THE STRUCTURE OF ENERGY-EXTRACTING BLACK HOLE MAGNETOSPHERES

by

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Doctor of Philosophy in

Physics

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For Peter
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NOMENCLATURE

Newtonian (non-relativistic flat space) vectors will be denoted with an arrow; for example \( \vec{B} \) denotes the magnetic field as typically considered in three dimensions. Fully relativistic quantities will be denoted using index notation; for example \( u^\alpha \) denotes a four velocity. The Einstein summation convention is used throughout and repeated indices are summed over. When indexing relativistic quantities Greek indices \((\alpha, \beta, \gamma, \ldots)\) will be used to denote temporal and spatial indices, lowercase Latin indices \((a, b, c, \ldots)\) will denote spatial indices, uppercase Latin indices near the beginning of the alphabet \((A, B, C)\) will denote poloidal indices, and uppercase Latin indices near the end of the alphabet \((X, Y, Z)\) will denote temporal and azimuthal indices:

\[
\begin{align*}
  u_\alpha u^\alpha &= u_t u^t + u_r u^r + u_\theta u^\theta + u_\phi u^\phi, \\
  u_a u^a &= u_r u^r + u_\theta u^\theta, \\
  u_A u^A &= u_r u^r + u_\theta u^\theta, \\
  u_Z u^Z &= u_t u^t + u_\phi u^\phi.
\end{align*}
\]

(0.1)

Partial derivatives will either be denoted with a comma or the symbol \( \partial \); \( t,\alpha = \partial_\alpha t \). Covariant derivatives will be denoted with either a semicolon or the symbol \( \nabla \), with the appropriate connection coefficients (Christoffel symbols) written as \( \Gamma^\gamma_{\alpha\beta} \); for example \( u^\alpha \beta = D_\beta u^\alpha \) and \( u^\alpha \beta = u^\alpha \beta + \Gamma^\alpha_{\beta\gamma} u^\gamma \). Field-aligned derivatives will typically be denoted with a prime; for example if in three dimensions \( \hat{b} \) is a unit vector in the direction of the magnetic field, then \( \rho' = \hat{b} \cdot \vec{\nabla} \rho \). The individual components of a vector (or covector) will be written in order as \((t, r, \theta, \phi)\); for example \( u^a = (u^t, u^r, u^\theta, u^\phi) \) and \( u_t = (u_t, u_r, u_\theta, u_\phi) \). That convention extends to tensors, which will be written as (for both lowered and raised indices):

\[
F^{\alpha\beta} = \begin{pmatrix}
  F^{tt} & F^{tr} & F^{t\theta} & F^{t\phi} \\
  F^{rt} & F^{rr} & F^{r\theta} & F^{r\phi} \\
  F^{\theta t} & F^{\theta r} & F^{\theta \theta} & F^{\theta \phi} \\
  F^{\phi t} & F^{\phi r} & F^{\phi \theta} & F^{\phi \phi}
\end{pmatrix}.
\]

(0.2)

Geometrized units will be primarily used; specifically the speed of light \( c \) and gravitational constant \( G \) are generally set equal to unity. They will be written explicitly only in cases where their exclusion would be harmful to clarity, such as when making a series expansion in \( c \) of a relativistic quantity in order to show correspondences to non-relativistic quantities. A \((+,−,−,−)\) metric signature will be used throughout; for example the velocity normalization condition for a timelike observer is \( u_\alpha u^\alpha = 1 \). When specific values are required, centimeter–gram–second (CGS) units will be used.
Selected Variable Reference

Unless otherwise explicitly noted, the following variable definitions are used throughout. When explicit components are given, Boyer-Lindquist coordinates of the form shown in Chapter 2 should be assumed. When equivalences are given, they should be understood to be valid within the assumptions applicable to this work as stated in Chapter 2. When vector components are given, they are listed in order as \((t, r, \theta, \phi)\). Appropriate explanations and derivations of the variables will be provided in the main text; this table is only intended to provide a convenient reference for commonly occurring variables and is not exhaustive.

\[
\begin{align*}
g_{\alpha\beta} & \quad \text{Metric Tensor} \\
F^{\alpha\beta} & \quad \text{Maxwell Field Strength Tensor} \\
\mathcal{F}^{\alpha\beta} & \quad \text{Dual Field Strength Tensor; } \mathcal{F}^{\alpha\beta} = (1/2\sqrt{-g})\epsilon^{\alpha\beta\mu\nu}F_{\mu\nu}. \\
T^{\alpha\beta} & \quad \text{Stress Energy Tensor} \\
u^{\alpha} & \quad \text{Plasma Four Velocity} \\
N^{\alpha} & \quad \text{Particle Number Flux Vector; } N^{\alpha} = nu^{\alpha} \text{ and } N^{\alpha}_{,\alpha} = 0. \\
J^{\alpha} & \quad \text{Four Current} \\
A_{\alpha} & \quad \text{Vector Potential; } A_{\alpha} = A_{\beta,\alpha} - A_{\alpha,\beta}. \\
B^{\alpha} & \quad \text{Magnetic Field Four Vector; } B^{\alpha} = (1/2\sqrt{-g})\epsilon^{\alpha\beta\gamma\delta}t_{\beta}F_{\gamma\delta}. \\
k^{\alpha} & \quad \text{Temporal Killing Vector; } k^{\alpha} = (1, 0, 0, 0). \\
l^{\alpha} & \quad \text{Azimuthal Killing Vector; } l^{\alpha} = (0, 0, 0, 1). \\
t^{\alpha} & \quad \text{Coordinate Time Gradient; } t^{\alpha} = (1, 0, 0, 0). \\
r^{\alpha} & \quad \text{Rotation Vector; } r^{\alpha} = k^{\alpha} + \Omega^{\alpha}_{F}l^{\alpha}. \\
\epsilon^{\alpha\beta\mu\nu} & \quad \text{Fully Antisymmetric Symbol; } \epsilon^{tr\theta\phi} = 1, \epsilon^{rt\theta\phi} = -1. \\
X^{\alpha} & \quad \text{Momentum Flux Vector; } T^{\alpha\beta}_{,\beta} = X^{\alpha}. \\
\vec{B} & \quad \text{Magnetic Field, Newtonian Context} \\
\vec{E} & \quad \text{Electric Field, Newtonian Context} \\
\vec{J} & \quad \text{Electric Current, Newtonian Context} \\
\rho & \quad \text{Relativistic Plasma Energy Density} \\
p & \quad \text{Proper Plasma Pressure} \\
\sigma & \quad \text{Plasma Conductivity} \\
n & \quad \text{Plasma Number Density} \\
\mu & \quad \text{Relativistic Plasma Enthalpy; } \mu = (\rho + p)/n. \\
\mathcal{M}^2 & \quad \text{Plasma Alfvén Mach Number; } \mathcal{M}^2 = 4\pi(\mu\eta)^2/\mu n. \\
u_p & \quad \text{Poloidal Plasma Velocity; } u_p^2 = -u_A u^A. \\
B_p & \quad \text{Poloidal Magnetic Field Strength; } B_p^2 = B_A B^A. \\
\Omega_F & \quad \text{Field Line Angular Velocity; } F_{r\phi}\Omega_F = F_{r\theta}, \ F_{\phi\theta}\Omega_F = F_{t\theta}. \\
\eta & \quad \text{Conserved Particle Flux; } \eta B^A = -N^A, \ \eta B^\phi = -N^\phi + N^t\Omega_F. \\
E & \quad \text{Conserved Energy Flux; } \eta E = \mu n u_t + (1/4\pi)\sqrt{-g}F^{\theta\phi}\Omega_F.
\end{align*}
\]
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<td>( r_H )</td>
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<td>( \omega_H )</td>
<td>Horizon Angular Velocity; ( \omega_H = a/2mr_H ).</td>
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<tr>
<td>( \Delta )</td>
<td>Metric Component; ( \Delta = r^2 + a^2 - 2mr, \Delta(r_H) = 0 ).</td>
</tr>
<tr>
<td>( \Sigma )</td>
<td>Metric Component; ( \Sigma = r^2 + a^2 \cos^2 \theta ).</td>
</tr>
<tr>
<td>( g )</td>
<td>Metric Determinant; ( \sqrt{-g} = \Sigma \sin \theta ).</td>
</tr>
<tr>
<td>( \rho_\omega )</td>
<td>Cylindrical Radius; ( \rho_\omega^2 = g_{\omega \phi} - g_{tt} g_{\phi \phi} = \Delta \sin^2 \theta ).</td>
</tr>
<tr>
<td>( G_t )</td>
<td>Metric and Field Component; ( G_t = g_{tt} + g_{t \phi} \Omega_F ).</td>
</tr>
<tr>
<td>( G_\phi )</td>
<td>Metric and Field Component; ( G_\phi = g_{\phi \phi} + g_{\phi \phi} \Omega_F ).</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>Gravitational Lorentz Factor; ( \alpha = G_t + G_\phi \Omega_F ).</td>
</tr>
</tbody>
</table>
ABSTRACT

