propagation of scalar, vectorial, and tensorial metric perturbations in compact binary systems [95, 101, 102]. Recent gravitational wave observations have stringently constrained the speed of propagation of tensor modes [103], but they could not say anything about the other modes, which are only constrained by Solar System and binary pulsar observations much more weakly [2, 101, 102].

Detailed GW tests of Æ-theory require the construction of GW models in the theory, which in turn requires an understanding of coalescing binaries. During the inspiral, their trajectory can be modeled as a sequence of osculating Keplerian orbits, with adiabatically changing semi-major axis and orbital eccentricity, within the post-Newtonian (PN) approximation [104, 105]. These adiabatic changes are controlled by the rate at which the binary loses orbital energy and angular momentum. In Æ-theory, these rates are modified due to the activation of scalar and vectorial tensor perturbations that propagate away from the binary and carry energy and angular momentum away with them [1, 106]. This, in turn, modifies the chirping rate of GWs, leaving a signature in the GW phase that could be used to constrain the theory if it is found to be absent from the data.

We here study the energy and the angular momentum loss rate of an eccentric binary system in Æ-theory to, for the first time, derive the rate at which the semi-major axis and the orbital eccentricity decays due to GW emission. We first compute the GW, stress-energy pseudo-tensor (SET) by expanding the field equations to second order in perturbations about a flat background, a method developed by Isaacson [55, 56]. We then use this SET to compute the energy and the angular momentum flux carried by all propagating modes in
terms of derivatives of these fields [64]. With this at hand, we use a multipolar expansion of the fields to calculate these fluxes in terms of derivatives of multipole moments. Lastly, we use a Keplerian parametrization to present the energy and angular momentum flux as a function of the orbital parameters of the binary. In doing so, we verify the energy flux calculations of [2], extending them to include the angular momentum flux for the first time. This last result allows us to calculate the rate of decay of the semi-major axis and the orbital eccentricity of the binary.

We find that generically the semi-major axis and the eccentricity decay rate is faster in $\mathcal{AE}$-theory than in GR when dipole emission is present. Figure 3.1 shows these evolutions for different choices of the $c_{14}$ combination of coupling constants of the theory. All other coupling constants are chosen such that all propagating modes travel at the speed of light, thus satisfying stability and Cherenkov constraints [107], and leaving $c_{14}$ as the only free parameter of the theory that allows us to continuously approach the GR limit [108]. Observe that the evolution of the semi-major axis and the orbital eccentricity becomes faster than in GR ($c_{14} = 0$ case) as the combination of coupling constants $c_{14}$ is increased. Such an increased rate of decay will imprint onto the chirping rate of the emitted GWs, which could be constrained in the future with GW observations.

In the remainder of this paper, we use the standard $(-,+,+,+)$ metric signature as described in [90], as well as units in which $c = 1$.

**Einstein-Æther Theory**

We begin with a brief introduction to $\mathcal{AE}$-theory, which was developed by Jacobson and Mattingly in 2001 by coupling a Lorentz violating vector field to GR [45]. We here follow mostly the presentation [1], where the full action of the theory can be written as a
Figure 3.1: Temporal evolution of the semi-major axis (a) and the orbital eccentricity (e) of a 1.4\(M_\odot\)-2\(M_\odot\) neutron star binary due to the modified decay rates of the binary’s orbital energy and angular momentum. To construct this figure, we modeled the sensitivities with the weak-field expansion of [1, 2], and we used an initial orbital frequency of 10 Hz and an initial orbital eccentricity of 0.9. Observe that the semi-major axis and the orbital eccentricity decay faster than in GR.
sum of a gravitational term and a matter term

\[ S = S_\mathcal{A} + \sum_A S_A, \quad (3.1) \]

for \( A \) bodies in the system.

**Gravitational Action**

The gravitational action in \( \mathcal{A} \)-theory is

\[ S_\mathcal{A} = \frac{1}{16\pi G_\mathcal{A}} \int d^4x \sqrt{-g} \left( R - K^{\alpha\beta}_{\gamma\epsilon} \nabla_\alpha u^\gamma \nabla_\beta u^\epsilon + \lambda (u^\alpha u_\alpha + 1) \right), \quad (3.2) \]

where

\[ K^{\alpha\beta}_{\gamma\epsilon} = c_1 g^{\alpha\beta} g_{\gamma\epsilon} + c_2 \delta^\alpha_\gamma \delta^\beta_\epsilon + c_3 \delta^\alpha_\epsilon \delta^\beta_\gamma - c_4 u^\alpha u^\beta g_{\gamma\epsilon}, \quad (3.3) \]

\( R \) is the Ricci tensor associated with the metric \( g_{\alpha\beta} \), and \( u^\alpha \) is the unit timelike æther field constrained via a Lagrange multiplier \( \lambda \). The coupling constants \( c_i \) control the various ways in which the æther field couples to the metric. For future convenience, we introduce the following combinations of \( c_i \):

\[ c_\pm = c_1 \pm c_3, \quad (3.4a) \]

\[ c_{14} = c_1 + c_4, \quad (3.4b) \]

\[ c_{123} = c_1 + c_2 + c_3. \quad (3.4c) \]
The term $G_{AE}$ in Eq. (3.2) is the modified gravitational constant which is related to the Newtonian one through the relation [2,109]

$$G_N = \frac{2}{2 - c_{14}} G_{AE}.$$  \hfill (3.5)

**Matter Action**

For compact objects modeled as effective point-particles, the matter action of the $A$th particle is given by [106]

$$S_A = - \int d\tau_A \tilde{m}_A(\gamma_A)$$  \hfill (3.6a)

$$= -\tilde{m}_A \int d\tau_A \left[ 1 + \sigma_A (1 - \gamma_A) + \frac{1}{2} \sigma_A' (1 - \gamma_A)^2 + \cdots \right],$$  \hfill (3.6b)

where $d\tau_A$ is the proper time along the particle’s world-line, $\gamma_A = -u_\alpha v_\alpha^A$ with $v_\alpha$ the particle’s four-velocity, and $\tilde{m}_A$ is the bare mass, which is related to the particle’s active gravitational mass via

$$m_A = (1 + \sigma_A) \tilde{m}_A.$$  \hfill (3.7)

We define the $\sigma_A$ and $\sigma_A'$ charges in Eq. (3.6b) as

$$\sigma_A = -\frac{d \ln(\tilde{m}_A)}{d \ln(\gamma_A)}|_{\gamma_A=1},$$  \hfill (3.8a)

$$\sigma_A' = \sigma_A + \sigma_A^2 + \frac{d^2 \ln(\tilde{m}_A)}{d (\ln(\gamma_A))}|_{\gamma_A=1},$$  \hfill (3.8b)

and relate them to a rescaled sensitivity parameter by [2]

$$s_A = \frac{\sigma_A}{1 + \sigma_A}.$$  \hfill (3.9)
Field Equations

The variation of the full action in Eq. (3.1) with respect to the metric yields the AE modification to the Einstein equations [88]

\[ G_{\alpha\beta} - S_{\alpha\beta} = 8\pi T_{\alpha\beta}, \]  

(3.10)

where \( G_{\alpha\beta} \) is the Einstein tensor, and \( S_{\alpha\beta} \) is given by

\[ S_{\alpha\beta} = \nabla_\gamma \left( K_\beta^\gamma (\alpha u_\beta) + K_{(\alpha\beta)} u_\gamma - K_{(\alpha} u_{\beta)} \right) + c_1 \left( \nabla_\alpha u_\gamma \nabla_\beta u_\gamma - \nabla_\gamma u_\alpha \nabla_\gamma u_\beta \right) + c_4 \left( u_\gamma \nabla_\gamma u_\alpha \nabla_\delta u_\beta + \lambda u_\alpha u_\beta - \frac{1}{2} g_{\alpha\beta} \left( K^\gamma_\delta \nabla_\gamma u^\delta \right) \right). \]  

(3.11)

with the contracted \( K \)-tensor

\[ K_\alpha^\gamma = K_{\alpha\beta\gamma\delta} \nabla_\beta u^\delta. \]  

(3.12)

The matter stress-energy tensor \( T_{\alpha\beta} \) is defined via

\[ T_{\alpha\beta} = -\frac{2}{\sqrt{-g}} \frac{\partial S_m}{\partial g^{\alpha\beta}}, \]  

(3.13)

which can be calculated through the matter action in Eq. (3.6b):

\[ T^{\alpha\beta} = \sum_A \tilde{m}_A \tilde{\delta}_A \left[ \left( 1 + \sigma_A - \frac{1}{2} \sigma'_A \left( \gamma_A^2 - 1 \right) \right) v_\alpha_A v_\beta^A - 2 \left( \sigma_A + \sigma'_A \left( \gamma_A + 1 \right) \right) u^{(\alpha} A v_{\beta)} \right], \]  

(3.14)

where

\[ \tilde{\delta}_A = \frac{1}{\nu_A^3} \delta^3 (\vec{x} - \vec{x}_A), \]  

(3.15)

is short hand for a renormalized Dirac delta function.

The variation of the full action in Eq. (3.1) with respect to the æther field \( u^\alpha \) yields
the æther field equation

\[ \nabla_\alpha K^{\alpha \beta} = \lambda u^\beta + c_4 \left( u^\alpha \nabla_\alpha u_\gamma \right) \nabla^\beta u_\gamma + 8\pi G_\Lambda \Upsilon^\beta, \]  

(3.16)

where

\[ \Upsilon^\alpha = - \sum_A \bar{m}_A \delta_A \left[ \sigma_A + \sigma'_A (\gamma_A + 1) \right] v^\alpha_A. \]  

(3.17)

In addition to the modified Einstein equations and the æther field equation, AE-theory also possesses the timelike constraint

\[ g_{\alpha \beta} u^\alpha u^\beta = -1 \]  

(3.18)

which can be derived by varying the full action in Eq. (3.1) with respect to the Lagrange multiplier \( \lambda \). With this we can solve Eq. (3.16) for \( \lambda \) to find

\[ \lambda = 8\pi G_\Lambda \Upsilon_\alpha u^\alpha + c_4 \left( u^\beta \nabla_\beta u_\gamma \right) \left( \nabla^\alpha u_\gamma \right) u^\alpha - \nabla^\beta K^\beta_\alpha u^\alpha. \]  

(3.19)

Field Decomposition

We conclude this section by carrying out an irreducible decomposition of the perturbations of all the fields about a fixed background, as done for example in [1]. We begin by expanding the metric

\[ g_{\alpha \beta} = \eta_{\alpha \beta} + h_{\alpha \beta}, \]  

(3.20)

about Minkowski spacetime \( \eta_{\alpha \beta} \), with \( |h_{\alpha \beta}| \ll |\eta_{\alpha \beta}| \), and expanding the æther field

\[ u^\alpha = t^\alpha + \omega^\alpha, \]  

(3.21)

about \( t^\alpha = (1, 0, 0, 0) \) a timelike background unit vector, with \( |\omega^\alpha| \sim O (h) \).
Next, we perform two more decompositions. We begin with a $3+1$ decomposition of the metric perturbation about a spatial hypersurface normal to the timelike background vector

$$h^\alpha{}^\beta = t^\alpha{}t^\beta h_{00} + 2\mathcal{P}_i{}^\alpha{}t^\beta h^{0i} + \mathcal{P}_i{}^\alpha\mathcal{P}_j{}^\beta h^{ij}, \quad (3.22)$$

where $\mathcal{P}_i{}^\alpha$ is the background spatial projector

$$\mathcal{P}^\alpha{}^\beta = \eta^\alpha{}^\beta + t^\alpha{}t^\beta. \quad (3.23)$$

We then further decompose the $3+1$-decomposed metric tensor into transverse and longitudinal pieces

$$h^{0i} = \gamma^i + \partial^i \gamma, \quad (3.24a)$$
$$h^{ij} = \phi^{ij}_{\text{TT}} + \frac{1}{2} F_{ij} [f] + 2\partial_{(i}\phi_{j)} + \partial_i \partial_j \phi, \quad (3.24b)$$

where $\phi^{ij}_{\text{TT}}$ is a transverse-traceless spatial tensor, $F_{ij}[f] \equiv \eta_{ij} F - \partial_i \partial_j f$ with $F = \partial_k \partial^k f$, and $\gamma^i$ and $\phi^i$ are longitudinal so that $\gamma^i_{,j} = \phi^i_{,j} = 0$. We similarly decompose the æther field into transverse and longitudinal modes through

$$\omega^\alpha = t^\alpha \omega_0 + \mathcal{P}_i{}^\alpha \left( \nu^i + \partial^i \nu \right), \quad (3.25)$$

with $\nu^i$ longitudinal so that $\nu^i_{,j} = 0$.

We can simplify the above expressions somewhat as follows. First, we use the timelike constraint on the æther field in Eq. (3.18) to find

$$\omega_0 = -\frac{h_{00}}{2}, \quad (3.26)$$

to leading order in the metric perturbation. Next, we use the remaining gauge freedom to
set

\[ \partial_i \omega^i = 0, \quad (3.27a) \]
\[ \partial_i h^{0i} = 0, \quad (3.27b) \]
\[ \partial_i \partial_{[j} h_{k]}^i = 0, \quad (3.27c) \]

which grants us \( \nu = \gamma = \phi_i = 0 \) upon simplification. Notice this is not the harmonic gauge typically used in post-Newtonian theory. Through the evaluation of the field equations given in Eq. (3.10) and (3.16), we are able to find relations which exist between the various remaining fields. By looking at the tensor, vector, and scalar modes of the equations separately, we are able to obtain\(^1\)

\[ \gamma^i = -c_+ \nu^i, \quad (3.28a) \]
\[ h_{00} = \frac{1}{c_{14}} F, \quad (3.28b) \]
\[ \partial_i \partial^i \phi = -\frac{1 + c_2}{c_{123}} F. \quad (3.28c) \]

With these simplifications at hand, one finds that \( \mathcal{A} \)-theory only presents 3 non-vanishing perturbations: \( \phi_{ij}^{TT} \), \( \nu^i \), and \( F \). These modes propagate as waves about the background spacetime, satisfying modified wave equations in vacuum of the form

\[ \left( \frac{1}{v_T^2} \partial_0^2 - \partial^i \partial_i \right) \phi_{jk}^{TT} = 0, \quad (3.29a) \]
\[ \left( \frac{1}{v_V^2} \partial_0^2 - \partial^i \partial_i \right) \nu^k = 0, \quad (3.29b) \]
\[ \left( \frac{1}{v_S^2} \partial_0^2 - \partial^i \partial_i \right) F = 0, \quad (3.29c) \]

\(^1\)For details on the calculation of these relations see [1, 110].
where the various propagation speeds are given by

\begin{align}
  v_T^2 &= \frac{1}{1 - c_+}, \quad (3.30a) \\
  v_V^2 &= \frac{2 c_1 - c_+ c_-}{2 c_{14} (1 - c_+)} , \quad (3.30b) \\
  v_S^2 &= \frac{c_{123} (2 - c_{14})}{c_{14} (1 - c_+) (2 + 3 c_2 + c_+)} . \quad (3.30c)
\end{align}

The recent coincident electromagnetic and GW observation of a neutron star merger [103] has placed a severe constraint on $v_T$, effectively forcing $c_+ \lesssim 10^{-15}$, but this observation places no constraint on $v_V$ or on $v_S$.

**Angular Momentum Loss Rate**

In this section, we calculate the angular momentum loss rate in $\AE$-theory. We begin with a generic calculation that leads to a result in terms of derivatives of the decomposed fields. We then specialize these results to an N-body system and provide the angular momentum loss in terms of derivatives of the multipole moments of the system. In both cases, we also present previously-derived expressions for the energy loss for the sake of completeness.

**General Calculation**

The angular momentum loss can be calculated from [64, 65]

\[ \dot{L}^i \equiv \frac{1}{2} \epsilon^{ijk} \dot{L}^{jk}, \quad (3.31) \]

where $\epsilon^{ijk}$ is the Levi-Civita symbol and where we have defined

\[ \dot{L}^{ij} = -2 \int x^{[i} \Theta^{jk]} dS_k , \quad (3.32) \]
with \( \Theta^{ij} \) the spatial components of the GW SET. The latter can be calculated in \( \mathcal{AE} \)-theory using the techniques outlined in \([55,56,110]\) by expanding the field equations to second order in the metric perturbation. However, care must be taken because \( \Theta^{ijk} \) needs to be expanded to \( 1/r^3 \) prior to performing a Brill-Hartle average \([111]\). When one computes \( \dot{E} \), one need only expand \( \Theta^{ijk} \) to \( 1/r^2 \), since the integrand does not explicitly depend on \( x^i \) \([110]\).