Spinning black holes can store enormous amounts of rotational energy. Efficiently extracting that rotational energy can lead to significant energy outflows capable of powering very high energy astrophysical phenomena, such as gamma-ray bursts and active galactic nuclei. Black holes are unique in that they do not exist as physical objects in the same way a rock, planet, or star exists; instead, black holes exist only as spacetime curvature. As such processes for extracting a black hole's rotational energy are largely unique to black holes. This work explores one such process, the extraction of a black hole's rotational energy via an appropriately configured magnetosphere. Both analytic perturbation techniques and numerical codes are developed in order to solve for thousands of energy-extracting black hole magnetospheres. Those magnetospheres broadly sample the relevant solution space, allowing correlations to be drawn between different rates of black hole rotational energy and angular momentum extraction and global magnetosphere structure. The most fundamental behavior discovered is that magnetospheres that extract the most energy per unit angular momentum direct that energy away from the black hole's rotational axis, while magnetospheres that extract the least amount of energy per unit angular momentum direct that energy into jet-like structures aligned with the black hole's rotational axis. Exploration of the solutions obtained also suggests that magnetospheres most compatible with nearby accreting matter can very naturally launch jets, implying that black hole energy extraction and jet launching are likely to be concurrent and common features of astrophysical black hole magnetospheres.
INTRODUCTION

In this chapter a brief summary and background of the physical principles and prior work relevant to our study of energy-extracting black hole magnetospheres will be given. We will begin with a short discussion of historical background, theoretical energy levels, and astrophysical phenomena associated with black hole energy extraction. Next we will discuss the mechanical Penrose process, a simple method of extracting a black hole’s energy that illuminates the fundamental physical principles involved in black hole energy extraction. The Blandford-Znajek mechanism will then be discussed as a more practical method of extracting a black hole’s rotational energy via an appropriately configured magnetospheres. The generalization of the Blandford-Znajek mechanism to ideal magnetohydrodynamics will then be explored as the central physical mechanism on which this work relies. Finally the existing solution space of energy-extracting black hole magnetospheres known prior to this work will be outlined, followed by a description of the enhancements to that solution space that we have provided.

**Brief Background**

In 1687, Newton hypothesized in the third book of his *Principia* that two objects will be attracted to one another with a gravitational force proportional to the product of their masses and inversely proportional to the square of the distance separating them [67, 98]. If the two objects are initially touching, there is a minimum initial “escape velocity” required for one object to overcome the gravitational pull of the other and travel to asymptotic infinity. As one of the objects increases in mass, such
as in a transition from a pebble to a boulder to a planet, the velocity required to
ecape from the gravitational pull of that object also increases. In 1783, Michell used
that relationship to hypothesize the existence of stars so massive that their escape
velocity was larger than the speed of light; if light could not escape from their surface,
such stars would be dark and effectively invisible to astronomers [13, 62]. In 1916
Schwarzschild [95, 96] made the concept of such an object robust within the framework
of Einstein’s theory of general relativity, published one month earlier [21, 92]. Such
an object is now referred to as a “black hole”. Rather than having a physical surface
from which it is impossible to escape, a Schwarzschild black hole present itself to
the outside universe as nothing more than spacetime curvature, which is to say as
a gravitational force. The only “surface” a Schwarzschild black hole possesses is an
“event horizon”, which is not a physical object or surface in any conventional sense,
but rather merely the point in space at which light can no longer travel outward and
away from the black hole.

A Schwarzschild black hole is defined purely in terms of its mass. However black
holes must conserve any electric charge and angular momentum that crosses their
event horizon, so a more general black hole is described by its mass, net electric
charge, and “spin”. It is generally considered unlikely that astrophysical black holes
would possess significant amounts of electric charge, as any such charge would be
quickly neutralized, though it has been argued [88] that there might be exceptions.
Angular momentum, however, is likely to be a common feature of black holes; such a
spinning black hole can be described by the Kerr metric [39]. Unlike a Schwarzschild
black hole, a Kerr black hole has two relevant features: an event horizon and an
ergosphere. The ergosphere is the region outside the event horizon where light can
no longer counter-rotate; the spacetime is rotating (“frame-dragging”) to such an
extreme degree that light only rotates with the black hole. As such the the ergosphere
may be thought of as being akin to a rotating, one-dimensional event horizon. The descripti
dive principles at work are roughly the same, the most significant difference is that par
ticles and information can escape from inside the ergosphere, but they cannot escape from inside the event horizon. Ultimately it is the special properties of the ergosphere that allow a black hole’s rotational energy to be extracted.

The event horizon of a non-rotating black hole has a greater surface area than the event horizon of a rotating black hole of equivalent mass. Crudely speaking, this is because the radius of the event horizon of a non-rotating black hole is larger than that of a rotating black hole of equivalent mass, as in the case of the rotating black hole effective centripetal forces from spacetime rotation work to counteract effective gravitational forces and allow particles and light to get closer to the black hole before becoming trapped. Within the limit of classical processes, the area of a black hole’s horizon can never shrink \cite{34}. Because a rotating black hole has a smaller event horizon area than a non-rotating black hole, however, a rotating black hole can in theory be “spun down” to a less massive non-rotating black hole with an event horizon of equivalent area. That lesser mass is the “irreducible” mass of a rotating black hole \cite{14}. The “reducible” mass (the difference between the irreducible mass and the initial mass) is, at least in theory, rotational energy that could be extracted from a rotating black hole and transmitted to distant observers.

Due to their potentially enormous mass, rotating black holes can carry concurrently enormous amounts of theoretically extractable energy; by the early 1970’s they were widely recognized as being possibly the “the largest storehouse of energy in the universe” \cite{15}. The spin of a rotating black hole, weighted to its mass, ranges from $-1$ to $1$ within the description of the Kerr metric. The extremes are analogous to the break up velocity of a star, the rotational rate at which the outer layers of a star will be flung away as outward centripetal forces overpower inward gravitational forces.
The sign denotes the direction of rotation, so ignoring spatial orientation the spin of a Kerr black hole effectively ranges from 0 (a non-rotating Schwarzschild black hole) to 1 (referred to as extremal spin). The reducible mass of a rotating black hole similarly ranges from 0 for a non-rotating black hole to around 29% (specifically $1 - 1/\sqrt{2}$) of its total mass in the case of extremal spin; on average around 6% of a rotating black hole’s mass is reducible. If processes were available to convert those reducible masses to energy, it is immediately apparent that rotating black holes could easily provide enough energy to power high luminosity phenomena ranging from gamma-ray bursts to active galactic nuclei. For example, the maximum reducible mass of a black hole of mass $m$ approaches $10^{54}(m/M_\odot)^2$ erg. The initial (prompt) phase of gamma-ray bursts (generally lasting less than 100 s) typically releases an energy content of around $10^{51}$ erg [26], implying that a rotating stellar mass black hole could easily serve as the central engine of a gamma-ray burst. Active galactic nuclei typically have central supermassive black holes (masses $\gtrsim 10^6M_\odot$) and luminosities up to $\sim 10^{48}$ erg/s [69]; an initially extremal $\sim 10^6M_\odot$ black hole could in theory provide that luminosity for around 30 billion years before being completely spun down. Gamma-ray bursts and active galactic nuclei are both highly heterogeneous sets of phenomena that have a variety of potential physical sources beyond black hole energy extraction, but none of those sources could in theory provide as much energy as a rapidly rotating black hole. That places rotating black holes as the leading, if not only, candidate for powering some of the most electromagnetically luminous transient and persistent astrophysical phenomena observed.

Rotating black holes can carry enormous amounts of extractable energy that might provide a useful power source for highly luminous astrophysical phenomena, but the question still remains if there are any physically reasonable processes that might serve to naturally and productively extract that energy. Ultimately the
reasonableness of any proposed processes will necessarily heavily rely on a given black hole’s environment, which is to say the astrophysical phenomena being considered. The black hole itself is nothing but spacetime curvature described two scalars, a mass and a spin, so additional mechanisms need to be invoked in order to extract a black hole’s rotational energy that might be more or less applicable or relevant to different contexts. That means that there are two general approaches (not necessarily exclusive or opposed) to exploring black hole energy extraction. The first is to assume a specific astrophysical context and see if that context leads to black hole energy extraction. The second is to generically explore processes that might be capable of extracting a black hole’s rotational energy without explicitly invoking a specific mechanism for driving that process; for example, magnetic fields and a plasma might be assumed to exist in a specific configuration near the horizon without significant concern as to how those fields and plasma might be sourced. In this work we largely take the latter approach, and explore black hole energy extraction in a way that is largely agnostic to astrophysical setting. When appropriate we will connect our results and assumptions to various astrophysical scenarios, but our goal is not to directly describe any specific astrophysical object; instead our goal is to study black hole energy extraction in as isolated and independent a manner as is practical and reasonable.

The Penrose Process

The basic principles of extracting an isolated black hole’s rotational energy were first discussed by Penrose [83, 84]. Although the specific example mechanism invoked was acknowledged to be impractical, its basic features are common to the more practical methods we will study (which are sometimes placed under the label of “generalized Penrose processes”). As such it is useful to briefly explore the more specific “Penrose process”, as its relative simplicity can better elucidate the processes
we will study.