The resulting expression can be simplified further through the shortwave approximation \([65,90]\). This assumes the observer is in a region far removed from the source of radiation, such that the observer’s local radius of curvature \( R_c \) is much larger than the characteristic wavelength of the gravitational radiation, \( \lambda_c \). This allows us to use \( \lambda_c/R_c \ll 1 \) as an expansion parameter and simplify partial spatial derivatives on any field \( \Phi \) via

\[
\partial_i \Phi = -\frac{N_i}{v_\Phi} \partial_\tau \Phi + \tilde{\partial}_i \Phi, \tag{3.33}
\]

where \( v_\Phi \) is the speed of the field in question and the special derivative \( \tilde{\partial} \) operates only on the spatial (typically radial) part of the field’s argument. It should be noted that when \( \tilde{\partial} \) acts on the field \( \Phi \), it generates terms of \( O \left( \lambda_c^2/R^2 \right) \), while \( \partial_\tau \) generates terms of \( O \left( \lambda_c/R \right) \). We will aim to keep our expressions to \( O \left( \lambda_c^3/R^3 \right) \) to ensure the results of Eq. (3.32) do not diverge as \( R \rightarrow \infty \).

When we combine this approximation with the properties of the decomposed fields, we can further simplify the resulting expressions. For instance, we have that

\[
N_i \phi_{TT}^{ij} = N_i \nu^i = 0, \tag{3.34a}
\]

\[
N_i \partial_\tau \phi_{TT}^{ij} = N_i \partial_\tau \nu^i = 0, \tag{3.34b}
\]

\[
N_i \tilde{\partial}_j \phi_{TT}^{ik} = -\frac{\phi_{TT}^{ik}}{r}, \tag{3.34c}
\]

\[
\tilde{\partial}_i N^i = \frac{(3 \Phi)_j}{r}, \tag{3.34d}
\]
where \((3)P_{ij} = \delta_{ij} - N_i N_j\) is a projector to a 2-surface orthogonal to the wave’s propagation vector \(N^i\), and \(r\) is the distance to the source. The hypersurface orthogonal to \(t^\alpha\) need not coincide with that orthogonal to \(N^i\) and thus \((3)P_{ij} \neq P_{ij}\). We are required to utilize the second term in Eq. (3.33) in the \(\dot{L}^{ij}\) calculation due to the presence of an extra radial term present in Eq. (3.32).

After using all of these simplifications, the end result for \(\dot{L}^{ij}\) is

\[
\dot{L}^{ij} = \dot{L}_{(T)}^{ij} + \dot{L}_{(V)}^{ij} + \dot{L}_{(S)}^{ij},
\]

with

\[
\dot{L}_{(T)}^{ij} = -\frac{1}{8\pi v_T G_{AE}} \int r^2 \left[ \phi_T^{ai} \left( \partial_T \phi_T^{aj} \right) - \frac{1}{2} \left( \partial_T \phi_T^{ab} \right) x^i \delta^{[ij]} \phi_T^{ab} \right] \, d\Omega,
\]

\[
\dot{L}_{(V)}^{ij} = -\frac{(1 - c_+)(2c_1 - c_+ - c_-)}{8\pi v_V G_{AE}} \int r^2 \left[ v^i \left( \partial_T v^{ij} \right) - \left( \partial_T v^a \right) x^i \delta^{[ij]} v_{a} \right] \, d\Omega,
\]

\[
\dot{L}_{(S)}^{ij} = -\frac{(2 - c_{14})}{32\pi v_S c_{14} G_{AE}} \int r^2 \left( \partial_T F \right) x^i \delta^{[ij]} F \, d\Omega.
\]

and the analogous result for the angular momentum loss is

\[
\dot{L}^i = \dot{L}_{(T)}^i + \dot{L}_{(V)}^i + \dot{L}_{(S)}^i,
\]

with

\[
\dot{L}_{(T)}^i = -\frac{1}{16\pi v_T G_{AE}} \int r^2 \epsilon_{jk} \left[ \phi_T^{ai} \left( \partial_T \phi_T^{kj} \right) - \frac{1}{2} x^j \left( \partial^k \phi_T^{ab} \right) \left( \partial_T \phi_T^{ab} \right) \right] \, d\Omega,
\]

\[
\dot{L}_{(V)}^i = -\frac{(1 - c_+)(2c_1 - c_+ - c_-)}{16\pi v_V G_{AE}} \int r^2 \epsilon_{jk} \left[ v^j \left( \partial_T v^k \right) - x^j \left( \partial^k v^a \right) \left( \partial_T v_a \right) \right] \, d\Omega,
\]

\[
\dot{L}_{(S)}^i = -\frac{(2 - c_{14})}{64\pi v_S c_{14} G_{AE}} \int r^2 \epsilon_{jk} \left[ x^j \left( \partial_T F \right) \left( \partial_T F \right) \right] \, d\Omega.
\]

Notice that the tensor term in Eq. (3.38) matches that given in GR except for the prefactor
in front of the integral. The largest difference between these expressions and those in GR is the presence of the vector and scalar modes, which contribute to the total loss of angular momentum of the system. Even if one sets \( c_+ = 0 \) to satisfy the tensor propagation bounds of [103], the vector and scalar contributions still remain.

For completeness, we show the rate of energy loss in terms of our fields\(^2\)

\[
\dot{E}_{(T)} = -\frac{1}{32\pi v_T G_{AE}} \int r^2 \left( \partial_\tau \phi_{ij}^{TT} \right) \left( \partial_\tau \phi_{ij}^{II} \right) d\Omega, 
\]
\[
\dot{E}_{(V)} = -\frac{(1 + c_+)(2c_1 - c_+ c_-)}{16\pi v_V G_{AE}} \int r^2 \left( \partial_\tau v_i \right) \left( \partial_\tau v^i \right) d\Omega, 
\]
\[
\dot{E}_{(S)} = -\frac{(2 - c_{14})}{64\pi v_S c_{14} G_{AE}} \int r^2 \left( \partial_\tau F \right) \left( \partial_\tau F \right) d\Omega. 
\]

It is a curious occurrence that all factors of our coupling constants are identical in the prefactors for both \( \dot{L}_i \) and \( \dot{E} \) for their respective modes. We will make further use of Eq. (3.39) below.

Specialization to an N-Body System

The solution to the \( \ae \) field equations for a PN binary system were first calculated in [1] and found to be

\[
\phi_{ij}^{TT} = \frac{2G_{AE}}{r} \tilde{Q}_{ij}^{TT} \left( t - \frac{r}{v_T} \right), 
\]
\[
\nu^i = -\frac{2G_{AE}}{(2c_1 - c_+ c_-) r} \left[ \frac{1}{v_V} \left( \frac{c_+}{1 - c_+} \tilde{Q}_j^i + \tilde{Q}_j^i + \nu^j \right) N^j - 2 \Sigma^i \right]^T, 
\]
\[
F = \frac{2c_{14} G_{AE}}{(2 - c_{14}) r} \left[ 3 (Z - 1) N^i N^j \tilde{Q}_{ij} + Z \tilde{I}^k_k + \frac{2}{c_{14} v_S} N^i \Sigma_i 
- \frac{1}{c_{14} v_S} \left( \tilde{Q}_{ij} + \frac{1}{3} \delta_{ij} \tilde{I}^k_k \right) N^i N^j \right], 
\]

\(^2\)See [1, 2, 110] for a calculation on how \( \dot{E} \) was obtained.
where over-head dots stand for partial time derivatives, the superscript T and TT stand for the transverse and the transverse-traceless part (relative to the direction of propagation) of a given object, and we have defined the short-hands

\[ Q_{ij} = I_{ij} - \frac{1}{3} \delta_{ij} I^k_k, \]  
\[ Q_{ij} = \tilde{I}_{ij} - \frac{1}{3} \delta_{ij} \tilde{I}^k_k. \]  

These shorthands depend on the multipole moments of a system of \( A \) bodies

\[ I^{ij} = \sum_A m_A x^i_A x^j_A, \]  
\[ \dot{I}^{ij} = \sum_A \sigma_A \dot{m}_A x^i_A x^j_A, \]  
\[ \Sigma^i = -\sum_A \sigma_A \tilde{m}_A v^i_A, \]  
\[ \mathcal{V}^{ij} = 2 \sum_A \sigma_A \tilde{m}_A \dot{v}^i_A x^j_A, \]

where \( x^i_A \) labels their spatial trajectories and \( v^i_A = \dot{x}^i_A \) their orbital velocities, and we have defined the constants [102]

\[ Z = \frac{2 (1 - c_+) (\alpha_1 - 2 \alpha_2)}{3 (2c_+ - c_14)}, \]  
\[ \alpha_1 = \frac{8 \left( c_1 c_4 + c_3^2 \right)}{2c_1 - c_+ c_-}, \]  
\[ \alpha_2 = \frac{\alpha_1}{2} - \frac{(c_1 + 2c_3 - c_4)(2c_2 + c_123 + c_14)}{c_123 (2 - c_14)}. \]

All expressions here are to be evaluated at the retarded time \( t - r/v \), where \( v \) is to be replaced with the propagation velocity of the corresponding mode.

With this at hand, we may utilize the Brill-Hartle averaging scheme [111] and write
the angular momentum loss rate as

$$
\dot{L}_T^i = -\frac{2G_{AE}}{5v_T} \epsilon^{ijk} \left( \ddot{Q}^{ja} \dot{Q}^{k} a \right).
\tag{3.52}
$$

$$
\dot{L}_V^i = -\frac{4 c_{14} c_+^2 G_{AE}}{5 (2 c_1 - c_+ c_-)^2 v_V} \epsilon^{ijk} \left( \ddot{Q}^{ja} \dot{Q}^{k} a \right) - \frac{2 (1 - c_+) G_{AE}}{5 (2 c_1 - c_+ c_-) v_V^3} \epsilon^{ijk} \left( \ddot{Q}^{ja} \dot{Q}^{k} a \right)
- \frac{2 c_+ G_{AE}}{5 (2 c_1 - c_+ c_-) v_V^3} \epsilon^{ijk} \left( \ddot{Q}^{ja} \dot{Q}^{k} a + \dddot{Q}^{ja} \dot{Q}^{k} a \right) - \frac{2 (1 - c_+) G_{AE}}{3 (2 c_1 - c_+ c_-) v_V^3} \epsilon^{ijk} \left( \dot{V}^{ja} \dot{V}^{k} a \right)
- \frac{8 (1 - c_+) G_{AE}}{3 (2 c_1 - c_+ c_-) v_V} \epsilon^{ijk} \left( \Sigma^{j} \Sigma^{k} \right),
\tag{3.53}
$$

$$
\dot{L}_S^i = -\frac{3 c_{14} G_{AE}}{5 (2 c_{14}) v_S} (Z - 1)^2 \epsilon^{ijk} \left( \ddot{Q}^{ja} \dot{Q}^{k} a \right) - \frac{15 (2 - c_{14})}{15 (2 - c_{14}) c_{14} v_S^3} \epsilon^{ijk} \left( \ddot{Q}^{ja} \dot{Q}^{k} a \right)
+ \frac{2 G_{AE}}{5 (2 - c_{14}) v_S^3} (Z - 1) \epsilon^{ijk} \left( \ddot{Q}^{ja} \dot{Q}^{k} a + \dddot{Q}^{ja} \dot{Q}^{k} a \right) - \frac{2 G_{AE}}{3 (2 - c_{14}) c_{14} v_S^3} \epsilon^{ijk} \left( \Sigma^{j} \Sigma^{k} \right).
\tag{3.54}
$$

The above expressions can be combined and simplified to write the total angular momentum loss as follows

$$
\dot{L} = -G_{AE} \epsilon^{ijk} \left( \frac{2 \mathcal{A}_1}{5} \dddot{Q}^{ja} \dot{Q}^{k} a + \frac{\mathcal{A}_2}{5} \left( \dddot{Q}^{ja} \dot{Q}^{k} a + \dddot{Q}^{ja} \dot{Q}^{k} a \right)
+ \frac{2 \mathcal{A}_3}{5} \dddot{Q}^{ja} \dot{Q}^{k} a + C \Sigma^{j} \Sigma^{k} + 2 \mathcal{D} \dot{V}^{ja} \dot{V}^{k} a \right),
\tag{3.55}
$$

where we have introduced the new constants

$$
\mathcal{A}_1 \equiv \frac{1}{v_T} + \frac{2 c_{14} c_+^2}{(2 c_1 - c_+ c_-) v_V} + \frac{3 c_{14} (Z - 1)^2}{2 (2 - c_{14}) v_S},
\tag{3.56a}
$$

$$
\mathcal{A}_2 \equiv \frac{2 c_+}{(2 c_1 - c_+ c_-) v_V^3} - \frac{2 (Z - 1)}{(2 - c_{14}) v_S^3},
\tag{3.56b}
$$

$$
\mathcal{A}_3 \equiv \frac{1}{2 c_{14} v_V^5} + \frac{3 c_{14} (Z - 1)}{2 (2 - c_{14}) v_S^3},
\tag{3.56c}
$$

$$
C \equiv \frac{4}{3 c_{14} (2 - c_{14}) v_S^3} + \frac{4}{3 c_{14} v_V^3},
\tag{3.56d}
$$

$$
\mathcal{D} \equiv \frac{1}{6 c_{14} v_V^5}.
\tag{3.56e}
$$
The first term in the $A_1$ quantity represents the GR term (with the modified propagation velocity). All other terms lead to deviations away from GR due to contributions of the æther field. As with the discussion above, in the limit $c_+ \to 0$ the contributions of the additional modes of propagation persist.

For completeness, we also present here the energy loss for an N-Body system, computed e.g. in [2]

$$\dot{E} = -G_\mathcal{A} \left\{ \frac{A_1}{5} \ddot{Q}^i \ddot{Q}_{ij} + \frac{A_2}{5} \dot{Q}^i \dddot{Q}_{ij} + \frac{A_3}{5} \dot{Q}^i \dddot{Q}_{ij} ight. $$
$$\left. + B_1 \dot{I}^i \dot{I}^i + B_2 \dot{L}^i \dot{I}^i + B_3 \dot{L}^i \dot{L}^i + C \dot{\Sigma}^i \dot{\Sigma}_i + D \dot{\mathcal{V}}^{ij} \dot{\mathcal{V}}_{ij} \right\}, \quad (3.57)$$

where

$$B_1 \equiv \frac{c_{14} Z^2}{4 (2 - c_{14}) v_s}, \quad (3.58a)$$
$$B_2 \equiv \frac{Z}{3 (c_{14} - 2) v_s^3}, \quad (3.58b)$$
$$B_3 \equiv \frac{1}{9 c_{14} (2 - c_{14}) v_s^5}. \quad (3.58c)$$

This result is important for use in the following section. As with the $\dot{L}^i$ term, the deviations away from the GR result for $\dot{E}$ are due to the presence of the alternative modes whose velocities are unbounded.