Within the ergosphere, where particles must rotate with the black hole, it is possible for a particle to have negative energy (with respect to infinity); energy is defined here as the contraction of the particle’s four momentum with the spacetime’s temporal Killing vector. With respect to zero angular momentum observers such particles are typically very rapidly counter-rotating (Appendix A), and after crossing the event horizon would reduce both the spin and energy of the black hole. Viewed from spatial infinity, a stream of such particles would be seen as a negative energy flux crossing the horizon; near horizon observers would see a stream of rapidly counter-rotating positive energy particles crossing the horizon.

Particle collisions can be invoked to create such negative energy particles. For example, if two positive energy particles fall into the ergosphere and collide, post collision one could have negative energy and the other positive energy. Applying energy conservation, if the positive energy particle then escapes the ergosphere it will carry more energy out than the two particles initially carried in; some of the black hole’s rotational energy will have been extracted. That type of process (two particles falling into the ergosphere, one higher energy particle leaving) is typically what is meant when the phrase “Penrose process” is used without qualification.

There are three things to note about a Penrose process that are common to other mechanisms for extracting a black hole’s rotational energy (those commonalities occasionally lead to them being placed under the umbrella of “generalized Penrose processes” with “mechanical Penrose process” specifying particle collisions). First, something must take place inside the ergosphere. In the example of colliding particles, it is impossible for negative energy particles to exist outside the ergoregion (they would be moving faster than the speed of light if they escaped), so any creation of negative energy particles must occur within the ergosphere. Second, the horizon is
somewhat irrelevant. In the case of colliding particles it is tacitly assumed that the negative energy particle will eventually cross the horizon, but there is no requirement that it immediately do so. The extraction of a black hole’s rotational energy occurs purely as a result of interactions within the ergosphere; there is no rotating surface (horizon or otherwise) that is directly slowed down. The horizon does play a role in the stability of an ergosphere, however; an object with an ergosphere but no horizon would be unstable [24]. Third, and lastly, a black hole cannot spin itself down; external mechanisms must be applied, such as the generation of appropriately configured particles and trajectories.

Ultimately the Penrose process is fairly impractical, as collisions resulting in negative energy particles are generally somewhat contrived (physically related to the fact that negative energy particles are typically extremely energetic from the perspective of zero angular momentum observers). Nonetheless such processes have been suggested as sources for very high energy cosmic rays [3] due to the fact that the positive energy particle that escapes the ergoregion could potentially be extremely energetic (but see [93, 126] for some objections). In any reasonable context productive particle collisions would likely be rare enough that any energy outflow would be dwarfed by the energy inflow of particles that failed to collide productively, however, and as such significant black hole energy extraction via a Penrose process would be unlikely to occur naturally. Nonetheless the Penrose process illustrates that black hole energy extraction is possible, however impractical its specifics might be.

**The Blandford-Znajek Mechanism**

The first demonstration of the possibility of practical black hole energy extraction was given by Blandford and Znajek in 1977 [6]. Rather than relying on the creation of negative energy particles, they proposed a negative energy Poynting
flux through an appropriately configured magnetosphere, directly analogous to the way in which a pulsar can be spun down [28]. For the magnetic field, they suggested immersing a rotating black hole into a split-monopolar magnetic field (that monopolarity being a useful example, not a requirement). In order to create the plasma, they suggested a pair production cascade [12, 59, 89]. The effects of spacetime rotation would cause observers near the horizon to see a concurrent electric field (a “spark gap”) which could accelerate electrons (or positrons) to energies sufficient to radiate high-energy photons and create electron-positron pairs, which would in turn be accelerated and lead to a pair production cascade. If the initial magnetic field is appropriately configured, such cascades will lead to the creation of a force-free magnetosphere.

Once a force-free plasma was presumptively established, Blandford & Znajek [6] demonstrated that if the magnetosphere co-rotated more slowly than the horizon then a negative energy field-aligned conserved Poynting flux directed into the horizon could emerge (local horizon observers would see a positive energy Poynting flux into the horizon). The magnitude of the equivalent outward Poynting flux of positive energy is largely a function of the strength of the magnetic field, allowing for almost arbitrarily large energy outflows.

In order to demonstrate that an appropriately configured magnetosphere for energy extraction might emerge naturally, Blandford & Znajek [6] extended an initially non-rotating monopolar magnetosphere around a static black hole to a slowly rotating black hole via a perturbation in black hole spin. The conclusion was that the initially non-rotating magnetosphere would begin to rotate at half the rate at which the black hole rotated, demonstrating that under at least some circumstances a black hole immersed in a force-free plasma could be efficiently spun down and have its energy extracted. A magnetosphere rotating at half the rate of the black hole is still to
this day widely assumed to be synonymous with the “Blandford-Znajek mechanism”, although (as we will show) that rotational rate is correlated to a monopolar geometry and should not be generically expected in all magnetospheres.

Although Blandford & Znajek [6] did show that efficient black hole energy extraction was possible, many open questions remained. Among other things, it was unclear if the assumed flux of magnetic field threading the horizon was completely reasonable, if a force-free plasma could actually be developed, if a magnetosphere configuration corresponding to energy extraction would be stable, what physical relevance the rate of magnetosphere rotation might have, and, perhaps most importantly, if an appropriately configured magnetosphere would actually emerge naturally in relevant astrophysical contexts. Such questions are still under active exploration today [38, 50, 53, 64, 70, 78, 91, 97, 108], although the general consensus is that energy-extracting magnetospheres are a reasonable expectation in many astrophysical contexts, although the relative importance of the energy extracted (as compared to other sources, such as from neutrino annihilation and accretion disk physics) is debated.

Most of the open questions noted above require a specific astrophysical context to study, and as such are less fundamental questions of black hole energy extraction and more questions specific to a given astrophysical phenomena. The exception is the physical relevance and significance of magnetosphere rotation. Exploring the effects and consequences of that rotation is a large focus of this work, and the physical significance of that rotation is most readily apparent by generalizing from a force-free plasma to an ideal plasma, as we will now discuss.
In 1990, Takahashi et al. [101] explored the nature of black hole energy extraction via a magnetosphere composed of an ideal plasma (as opposed to the simpler case of a force-free plasma in Blandford & Znajek [6], where plasma inertial effects are ignored). The equations governing force balance across magnetic field lines (typically referred to as either the general relativistic Grad-Shafranov or transfield equations) were far too complex to be solved, but under the assumption of a single roughly monopolar field line it was possible to further elucidate the nature of black hole energy extraction via an appropriately configured magnetosphere.

The conclusion most relevant to the current discussion was that Blandford and Znajek’s discovery that a magnetosphere must co-rotate more slowly than the black hole in order to extract the black hole’s rotational energy is equivalent to the statement that a plasma inflow must become super-Alfvénic inside the ergoregion if energy is to be extracted. The limits found by Blandford and Znajek coincide with ingoing Alfvén points (or surfaces) along the horizon or the outer limit of the ergosphere, as in the limit of a force-free plasma the specification of a magnetosphere’s rotational profile is equivalent to specifying the location of its ingoing and outgoing Alfvén surfaces. That corresponds to the fundamental conclusion of the Penrose process, in that something must occur within the ergoregion (and that the horizon is somewhat irrelevant outside of its role in allowing for a stable ergoregion).

Later studies [25, 27, 36, 99, 100, 102, 103, 104, 115, 116] greatly expanded on the roles and behaviors of different plasma parameters in black hole energy extraction, explored forbidden and allowed parameter space regions, plasma acceleration behaviors, shocks, and other effects. However all such studies have to some extent relied upon an assumed magnetosphere structure, either by direct imposition of an
assumed structure or by restricting their explorations to behaviors along a single assumed magnetic field line.

The fundamental goal of this work was to address that deficiency and find self-consistent global magnetosphere structures correlated with the fundamental attributes of energy-extracting black hole magnetospheres. Specifically, we sought to correlate the location of an ingoing Alfvén surface (which determines whether or not energy is extracted) with global magnetosphere structures. Doing so would allow the largely one-dimensional studies already conducted to be placed in fuller context, enhancing their value by allowing for more robust application. Additionally, any such solutions could serve as useful backgrounds for studying additional effects, such as the propagation of magnetohydrodynamic waves or radiative and neutrino processes, that are often assumed to be correlated with energy extracting black hole magnetospheres.

Prior Solutions

The equations governing transverse force balance that determine the global structure of a magnetosphere are highly non-linear and have very few known solutions, either analytically or numerically. Analytically, in the force free limit, four primary solutions were known prior to this work. Two were found perturbatively in the original work of Blandford & Znajek [6], which took fields valid around non-rotating black holes and extended them to slowly rotating black holes via perturbations in black hole spin. The first of those was a split-monopolar solution, the second a paraboloidal solution intended to correspond to an accretion disk. The third primary solution is the result of proposing a uniform magnetic field of some kind that connects to a more monopolar field near the horizon [4, 76, 122]. The fourth and final primary analytic solution is the only known exact solution [60], but it does not extract energy and is physically problematic near the axis, so it is mostly a mathematical curiosity.
The split-monopolar solution is by far most widely used and has experienced extensive study that still continues today [4, 29, 31, 33, 51, 58, 74, 75, 105], both for its simplicity and because the poloidal magnetic fields near the horizon are generally expected to be roughly monopolar in the sense of being primarily radial in the poloidal plane. The radial nature of the poloidal magnetic field is a consequence of the effects of spacetime curvature in that ingoing observers (or plasma) must observe finite electromagnetic fields on the horizon [36, 68, 127]. Vacuum solutions such as the Wald field [121] also exhibit the same roughly monopolar behavior near the horizon, though in the limits of high black hole spin such fields can be completely expelled from the horizon [40] as a result of the “black hole Meissner effect”. This has led to some concern that rapidly spinning black holes might expel magnetic flux and shut down black hole energy extraction via a Blandford and Znajek type process. However, it can be shown that monopolar magnetic fields are not expelled by the Meissner effect [80], allowing such fields to continue extracting a black hole’s rotational energy even at very high spin.