**Binary Dynamics in $\mathcal{A}$-theory**

We now specialize the discussion further by considering a non-spinning binary system with active gravitational masses $m_1$ and $m_2$ in an elliptical orbit of separation $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$, as shown schematically in Fig. 3.2. Observe that the orbital plane is chosen here to coincide with the x-y plane, so that the orbital angular momentum is along the z-axis.
Neglecting radiation-reaction, such an orbit can still be described through Keplerian ellipses in $\mathcal{A}$-theory [2,106].

\[ r_{12} = \frac{a(1 - e^2)}{1 + e \cos(\phi)}, \]  
\hspace{1cm} (3.59)

where the azimuthal angle evolves according to

\[ \dot{\phi} = \frac{1}{r^{2}_{12}} \sqrt{G M a (1 - e^2)}, \]  
\hspace{1cm} (3.60)

and where $a$ is the semi-major axis, $e$ is the orbital eccentricity, $M = m_1 + m_2$ is the total mass and we have defined

\[ G \equiv \frac{G_N}{(1 + \sigma_1)(1 + \sigma_2)}. \]  
\hspace{1cm} (3.61)

Without radiation-reaction, the semi-major axis $a$ and the orbital eccentricity $e$ are constants, while $\phi$ is a function of time. The binary’s energy and angular momentum are then also
constant, and related to the semi-major axis and the eccentricity via

\[ E = -\frac{G m_1 m_2}{2a}, \]  
\[ L^2 = \frac{G m_1^2 m_2}{M} a(1 - e^2). \]

Once we include radiation-reaction, the energy and angular momentum are no longer constant, as we found in the previous section. Instead, for a binary system, Eqs. (3.57) and (3.55) are given by

\[ \dot{E} = -C G_{\mathcal{AE}} G^2 \left( 2 - e^2 - e^4 \right) S^2 M^4 \eta^2 \frac{a^4}{2 (1 - e^2)^{7/2}} - \frac{G_{\mathcal{AE}} G^3 M^5 \eta^2}{30 (1 - e^2)^{7/2} a^5} \times \left[ 2 \left( 96 + 292 e^2 + 37 e^4 \right) \left( \mathcal{A}_1 + S \mathcal{A}_2 + S^2 \mathcal{A}_3 \right) + 15 e^2 \left( 4 + e^2 \right) \left( \mathcal{B}_1 + S \mathcal{B}_2 + S^2 \mathcal{B}_3 \right) \right], \]

and

\[ \dot{L}^i = -C G_{\mathcal{AE}} G^{3/2} S^2 M^{7/2} \eta^2 \frac{a^{5/2}}{(1 - e^2)} - \frac{4 \left( 8 + 7 e^2 \right) G_{\mathcal{AE}} G^{5/2} M^{9/2} \eta^2}{5 (1 - e^2)^2 a^{7/2}} \left( \mathcal{A}_1 + S \mathcal{A}_2 + S^2 \mathcal{A}_3 \right) \]

where we introduced the symmetric mass ratio \( \eta = m_1 m_2/M^2 \), the mass-weighted symmetric sensitivity \( S = m_1 s_2/M + m_2 s_1/M \), and the anti-symmetric sensitivity \( S_- = s_1 - s_2 \).

The above expressions shown in Eqs. (3.63) and (3.64) do not contain angle-brackets because we have explicitly carried out the orbit averaging. By the latter, we mean specifically

\[ \langle X(t) \rangle = \frac{1}{P} \int_0^P X(t) \, dt, \]

for any quantity \( X(t) \) with \( P \) the orbital period. This integral can of course be rewritten in
terms of the orbital angle ($\varphi$ in Fig. 3.2), as

$$
\langle X(\varphi) \rangle = \frac{(1 - e^2)^{3/2}}{2\pi} \int_0^{2\pi} \frac{X(\varphi)}{(1 + e \cos(\varphi))^2} d\varphi.
$$

(3.66)

where we have used

$$
\frac{d\varphi}{dt} = \sqrt{\frac{G M S}{a^3}} \left(1 - e^2\right)^{-3/2} (1 + e \cos(\varphi))^2.
$$

(3.67)

We can now combine the above equations with the time derivative of Eq. (3.62) to find how the semi-major axis and the orbital eccentricity evolve. We find that

$$
\dot{a} = -\frac{C G M S^2 M^2 \eta}{a^2} h_1(e)
$$

$$
- \frac{G G^2 M^3 \eta}{15 a^3} \left[2 \left(\mathcal{A}_1 + S\mathcal{A}_2 + S^2\mathcal{A}_3\right) f_1(e) + 15 \left(\mathcal{B}_1 + S\mathcal{B}_2 + S^2\mathcal{B}_3\right) g_1(e)\right],
$$

(3.68)

$$
\dot{e} = -\frac{C G M S^2 M^2 \eta}{a^3} h_2(e)
$$

$$
- \frac{G G^2 M^3 \eta}{30 a^4} \left[2 \left(\mathcal{A}_1 + S\mathcal{A}_2 + S^2\mathcal{A}_3\right) f_2(e) + 15 \left(\mathcal{B}_1 + S\mathcal{B}_2 + S^2\mathcal{B}_3\right) g_2(e)\right].
$$

(3.69)

The above expressions depend on certain enhancement factors that are only functions of eccentricity, namely

$$
f_1(e) \equiv \frac{96 + 292e^2 + 37e^4}{(1 - e^2)^{7/2}},
$$

(3.70a)

$$
g_1(e) \equiv \frac{e^2(4 + e^2)}{(1 - e^2)^{7/2}},
$$

(3.70b)

$$
h_1(e) \equiv \frac{2 + e^2}{(1 - e^2)^{5/2}},
$$

(3.70c)

$$
f_2(e) \equiv \frac{e \left(304 + 121e^2\right)}{(1 - e^2)^{5/2}}.
$$

(3.70d)
\[ g_2(e) \equiv \frac{e(4 + e^2)}{(1 - e^2)^{5/2}}, \quad (3.70e) \]

\[ h_2(e) \equiv \frac{3e}{2(1 - e^2)^{3/2}}. \quad (3.70f) \]

The evolution of the semi-major axis and the orbital eccentricity is clearly faster than in GR due to the dipole term proportional to \( C \). As was shown earlier, \( C \) is produced by terms that come from the vector and scalar modes of propagation. Therefore, it is these additional propagation modes that are responsible for the contribution to the dipole term in the decay rates, and the latter is thus not affected by bounds on the propagation speed of the tensor mode.

Figure 3.1 shows the temporal evolution of the semi-major axis and the orbital eccentricity for a neutron star binary in the inspiral. This evolution was obtained by numerically solving Eqs. (3.68) and (3.69) for a binary with masses \( m_1 = 1.365M_\odot \) and \( m_2 = 2.040M_\odot \). The sensitivities are modeled through a weak-field expansion of [1] with stellar radii \( R_1 = 12.214\text{km} \) and \( R_2 = 11.966\text{km} \) and are given by

\[ s_A = \left(\frac{\alpha_1 - \frac{2}{3}\alpha_2}{\alpha_2}\right) \frac{m_A}{R_A}. \quad (3.71) \]

This approximation suffices to roughly model the temporal evolution of the semi-major axis and the orbital eccentricity. A proper GW model in Æ-theory, however, ought to employ the strong-field representation of the sensitivities found in [2].

**Conclusion and Future**

We have here calculated the rate of change of the orbital angular momentum due to GW emission in Æ-theory, accounting for all modes of propagation. This calculation, together with the energy flux of [1,2], allowed us to compute the rate of change of a binary’s
semi-major axis and its orbital eccentricity. We found that these quantities decay much more rapidly than in GR due to the emission of dipole radiation by the vector and scalar modes of the theory. These modifications persist even when one forces the tensor modes to propagate at the speed of light, thus by-passing the recent GW bounds of [103].

One could use these results in a variety of ways. One possibility is to construct a leading order GR deviation model for the GWs emitted in the inspiral of eccentric compact binaries. This model could be based on the recent work of [112], which built a model in GR that is valid for binaries with arbitrary eccentricity. Alternatively, one could work in a small-eccentricity expansion, as suggested in [113], or in a large-eccentricity expansion, as suggested in [114,115]. Such a model could then be used to inspiral a model-independent parameterization of modified GW models, as done for quasi-circular inspirals through the parameterized post-Einsteinian framework [116]. With this at hand, one could use future GW observations by advanced LIGO, or by third generation GW detectors or space-based detectors [62], to place interesting constraints on $\mathcal{A}$-theory.

Another possibility for future work is to revisit the strong-field sensitivity calculations of [2] and repeat the computation of binary pulsar constraints in light of the recent tensor propagation speed constraints of [103]. The results on the $\mathcal{A}$ constraints found in [2] are not valid any longer because that work made certain assumptions regarding the magnitude of the coupling constants, which is violated by the constraints in [103]. Once this work is repeated, however, one could redo the computation of binary pulsar constraints using not only binaries with very small eccentricity, but also much more eccentric ones. Such an extension would require the use of the angular momentum flux calculated in this paper.

Acknowledgments

N.Y and A.S. acknowledge support through the NSF CAREER grant PHY-1250636 and NASA grants NNX16AB98G and 80NSSC17M0041.
CHAPTER FOUR

THE EXTERIOR SPACETIME OF RELATIVISTIC STARS IN
SCALAR-GAUSS-BONNET GRAVITY

Contribution of Authors and Co-Authors

Manuscript in Chapter 4

Author: Alexander Saffer
Contributions: Thought of the problem, performed mathematical analysis of problem, compared results, and wrote the manuscript.

Author: Hector O. Silva
Contributions: Thought of the problem, aided in calculations, discussed implications, and edited manuscripts.

Co-Author: Nicolás Yunes
Contributions: Thought of the problem, discussed implications, and edited manuscripts.
Alexander Saffer, Hector O. Silva, and Nicolás Yunes

Submitted to Physical Review D

Status of Manuscript:

_____ Prepared for submission to a peer-reviewed journal

X  Officially submitted to a peer-reviewed journal

_____ Accepted by a peer-reviewed journal

_____ Published in a peer-reviewed journal

Submitted to: Phys. Rev. D

Date of Submission: 19 March 2019

Abstract

The spacetime around compact objects is an excellent place to study gravity in the strong, non-linear, dynamical regime where solar system tests cannot account for the effects of large curvature. Understanding the dynamics of this spacetime is important for testing theories of gravity and probing a regime which has not yet been studied with observations. In this paper, we construct an analytical solution for the exterior spacetime of a neutron star in scalar-Gauss-Bonnet gravity that is independent of the equation of state chosen. The aim is to provide a metric that can be used to probe the strong field regime near a neutron star and create predictions that can be compared with future observations to place possible constraints on the theory. In addition to constructing the metric, we examine a number of physical systems in order to see what deviations exist between our spacetime and that of general relativity. We find these deviations to be small and of higher post-Newtonian order than previous results using black hole solutions. The metric derived here can be used to further the study of scalar-Gauss-Bonnet gravity in the strong field, and allow for constraints on corrections to general relativity with future observations.

Introduction

Einstein’s general relativity (GR) has proved to be an exceptional theory to describe gravitational phenomena in Nature. From its early success in explaining the hitherto mysterious advance of the perihelion of Mercury’s orbit around the Sun [35] to its consistency with the gravitational wave observations of merging black hole (BH) and neutron star (NS) binaries by the LIGO/Virgo collaboration [117], GR has passed – with flying colors – all experiments it has been confronted with.

Given the continual success of the theory, it is natural to ask: should we consider GR as the final theory of the gravitational interaction? Is it worth the effort to keep developing
further tests, seeking glimpses of a more complete theory? Regarding the first question, field-theoretic considerations have shown that GR is non-renormalizable, placing a major obstacle to its quantization, and indicating that the theory must be modified in the ultraviolet regime. Indeed, a generic prediction of the low-energy limits of quantum gravity theories, such as string theory and loop quantum gravity, is that GR ought to be augmented by both additional fields and higher-order curvature scalars. Regarding the second question, GR’s firm place in our vault of fundamental physical theories implies that experimental evidence for a deviation would shake the foundations of this vault, possibly igniting a scientific revolution.

Where should we search for signatures of beyond GR phenomenology? Compact objects, NSs and BHs, provide a strong-field arena on which to put GR to the test in a regime beyond the weak-fields and low-velocities of our Solar System. The prototypical example are radio observations of binary pulsars, which through the detailed and careful monitoring of received pulses can reconstruct the orbital motion of relativistic binaries to stupendous precision [118–120]. Another example of tests of GR with compact objects is through the observation of electromagnetic radiation emitted by the accretion disks that surround black holes, although these tests are more challenging because of the complex astrophysics in play during such observations [79,121]. A final and more recent example is through the observation of the x-ray pulse profile emitted by hot spots on the surface of rapidly rotating stars [122–126].

All of the tests mentioned probe the exterior spacetime of compact objects in one way or another. Therefore, the construction of spacetimes close to these compact objects are required to place constraints on theories that go beyond GR. While there are many modified theories that attempt to explain anomalies between observations and the theoretical predictions of GR [127,128], a particularly interesting one is Einstein-dilaton-Gauss-Bonnet (EdGB) gravity. This theory is interesting because it emerges in the low energy limit of
heterotic string theory [129], and it agrees well with GR in the weak field region [130]. EdGB gravity modifies the Einstein-Hilbert action through the coupling of the Gauss-Bonnet invariant and a dynamical (dilaton) scalar field [131]. BH solutions in this theory have already been developed [53, 132–134], but until now, NS solutions had only been obtained numerically [135–137].

In this paper, we present the first analytical solution of the field equations in the small-dilaton expansion of EdGB gravity (i.e. of scalar-Gauss-Bonnet (sGB) gravity) that represents the exterior spacetime of non-rotating NSs, working in the small coupling approximation. Remarkably, these solutions depend only on the mass of the NS and the strength of the sGB coupling parameter, without any dependence on the dilaton scalar charge or any additional constants of integration. We study the properties of this spacetime by considering (timelike) geodesics, and derive sGB corrections to the innermost stable circular orbit (ISCO), to the (circular) orbital frequency and to the epicyclical radial frequencies of perturbed circular orbits. We also consider null geodesics and derive sGB corrections to the visible fraction of a NS hot spot as observed from spatial infinity.

A representative result is shown in Fig. 4.1, where we present the sGB-corrected mass-radius relation of NS for different values of the sGB coupling parameter and different equations of state (EoSs). Each point in this plane represents a NS solution of a given total gravitational mass and a given total radius, fixed through a numerical integration of a given central density that requires the metric be asymptotically flat at spatial infinity and $C^1$ everywhere. Observe that the largest deviations arise in the high compactness regime of the mass-radius relation, where the central densities are highest. This makes sense given that sGB gravity introduces higher curvature corrections to GR, which are bound to be largest when the compactness is as large as possible.

In the remainder of this paper, we use the $(-,+,+,+)$ metric signature and the conventions of [90], as well as units in which $c = 1 = G$. 
Scalar Gauss-Bonnet gravity

In this section, we present the action of sGB gravity and its field equations. We then introduce the perturbative scheme we will employ to analytically solve the field equations, and we conclude by presenting the perturbatively expanded field equations.

Action

We start by considering the action of a class of theories that contain modifications proportional to the Gauss-Bonnet invariant, whose taxonomy was described in [138]:

\[ S = S_{EH} + S_{\psi} + S_{GB} + S_{m} \]
where $S_{EH}$ is the Einstein-Hilbert action given by

$$S_{EH} \equiv \kappa \int d^4 x \sqrt{-g} \, R,$$

(4.2)

with $\kappa \equiv (16\pi)^{-1}$, $g$ is the determinant of the metric $g_{ab}$, $R \equiv g^{ab} R_{ab} = g^{ab} R_{acb}^\ c$ is the Ricci scalar (with $R_{ab}$ and $R_{abcd}$ being the Ricci and Riemann tensors respectively) and

$$S_\varphi \equiv -\frac{1}{2} \int d^4 x \sqrt{-g} \left[ \nabla_a \varphi \nabla^a \varphi + 2 U (\varphi) \right].$$

(4.3)

is the action of a canonical scalar field $\varphi$ with potential $U$. The coupling between the scalar field and the Gauss-Bonnet density

$$\mathcal{G} \equiv R^2 - 4 R_{ab} R^{ab} + R_{abcd} R^{abcd}$$

(4.4)

is given by

$$S_{GB} \equiv \int d^4 x \sqrt{-g} \, \alpha \, f(\varphi) \, \mathcal{G},$$

(4.5)

where $f(\varphi)$ specifies the functional form of the coupling and $\alpha$ (with dimensions of $[\text{length}]^2$) its strength. Finally, $S_m$ is the action of matter fields minimally coupled to the metric.

The choice of $f(\varphi)$ defines the particular member in the class of Gauss-Bonnet theories. For example, EdGB gravity is defined via $f(\phi) = e^{\phi}$, with typically a massless dilaton so $U(\varphi) = 0$. Other coupling function $f(\phi)$ were also introduced in [139, 140] and in the context of spontaneous black hole scalarization in [137, 141–143]. Hereafter, we expand $f(\varphi)$ in a Taylor series $f(\varphi) = f(0) + f_\varphi(0) \varphi + \mathcal{O} (\varphi^2)$ and work in the so-called decoupling limit of the theory [138]. Since $\mathcal{G}$ is a topological density, the first term in the series yields a boundary term to the action which does not contribute to the equations of
motion. In the second term, $f_\varphi(0)$ can be absorbed into the definition of $\alpha$ and we obtain:

$$S_{GB} \equiv \int d^4x \sqrt{-g} \, \alpha \, \varphi \, G.$$  \hspace{1cm} (4.6)

The action in Eq. (4.1) with $S_{GB}$ given by Eq. (4.6), is sometimes called decoupled dynamical Gauss-Bonnet gravity (D$^2$GB) [138] or more simply sGB gravity in this paper. This theory is invariant under constant shifts $\varphi \to \varphi + c$ when $U = 0$ [144–146], and thus, it belongs to shift-symmetric Horndeski gravity [147]. In this paper, we will restrict attention to sGB gravity with $U(\varphi) = 0$.

We here work in the small-coupling approximation in which sGB modifications are small relative to GR predictions. This approximation can be enforced by requiring that $\alpha/\ell^2 \ll 1$, where $\ell$ is the characteristic length of our system. For isolated NSs, the characteristic length scale is $\ell = \sqrt{R^3/M} = R/\sqrt{C}$, where $R$ is the radius of the star, $M$ its mass and $C$ its compactness. This length scale suggests the introduction of the dimensionless coupling parameter

$$\bar{\alpha} = \frac{\alpha}{\kappa M^2},$$ \hspace{1cm} (4.7)

in terms of which the small coupling approximation reduces to $\bar{\alpha} \ll \kappa^{-1}(R/M_\odot)^2C^{-1}$. For NSs with $R \sim 11$ km and $C \sim 0.2$, the small coupling approximation then requires that $\alpha \ll 600$ km$^2$ or equivalently $\bar{\alpha} \ll 1.5 \times 10^4$. The small-coupling approximation is well-justified because of current constraints on $\alpha$. Observations of the orbital decay of low-mass x-ray binaries [148] require that $\alpha < 9$ km$^2$, or $\bar{\alpha} < 220$. When making plots, we will here work with $\bar{\alpha} \in (0, 30)$, which satisfies both the small coupling approximation and current constraints from low-mass x-ray binary observations.

\footnote{Note that this dimensionless coupling parameter is different from that chosen in other work [148], since here we normalize $\alpha$ by $M_\odot$ instead of $M$.}
Field equations

We can obtain the field equations of the theory by varying the action $S$ with respect to the metric and the scalar field, with the result

$$G_{ab} = -\frac{\alpha}{\kappa}K_{ab} + \frac{1}{2\kappa} \left( T_{ab}^{m} + T_{ab}^{\varphi} \right), \quad (4.8a)$$

$$\Box \varphi - U \varphi = -\alpha G, \quad (4.8b)$$

where $G_{ab}$ is the Einstein tensor,

$$K_{ab} = -2R\nabla_a \nabla_b \varphi + 2 \left( g_{ab} R - 2R_{ab} \right) \Box \varphi + 8R_{(a(c} \nabla^c \nabla_{b))} \varphi$$

$$- 4g_{ab} R^{c}d \nabla_c \nabla_d \varphi + 4R_{acbd} \nabla^c \nabla^d \varphi, \quad (4.9)$$

while the stress-energy for the scalar field is

$$T_{ab}^{\varphi} = \nabla_a \varphi \nabla_b \varphi - \frac{1}{2} g_{ab} \left[ \nabla_c \varphi \nabla^c \varphi - 2U(\varphi) \right], \quad (4.10)$$

As we stated before, we will here choose the scalar field to be massless and not self-interacting, meaning that we can set $U = 0 = U_{\varphi}$ in the field equations.

Since we are interested in obtaining NS solutions in this theory, we assume that matter is described by a perfect fluid, whose stress-energy tensor is

$$T_{m}^{ab} = (\varepsilon + p) u^a u^b + p g^{ab}, \quad (4.11)$$

where $u^a$ is the four-velocity of the fluid (with pressure $p$ and total energy density ($\varepsilon$) subject to the constraint $u^a u_a = -1$. Due to the diffeomorphism invariance of the theory, $T_{m}^{ab}$ satisfies the conservation law

$$\nabla_a T_{m}^{ab} = 0, \quad (4.12)$$
as can be verified directly by taking the divergence of the field equations and using the
equations of motion for the scalar field.

The EoS of cold nuclear matter characteristic of old NSs can be well approximated
by a barotropic EoS, that is \( p = p(\epsilon) \). The large uncertainties on the properties of matter
in NS interiors result in a wide variety of competing EoS models [149]. Here, to remain
agnostic on which EoS correctly describes NS interiors we consider eight different EoSs,
which cover a wide range of underlying nuclear physics models. In increasing order of
stiffness we use: FPS [150], SLy [151], WFF1 [152], WFF2 [152], AP4 [150], ENG [153],
AP3 [150], and MPA1 [154].

Perturbative expansion for the metric and fluid variables

Having obtained the field equations, we now present the perturbative scheme that we
will use throughout this work. This approach was first introduced in [131] and was used in
a number of studies involving BHs [53, 145, 155, 156]. Here, we apply this scheme for the
first time to relativistic stars.

Let us consider a static, spherically symmetric star with spacetime described by the
line element
\[
\mathrm{d}s^2 = -e^{2\tau}\mathrm{d}t^2 + e^{2\sigma}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2,
\]
(4.13)
where the metric functions \( \tau \) and \( \sigma \) contain only radial dependence, and \( \mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 \)
is the line element of the unit two-sphere. The first step in the small-coupling approximation
is to expand all variables \( \tilde{z} \in \{\tau, \sigma, \varphi, \epsilon, p\} \) in a power series in \( \tilde{\alpha} \) as follows
\[
\tilde{z}(r) = \sum_{n=0}^{N} \tilde{z}_n(r),
\]
(4.14)
where the subscript \( n \) determines the power of \( \tilde{\alpha} \) associated with \( \tilde{z}_n \), i.e. \( \tilde{z}_n = O(\tilde{\alpha}^n) \).

With these expansions, we can immediately make a few observations. First, at \( O(\tilde{\alpha}^0) \),
the scalar field is everywhere constant, because its source is zero [cf. Eq. (4.8b)]. We can then exploit shift-symmetry to impose \( \varphi_0 = 0 \). Second, at \( O(\bar{\alpha}^1) \), we have \( \tau_1 = \sigma_1 = 0 \). This follows from the fact that \( \varphi_0 = 0 \) and by Eq. (4.8a), the Einstein equations are identical to those of GR at this order. Furthermore, since the metric is unaffected to this order and there is no direct coupling between \( \varphi \) and matter, we also have that \( \varepsilon_1 = p_1 = 0 \).

In this paper, we will obtain solutions for all variables \( \vec{z} \) up to \( O(\bar{\alpha}^2) \). From the proceeding discussion, we can outline the steps of the calculation ahead as follows:

1. at \( O(\bar{\alpha}^0) \), the problem is identical to GR and we have to calculate \( \{p_0, \varepsilon_0, \tau_0, \sigma_0\} \);

2. at \( O(\bar{\alpha}^1) \), we have to determine \( \varphi_1 \) on the background of a GR star obtained in the previous step;

3. at \( O(\bar{\alpha}^2) \), we must take into account the backreaction of the scalar field \( \varphi_1 \) onto the star to calculate \( \vec{z}_2 \in \{p_2, \varepsilon_2, \tau_2, \sigma_2\} \). The first two quantities tell us how the fluid is redistributed, while the latter how the spacetime is modified relative to the background GR metric.

The perturbative scheme outlined above could be carried out to higher orders. For instance, at \( O(\alpha^3) \) we would need to calculate \( \varphi_3 \) using the solutions for \( \vec{z}_2 \). Then, at \( O(\alpha^4) \), \( \varphi_3 \) would be used to obtain \( \vec{z}_4 \). We here stop our calculations at \( O(\bar{\alpha}^2) \) because this is the lowest order at which the metric is modified, and therefore, the lowest-order we must have at hand if we want to calculate sGB corrections to astrophysical observables.

---

**Perturbative expansion of the field equations**

At \( O(\bar{\alpha}^0) \), the GR limit of the field equations give

\[
G^0_{ab} = \frac{1}{2\kappa} \left[ (\varepsilon_0 + p_0) u_a u_b + p_0 g_{ab} \right].
\] (4.15)
As usual [157], it is convenient to introduce a mass function \( m_0 = (r/2)[1 - \exp(-2\sigma_0)] \), and then from the \((t, t)\) and \((r, r)\)-components of Eq. (4.15), we find

\[
\begin{align*}
\dot{m}_0' &= 4\pi\varepsilon_0 r^2, \\
\dot{\tau}_0' &= \frac{4\pi p_0 r^3 + m_0}{r(r - 2m_0)},
\end{align*}
\]

Additionally, we can use the conservation law of Eq. (4.12) to obtain

\[
\dot{p}_0' = \frac{(\varepsilon_0 + p_0)(4\pi p_0 r^3 + m_0)}{r(2m_0 - r)},
\]

The system of equations (4.16) and (4.17) are known as the Tolman-Oppenheimer-Volkoff (TOV) equations [158, 159] and they are valid inside the star. The field equations outside the star can be obtained from the set above through the limits \((\varepsilon_0, p_0) \to 0\).

At \( O(\bar{\alpha}^1) \), we have to solve the following equation

\[
\Box_0 \varphi_1 = -\alpha \mathcal{G}_0,
\]

both inside and outside the star, where the d’Alembertian operator and the Gauss-Bonnet curvature invariant are constructed from the metric functions found at \( O(\bar{\alpha}^0) \), i.e. \( \tau_0 \) and \( \sigma_0 \). Thus, equation (4.18) can be rewritten explicitly as

\[
\frac{e^{-2\sigma_0}}{r} \left[ \varphi_1'' r + \varphi_1' \left( \tau_0' r - \sigma_0' r + 2 \right) \right] = -\alpha \mathcal{G}
\]

At \( O(\bar{\alpha}^2) \), the \((t, t)\) and \((r, r)\) components of the field equations yield

\[
\begin{align*}
&\left\{-64\pi \alpha \varphi_1'' + \left[ 64\pi \alpha \varphi_1' + 2r(2\tau_2 - 2\sigma_2 + 1) \right] \sigma_0' + 2r \sigma_2' - 2\tau_2 + 2\sigma_2 - 1 \right\} e^{-2\sigma_0} - 192\pi \alpha \varphi_1' \sigma_0' e^{-4\sigma_0} + 64\pi \alpha \varphi_1'' e^{-4\sigma_0} + 2\tau_2 + 1
\end{align*}
\]
while the conservation law of Eq. (4.12) gives

\[
p_2' = -\frac{1}{r^2} \left[ \left( \varphi_1'' r^2 + (\tau_0' r^2 - \sigma_0' r^2 + 2 r) \varphi_1' - 8 \alpha \tau_0'' \right) \varphi_1' e^{-2\sigma_0} + 8 \alpha \left( \tau_0'' + r^2 - 3 \tau_0' \sigma_0' \right) \varphi_1' e^{-2\sigma_0} \right] - \left( p_0 + p_2 + \varepsilon_0 + \varepsilon_2 \right) \tau_0' + \left( p_0 + \varepsilon_0 \right) \tau_2' + p_0'
\]

(4.21)
in the stellar interior. The equations in the exterior can be found through the limits \((\varepsilon_0, \varepsilon_2, p_0, p_2) \to 0\).

**Solutions of the field equations outside the star**

In this section, we first solve analytically, in vacuum, the equations presented order by order in \(\bar{\alpha}\). The general solutions to these equations will depend on integration constants. These constants can be fixed by examining the solutions’ asymptotic behavior at spatial infinity and imposing that (i) the spacetime is asymptotically flat and that (ii) the scalar field approaches zero at spatial infinity.

**\(O(\bar{\alpha}^0)\) equations**

At this order, the solutions of Eqs. (4.16) have the usual Schwarzschild form

\[
e^{2\tau_0} = e^{-2\sigma_0} = 1 - \frac{a}{r},
\]

(4.22)
where \( a \) is an integration constant which (as we will see shortly) is related with the gravitational mass \( M \) of the star. In obtaining this solution, we required that the metric be asymptotically flat near spatial infinity.

\( \mathcal{O}(\bar{a}^1) \) equations

At this order, we need to consider Eq. (4.18). To solve it, we first calculate \( \mathcal{G}_0 \) which can easily be found using Eqs. (4.22) to be

\[
\mathcal{G}_0 = \frac{12 a^2}{r^6},
\]

(4.23)

and in turn Eq. (4.19) becomes

\[
 r (a - r) \varphi''_1 + (a - 2 r) \varphi'_1 = \alpha \frac{12 a^2}{r^4},
\]

(4.24)

where Eq. (4.22) was used once again.

Equation (4.24) can be solved analytically to find

\[
\varphi_1 = \frac{c_1}{a} \ln \left(1 - \frac{a}{r}\right) + \frac{4 \alpha}{a^2} \ln \left(1 - \frac{a}{r}\right) + \frac{2 \alpha}{r} \left(\frac{2}{a} + \frac{1}{r} + \frac{2 a}{3r^2}\right) + c_2,
\]

(4.25)

where \( c_1 \) and \( c_2 \) are two integration constants. Requiring that the field vanishes at spatial infinity (i.e. that the cosmological background value of the scalar field is zero), we set \( c_2 = 0 \). Expanding \( \varphi_1 \) about spatial infinity, we find that

\[
\varphi_1 = -\frac{c_1}{r} - \frac{a c_1}{2r^2} - \frac{a^2 c_1}{3r^3} + \mathcal{O}(r^{-4}),
\]

(4.26)

which shows that \( c_1 \) is the scalar monopole charge. Reference [138] showed that this charge vanishes for all stars, and therefore, we can set \( c_1 = 0 \). The final expression for the scalar
field outside the star is then

\[ \varphi_1 = \frac{4 \alpha}{a^2} \ln \left(1 - \frac{a}{r} \right) + \frac{2 \alpha}{r} \left( \frac{2}{a} + \frac{1}{r} + \frac{2}{3r^2} \right) . \] (4.27)

\( O(\bar{a}^2) \) equations

At this order, we can substitute \( \varphi_1 \) [cf. Eq. (4.27)] into Eqs. (4.20). The resulting system of differential equations can be solved to find

\[ \tau_2 = -\frac{3 \zeta}{4} \left(1 - \frac{7a}{6r} \right) \left(1 - \frac{a}{r} \right)^{-1} \ln \left(1 - \frac{a}{r} \right) - d_1 \zeta \frac{a}{2r} \left(1 - \frac{a}{r} \right)^{-1} + d_2 \]

\[ -\zeta \frac{a}{r} \left(1 - \frac{a}{r} \right)^{-1} \left( \frac{3}{4} - \frac{a}{2r} - \frac{3a^2}{16r^2} - \frac{5a^3}{48r^3} - \frac{11a^4}{160r^4} - \frac{a^5}{20r^5} + \frac{5a^6}{48r^6} \right) , \] (4.28a)

\[ \sigma_2 = -\frac{\zeta}{8} \frac{a}{r} \left(1 - \frac{a}{r} \right)^{-1} \ln \left(1 - \frac{a}{r} \right) + d_1 \zeta \frac{a}{2r} \left(1 - \frac{a}{r} \right)^{-1} \]

\[ -\zeta \frac{a^2}{r^2} \left(1 - \frac{a}{r} \right)^{-1} \left( \frac{1}{8} + \frac{a}{16r} + \frac{a^2}{24r^2} + \frac{a^3}{32r^3} + \frac{a^4}{40r^4} - \frac{23a^5}{48r^5} \right) , \] (4.28b)

where \( d_1 \) and \( d_2 \) are integration constants and we defined the dimensionless parameter \( \zeta \) via

\[ \zeta \equiv \frac{256\pi a^2}{a^4} . \] (4.29)

The constants of integration can be determined by studying the asymptotic behavior of the metric functions about spatial infinity. For the \( g_{tt} \) metric component we find

\[ g_{tt} = e^{2d_2} - \frac{a}{r} \left(1 + d_1 \zeta \right) e^{2d_2} + O \left(r^{-2} \right) , \] (4.30)

and thus, we set \( d_2 = 0 \) without loss of generality, as any other choice corresponds to a simple rescaling of the time coordinate \( t \to t \exp(d_2) \). From the \( 1/r \) term we identify

\[ M \equiv \frac{a}{2} \left(1 + d_1 \zeta \right) . \] (4.31)
as a *renormalized mass*: the gravitational mass of the star that would be measured by an observer at spatial infinity when performing a Keplerian observation. Decomposing the mass via \( M = M_0 + M_2 \), we can identify \( M_0 = a/2 \) as the gravitational mass of a GR NS, and \( M_2 = \zeta d_1 M_0 \) as the sGB correction to it. A similar mass renormalization occurs for black holes [53].