Numerically, in the force free limit, only a small number of distinct force-free solutions are known to exist. They primarily consist of monopolar solutions [11, 18, 44, 55, 65, 71, 107] (corresponding to the analytic monopolar solution), solutions with artificial “walls” that force the magnetic field into a given configuration [55, 65, 107], solutions corresponding to the presence of a vertical field at large distances (i.e. a Wald field) [11, 55, 65, 71, 73, 76] and a small handful of accretion disk solutions with various external forcings and boundary conditions applied [22, 117, 124].

In the more general limit of ideal magnetohydrodynamics, no self-consistent solutions are known, numerical or otherwise. Many time dependent simulations have been conducted under a wide range of plasma assumptions and initial conditions [19, 42, 43, 46, 57, 58, 77, 78, 79, 81, 82, 90, 91, 108] from which some behaviors
and tendencies may be intuited. However, when considering the problem of black hole energy extraction in general such simulations will at best give a single set of plasma parameters with differing degrees of robust measurement and independence from confounding effects.

In short, while much is known (or assumed) about the general nature of black hole energy extraction via a Blandford and Znajek type process, even within the relatively simple limit of a force-free plasma no studies have been conducted to explore what relationships, if any, might exist between the fundamental parameters governing black hole energy extraction and global magnetosphere structure. That is largely due to the paucity of known solutions, which are simply too few in number and restricted in nature to sample the relevant solution space in a robust or meaningful way. One of the fundamental unknown open questions regarding black hole energy extraction is if ingoing Alfvén surface location (which directly determines if a magnetosphere is energy extracting) is coupled in any meaningful way to global magnetosphere structure. The fundamental goal of this work can be stated as a desire to find more solutions that more widely sample the relevant solution space to remedy the deficiencies in understanding caused by the scarcity of known solutions prior to this work.

Accomplishments

In this work we accomplished three primary things. First, we analytically demonstrated that the monopolar solution originally arrived at by Blandford & Znajek [6], extended to higher order by others, and still under active study today is in fact a special case of a broad class of magnetospheres, and is a separatrix between distinct magnetosphere structures functionally sourced by magnetosphere rotation. Second, we numerically solved for thousands of force-free magnetospheres, allowing
for an exploration of correlations between magnetosphere rotation, black hole spin, and magnetosphere structure. Third, and finally, we developed numerical techniques capable of solving for the structure of energy extracting magnetospheres outside of the force-free limit, for the first time allowing plasma parameters to be explicitly coupled to energy extracting black hole magnetospheres in the more general limit of ideal magnetohydrodynamics.

The primary conclusion that may be drawn from our results is that field line angular velocity (as a proxy for Alfvén surface location) does in fact couple to magnetosphere structure. More slowly rotating magnetospheres with Alfvén surfaces near the outer limits of the ergosphere naturally form jet-like structures aligned with the azimuthal axis, while more rapidly rotating magnetospheres with Alfvén surfaces near the event horizon can naturally connect to nearby accreting matter in the equatorial plane. When combined, those effects suggest that a factor in excess of 1000 might be expected in the luminosity of a Poynting-flux dominated jets directly powered by black hole energy extraction. Concurrent fluxes of energy and angular momentum into nearby accreting matter simultaneously place upper limits on the practical luminosity of any such jets, as that matter would need to absorb and transport concurrently large amounts of angular momentum away from the black hole. Any failure to do so could lead to that matter being blown away and halting both black hole energy extraction and the jet, suggesting that as such jets become more luminous they will also become intrinsically variable.
ASSUMPTIONS

In this chapter we outline the primary assumptions upon which this work rests. We conclude with a description of the fundamental system of equations that derive from those assumptions and from which everything in this work ultimately follows.

Background Geometry

We begin by assuming a background spacetime that is adequately described by the Kerr metric. Using Boyer-Lindquist coordinates with a (+, −, −, −) metric signature and units where \( c = G = 1 \), that corresponds to the line element:

\[
ds^2 = \left(1 - \frac{2mr}{\Sigma}\right)dt^2 + \frac{4mar\sin^2\theta}{\Sigma}dtd\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - \frac{A\sin^2\theta}{\Sigma}d\phi^2,
\]

(2.1)

where

\[
\Sigma = r^2 + a^2 \cos^2 \theta,
\]
\[
\Delta = r^2 - 2mr + a^2,
\]
\[
A = \left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2 \theta.
\]

(2.2)

This assumption implies that the black hole does not possess a gravitationally significant electromagnetic charge, any electromagnetic fields or plasma surrounding the black hole do not possess sufficient energy density to significantly modify the structure of the spacetime, and that the self-gravity of any matter near the black hole is ignorable.

The assumption of the absence of a gravitationally significant electromagnetic charge is not severe, as any such charge would generally be expected to quickly
neutralize in most astrophysical contexts. In the most extreme cases the energy density of the electromagnetic fields might be expected to slightly modify the structure of the spacetime, but in general ignoring such effects should be a reasonable approximation. The lack of a self-gravitating plasma is perhaps most significant, but outside of extreme cases such effects might primarily be expected to affect the nature of nearby matter structures (such as an accretion disk) that would generally enter into the problem as an assumed boundary condition, not as an independent variable.

**Ideal MHD**

The next assumption we make is that the plasma surrounding the black hole allows for the application of ideal magnetohydrodynamics. If we assume that the classical form of Ohm’s Law, $\vec{J} = \sigma \vec{E}$ (where $\vec{J}$ is electric current, $\sigma$ conductivity, and $\vec{E}$ electric field) is valid in the plasma rest frame, then for an arbitrary observer $O$ we must have a four-current that is given by:

$$J^\alpha = \rho_e u^\alpha + \sigma F^{\alpha\beta} u_\beta.$$  \hfill (2.3)

Here $\rho_e$ is the charge density and $u_\alpha$ is the plasma four-velocity, and all quantities are as measured by observer $O$. The assumption of an ideal plasma is a demand for infinite conductivity, $\sigma \to \infty$, which for finite $\rho_e$, $u^\alpha$, and $J^\alpha$ in turn demands that:

$$u_\beta F^{\alpha\beta} = 0.$$  \hfill (2.4)

The assumption of ideal magnetohydrodynamics is completely encapsulated by this equation; physically, it is a demand that the electric field vanish in the plasma’s rest frame and that the plasma be “frozen” into the magnetic field $F^{\alpha\beta}$. In the
force-free limit where explicit references to plasma parameters are discarded and $u^\alpha$ doesn’t enter into the problem, the above assumption should be replaced by the complementary assumptions of perpendicular electric and magnetic fields and a magnetically dominated magnetosphere:

$$F^{\alpha\beta} F_{\alpha\beta} = 4 \vec{E} \cdot \vec{B} = 0,$$
$$F^{\alpha\beta} F_{\alpha\beta} = 2 \left( |\vec{B}|^2 - |\vec{E}|^2 \right) > 0. \quad (2.5)$$

The assumption of ideal magnetohydrodynamics is primarily made for its simplicity. Astrophysical plasmas near black holes can generally be assumed to be very good conductors \[45\], so avoiding the significant additional complications introduced by resistive magnetohydrodynamics \[72\] is desirable as a first approximation.

**Perfect Fluid**

We also assume that the plasma is a perfect fluid, such that the plasma component of the stress-energy tensor may be described by:

$$T^{\alpha\beta}_{\text{Plasma}} = (\rho + p) u^\alpha u^\beta - pg^{\alpha\beta}. \quad (2.6)$$

Here $\rho$ and $p$ are the proper energy density and pressure of the plasma. This is a statement that the plasma primarily interacts with the magnetosphere via pressure and inertial forcings; effects such as viscosity and heat conduction are not present. With particle conservation this immediately implies that the plasma flow is adiabatic with a polytropic equation of state.

A perfect fluid might be viewed as one of the most severe assumption that we make. However, efficient black hole energy extraction can typically be expected to
be an electromagnetically driven process; negative energy particle inflows require mechanisms for generating extremely large counter-rotating velocities (as referenced to zero angular momentum frames), while outward-directed Poynting fluxes are relatively straightforward to accomplish. In other words, in energy-extracting black hole magnetospheres the primary role of the plasma is to support and maintain perpendicular electric and magnetic fields and a concurrent outward Poynting flux (stationary and axisymmetric vacuum fields cannot extract a black hole’s rotational energy [45, 47]). A perfect fluid can provide that support while also reasonably describing any transfers of energy between the electromagnetic fields and accreting plasma (at least in a net global sense). So while there might be significant fine structure that is not captured by a perfect fluid, such as from heat conduction, viscosity, radiation, neutrino production, or other effects, the overall nature of efficient black hole energy extraction should be adequately approximated by a perfect fluid, making its simplicity an attractive first choice.