We can now re-express our exterior solution in terms of the renormalized mass. First, we eliminate \( a \) in favor of \( M \) in Eq. (4.31) and substitute the resulting equation into Eq. (4.28). The resulting expressions for \( \tau_2 \) and \( \sigma_2 \) can now be inserted in \( g_{tt} = -\exp[2(\tau_0 + \tau_2)] \) and \( g_{rr} = \exp[2(\sigma_0 + \sigma_2)] \) and then, after an re-expansion in powers of \( \zeta \), we obtain our final expressions for the metric up to \( O \left( \bar{\alpha}^2 \right) \):

\[
g_{tt} = -\left(1 - \frac{2M}{r}\right) + \left[\frac{3}{2} \left(1 - \frac{7M}{3r}\right) \ln \left(1 - \frac{2M}{r}\right) \right] + \frac{M}{r} \left[3 - \frac{4M}{r} - \frac{3M^2}{r^2} - \frac{10M^3}{3r^3} - \frac{22M^4}{5r^4} - \frac{32M^5}{5r^5} + \frac{80M^6}{3r^6}\right] \zeta,
\]

\[
g_{rr} = \left(1 - \frac{2M}{r}\right)^{-1} + \frac{M}{r} \left(1 - \frac{2M}{r}\right)^{-2} \left[\frac{1}{2} \ln \left(1 - \frac{2M}{r}\right) \right] + \frac{M}{r} + \frac{M^2}{r^2} + \frac{4M^3}{3r^3} + \frac{2M^4}{r^4} + \frac{16M^5}{5r^5} - \frac{368M^6}{3r^6} \zeta.
\]

The equations (4.32) are independent of \( d_1 \), and are instead fully determined by the mass \( M \) of the star and the strength of the coupling constant (through \( \zeta \)) only.

For consistency, let us now re-express the scalar field also in terms of the renormalized mass. The astute reader will notice that to \( O \left( \bar{\alpha}^1 \right) \) we can simply replace \( a \to 2M \) in Eq. (4.27) to obtain:

\[
\varphi_1 = \frac{\alpha}{M^2} \ln \left(1 - \frac{2M}{r}\right) + \frac{2\alpha}{r} \left(1 + \frac{1}{M} + \frac{4M}{3r^2}\right),
\]

which is our final expression for the scalar field at \( O \left( \bar{\alpha}^1 \right) \). The sGB corrections to \( a \) can
be ignored in the scalar field, as they would enter at $O(\tilde{\alpha}^3)$.

**Comparison with black holes spacetimes**

Before proceeding with the interior solution, let us compare the solutions obtained above to their counterparts for BHs [53], focusing first on the scalar field solution. The only difference between the calculation performed here and the one carried out for BHs is that in the latter case $\varphi_1$ must be regular at the event horizon. This results in a nonzero value of $c_1$ that yields

$$\varphi_1^{BH} = \frac{2 \alpha}{r} \left( \frac{1}{M_\bullet} + \frac{4 M_\bullet}{3 r^2} \right),$$

which is identical to the second term in Eq. (4.33) with $M$ replaced by the hole’s mass $M_\bullet$. In a sense then, $\varphi_1$ is equal to $\varphi_1^{BH}$ plus a correction that arises from its continuity across the stellar surface. We also observe that the BH limit of the NS solution for $\varphi_1$ is discontinuous. One can see this easily by evaluating Eq. (4.33) at the surface of the star $R_0$ and taking the BH limit, $M/R \sim M_0/R_0 \to 1/2$, which is possible for certain anisotropic fluids in GR [160, 161].

Let us now compare the NS and BH exterior solutions for the exterior metric. As in the case of the scalar field, requiring that the metric tensor be regular at the horizon yields

$$g_{tt}^{BH} = - \left( 1 - \frac{2M_\bullet}{r} \right) - \frac{1}{3} M_\bullet^3 \left( 1 + \frac{26 M_\bullet}{r} + \frac{66 M_\bullet^2}{5 r^2} + \frac{96 M_\bullet^3}{5 r^3} - \frac{80 M_\bullet^4}{r^4} \right) \zeta,$$

$$g_{rr}^{BH} = \left( 1 - \frac{2M_\bullet}{r} \right)^{-1} - \frac{M_\bullet^2}{r^2} \left( 1 - \frac{2M_\bullet}{r} \right)^{-2} \times \left( 1 + \frac{52 M_\bullet^2}{3 r^2} + \frac{2 M_\bullet^3}{r^3} + \frac{16 M_\bullet^4}{5 r^4} - \frac{368 M_\bullet^5}{3 r^5} \right) \zeta. \quad (4.35a)$$

As in the case of the scalar field, the BH solution contains no logarithmic terms, implying that the BH limit of the NS solution is singular. This is because of the different choice of constants of integration in the NS and BH cases. Notice also that in the BH case the metric
differs from GR through terms of $\delta g_{tt}^{\text{BH}} = O(M^3/r^3)$ and $\delta g_{rr}^{\text{BH}} = O(M^2/r^2)$ in the far field, while in the NS case it differs through terms of $\delta g_{tt} = O(M^7/r^7) = \delta g_{rr}$. This occurs due to a striking cancellation of terms in Eq. (4.32) upon expansion about spatial infinity.

Solutions of the field equations inside the star

For completeness, let us now tackle the problem of solving for the fluid variables, scalar field, and metric components inside the star. This step will inevitably require numerical integrations, for a relationship between pressure $p$ and energy density $\varepsilon$ (i.e. the EoS) must be given and the resulting equations cannot be solved analytically. In this section, we present the numerical scheme and the numerical solutions for the interior fields. We stress however that the exterior solutions found in the previous section do not require these interior numerical solutions.

$O(\tilde{a}^0)$ equations

Here, we need to solve the TOV equations of GR, i.e. Eqs. (4.16) and (4.17). We start by choosing an EoS from our catalog for which, given a central total energy density $\varepsilon_c$, gives the corresponding central pressure $p_c = p(\varepsilon_c)$. We can then integrate Eqs. (4.16) and (4.17) from $r = 0$ up to a point where $p_0(R_0) = 0$, which determines the star’s radius $R_0$.

In practice, we do this integration starting from a small, finite value of $r_c$ and using a series solution valid in this region

\begin{align}
  m_0(r_c) &= \frac{4\pi}{3} \varepsilon_c r_c^3 + O\left(r_c^5\right), \quad (4.36a) \\
  p_0(r_c) &= p_c - \frac{2}{3} \left(3\pi p_c^2 + 4\pi p_c \varepsilon_c \varepsilon_c^3\right) r_c^2 + O\left(r_c^4\right), \quad (4.36b) \\
  \tau_0(r_c) &= \tau_{0c} + \left(2\pi p_c + \frac{2\pi}{3} \varepsilon_c\right) r_c^2 + O\left(r_c^4\right). \quad (4.36c)
\end{align}
We terminate all integrations at the location where \( p_0/p_{0c} = 10^{-11} \). The constant \( \tau_{0c} \) in the series solution of the metric is arbitrary and is fixed a posteriori.

At the star’s surface \( R_0 \) we impose that the metric functions \( \tau_0 \) and \( \sigma_0 \) are continuous, that is:

\[
\tau_0^{\text{in}}(R_0) = \tau_0^{\text{ext}}(R_0), \quad (4.37a)
\]
\[
\sigma_0^{\text{in}}(R_0) = \sigma_0^{\text{ext}}(R_0). \quad (4.37b)
\]

We can analytically match Eqs. (4.16a) and (4.22) at \( R_0 \) to find

\[
a = 2m_0(R_0) \equiv 2M_0, \quad (4.38)
\]

where \( m_0(R_0) \) is the mass of the star enclosed inside the radius \( R_0 \). Furthermore, Eq. (4.37a) fixes the value of the constant \( \tau_{0c} \). Our final numerical solution for \( \tau_0 \) correspond to simple shift \( \tau_0 \rightarrow \tau_0 + \tau_{0c} \).

The outcome of these integrations can be summarized in a mass-radius relation, shown in Fig. 4.1. In this figure, the solid lines correspond to various mass-radius curves for the EoSs in our catalog.

\( O(\tilde{\alpha}^{-1}) \) equations

At \( O(\tilde{\alpha}^{-1}) \) we only need to solve Eq. (4.18). From the \( O(\tilde{\alpha}^{-0}) \) solution, we know \( a \) and \( R_0 \), which fully determines \( \varphi_1^{\text{ext}} \) and its derivative at \( R_0 \) [cf. Eq. (4.33)]. This information can be used as initial conditions to integrate Eq. (4.18) inside the star: we start our integration at \( r = R_0 \) and move in toward \( r = 0 \). In this calculation, it is useful to note that \( G_0 \) is given by

\[
G_0 = \frac{48 m_0^2}{r^6} - \frac{128\pi(m + 2\pi r^2 p_0)}{r^6}, \quad (4.39)
\]
inside the star [142], where the functions $m_0$, $p_0$ and $\varepsilon_0$ are all known from the $O(\bar{\alpha}^0)$ calculation.

The radial profiles of $G_0$ and $\varphi_1$ are shown in Fig. 4.2 using the FPS EoS with the scalar-Gauss-Bonnet coupling fixed to $\bar{\alpha} = 15$. In the top-panel, we see that $G_0$ is mostly negative within the star, except near the surface (indicated by the dashed vertical line) where it changes sign and then matches smoothly to its exterior form, given in Eq. (4.23). We also observe that $G_0$ has a larger magnitude for stars with larger values of $\varepsilon_{0c}$. This is can be seen by substituting the expansions of Eq. (4.36) in Eq. (4.39). We find that $G_0$ is negative and nearly constant close to the center of the star at $r \approx 0$, with its magnitude proportional to $\varepsilon_{0c}$. In the bottom-panel, we see that NSs with larger central energy densities $\varepsilon_{0c}$ have larger amplitudes of $\varphi_1$ at their cores. This is unsurprising given the fact that the source of the scalar, i.e. $G_0$, has a larger magnitude near the stellar center. At the surface, $\varphi_1$ connects smoothly with its exterior solution, given by Eq. (4.25) (at this order in $\alpha$). The results for other EoSs are qualitatively the same as the ones shown here.

Let us now investigate how the central values of the scalar field $\varphi_1$ vary as a function of both $\varepsilon_{0c}$ and of $\bar{\alpha}$. This dependence is shown in Fig. 4.3 for four representative values of $\bar{\alpha} = \{5, 10, 15, 20\}$ covering a range of central energy densities $\varepsilon_{0c}$ that span stars with masses $0.41 \, M_\odot$ to $1.75 \, M_\odot$ using the FPS EoS. We see that for small $\varepsilon_{0c}$ (i.e. low-mass stars) all values of $\varepsilon_{1c}$ converge towards zero regardless of the strength of the coupling. This is can understood by noticing that in this limit $G_0$ is very small and nearly flat (cf. Fig. 4.2), thus sourcing $\varphi_1$ weakly. For larger $\varepsilon_{0c}$, the situation is different and we see a stronger dependence of the central value of $\varphi_1$ on $\bar{\alpha}$. Unsurprisingly, the magnitude of $\varphi_1$ is larger at the stellar core the larger the strength of the coupling $\bar{\alpha}$. 
$O\left(\tilde{\alpha}^2\right)$ equations

At $O\left(\tilde{\alpha}^2\right)$ we need to solve Eqs. (4.20) and (4.21). The boundary conditions are similar to those at $O\left(\tilde{\alpha}^0\right)$. Imposing continuity at the surface gives us

$$g_{\alpha \beta}^{\text{in}}(\alpha^0, \alpha^2, R_2) = g_{\alpha \beta}^{\text{ext}}(\alpha^0, \alpha^2, R_2),$$

(4.40)

where $R_2$ is the radius of the NS at $O\left(\tilde{\alpha}^2\right)$ given by the condition:

$$p_0(R_2) + p_2(R_2) = 0.$$  

(4.41)
Figure 4.3: Central values of the scalar field $\varphi_1$ for various values of $\tilde{\alpha}$ as a function of the central densities of the star with an FPS EoS. Observe how the central value of the scalar field converges toward zero at small central densities irrespective of the coupling constant.

As in the $O(\tilde{\alpha}^0)$ integrations, we start from $r_c$ and integrate outwards until the point $R_2$ where the condition $(p_2 + p_0)/p_{0c} = 10^{-11}$ is met. Equation (4.40) allow us to determine the numerical value of $d_1$, which in turn allows us to calculate the renormalized mass $M$ [Eq. (4.31)] and thereby determine the exterior metric in terms of interior quantities.

When integrating Eqs. (4.20)-(4.21) we need to be careful on how we calculate the perturbed density $\epsilon_2$. To do this, we take our total density $\epsilon(p) = \epsilon_0 + \epsilon_2$, and solve for $\epsilon_2$ as

$$
\epsilon_2 = \epsilon(p_0 + p_2) - \epsilon_0(p_0),
$$

(4.42)

where $\epsilon(p_0 + p_2)$ is a spline interpolation of our EoS table. This allows us to eliminate the variable $\epsilon_2$ in favor of the perturbed pressure $p_2$. With Eq. (4.42) and the solutions to all fields up to $O(\tilde{\alpha})$, we can solve our system of equations given by Eqs. (4.20) and (4.21).

The dashed curves in Fig. 4.1 show the mass-radius relations calculated to $O(\tilde{\alpha}^2)$ for
the various EoSs of our catalog. We see that for a fixed value of $\bar{\alpha}$ the deviations from the GR mass-radius relation occur at larger masses. This is consistent with our previous observations on $\varphi_1$, which had a larger magnitude for larger masses. Consequently, these large scalar fields backreact more strongly onto the GR solution, causing larger changes to the mass and the radius. The sGB corrections typically lead to less massive NSs regardless of the EoS considered, as result consistent with those of [135]. For clarity, in Fig. 4.1 we only showed curves with a fixed $\bar{\alpha} = 15$, but how do the mass-radius curves change (for a fixed EoS) as we vary $\bar{\alpha}$? This is shown in Fig. 4.4 for EoS SLy. As expected, from our previous discussion of the $O\left(\bar{\alpha}^{-1}\right)$ results, an increase in $\bar{\alpha}$ causes larger deviations in the mass-radius curve. Indeed, the large the value of the sGB coupling, the smaller the maximum NS mass that is allowed for a given EoS.

Figure 4.1 also shows vividly the difficulties of testing modified theories of gravity with masses and radii measurements of NSs. In the absence of a complete understanding of matter in the NS interior, the various competing EoS models, predict NSs that cover a wide portion of the $(M, R)$-plane. But this problem could be averted if, in the future, the EoS is constrained through the NICER [122–124] and/or LIGO/VIRGO [25]. Let us imagine, for example, that the SLy EoS is favored by observations. If so, the observation of a $\approx 2M_\odot$ NS (see e.g. [162]) would place the stringent constraint $\bar{\alpha} \lesssim 10$ (roughly one order of magnitude more stringent than current bounds), since for larger values a SLy EoS could not predict such a massive NS. This constraint would be weaker if the true EoS is stiffer (e.g. MPA1 and AP3), as larger values of $\bar{\alpha}$ would be required to pull the mass-radius curve below $\approx 2M_\odot$, but stiffer EoSs are disfavored by recent tidal deformability constraints from the GW170817 gravitational-wave event [163].
Figure 4.4: Mass-radius curves with a SLy EoS for varying couplings $\tilde{\alpha}$. Observe that greater couplings lead to a decrease in the maximum mass of NSs, which can aid in constraining the theory with observations of massive pulsars.

Astrophysical applications

Now that we have an analytic solution for the exterior spacetime of a NS in sGB gravity, let us explore some astrophysical applications to investigate the physical effects of the corrections on observables.

Probing astrophysical phenomena in the vicinity of NSs naturally requires that one first analyze the geodesic motion of massive test particles and of light in the stellar exterior. Since our metric is static and axisymmetric, we know it possesses a timelike and azimuthal Killing vector, which imply the existence of two conserved quantities: the specific energy
\(E\) and the specific angular momentum \(L\)

\[
E = -g_{tt} \dot{t}, \quad L = g_{\phi \phi} \dot{\phi}.
\]  

(4.43)

where the dots indicate differentiation with respect to proper time. From normalization condition of the four-velocity, \(u^a u_a = \epsilon\), where \(\epsilon = (-1 \text{ or } 0)\) for time-like or null trajectories respectively, we obtain

\[
\frac{\dot{r}^2}{2} = V_{\text{eff}}(r),
\]  

(4.44)

which describes the radial motion of the particle in terms of the effective potential

\[
V_{\text{eff}}(r) = -\frac{1}{2g_{rr}} \left(\frac{E^2}{g_{tt}} + \frac{L^2}{g_{\phi \phi}} - \epsilon\right).
\]  

(4.45)

Because of spherical symmetry, we can set \(\theta = \pi/2\) (and therefore \(g_{\phi \phi} = r^2\)) without loss of generality.

Circular orbits around the star

Let us study the circular motion of massive test particles (\(\epsilon = -1\)) around a NS with exterior metric given by Eqs. (4.32). For a circular orbit at \(r = r_+\), the conditions \(V_{\text{eff}}(r_+) = 0\) and \(V'_{\text{eff}}(r_+) = 0\) must be satisfied. Using Eq. (4.45), we can solve for \(E\) and \(L\), and expand in powers of \(\zeta\) to obtain

\[
E = E_0 + \zeta E_2 + O(\zeta^2),
\]  

(4.46a)

\[
L = L_0 + \zeta L_2 + O(\zeta^2),
\]  

(4.46b)
where $E_0$ and $L_0$ are the GR specific energy and angular momentum for circular orbits [157]

\[
E_0 = \left(1 - \frac{2M}{r_*}\right) \left(1 - \frac{3M}{r_*}\right)^{-1/2},
\]

\[E_0 = (Mr_*^{1/2}) \left(1 - \frac{2M}{r_*}\right) E_0, \tag{4.47b}\]

and $E_2$ and $L_2$ are modifications of $O\left(\tilde{\alpha}^2\right)$. The latter are given by

\[
E_2 = -\left(1 - \frac{3M}{r_*}\right)^{-3/2} \left[\frac{3}{4} - \frac{31M}{8r_*} + \frac{21M^2}{4r_*^2}\right] \ln \left(1 - \frac{2M}{r_*}\right)
\]

\[+ \frac{M}{r_*} \left(1 - \frac{3M}{r_*}\right)^{-3/2} \left[\frac{3}{2} - \frac{25M}{4r_*} + \frac{19M^2}{4r_*^2}\right]
\]

\[+ \frac{19}{6} \frac{M^3}{r_*^3} + \frac{33M^4}{10r_*^4} + \frac{21M^5}{5r_*^5} - \frac{596M^6}{15r_*^6} + \frac{40M^7}{r_*^7}\right), \tag{4.48}\]

and

\[
L_2 = \frac{(Mr_*^{1/2})}{8} \left(1 - \frac{3M}{r_*}\right)^{-3/2} \ln \left(1 - \frac{2M}{r_*}\right)
\]

\[+ (Mr_*^{1/2}) \left(1 - \frac{3M}{r_*}\right)^{-3/2} \left[\frac{M}{4r_*} + \frac{M^2}{4r_*^2} + \frac{M^3}{3r_*^3}\right]
\]

\[+ \frac{M^4}{2r_*^4} + \frac{4M^5}{5r_*^5} - \frac{188M^6}{3r_*^6} + \frac{80M^7}{r_*^7}\right). \tag{4.49}\]

We may now make use of Eqs. (4.44) and (4.45) along with our circular orbit conditions to find the sGB modifications to the location of the ISCO. Doing so, we find that the ISCO radius is

\[
R_{\text{ISCO}} = 6M - \frac{3M}{2} \left[\frac{5047}{14580} + \ln \left(\frac{2}{3}\right)\right] \zeta + O\left(\zeta^2\right), \tag{4.50}\]

which reproduces the well-known GR result when $\zeta = 0$. Notice that the sGB correction pushes the ISCO location farther away from the stellar surface (assuming the star is
sufficiently compact so that the ISCO is outside the surface in GR in the first place) by a small amount $R_{\text{ISCO}} - 6M \approx +0.089M\zeta$.

**Modified Kepler’s Third Law**

Now let us derive an expression for the orbital frequency $\Omega_\phi = d\phi/dt$ of a massive particle in circular orbit at radius $r_*$ as measured by an observer at infinity. Using Eqs. (4.43) and (4.46) we find

$$\frac{\Omega_\phi^2}{\Omega_0^2} - 1 = -\frac{7}{4} \ln \left(1 - \frac{2M}{r_*}\right) \zeta - \left(1 - \frac{2M}{r_*}\right)^{-1} \times \left(\frac{7M}{2r_*} - \frac{7M^2}{2r_*^2} - \frac{7M^3}{3r_*^3} - \frac{7M^4}{3r_*^4} - \frac{14M^5}{5r_*^5} - \frac{1976M^6}{15r_*^6} + \frac{560M^7}{3r_*^7}\right) \zeta + O\left(\zeta^2\right),$$

(4.51)

where $\Omega_0^2 = M/r_*^3$ is the usual GR result. Expanding Eq. (4.51) in the far field limit we find to leading order in $\zeta$

$$\Omega_\phi^2 \approx \Omega_0^2 \left(1 + \frac{128M^6}{r_*^6} \zeta\right),$$

(4.52)

which is consistent with our expansions of the sGB metric deformation in our comparison with the BH solutions. Unlike the case for BHs, where the correction to the frequency occurs as $O\left(M^2/r_*^2\right)$ [53], the presence of the logarithmic term in Eq. (4.51) gives a small correction. This suggests that weak-field observables will be very poor probes of sGB gravity.

**Quasiperiodic oscillations**

Let us now focus on the frequencies of quasi-periodic oscillations (QPOs). There are a number of models which have been proposed as possible causes of QPOs including the relativistic motion of matter [4] and resonance between orbital and epicyclic motion [164]. Regardless of the model in question, it may be interesting to calculate the sGB corrections
the QPO frequencies to study what effect, if any, this modification to GR has.

The orbital frequency was already calculated in Eq. (4.52), so let us now calculate the epicyclic frequency for timelike geodesics. This frequency is determined by a radially-perturbation to the circular orbit equation [Eq. (4.44)], which yields

$$\Omega_r^2 = -\frac{1}{2i} \frac{\partial^2 V_{\text{eff}}(r)}{\partial r^2}.$$  \hspace{1cm} (4.53)

Solving Eq. (4.53) with the $V_{\text{eff}}(r)$ defined in Eq. (4.45) gives us

$$\frac{\Omega_r^2}{\Omega_0^2} = 1 - \frac{6M}{r} - \frac{7}{4} \left( 1 - \frac{48M}{7r} \right) \ln \left( 1 - \frac{2M}{r} \right) \zeta
+ \frac{M}{r} \left( 1 - \frac{2M}{r} \right)^{-1} \left( \frac{7}{2} - \frac{55M}{2r} + \frac{65M^2}{3r^2} + \frac{41M^3}{3r^3} 
+ \frac{66M^4}{5r^4} + \frac{9832M^5}{15r^5} - \frac{15056M^6}{15r^6} - \frac{256M^7}{r^7} \right) \zeta + O(\zeta^2).$$  \hspace{1cm} (4.54)

If one were to asymptotically expand this frequency about spatial infinity, one would again find that the sGB corrections are highly suppressed. As we will see below, however, QPOs are sensitive to physics near the ISCO, and in this regime, the sGB corrections are not nearly as suppressed.

In addition to these two frequencies, there is often a third one that is important in QPOs and measures the rate of periastron precession of the orbit. This precession frequency can be found via

$$\Omega_{\text{per}} = \Omega_{\text{sGB}} - \Omega_r,$$  \hspace{1cm} (4.55)

and it is usually important in lower frequency QPOs\(^2\) [167]. With these three frequencies in hand, one could imagine using the observation of QPOs to place constraints on sGB.

\(^2\)Some models treat this frequency as stemming from inhomogeneities near the inner accretion disk boundary, causing a beat frequency [165]. However, this was found to be inconsistent with observations [166].
Figure 4.5: Orbital frequencies $\Omega_\ell$ versus $\Omega_{sGB}$ for a NS of mass $1.4 M_\odot$ to leading order in $\zeta$.

Figure 4.5 depicts two of our frequencies against one another (in dimensionless units) and illustrates how there are noticeable deviations from the GR predictions as $\alpha$ increases. Observe that the frequencies approach each other when either of them is small, since here one approaches the weak-field regime described in Eq. (4.52).

One may present these frequencies in terms of an observable quantity, namely the dimensionless linear orbital velocity $v$, as done in [167,168]. By introducing the orbital velocity as $v = (M\Omega_{sGB})^{1/3}$, we may re-express the ratio of the precession frequency to the
orbital frequency as a series in velocity to obtain

$$\frac{\Omega_{\text{per}}}{\Omega_{\text{sGB}}} = 3 v^2 + \frac{9}{2} v^4 + \frac{27}{2} v^6 + \frac{405}{8} v^8 + \frac{1701}{8} v^{10} + \left(\frac{15309}{16} + 384 \zeta \right) v^{12} + \mathcal{O}(v^{14}) ,$$

(4.56)

where the modification to the GR solution again is suppressed by a high power of velocity that is consistent with the expansion of Eq. (4.52). As before, the largest deviations will then occur for observables that are sensitive to physics near the surface of the NS, i.e. where the orbital velocity is not extremely small.

**Light bending**

![Diagram](null)

Figure 4.6: Diagram of emitted photon trajectory. A photon emitted in the direction $\vec{k}$ from the surface of the star orthogonal to $\vec{n}$ will have its trajectory bent by an angle $\psi - \gamma$ to an impact parameter of $b$.

Let us now consider photon motion in the sGB exterior spacetime, as depicted in Fig. 4.6. Imagine then a photon leaving the surface of the NS along the unit vector $\vec{k}$, which makes an angle $\gamma$ with the unit vector $\vec{n}$ normal to the star’s surface. The angle $\psi$, between
and the line of sight, is an important quantity in astrophysical applications. For instance, when \( \gamma = \pi/2 \), \( \psi = \psi_{\text{crit}} \) is the critical angle between the line of sight and the normal to the surface beyond which the photon cannot reach the observer. This allows one to define a visible fraction of the star as

\[
\varsigma \equiv \frac{1}{2} \left[ 1 - \cos(\psi_{\text{crit}}) \right].
\]  (4.57)

Moreover, in the context of pulse profile modeling, photons emitted by the hot spot can only reach the observer when they are emitted if emitted with \( \cos \psi > \cos \psi_{\text{crit}} \) [169].

Let us now derive an expression for \( \psi \). We again restrict attention to equatorial orbits, such that \( \theta = 0 \) and \( \dot{\theta} = 0 \), and change notation \( \phi \rightarrow \psi \) in Eqs. (4.43) and (4.44) with \( \epsilon = 0 \). Solving for the fraction \( d\psi / dr \) yields

\[
\frac{d\psi}{dr} = \frac{1}{g_{\phi\phi}} \left[ -\frac{1}{g_{rr}} \left( \frac{E^2}{L^2} g_{tt} + \frac{1}{g_{\phi\phi}} \right) \right]^{-1/2}. 
\]  (4.58)

Since \( E \) and \( L \) are constant, we can simplify the above expression through the emission angle \( \gamma \), defined via [169]

\[
\tan^2(\gamma) = \frac{u^\phi u_\psi}{u^\nu u_r}. 
\]  (4.59)

The above expression allows us to find a relation between \( E \), \( L \), and \( \gamma \), namely

\[
\frac{L}{E} = \sqrt{-\frac{g_{tt}(R)}{g_{\phi\phi}(R)} \sin(\gamma)}, 
\]  (4.60)

where we evaluate the metric functions at the stellar surface. Substituting Eq. (4.60) into Eq. (4.58) gives a direct relation between \( \psi \) and \( \gamma \) for a given \( R \), which can be solved to

\[^3\text{The ratio of } L/E \text{ is also called the impact parameter [170], which is denoted as } b \text{ in Fig. 4.6.} \]
obtain
\[
\psi(R, \gamma) = \int_{R}^{\infty} \frac{dr}{g_{\psi\psi}} \left[ -\frac{1}{g_{rr}} \left( \frac{1}{g_{\psi\psi}} - \frac{g_{\phi\phi}(R) \csc^2(\gamma)}{g_{tt}(R) g_{tt}} \right) \right].
\] (4.61)

The integral in Eq. (4.61) may not be straightforward to solve, even numerically, but following [171,172], we can rewrite it in terms of the compactness \( M/R \) and a new variable \( x = \sqrt{1 - R/r} \) to ease the numerical integration.

The results of evaluating Eq. (4.61) as a function of the compactness are shown in Fig. 4.7. Observe that there is a greater deflection of light for NSs of greater compactness.

This is apparent even in the GR limit, and it is due to the effects of curvature near compact objects. However, this effect is enhanced in sGB gravity, increasing with larger \( \zeta \), which dictates how strongly the \( \mathcal{G} \) correction contributes to the system. For stars with smaller

\footnote{The relation between \( \zeta \) and \( \alpha \) depends on the mass of the NS, which is not specified here. As a reference, for a 1.4 \( M_\odot \) NS, \( \tilde{\alpha} = (10, 20, 30) \) corresponds to \( \zeta \approx (0.5, 2.1, 4.6) \).}
masses and larger radii, there is a negligible change in the deflection of light, regardless of the strength of $\zeta$. The curvature of spacetime near the surface of these NS is simply not large enough even with the quadratic curvature nature of our theory to cause any deviations that may be detectable in future observations.

We may also look at how light bending in sGB gravity compares to light bending in GR, as shown in Fig. 4.8 for various choices of $\zeta$ values and two fixed compactnesses. As with Fig. 4.7, there are only tiny deviations when the compactness is small. However, NSs with larger compactnesses do present sGB corrections to light bending that make it stronger relative to GR.

![Figure 4.8: Light bending in sGB gravity. The solid lines represent the GR solution, while the dashed (dotted) lines correspond to $\zeta = 2.5$ ($\zeta = 5$). Deviations from GR are more noticeable when the compactnesses is large and and $\zeta$ increases light bending, at fixed emission angle $\gamma$.](image)

As a final calculation, we can also find the visible fraction of the NS surface, given in
Eq. (4.57). This is shown in Fig. 4.9. In agreement with our previous results, there is little to be learned about sGB gravity from observations of low compactness stars. However, as the compactness increases, so does the effects of the coupling with the Gauss-Bonnet invariant. Likewise, larger values of the coupling constant lead to larger changes in the visible fraction. In GR, it is known that for NSs with $M/R \approx 0.28$, strong gravitational light bending can make the whole surface of the star visible [70]. The effect of the scalar-Gauss-Bonnet coupling is to reduce the necessary compactness the whole surface of the star to become visible. For instance, when $\zeta = 5$, this compactness is 0.264.

![Figure 4.9: Visible fraction of a star as a function of compactness for various coupling strengths. The lines terminate at the value of compactness for which the whole surface of the star becomes visible. Large values of $\zeta$ require smaller values of compactness for this to happen.](image-url)
Conclusions and outlook

In this paper, we obtained an analytical metric that represents the exterior spacetime of a NS in sGB gravity as well as an analytical expression for the scalar field. The metric was derived through a small-coupling perturbative scheme and depends only on the mass of the NS in question and the desired strength of the coupling constant. Our metric is valid to $O(\tilde{\alpha}^2)$ and we have outlined how higher-order corrections can be obtained. We applied the new spacetime to a sample of astrophysical applications, including the motion of test particle (which is important for instance to model QPOs) and light bending (which is important to model x-ray pulse profiles generate by hot spots at the surface of rotating NSs).