**Stationarity and Axisymmetry**

The last primary assumption that we make is that the magnetosphere is both stationary and axisymmetric, with an axis of symmetry corresponding to that of the black hole. In general we would expect stationary solutions to be able to adequately describe a wide range of astrophysical scenarios, though transient and highly variable objects would of course limit the timescales over which any one solution would be applicable. Aligned axisymmetry is perhaps more restrictive, but we would not expect the emergence of fundamentally different physical effects in non-axisymmetric magnetospheres, merely greater complexity. As with our previous assumptions, we favor the simplicity of stationarity and axisymmetry over the complications of fine structure.
Fundamental Equations

Our treatment of energy-extracting black-hole magnetospheres rests on several basic equations, influenced by both fundamental physics and the assumptions we have outlined above. The first set of those equations, and perhaps the most fundamentally important, are Maxwell’s equations, which may be written as (taking advantage of our current coordinate basis to write Maxwell’s homogeneous equations in terms of partial derivatives):

\[ F^\alpha_{\beta;\gamma} = -4\pi J^\alpha, \]
\[ F_{\alpha\beta;\gamma} = F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0. \]  

(2.7)

Next, we demand an overall stress energy tensor compatible with our assumption of a perfect fluid:

\[ T^{\alpha\beta} = T_{\text{Plasma}}^{\alpha\beta} + T_{\text{EM}}^{\alpha\beta} = (\rho + p) u^\alpha u^\beta - pg^{\alpha\beta} + \frac{1}{4\pi} g^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{1}{16\pi} g^{\alpha\beta} F_{\mu\lambda} F^{\mu\lambda}. \]  

(2.8)

We then demand that no external forcings be present such that \( T^{\alpha\beta}_{;\beta} = 0 \), application of the ideal MHD condition \( u_\alpha F^{\alpha\beta} = 0 \), the velocity normalization condition \( u^\alpha u^\alpha = 1 \), stationarity and axisymmetry such that \( f, t = 0 \) and \( f, \phi = 0 \) for any scalar quantity \( f \), and a background spacetime geometry described by the Kerr metric in Boyer-Lindquist coordinates.

All of our results rely exclusively on the self-consistent application of the above equations. In their present form they are not immediately useful, however; in Chapter 4 we recast them into forms more practical for studying black hole energy extraction. Before doing so, however, in Chapter 3 we first explore what the most useful problem variables might be.
In this chapter we take the fundamental equations from the previous chapter and arrive at the problem variables that are most useful in describing an energy-extracting black hole magnetosphere under the assumptions we have made. From the outset we caution that the proofs and physical interpretations of these variables are somewhat intertwined. As such we suggest that the variables we outline below might be most productively viewed as a set, rather than as disjoint independent quantities (as potentially and unfortunately implied by the necessarily successively linear organization of this chapter).

Poloidal Flux Function

The first thing we define is a useful flux function for the poloidal magnetic field. For a flux function to be useful we demand that it be both constant along poloidal magnetic field lines (such that contours of the flux function trace poloidal magnetic field lines) and that it be intrinsically related to the poloidal magnetic field. To find such a function, we first define a poloidal magnetic field four-vector in terms of the field strength tensor as (see Appendix B for additional discussion of this form):

\[
F_{\alpha\beta} \equiv -\sqrt{-g} \epsilon_{\alpha\beta\mu\nu} (k^\mu + \Omega_F l^\mu) B^\nu,
\]

\[
B^\alpha \equiv \frac{1}{2\sqrt{-g}} \epsilon^{\alpha\beta\gamma\delta} t_\beta F_{\gamma\delta}.
\]  

Here \(k^\alpha\) and \(l^\alpha\) are the Killing vectors associated with stationarity and axisymmetry and \(t_\alpha = \partial_\alpha t\); in component form \(k^\alpha = (1, 0, 0, 0)\), \(l^\alpha = (0, 0, 0, 1)\), and \(t_\alpha = (1, 0, 0, 0)\). The field line angular velocity \(\Omega_F\) will be defined in the next section. The completely antisymmetric symbol is written as \(\epsilon^{\alpha\beta\mu\nu}\) and \(\epsilon_{\alpha\beta\mu\nu}\); if the factor of the
metric determinant $\sqrt{-g}$ and appropriate sign were absorbed into them they would correspond to the standard Levi-Civita tensor as defined in [63] (we have selected the above notation in order to make the metric factors related to a tensor density explicit).

Alternative definitions of a magnetic field four-vector exist in the literature (such as Takahashi et al. [101], on which this work is substantively based); we have developed the above form because it may be arrived at by transforming from a Cartesian basis in flat space to spherical coordinates and then generalizing to a curved spacetime, while simultaneously maintaining the restriction $B^\alpha;\alpha = 0$ (Appendix B). For our purposes in this section it is sufficient to note that $\vec{B}^r \sim F_{\theta\phi}$ and $\vec{B}^\theta \sim F_{r\phi}$ (Appendix C), a common feature of all definitions. Ultimately any such definition should generally not be taken to necessarily correspond to the magnitude of the physical magnetic field from the perspective of any observer, but should rather be interpreted as a convenient mathematical construct aligned with such a magnetic field.

Next, suppose that there is a scalar quantity $f$ that is conserved along poloidal magnetic field lines; then we must have $\vec{B} \cdot \nabla f = 0$, or in relativistic language:

$$B^\alpha f_{,\alpha} = 0. \quad (3.2)$$

Applying the assumptions of stationarity and axisymmetry we must also have $f_t = 0$ and $f_{,\phi} = 0$, so we find:

$$B^\alpha f_{,\alpha} = \left( \frac{1}{2\sqrt{-g}} \epsilon^{\alpha\beta\gamma\delta} t_{\beta} F_{\gamma\delta} \right) f_{,\alpha}$$

$$= \frac{1}{\sqrt{-g}} \left[ (-F_{\theta\phi}) f_r + (F_{r\phi}) f_\theta \right]. \quad (3.3)$$
We next introduce the vector potential $A_\alpha$, defined in terms of the field strength tensor as:

$$F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}.$$  \hfill (3.4)

We have used partial derivatives here because we have specified a coordinate basis in demanding Boyer-Lindquist coordinates, making covariant derivatives unnecessary. In terms of the vector potential, the conservation of the scalar $f$ then becomes:

$$B_\alpha^\alpha f_{,\alpha} = \frac{1}{\sqrt{-g}} [-A_{\phi,\theta} f_{,r} + A_{\phi,r} f_{,\theta}].$$  \hfill (3.5)

If we now take $f = A_\phi$, it can be seen that $B_\alpha^\alpha A_{\phi,\alpha} = 0$. Therefore we take $A_\phi$ to be a useful flux function for poloidal magnetic field lines; it is both constant along magnetic field lines and is intrinsically tied to the poloidal magnetic field via $\vec{B}_r \sim F_{\theta\phi}$ and $\vec{B}_\theta \sim F_{r\phi}$. For completeness we still need to demonstrate that our definition of $B^A$ is parallel to $u^A$ (i.e. is compatible with the assumption of an ideal plasma); that will be accomplished below when we discuss conserved particle flux. For now it will simply be taken as a given that $A_\phi$ is a valid flux function and that any scalar function $f$ that is conserved along magnetic field lines must satisfy $A_{\phi,r} f_{,\theta} = A_{\phi,\theta} f_{,r}$.

**Conserved Field Line Angular Velocity**

The assumption of an ideal plasma results in the plasma being “frozen” to magnetic field lines, implying that a flux tube may be conceptualized as a string, wire, or similar physical object running through the magnetosphere. The assumption of stationarity and axisymmetry further implies that if such an object rotates it must do so rigidly, suggesting the existence of a conserved “field line angular velocity” describing any such rotation. That is indeed the case, and in this section we derive
We begin by noting that the assumption of an ideal plasma demands the existence of a frame (the plasma rest frame) where the electric field vanishes, so the invariant contraction of the field strength tensor with its dual must vanish:

$$\mathcal{F}^{\alpha\beta} F_{\alpha\beta} = 4 E_i B^i = 0. \quad (3.6)$$

Expanding that contraction, we find:

$$\mathcal{F}^{\alpha\beta} F_{\alpha\beta} = \frac{1}{2\sqrt{-g}} \epsilon^{\alpha\beta\mu\nu} F_{\mu\nu} F_{\alpha\beta}$$

$$= \frac{4}{\sqrt{-g}} \left( F_{tr} F_{\theta\phi} - F_{t\theta} F_{r\phi} \right). \quad (3.7)$$

Therefore we have $F_{tr} F_{\theta\phi} = F_{t\theta} F_{r\phi}$. We next define the field line angular velocity $\Omega_F$ as:

$$F_{r\phi} \Omega_F = F_{tr},$$

$$F_{\theta\phi} \Omega_F = F_{t\theta}. \quad (3.8)$$

We define it in this way to avoid division by zero in cases where either $F_{r\phi}$ or $F_{\theta\phi}$ vanish (if they both vanished simultaneously there would be no poloidal magnetic field); the vanishing contraction of the field strength tensor with its dual shows that they are compatible definitions. To show that field line angular velocity is conserved, we consider Maxwell’s homogeneous equations to find:

$$F_{\alpha\beta,\gamma} + F_{\beta,\gamma,\alpha} + F_{\gamma,\alpha,\beta} = F_{t\theta,r} + F_{\theta,r,t} + F_{rt,\theta}$$

$$= (F_{\theta\phi} \Omega_F)_r - (F_{r\phi} \Omega_F)_\theta$$
This must vanish if Maxwell’s homogeneous equations are to be satisfied, so we have 

\[ A_{\phi, r} \Omega_{F, r} = A_{\phi, \theta} \Omega_{F, \theta} \]

and by virtue of the results of the previous section the field line angular velocity \( \Omega_F \) is conserved along poloidal magnetic field lines.