Our work opens the door for number a future studies with NSs in sGB gravity, with the convenience of now being able to treat the metric analytically. One application could be the development of an effective-one-body (EOB) formalism, to model NS binaries in sGB, along the lines of the recent work in scalar-tensor theories [173]. Having the theory expressed in an EOB framework allows one to understand the dynamics of the two-body system, the radiation-reaction components of the system, and knowledge of the gravitational-waveform emitted from a coalescing binary [174].

Another possible use for the exterior metric is in the modeling of x-ray pulse profiles as a possible testbed for sGB gravity. These pulse profiles are generated by the x-ray emission from hot spots on the surface of rotating NS [70] (see [175–177] for reviews). As the photons propagate from the surface towards the observer, they probe the spacetime around the NS which, in principle, can leave detectable deviations in the observed pulse profile relative to what is predicted in GR. This possibility was recently explored in the context of scalar-tensor theories [125,126,178]. In particular, [126] showed that in principle observations made by NICER can constrain these theories. It would be interesting to see if
the same is true in sGB gravity.

A final observation of interest is the absence of any sGB gravity integration constants in the final expression for the exterior metric presented in Eq. (4.32). This is rather unexpected because in other theories (such as in scalar tensor theories) the exterior metric does depend on charges that must be computed numerically. In sGB gravity, however, the exterior metric is fully determined in terms of the mass of the star $M$ and the coupling constant of the theory $\alpha$. A deeper physical or mathematical understanding of why this is the case in sGB gravity would be most interesting and will be studied elsewhere.

Such a charge-independence in the metric is clearly advantageous for data analysis applications. For instance, in a Bayesian parameter estimation study of any of the astrophysical scenarios outlined above, millions of pulse profiles are required to probe the likelihood surface. Being able to compute these pulse profiles without the prior construction of numerical NS spacetimes is clearly computational advantageous. This is in sharp contrast with what happens in scalar-tensor theories, where the exterior of NSs is described by the Just metric [74,179]; in this spacetime, the metric depends explicitly on the scalar charge $Q$, whose numerical value can only be determined by integrating the theory’s equations of stellar structure numerically, increasing the computational cost of parameter estimation studies.

Acknowledgments

This work was supported by NSF Grant No. PHY-1250636 and PHY-1759615, as well as NASA grants NNX16AB98G and 80NSSC17M0041. We thank Alejandro Cárdenas-Avendaño, Paolo Pani, Thomas Sotiriou and Kent Yagi for helpful discussions. Computational efforts were performed on the Hyalite High Performance Computing System, operated and supported by University Information Technology Research Cyberinfrastructure at Montana State University.
Addendum to sGB Work

Following the submission of the previous paper, we investigated the results of the sGB metric in more depth in a few areas.

On the maximum mass of the neutron star

We found in Fig. 4.4 that the maximum mass for a NS decreases for a given radius as our coupling parameter $\alpha$ is increased. This would seem to indicate that the coupling between the scalar field and Gauss-Bonnet invariant has an effect on the gravity of the system which causes the mass of the neutron star to decrease for a given radius. The question becomes, why is this? Would we truly observe a less massive neutron star in sGB than we would in GR?

For the sake of argument, let us assume our neutron star is of constant density $\varepsilon_0 \rightarrow \varepsilon_c$. In GR, the TOV equations in Eq. (4.17) can be integrated to show

$$p_0 = \varepsilon_c \frac{\sqrt{1 - R r^2/R_0^3} - \sqrt{1 - R/R_0}}{3 \sqrt{1 - R/R_0} - \sqrt{1 - R^2/R_0^3}},$$

(4.62)

where $R_0$ is the radius of the star and $R$ is the Schwarzschild radius. At the center of the star, the pressure takes the form

$$p_{0c} = \varepsilon_c \frac{1 - \sqrt{1 - R/R_0}}{3 \sqrt{1 - R/R_0} - 1}.$$

(4.63)

Notice that this term diverges when we have

$$R_{0,\text{crit}} = \frac{9}{8} R.$$  

(4.64)
If we are to solve for the critical mass of our NS using \( M_{\text{crit}} = 4\pi \varphi_c R_{0,\text{crit}}^3/3 \) we find

\[
M_{\text{crit}} = \sqrt{\frac{16}{243\pi G \varphi_c}},
\]

(4.65)

where we substituted the Schwarschild radius \( R = 2GM \).

It makes sense for us to perform this same calculation in sGB, however the results are not analytic. We must therefore turn to Fig. (4.1) and our numerical results to see that the mass changes. We find that when we include sGB terms, the mass decreases for a given radius. This would have the same effect as increasing the gravitational constant \( G \) in Eq. (4.65). We may therefore look at the field equations given in Eq. (4.8a), which we rewrite as

\[
G_{ab} + \frac{\alpha}{\kappa} K_{ab} = \frac{1}{2\kappa} \left( T_{ab}^m + T_{ab}^\varphi \right),
\]

(4.66)

where the left hand side of Eq. (4.66) contains all terms related to the curvature of the spacetime and the right hand side contains the matter and energy contributions from the mass and scalar field respectively. We expect that the left hand side should appear as terms which are greater than those found in GR, thus proving that we have additional gravitational contributions to the star.

Taking the trace allows us to rewrite Eq. (4.66) as

\[
G^a_{\ a} + \frac{\alpha}{\kappa} K^a_{\ a} = -R + \frac{\alpha}{\kappa} \left[ 2R \Box \varphi - 4R_{ab} \nabla^a \nabla^b \varphi \right],
\]

\[
\sim -R - 2\frac{\alpha}{\kappa} R \Box \varphi,
\]

\[
\sim -R + 2\frac{\alpha^2}{\kappa} R \mathcal{G},
\]

(4.67)

where we assumed \( R_{ab} \nabla^a \nabla^b \varphi \) was of the same order \( R \Box \varphi \). We know from Fig. 4.2 that \( \mathcal{G} \) is always negative at the center of the NS. Therefore, the gravitational terms of our field
equation take the form

\[ G^\alpha_\alpha + \frac{\alpha}{\kappa} \mathcal{K}^\alpha_\alpha \sim -R \left( 1 + 2\frac{\alpha^2}{\kappa} |\mathcal{G}| \right), \tag{4.68} \]

which we may understand as a strengthening of the gravitational contributions to the spacetime. This agrees with our original assumption that the sGB terms affect gravity by essentially “adding” to the gravitational strength and thus decreasing the mass of the NS for a given radius. This also matches with the results we see in Fig. 4.4. As $\alpha$ is increased, the resulting mass of the NS decreases for any radius.

One final comment has to do with $T^{\phi}_{ab}$. We note that the trace of this recovers terms proportional to $(\nabla \varphi)^2$. From Fig. 4.2, we see that the variation of $\varphi$ is small over the length of the NS. Therefore, terms which go as $(\nabla \varphi)^2$ will be suppressed due to their small contributions near the core. It is this reason why $T^{\phi}_{ab}$ was not included in Eq. (4.67) when we sought additional gravitational contributions to the NS.

On the study of quasi-periodic orbits

One model to explain QPOs \cite{4,80} makes the assumption that the matter within the accretion disk of a low-mass X-ray binary (LMXB) is under the influence of the accreting object’s gravity alone. Due to the relativistic effects caused by the curvature present in the spacetime surrounding the NS, the orbit of the accretion disk matter will precess as it orbits the star. In addition to this, the orbital frequency will be modified by terms proportional to the relativistic spacetime solution. Since the frequency separation $\Delta \nu$ can be measured from X-ray observations of a LMXB’s power spectrum, QPO’s provide an excellent way to study gravitational physics in the strong-field regime.

Within the paper, we derived an idealized model for the QPO frequencies (see Fig. 4.5), but did not go into detail on how these could be used in conjunction with observations to obtain constraints. Figure 4.10 shows an actual data set from QPO data obtained by
van der Klis, et. al. [3] showing the power spectrum for Scorpius (Sco) X-1, a LMXB. Note that Fig. 4.10 has two obvious peaks at frequency’s $\nu_1$ for the lower and $\nu_2$ for higher peak. These frequencies correspond to the periastron precession frequency and orbital frequency respectively [4, 167]. These frequencies are not stationary and change with the observation window employed, presumably because the mass-accretion rate is not constant [180]. Moreover, the distance between $\nu_1$ and $\nu_2$ also varies with time. Most often, the data for QPO’s is processed to plot the peak frequency separation $\Delta \nu = \nu_2 - \nu_1$ versus the highest frequency $\nu_2$. With the notation we discussed in our paper, this corresponds to $\Delta \nu = \Omega_{sGB} - \Omega_{\text{per}} = \Omega_r$ being plotted against $\Omega_{sGB}$.

From the assessment made above, individual observations will obtain a single $\Omega_r$ and $\Omega_{sGB}$ from the power frequency data. This yields one point on what we would call Fig. 4.5. As more measurements are made, more points on the plot would be filled in as seen in
Fig. 4.11. By observing these LMXBs over a long period of time, an observational shape to the ratio of $\Omega_r$ to $\Omega_{sGB}$ can be found and compared to theory to find constraints.

Figure 4.11: $\Delta \nu$ vs. $\nu_2$ ($\Omega_r$ vs. $\Omega_{sGB}$ in Fig. 4.5) for 10 LMXBs. The numbered lines correspond to the mass of a non-rotating NS in units of $M_\odot$ [4].
CHAPTER FIVE

CONCLUSION

Summary

The main objective of this thesis was to learn how to test theories of gravity which compete with GR. I did this in two ways, divided into three chapters.

Chapter 2 illustrated a variety of processes for deriving a SET for modified theories of gravity with arbitrary fields. The work here was mainly mathematical, but provided a catalog of methods for developing the terms needed for determination of radiative characteristics of compact binary systems. Of the four methods tested, those of the perturbed field equation and the perturbed action yielded identical results for all theories while also providing the equations of motion for all fields involved. The Landau-Lifshitz method also gave the correct results to all radiative quantities, however failed to generate the dynamics of any fields in question. Therefore, it must be known a priori how these fields propagate in a given spacetime to utilize this method to full effect. The last scheme tested, the Noether current method, worked well for theories which contained no Lorentz violation. Since $\mathcal{E}$-theory relies on particular symmetries of the system, breaking these symmetries has the unintended consequences of failing to generate the correct results. While the quantities derived do remain conserved, they hold no particular physical meaning. We have therefore shown that certain methods work better for particular theories.

Chapter 3 of this thesis expanded on the work of chapter 2 by applying the work to a compact binary system in $\mathcal{E}$-theory. The goal of this procedure was to find the rate of angular momentum loss given the additional influence of a time-like ether field which coupled to the metric. Once we derived this quantity, we made use of the orbital evolution equations to see how the semi-major axis and eccentricity for a binary system varied due
to the emission of energy and angular momentum. We found that coupling constants in $\mathcal{E}$-theory lead to a decay rate of the radius of the orbit and eccentricity that is faster than in GR.

Chapter 4 studied the spacetime of a NS in a modified gravity theory. The shape of the light curve associated with a pulsar is dependent on the surrounding spacetime. I studied NS spacetimes in sGB gravity, which couples a dynamical scalar field to a curvature squared term known as the Gauss-Bonnet invariant. The metric derived depended only on the mass of the NS and the strength of the Gauss-Bonnet coupling parameter $\alpha$. The results show deviations from the corresponding GR solutions for a number of calculations including the motion of both massive test particles and photons.

**Future Outlook**

From the results found within this thesis, there are a number of avenues which can be taken to continue researching. For the study involving binary systems, the SET can be found and applied to any number of modified theories (e.g. TeVeS [97] and MOG [98,99]). Once the SET is found, further analysis can be carried out to find the rates of energy and angular momentum carried away by any propagating degrees of freedom present. Of course, future observations of gravitational waves could then be used to constrain the strength of any alternative polarization modes found and place limits on the modified theory of gravity in question. In addition to this, one could derive a model for the GWs emitted during an inspiral to the leading order correction to GR for a given theory. These models could then be compared with future GW observations to place constraints on the theory.

For the NS work, the metric derived in this thesis is only the first step towards the development of light curves which could be compared to data. Now that the spacetime is known to leading order in $\zeta$ for sGB, it would be interesting to see whether or not light curve models could be developed and compared with the future results which NICER
aims to provide. Also, due to the fact that the spacetime found is entirely analytic, the
development of an EOB formalism is possible without the need to find particular solutions
for the metric for each EoS we wish to study. This will save time in analysis and can provide
a more general way to constrain the theory simply based on the coupling strength to the
Gauss-Bonnet invariant.
REFERENCES CITED


APPENDICES
APPENDIX A

BRILL-HARTLE AVERAGING SCHEME
This section will show some of the advantages of the averaging scheme used by Isaacson [181], which he called Brill-Hartle averaging, and we refer to as wavelength-averaging. The average of some tensor $X^{\alpha\beta}$ will be defined in the following way,

$$\left\langle X^{\alpha\beta}(x) \right\rangle \equiv \int d^4x \ g_{\alpha'}^{\alpha}(x,x') \ g_{\beta'}^{\beta}(x,x') \times X^{\alpha'\beta'}(x') \ f(x,x'), \quad \text{(A.1)}$$

where $g_{\alpha'}^{\alpha}(x,x')$ is the bivector of geodesic parallel displacement [182] that depends on the background geometry and $f(x,x')$ is the kernel of the integral satisfying

$$\int d^4x f(x,x') = 1. \quad \text{(A.2)}$$

There are four useful concepts to consider. The first will be the commutation of covariant derivatives. For the tensor $h^{\alpha\beta}$, the commutation of the covariant derivatives is by definition,

$$\left\langle \nabla_{\gamma} \nabla_{\delta} h^{\alpha\beta} \right\rangle = \left\langle \nabla_{\delta} \nabla_{\gamma} h^{\alpha\beta} - R_{\mu\alpha\beta\gamma} h^{\mu\delta} - R_{\mu\beta\gamma\alpha} h^{\mu\delta} \right\rangle, \quad \text{(A.3)}$$

where $R^{\alpha_{\beta\gamma\delta}}$ is the Riemann curvature tensor that corresponds to the background spacetime. From the field equations, we know that the curvature tensor is sourced by terms of $O(h^2)$. Therefore, we can see that the commutation of covariant derivatives goes as

$$\left\langle \nabla_{\gamma} \nabla_{\delta} h^{\alpha\beta} \right\rangle = \left\langle \nabla_{\delta} \nabla_{\gamma} h^{\alpha\beta} \right\rangle + O(h^3). \quad \text{(A.4)}$$

For our purposes, we neglect these higher order terms, allowing us to commute covariant derivatives without the addition of curvature terms.

The second useful point of emphasis is the vanishing of total divergences,

$$\left\langle \nabla_{\mu} X^{\mu} \right\rangle = 0, \quad \text{(A.5)}$$

for some $n$-rank tensor $X^{\mu a_1 \cdots a_{n-1}}$ that varies on the scale of the gravitational radiation wavelength. Substituting this quantity into the averaging integral generates four terms after integration by parts. The total divergence term will vanish since we turn it into a surface integral. The remaining three terms will be of $O(h)$ due to the bivector and kernel varying on scales larger than the wavelengths. We show this for one term below. Let $\lambda$ be the scale over which any perturbation varies while $\mathcal{R}$ is the scale length of fluctuations of the background. We will assume the high-frequency limit, which tells us that $\lambda \ll \mathcal{R}$. With this in mind, we see how the averaging modifies terms by looking at the scaling:

$$\left\langle \nabla_{\mu} X^A \right\rangle \sim g^{\mu}_{\mu'} \cdots g^{a_1}_{a_1'} \cdots g^{a_{n-1}}_{a_{n-1}'} X^{\mu' a_1' \cdots a_{n-1}'} \left( \nabla_{\mu} f \right),$$

$$\left\langle O \left( \frac{\partial X^A}{\partial \lambda} \right) \right\rangle \sim O \left( \frac{X^A}{\partial \mathcal{R}} \right).$$
\[ \sim O \left( \frac{\partial X^A}{\partial \lambda} \right) O \left( \frac{\partial \lambda}{\partial R} \right), \]  

(A.6)

where we have simplified notation such that \( A = (\mu, \alpha_1, \cdots, \alpha_{n-1}) \). We have made use of the fact that \( \partial f \) is of \( O(1) \) since it varies on scales related to the background geometry. Notice the averaging process added a term of \( O(\lambda/R) \) to the original tensor being averaged.

The third point to consider is the usefulness of integration by parts, which says that

\[ \langle \nabla_\mu (X^A) Y^B \rangle = -\langle X^A (\nabla_\mu Y^B) \rangle. \]  

(A.7)

Notice that the total divergence becomes of \( O(h) \) higher than the quantities remaining. Due to this, boundary terms do not need to be considered here.

The fourth and final point to consider is that the product of an odd number of the quantities being averaged over will vanish. This is straightforward when considering oscillatory functions which oscillate with a single frequency. An integral over an odd number of these quantities vanishes, which is again what happens here due to the fact that the averaging process involves integration.
APPENDIX B

ELECTROMAGNETIC CANONICAL SET
Consider the Lagrangian for classical electrodynamics in a source free spacetime,

$$\mathcal{L} = -\frac{1}{4} F_{\alpha \beta} F^{\alpha \beta}, \quad (B.1)$$

where $F_{\alpha \beta} \equiv 2 \delta_{[\alpha} A_{\beta]}$, with $A^{\alpha}$ being the 4-vector potential. Applying the canonical SET from Eq. (2.41) gives,

$$j^\alpha_\beta = -\frac{1}{4} \delta^\alpha_\beta F_{\mu \nu} F^{\mu \nu} - \partial_\beta A_\gamma F^{\gamma \alpha}. \quad (B.2)$$

Notice that the last term in Eq. (B.2) is not gauge invariant under the transformation $A^\alpha \rightarrow A^\alpha + \partial^\alpha \epsilon$. The solution to this is the Belinfante procedure [81, 82]. We here use the Guerrera and Hariton [83] implementation of the Belinfante procedure. The symmetric tensor is defined as

$$\Theta^{\alpha \beta}_{\text{GH}} = -\pi_\gamma^{(\alpha} \partial^{\beta) A^\gamma} + \partial_\delta \left( \pi_\gamma^{(\alpha} M^{\beta) \delta} A^\gamma \right) + g^{\alpha \beta} \mathcal{L}, \quad (B.3)$$

where $\pi_\gamma^{\alpha} \equiv \partial \mathcal{L} / \partial (\partial_\alpha A^\gamma)$ and $M^{\alpha \beta}$ is the spin tensor for the 4-vector potential. Applying Eq. (B.3) and using the Lorenz gauge and equations of motion in a source free medium,

$$\partial_\alpha A^\alpha = 0, \quad (B.4a)$$
$$\partial_\alpha \partial^\alpha A^\beta = 0, \quad (B.4b)$$

we arrive at the classical result for the electromagnetic SET

$$\Theta^{\alpha \beta}_{\text{GH}} = F^{\alpha \gamma} F_\gamma^\beta - \frac{1}{4} g^{\alpha \beta} F^{\gamma \delta} F_{\gamma \delta}. \quad (B.5)$$

Equation (B.3) was found using the assumption that the theory in question is Lorentz invariant. This was necessary in order to derive the correct $M^{\alpha \beta}$ tensors, which are the generators of infinitesimal Lorentz transformations.
APPENDIX C

DERIVATION OF JFBD REDUCED FIELD
In accordance with the procedures outlined in [40], we consider the notion of a “reduced field.” The reduced field is the field which enables the field equations to be written as

\[
\left( -\frac{1}{v_g^2} \frac{\partial^2}{\partial t^2} + \partial_i \partial^i \right) \mathcal{A} = -16\pi \mathcal{S},
\]  

where \( \mathcal{A} \) is the linear perturbation of the reduced field, \( v_g \) is the speed of propagation of the reduced field, and the source \( \mathcal{S} \) is the combination of matter and higher order perturbation effects. Notice that in GR, the “reduced field” is the trace-reversed metric perturbation.

For the work dealing with scalar-tensor theory in Chap. 2, the reduced field will need to be derived. This is accomplished through the use of decomposition. Let us assume that this decomposition takes the form

\[
\theta_{\alpha\beta} = h_{\alpha\beta} + C_1 \tilde{g}_{\alpha\beta} h + C_2 \tilde{g}_{\alpha\beta} \varphi,
\]

where \( \tilde{g}_{\alpha\beta} \) in Eq. (C.2) is any background metric. Since we need linear order terms, we expand the action in Eq. (2.52) to \( O(h) \) and vary with respect to the background metric to obtain the field equations in terms of \( h_{\alpha\beta} \) and \( \varphi \),

\[
0 = -\frac{1}{2} \Box \left( h_{\alpha\beta} - \tilde{g}_{\alpha\beta} h - 2\tilde{g}_{\alpha\beta} \frac{\varphi}{\phi_0} \right)
+ \frac{1}{2} \left( 2\partial_\gamma \partial_{(\alpha} h_{\beta)} \gamma - \partial_\alpha \partial_\beta h - \tilde{g}_{\alpha\beta} \partial_\gamma \partial_\delta h^{\gamma\delta} - \frac{2}{\phi_0} \partial_\alpha \partial_\beta \varphi \right).
\]  

Any derivative here is with respect to the background metric. At this point, we substitute in the reduced field using Eq. (C.2)

\[
0 = -\frac{1}{2} \phi_0 \left[ \theta_{\alpha\beta} + \left( 2 C_2 - \frac{2}{\phi_0} - \frac{8C_1 C_2}{1 + 4C_1} \right) \tilde{g}_{\alpha\beta} \varphi + \left( \frac{2C_1}{1 + 4C_1} - 1 \right) \tilde{g}_{\alpha\beta} \theta \right]
+ \phi_0 \partial_\alpha \partial_\beta \left[ \left( \frac{C_1}{1 + 4C_1} - \frac{1}{2} \right) \theta + \left( C_2 - \frac{1}{\phi_0} - \frac{4C_1 C_2}{1 + 4C_1} \right) \varphi \right]
+ \partial_\gamma \partial_{(\alpha} \theta_{\beta)} \gamma + \frac{\phi_0}{2} \tilde{g}_{\alpha\beta} \partial_\gamma \partial_\delta \theta^{\gamma\delta}.
\]  

We know we are looking for the only \( \Box \) term to be \( \Box \theta_{\alpha\beta} \). This allows for us to solve for \( C_1 \) and \( C_2 \) by eliminating the terms proportional to \( \theta \) and \( \varphi \) in the first bracket, leading to

\[
C_1 = -\frac{1}{2},
\]

\[
C_2 = -\frac{1}{\phi_0}.
\]
With the inclusion of these constants, the linearized reduced field equations become

\[
\Box \theta_{\alpha\beta} - \frac{2}{\phi_0} \partial_\gamma \partial_{(\alpha} \theta_{\beta)} \gamma + \tilde{g}_{\alpha\beta} \partial_\gamma \partial_\delta \theta^{\gamma\delta} = 0. 
\]

(C.6)

We are now free to impose the Lorenz gauge condition for the reduced field. This eliminates the final two terms in Eq. (C.6), leading to the field equations for the reduced field.
APPENDIX D

EXPANDED ACTION FOR EINSTEIN-ÆETHER
Here, we state the expanded action for Einstein-\AE ther. These results were obtained through the use of the xTensor package for Mathematica [183, 184].

\begin{align}
S^{(0)}_x &= \frac{1}{16\pi G} \int d^4 x \sqrt{-g} \left( \tilde{\mathcal{R}} - c_1 (\tilde{\nabla}_\alpha \tilde{u}_\beta) (\tilde{\nabla}^\alpha \tilde{u}^\beta) - c_2 (\tilde{\nabla}_\alpha \tilde{\nabla}^\alpha) (\tilde{\nabla}_\beta \tilde{\nabla}^\beta) \\
&\quad - c_3 (\tilde{\nabla}_\alpha \tilde{u}_\beta) (\tilde{\nabla}^\beta \tilde{u}^\alpha) + c_4 \tilde{u}^\alpha \tilde{u}^\beta (\tilde{\nabla}_\alpha \tilde{\nabla}^\alpha) (\tilde{\nabla}_\beta \tilde{\nabla}^\beta) \right), \\
S^{(1)}_x &= \frac{1}{16\pi G} \int d^4 x \sqrt{-g} \left[ \frac{1}{2} h \tilde{\mathcal{R}} - h^{\alpha \beta} \tilde{\mathcal{R}}_{\alpha \beta} + \tilde{\nabla}_\alpha \tilde{\nabla}_\beta h^{\alpha \beta} - \Box h \\
&\quad + c_1 \left( h_{\alpha \gamma} (\tilde{\nabla}^\alpha \tilde{u}^\beta) (\tilde{\nabla}_\beta \tilde{u}_\gamma) + 2 \tilde{u}^\alpha (\tilde{\nabla}_\gamma \tilde{h}_{\alpha \gamma}) (\tilde{\nabla}^\gamma \tilde{u}^\beta) - \frac{1}{2} h (\tilde{\nabla}_\gamma \tilde{u}_\alpha) (\tilde{\nabla}^\gamma \tilde{u}^\beta) \right) \\
&\quad - h_{\alpha \gamma} (\tilde{\nabla}_\beta \tilde{u}^\gamma) (\tilde{\nabla}^\alpha \tilde{u}^\beta) - \tilde{u}^\alpha (\tilde{\nabla}_\gamma h_{\alpha \beta}) (\tilde{\nabla}^\gamma \tilde{u}^\beta) - 2 (\tilde{\nabla}_\alpha \omega^\beta) (\tilde{\nabla}^\alpha \tilde{u}^\beta) \right] \\
&\quad + c_2 \left( \tilde{u}^\alpha (\tilde{\nabla}_\gamma \tilde{h}_{\beta \gamma}) (\tilde{\nabla}^\gamma \tilde{u}^\beta) - \tilde{u}^\alpha (\tilde{\nabla}_\gamma h_{\beta \gamma}) (\tilde{\nabla}^\gamma \tilde{u}^\beta) - \frac{1}{2} h (\tilde{\nabla}_\alpha \tilde{u}_\beta) (\tilde{\nabla}^\alpha \tilde{u}^\beta) - 2 (\tilde{\nabla}_\alpha \omega^\beta) (\tilde{\nabla}^\alpha \tilde{u}^\beta) \right) \\
&\quad + c_4 \left( \frac{1}{2} h \tilde{u}^\alpha \tilde{u}^\beta (\tilde{\nabla}_\gamma \tilde{h}_{\alpha \gamma}) (\tilde{\nabla}^\gamma \tilde{u}^\beta) + h_{\alpha \gamma} \tilde{u}^\alpha \tilde{u}^\beta (\tilde{\nabla}_\gamma \tilde{u}^\gamma) \right) \\
&\quad + 2 \tilde{u}^\alpha \tilde{u}^\beta (\tilde{\nabla}_\gamma \tilde{h}_{\gamma \beta}) (\tilde{\nabla}^\gamma \tilde{u}^\beta) - \tilde{u}^\alpha \tilde{u}^\beta (\tilde{\nabla}_\gamma \tilde{u}^\gamma) (\tilde{\nabla}_\delta \tilde{h}_{\gamma \beta}) \\
&\quad + 2 \omega^\alpha \tilde{u}^\beta (\tilde{\nabla}_\gamma \tilde{h}_{\gamma \beta}) (\tilde{\nabla}^\gamma \tilde{u}^\beta) + 2 \tilde{u}^\alpha \tilde{u}^\beta (\tilde{\nabla}_\gamma \tilde{u}^\gamma) (\tilde{\nabla}_\delta \tilde{h}_{\gamma \beta}) \right] \\
S^{(2)}_x &= \frac{1}{16\pi G} \int d^4 x \sqrt{-g} \left[ h^{\alpha \beta} h^{\gamma \delta} \tilde{\mathcal{R}}_{\alpha \beta \gamma \delta} - \frac{1}{2} h \tilde{\mathcal{R}}_{\alpha \beta} + \frac{1}{8} h^2 \tilde{\mathcal{R}} + \frac{1}{4} (\tilde{\nabla}_\alpha h) (\tilde{\nabla}^\alpha h) \\
&\quad - \frac{1}{2} (\tilde{\nabla}_\alpha h) (\tilde{\nabla}^\alpha h_{\beta \gamma}) + \frac{1}{2} (\tilde{\nabla}_\alpha h_{\gamma \beta}) (\tilde{\nabla}^\gamma h_{\alpha \beta}) - \frac{1}{4} (\tilde{\nabla}_\alpha h_{\gamma \beta}) (\tilde{\nabla}^\gamma h_{\alpha \beta}) \right] \\
&\quad + c_1 \left( \frac{1}{4} h_{\gamma \beta} (\tilde{\nabla}^\gamma \tilde{u}^\delta) (\tilde{\nabla}^\delta \tilde{u}^\gamma) - \frac{1}{4} \tilde{u}^\alpha (\tilde{\nabla}_\gamma \tilde{h}_{\alpha \gamma}) (\tilde{\nabla}^\gamma h_{\beta \gamma}) \right) \\
&\quad + \frac{1}{4} h_{\alpha \beta} h^{\gamma \delta} (\tilde{\nabla}^\gamma \tilde{h}_{\gamma \delta}) (\tilde{\nabla}^\delta \tilde{u}^\gamma) - \frac{1}{2} h^2 (\tilde{\nabla}_\alpha \tilde{u}_\beta) (\tilde{\nabla}^\alpha \tilde{u}^\beta) + \frac{1}{2} h_{\alpha \beta} (\tilde{\nabla}^\gamma \tilde{h}_{\gamma \beta}) (\tilde{\nabla}^\gamma \tilde{u}^\gamma) \\
&\quad - h_{\alpha \beta} (\tilde{\nabla}_\gamma \tilde{h}_{\delta \gamma}) (\tilde{\nabla}^\gamma \tilde{u}^\delta) + \frac{1}{2} h_{\alpha \beta} (\tilde{\nabla}_\gamma \tilde{u}_\gamma) (\tilde{\nabla}^\gamma \tilde{u}^\delta) - h \tilde{u}^\delta (\tilde{\nabla}_\gamma \tilde{h}_{\alpha \gamma}) (\tilde{\nabla}^\gamma \tilde{u}^\delta) \\
&\quad + 2 h_{\alpha \beta} (\tilde{\nabla}_\gamma \tilde{h}_{\gamma \beta}) (\tilde{\nabla}^\gamma \tilde{u}^\gamma) + \frac{1}{2} h \tilde{u}^\delta (\tilde{\nabla}_\gamma \tilde{h}_{\alpha \gamma}) (\tilde{\nabla}^\gamma \tilde{u}^\beta) + h_{\alpha \beta} (\tilde{\nabla}^\gamma \tilde{h}_{\gamma \beta}) (\tilde{\nabla}^\gamma \tilde{u}^\gamma) \\
&\quad + \tilde{u}^\delta (\tilde{\nabla}_\gamma \tilde{h}_{\alpha \gamma}) (\tilde{\nabla}^\gamma \tilde{u}^\gamma) - \tilde{u}^\delta (\tilde{\nabla}_\gamma \tilde{h}_{\alpha \gamma}) (\tilde{\nabla}^\gamma \tilde{u}^\gamma) - \tilde{u}^\delta (\tilde{\nabla}_\gamma \tilde{h}_{\alpha \gamma}) (\tilde{\nabla}^\gamma \tilde{u}^\gamma) \right] \\
&\quad + c_2 \left( \frac{1}{4} h_{\alpha \beta} h^{\gamma \delta} (\tilde{\nabla}^\gamma \tilde{h}_{\gamma \delta}) (\tilde{\nabla}^\delta \tilde{u}^\gamma) - \frac{1}{4} \tilde{u}^\alpha (\tilde{\nabla}_\gamma \tilde{h}_{\alpha \gamma}) (\tilde{\nabla}^\gamma h_{\delta \gamma}) + h^{\alpha \beta} (\tilde{\nabla}^\gamma \tilde{h}_{\gamma \beta}) (\tilde{\nabla}^\gamma \tilde{u}^\delta) \right)
\end{align}