So far we have only shown that \( \Omega_F \) is conserved; we have not fully justified its correspondence to a “field line angular velocity”. We will defer a more robust justification of that correspondence to Appendix E and merely state here that \( \Omega_F \) is indeed related to the boost velocity from the plasma rest frame, referenced to zero angular momentum observers. Such a result is already strongly suggested by inspecting the definition of \( \Omega_F \), however, in that it indicates that any electric fields (\( \sim F_{tr} \) and \( \sim F_{t\theta} \)) are directly sourced by some kind of rotational boost related to \( \Omega_F \).

**Conserved Particle Flux**

We have assumed an ideal plasma flow, so the plasma should be frozen to the magnetic field lines and as such the poloidal projection of plasma’s four-velocity should be everywhere parallel to the poloidal magnetic field. Further, due to stationarity we must demand that no particle build up take place anywhere, meaning that the particle flux should be constant along a magnetic field line (assuming particle number conservation). We therefore expect to find a conserved particle flux per unit flux tube \( \eta \) that is given by:

\[ \eta \vec{B} = -n \vec{v}. \]  

(3.10)

In other words the ratio \( v^i / B^i \) should be the same in every direction (the vectors should be parallel in an absolute sense), and the particle flux (per unit flux tube)
should be a constant determined by the product of the plasma density \( n \) and the plasma’s velocity down the flux tube. The negative sign is a convention.

In the language of four vectors, the particle flux may be cast as (explained in more detail in the next subsection):

\[
\eta B^\alpha = -n \left[ u^\alpha - t_\beta u^\beta (k^\alpha + \Omega_F l^\alpha) \right] = -N^\alpha + t_\beta N^\beta (k^\alpha + \Omega_F l^\alpha).
\] (3.11)

Here \( N^\alpha = n u^\alpha \) is the particle number flux four vector, with the property that \( N^\alpha;\alpha = 0 \) by virtue of conserved particle number. The trailing correction involving the Killing vectors \( k^\alpha \) and \( l^\alpha \) is necessary to correct for the rotation of magnetic field lines (the plasma velocity is with respect to the flux tube); it has vanishing divergence by virtue of stationarity, axisymmetry, and the properties of Killing vectors. Taking the divergence of both sides (and noting that \( B^\alpha;\alpha = 0 \) by construction) we therefore demand that:

\[
B^\alpha \eta;\alpha = 0.
\] (3.12)

Therefore particle flux as written above is conserved along poloidal magnetic field lines, in that \( A_{\phi, r} \eta; \theta = A_{\phi, \theta} \eta; r \).

Four Velocity Compatibility To make the above complete, at a minimum we must show that the ideal magnetohydrodynamics condition \( u_\alpha F^{\alpha \beta} \) implies that for a conserved quantity \( f \), the condition \( B^\alpha f;\alpha = 0 \) is equivalent to the condition \( u^\alpha f;\alpha = 0 \). Otherwise \( B^A \) and \( u^A \) are not parallel and the initial assumption of the definition of \( \eta \) is invalid, even when restricted to the poloidal plane. Further, due to the assumption of stationarity and axisymmetry (such that \( f; t = 0 \) and \( f; \phi = 0 \)), if the definition of \( \eta \) made above is to be fully valid as written (and not just in the poloidal plane) we must show that if \( B^\alpha Z_\alpha = 0 \) for arbitrary vector \( Z_\alpha \) then the contraction of \( Z_\alpha \) with
\( u^\alpha - t_\beta u^\beta (k^\alpha + \Omega_F l^\alpha) \) also vanishes.

This may be trivially accomplished via brute-force analysis, but it is slightly more elegant to consider a restricted cross-product to find (taking advantage of the definition of \( F_{\alpha\beta} \) in terms of \( B^\alpha \)):

\[
\frac{1}{\sqrt{-g}} \epsilon^{\alpha\beta\delta\sigma} t_\beta Z_\delta u^\omega F_{\omega\sigma} = \epsilon^{\alpha\beta\delta\sigma} \epsilon_{\omega\sigma\rho\gamma} t_\beta Z_\delta u^\omega (k^\rho + \Omega_F l^\rho) B^\gamma
\]

\[
= t_\beta B^\beta Z_\gamma (k^\gamma + \Omega_F l^\gamma) u^\alpha - Z_\beta B^\beta t_\gamma (k^\gamma + \Omega_F l^\gamma) u^\alpha
\]

\[
+ Z_\beta u^\beta t_\gamma (k^\gamma + \Omega_F l^\gamma) B^\alpha - t_\beta u^\beta Z_\gamma (k^\gamma + \Omega_F l^\gamma) B^\alpha
\]

\[
+ t_\beta u^\beta Z_\gamma B^\gamma (k^\alpha + \Omega_F l^\alpha) - Z_\beta u^\beta t_\gamma B^\gamma (k^\alpha + \Omega_F l^\alpha)
\]

\[
= \{ Z_\beta [u^\beta - t_\gamma u^\gamma (k^\beta + \Omega_F l^\beta)] \} B^\alpha. \quad (3.13)
\]

To arrive at the last line we applied \( B^\alpha Z_\alpha = 0 \) (while also noting that \( t_\alpha B^\alpha = 0 \)). If the condition of ideal magnetohydrodynamics is to be satisfied then this must vanish for arbitrary \( B^\alpha \), so the contraction of \( Z_\alpha \) with \( u^\alpha - t_\beta u^\beta (k^\alpha + \Omega_F l^\alpha) \) must vanish. Therefore if \( B^\alpha f_\alpha = 0 \) we also have \( u^\alpha f_\alpha = 0 \); \( u^\alpha \) and \( B^\alpha \) are parallel in the poloidal plane. They may be considered parallel in all spatial directions once a correction for uniform field line rotation is applied; \( B^\phi \sim u^\phi - u^t \Omega_F \).

In non-relativistic language, the above reduces to noting that the the condition of ideal magnetohydrodynamics is \( \vec{v} \times \vec{B} = 0 \), and then considering \( \vec{Z} \times (\vec{v} \times \vec{B}) \) under the constraint \( \vec{B} \cdot \vec{Z} = 0 \). In four dimensions the cross product as ill-defined, so in the above we flattened the temporal dimension via a contraction with \( t_\beta \).
A conserved energy flux is expected from the assumption of stationarity. To discover its form, we consider the contraction of a Killing vector $\xi^\alpha$ with the divergence of the stress energy tensor:

$$\xi^\alpha T^\beta_{\alpha \beta} = \left(\xi^\alpha T^\beta_{\alpha \beta}\right)_{,\beta} - T^\alpha_{\beta \gamma} \xi^\gamma_{,\alpha} = \left(\xi^\alpha T^\beta_{\alpha \beta}\right)_{,\beta}.$$  \hfill (3.14)

The second term dropped out because $T^\alpha_{\beta \gamma}$ is symmetric and $\xi^\gamma_{,\alpha}$ is antisymmetric. We then identify $k^\alpha T^\beta_{\alpha \beta}$ as an energy flux vector, whose divergence necessarily vanishes by virtue of the vanishing divergence of the stress energy tensor. To proceed further we must consider the exact form stress-energy tensor, which is given by:

$$T^\alpha_{\beta \gamma} = -\frac{\alpha}{4\pi} B^\alpha B^\beta - \frac{B^\gamma}{4\pi} r^\alpha r^\beta \frac{g_{t\phi} + g_{\phi \Omega_F}}{2\rho_\omega^2} \sqrt{-g} F^{\theta r} \left( B^\alpha r^\beta + r^\alpha B^\beta \right)$$

$$- \frac{(r^\gamma B_\gamma)^2}{8\pi} g^\alpha_{\beta \gamma} + \frac{\alpha B_\gamma B^\gamma}{8\pi} g^\alpha_{\beta \gamma} + \frac{\mu}{n} N^\alpha N^\beta - pg^\alpha_{\beta \gamma}.$$  \hfill (3.15)

This form is derived in Appendix F; for our present purposes it is sufficient to note that the $B^\alpha$ terms are the electromagnetic contribution and the $N^\alpha$ and $pg^\alpha_{\beta \gamma}$ terms are the fluid contribution, and that every vector or tensor involved is divergenceless (under the assumption of stationarity and axisymmetry). The rotation vector $r^\alpha$ was encountered in the previous section; $r^\alpha = k^\alpha + \Omega_F l^\alpha$. We then find the divergence of the energy flux vector to be given by (taking advantage of stationarity and axisymmetry):

$$\left( T^\beta_{\beta \gamma} \right)_{,\beta} = \left( -\frac{\alpha}{4\pi} B_t \right)_{,A} B^A + \frac{1}{4\pi} \left[ \frac{g_{t\phi} + g_{\phi \Omega_F}}{\rho_\omega^2} \sqrt{-g} F^{\theta r} (k_t + \Omega_F l_t) \right]_{,A} B^A + \left( \frac{\mu}{n} N_t \right)_{,A} N^A$$

$$= B^A \left[ \frac{1}{4\pi} \sqrt{-g} F^{\theta r} \left( -\frac{\alpha g_{t\phi}}{\rho_\omega^2} + \frac{g_{tt} g_{\phi \phi} \Omega_F^2 + g_{t\phi}^2 \Omega_F + g_{t\phi g_{\phi \phi} \Omega_F^2}}{\rho_\omega^2} \right) \right]_{,A}.$$
\(- \eta (\mu u_t)_A B^A\)

\[
= B^A \left[ \frac{1}{4\pi} \sqrt{-g} F^{\theta r} \left( \Omega_F g_{rt} g_{\phi\phi} - g_{tt} \rho^2 \right) - \eta \mu u_t \right]_A \\
= -B^A \left[ \eta \mu u_t + \frac{1}{4\pi} \sqrt{-g} F^{\theta r} \Omega_F \right]_A. \tag{3.16}
\]

As noted above this vanishes by virtue of the vanishing divergence of the stress energy tensor, so we identify a field-aligned conserved energy \(E\) as:

\[
\eta E = \eta \mu u_t + \frac{1}{4\pi} \sqrt{-g} F^{\theta r} \Omega_F. \tag{3.17}
\]

The sign of \(E\) and multiplicative factor of \(\eta\) are conventions. Physically \(E\) corresponds to the average energy per particle that is being transported by both the fluid and the electromagnetic fields in the poloidal plane. For an inflow, if \(E < 0\) then there is a net outward flow of energy; if \(E > 0\) there is a net inward flow of energy. We consider the non-relativistic limit of \(E\) in Appendix D as a way of making its physical content more clear.

### Conserved Angular Momentum

A conserved angular momentum flux is to be expected from the assumption of axisymmetry. Its form is derived identically to the way in which the conserved energy flux was derived in the previous section, using the Killing vector associated with axisymmetry \(l^\alpha\) instead of stationarity \(k^\alpha\). Under a procedure identical to that of the previous section, the divergence of the angular momentum flux vector is given
by:

\[
T^\beta_\phi \,^;\beta = \left( -\frac{\alpha}{4\pi} B_\phi \right)_A B^A + \frac{1}{4\pi} \left[ \frac{g_{t\phi} + \Omega_F g_{\phi\phi}}{\rho_\omega^2} \right. - g F^{\theta r} \left( k_\phi + \Omega_F l_\phi \right) \left]_A B^A + \left( \frac{\mu}{n} N_\phi \right)_A N^A \right.
\]

\[
= \left[ \frac{1}{4\pi} \sqrt{-g} F^{\theta r} \left( -\frac{\alpha g_{\phi\phi}}{\rho_\omega^2} + \frac{g_{t\phi}^2}{\rho_\omega^2} + 2 g_{t\phi} g_{\phi\phi} \Omega_F + \frac{g_{\phi\phi}^2 \Omega_F^2}{\rho_\omega^2} \right) \right]_A - \eta (\mu u_\phi)_A B^A
\]

\[
= \left[ \frac{1}{4\pi} \sqrt{-g} F^{\theta r} \left( \frac{g_{t\phi}^2}{\rho_\omega^2} - g_{t\phi} g_{\phi\phi} \right) - \eta \mu u_\phi \right]_A
\]

\[
= B^A \left[ -\eta \mu u_\phi + \frac{1}{4\pi} \sqrt{-g} F^{\theta r} \right]_A.
\] (3.18)

This vanishes by virtue of the vanishing divergence of the stress energy tensor, so we identify a field-aligned conserved angular momentum flux \( L \) as:

\[
\eta L = -\eta \mu u_\phi + \frac{1}{4\pi} \sqrt{-g} F^{\theta r}.
\] (3.19)

As with the energy \( E \), the sign of \( L \) and factor of \( \eta \) here are conventions. Physically \( L \) corresponds to the average angular momentum per particle that is being transported by both the fluid and electromagnetic fields in the poloidal plane. For an inflow, if \( L < 0 \) then there is a net outward flow of angular momentum; if \( L > 0 \) there is a net inward flow of angular momentum. We consider the non-relativistic limit of \( L \) in Appendix D as a way of making its physical content more clear.

**Conserved Entropy**

The assumption of a perfect fluid directly implies an adiabatic flow, so we can immediately assume a field-aligned conserved entropy \( S \) as a fifth conserved quantity (in addition to field line angular velocity, particle flux, energy, and angular momentum). In this work we will not explore any effects that would be driven by variations in entropy, but for the sake of completeness we will now briefly show how
such effects would enter into the problem.

Adiabatic exponents $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$ may be defined such that differential changes in pressure, volume, and temperature are related by (cf. [16]):

\[
\frac{dp}{p} + \Gamma_1 \frac{dV}{V} = 0,
\]
\[
\frac{dp}{p} + \frac{\Gamma_2}{1 - \Gamma_2} \frac{dT}{T} = 0,
\]
\[
\frac{dT}{T} + (\Gamma_3 - 1) \frac{dV}{V} = 0.
\]

These are defined for the sake of similarity to the ideal gas relations for changes at constant entropy $TV^{\gamma - 1} = C_1$, $pV^\gamma = C_2$, and $p^{1-\gamma}T^\gamma = C_3$ for constants $C_i$. We then take $dU = -pdV$ for an adiabatic flow to write $dU = -C_2V^{-\Gamma_1}dV$, which we integrate to find (demanding vanishing thermal energy density at zero pressure):

\[
U = \frac{1}{\Gamma_1 - 1} pV. \tag{3.21}
\]

The stress-energy tensor of the fluid is defined in terms of its energy density $\rho$ (encapsulating rest mass, kinetic energy, etc.) and its pressure $p$, which may be described by a relativistic enthalpy $\mu$ defined as $n\mu \equiv \rho + p$ (where $n$ is particle number density). In a momentarily co-moving reference frame the energy density $\rho$ reduces to the sum of the flow’s rest mass energy density and its thermal energy density $U/V$, and we find (cf. [94]):

\[
\mu_{\text{MCRF}} = \frac{\rho}{n} + \frac{p}{n} = m_p + \frac{U/V + p}{n} = m_p + \frac{\Gamma_1}{\Gamma_1 - 1} \frac{p}{n}. \tag{3.22}
\]

Here $m_p$ is the average rest mass per particle. If we now assume that the adiabatic exponents are not variable functions of the thermodynamic state, we may assume a
polytropic equation of state of the form \( p \sim (nm)^\Gamma \) to find:

\[
\frac{p}{n} = \frac{p_{\text{inj}}}{n_{\text{inj}}} \left( \frac{n}{n_i} \right)^{\Gamma - 1}.
\] (3.23)

Here the subscript text “inj” denotes “injection”, which is to say the point at which plasma is injected into a flux tube, but any point where the parameters are known would suffice - a speculative injection point is merely a likely convenience. From the definition of the conserved particle flux \( \eta \) we know that \( \eta/n = -u^A/B^A \); because \( \eta \) is conserved along a magnetic field line, we may therefore write:

\[
\frac{p}{n} = \frac{p_{\text{inj}}}{n_{\text{inj}}} \left( \frac{u^A_{\text{inj}} B^A}{B^A_{\text{inj}} u^A} \right)^{\Gamma - 1}.
\] (3.24)

The poloidal plane indexing (e.g. \( B^A \)) here is somewhat improper, and in a more thorough analysis would be replaced by something akin to \( |B_p|/|u_p| \) (with the “p” denoting a projection onto the poloidal plane), but is sufficient for our current purposes. The above immediately allows us to express the relativistic enthalpy as (dropping the restriction to a co-moving reference frame):

\[
\mu = m_p \left[ 1 + h_{\text{inj}} \left( \frac{u^A_{\text{inj}} B^A}{B^A_{\text{inj}} u^A} \right)^{\Gamma - 1} \right],
\] (3.25)

where:

\[
h_{\text{inj}} = \frac{\Gamma_1}{\Gamma_1 - 1} \frac{1}{m_p} \frac{p_{\text{inj}}}{n_{\text{inj}}}.
\] (3.26)

The constant \( h_{\text{inj}} \) may be interpreted as a ratio of enthalpy to rest mass density, and is a parameter that is freely specifiable along every field line. It is a direct consequence of the assumption of a perfect fluid (and therefore an adiabatic flow) and a polytropic equation of state, and as such is a conserved quantity directly related to a conserved...
entropy $S$. For this work we will not consider what effects $h_{\text{inj}}$ (as a stand-in for conserved entropy and a given polytropic equation of state) might have. Instead we will only consider the limit of a “cold flow” where pressure is assumed to be irrelevant, implying $h_{\text{inj}} = 0$ and $\mu = m_p$. This is largely for the sake of simplicity, as we expect finite temperature effects to typically be sub-dominant to electromagnetic field structure in driving black hole energy extraction. If more detail is desired, useful discussions of the principles exploited here may be found in [16, 94].

Concluding Remarks

We have shown that the toroidal component of the vector potential $A_\phi$ is a useful flux function for tracing the poloidal magnetic field, and that there are five field-aligned conserved quantities relevant to the problem: field line angular velocity $\Omega_F$, particle flux $\eta$, energy $E$, angular momentum $L$, and entropy $S$ (as related to $h_{\text{inj}}$).

Specifying a black hole spin $a$, vector potential $A_\phi(r, \theta)$, and functions $\Omega_F(A_\phi)$, $\eta(A_\phi)$, $E(A_\phi)$, $L(A_\phi)$ and $S(A_\phi)$ is equivalent to specifying a black hole magnetosphere. The spin $a$ determines the metric, the combination of $A_\phi$ and $\eta$ determines the poloidal velocity and plasma density $n$ (the overall weighting of $A_\phi$ also determines the relationship between magnetic field strength and black hole mass $m$), the energy $E$, angular momentum $L$, and velocity normalization condition $u_\alpha u^\alpha = 1$ determine the relativistic enthalpy $\mu$ and toroidal magnetic field $(1/\sqrt{-g})F^{\theta r}$ (using $\eta$ to determine the appropriate quadratic root from the normalization condition), and finally the entropy $S$ (specified by $h_{\text{inj}}$ and a polytropic index) decomposes $\mu$ into plasma energy density and pressure.

The entire difficulty of solving for the structure of black hole magnetospheres lies in the fact that the functions $\Omega_F(A_\phi)$, $\eta(A_\phi)$, $E(A_\phi)$, $L(A_\phi)$, and $S(A_\phi)$ are not
independent. They are immediately related by interlocking definitions, and for a self-consistent solution must satisfy both the velocity normalization condition and the vanishing divergence of the stress energy tensor, which reduce to satisfying the Bernoulli equation and transfield equation (explored in the next chapter).

In general, in the limit of a force-free magnetosphere one of three parameters ($\Omega_F$, $\eta E$, or $\eta L$) may to some extent be freely specified. In the limit of a cold flow two of four parameters ($\Omega_F$, $\eta$, $\eta E$, or $\eta L$) may to some extent be freely specified. In the most general case that our assumptions allow for three of six parameters ($\Omega_F$, $\eta$, $\eta E$, $\eta L$, $h_{\text{inj}}$, and a polytropic index) may be freely specified. It should be noted that there are situations and special cases where more (or less) freedom can emerge, however; those are only general rules.

In this work we focus on studying the behavior of force-free magnetospheres as a function of field line angular velocity $\Omega_F$ and the behavior of cold flows as a function of field line angular velocity $\Omega_F$ and particle flux $\eta$ (in terms of an accretion rate $\mu \eta$). From a mathematical perspective those choices are arbitrary; we select them under the assumption that field line rotation and accretion rate are likely to be the most immediately relevant free parameters to associate with astrophysical energy-extracting black hole magnetospheres.
GOVERNING EQUATIONS

In this chapter we discuss the two governing equations that must ultimately be solved for black hole magnetospheres: the field-aligned Bernoulli equation, which describes plasma acceleration along magnetic field lines, and the transfield or Grad-Shafranov equation, which describes force balance transverse to magnetic field lines. Before discussing those equations, however, we first discuss the significance of relativistic Alfvén Mach number, which is then used to cast the Bernoulli and transfield equations in useful forms.

The Alfvén Mach Number

In this section we explore the significance of the Alfvén Mach number $M$ of the plasma, which is defined as:

$$M^2 \equiv 4\pi \frac{(\mu \eta)^2}{\mu n}.$$  \hfill (4.1)

It is written in this way because $\mu \eta$ (roughly the plasma accretion rate) is a common factor throughout, as it ultimately determines the relative dominance of magnetic effects to fluid effects (suggested by the definition of $\eta E$ and $\eta L$ in the previous chapter, and explored in more detail in Appendix F). In order to more easily see why this is an “Alfvén” Mach number, we evaluate the magnitude of the poloidal magnetic field to find:

$$B_A B^A = g_{AC} B^A B^C = g_{AC} \left( -\frac{n}{\eta} u^A \right) \left( -\frac{n}{\eta} u^C \right) = \frac{n^2}{\eta^2} u_A u^A. \hfill (4.2)$$

If we then define the poloidal magnetic field strength as $B_p \equiv \sqrt{-B_A B^A}$ and the poloidal plasma speed as $u_p \equiv \sqrt{-u_A u^A}$ we find that $\eta B_p = -n u_p$. This reduces the
Alfvén Mach number to:

$$M^2 = 4\pi \mu n \frac{u_p^2}{B_p^2}. \quad (4.3)$$

Re-arranging, this immediately yields $u_p = \mathcal{M}(B_p/\sqrt{4\pi \mu n})$ and the nomenclature “Alfvén Mach number” becomes clear \[1\]. None of this is to say that $u_p$ and $B_p$ are necessarily the best measures of poloidal speed and field strength or that $B_p/\sqrt{4\pi \mu n}$ corresponds to the physical Alfvén wave speed for any given observer. Rather it is to say that $M^2$ ultimately emerges as a convenient and physically relevant quantity when solving the Bernoulli and transfield equations.

### The Bernoulli Equation

The Bernoulli Equation is a result of simplifying the velocity normalization condition $u_\alpha u^\alpha = 1$ by including restrictions imposed by conservation of field-aligned parameters. The usage of the definite article “the” in “the Bernoulli Equation” refers to the result of any such simplification process, not a specific result. There are in fact many different (but related) “Bernoulli Equations” in the literature that may be separated into two classes: integral forms that relate poloidal velocity ($\sim u_p$) to conserved parameters and the local metric, and differential forms that relate the change of poloidal velocity in the direction of a magnetic field line ($\sim u_p'$) to conserved parameters and the local metric. In either case the equations are most often expressed in terms of $M^2$, the square of the Alfvén Mach number.

#### Integral Bernoulli Equations

We will defer a more detailed discussion of deriving various forms of the Bernoulli equation to Appendix G. Here we will only outline a general approach, focusing on highlighting points that will be useful for later discussion. The core component of any Bernoulli equation is the restriction that the four velocity of the plasma be properly
normalized:

\[ 0 = 1 - u_t u^t - u_\phi u^\phi - (u_r u^r + u_\theta u^\theta) \]

\[ = 1 + \frac{1}{\rho_\omega^2} \left( g_{\phi\phi} u_t^2 - 2 g_{t\phi} u_t u_\phi + g_{tt} u_\phi^2 \right) + \frac{1}{16\pi^2 (\mu \eta)^2} \mathcal{M}^4 \frac{\Delta A_{\phi, r}^2 + A_{\phi, \theta}^2}{\Sigma \Delta \sin^2 \theta}. \]  \hspace{1cm} (4.4) 

To arrive at this result we exploited the definition of the conserved particle flux per unit flux tube \( \eta \) to replace poloidal velocity with poloidal magnetic field strength. If conserved parameters are already specified (e.g. \( E(A_\phi) \) and \( L(A_\phi) \) are already known) then this form can be used (together with the remaining definition of \( \eta \) in terms of the toroidal field) to find functions \( u_t(r, \theta) \) and \( u_\phi(r, \theta) \) that are compatible with those parameters. In the limit of a cold flow (i.e. vanishing plasma pressure \( p \)) such a procedure will effectively reduce to solving a quartic in \( \mathcal{M}^2 \); for a hot flow and an equation of state with a rational polytropic index it might be a polynomial with degree in the upper teens or larger for typical indices [8, 9, 10, 99].

If the conserved parameters \( E \) and \( L \) are not already known, then their definitions may be used to eliminate \( u_t \) and \( u_\phi \) via the relationships:

\[ \mu \eta u_t = \frac{G_\phi (\eta E - \eta \bar{L} \bar{\Omega}_F) - \eta E \mathcal{M}^2}{\alpha - \mathcal{M}^2}, \]

\[ \mu \eta u_\phi = \frac{G_\phi (\eta E - \eta \bar{L} \bar{\Omega}_F) + \eta L \mathcal{M}^2}{\alpha - \mathcal{M}^2}. \]  \hspace{1cm} (4.5) 

The definitions of \( G_\phi \) and \( \alpha \) may be found in nomenclature preface. Inserting these relationships into the expression for the normalization of the four velocity, it can be shown that:

\[ 1 - u_A u^A = \left( \frac{\eta E}{\mu \eta} \right)^2 \rho_\omega^2 \frac{(\alpha - 2 \mathcal{M}^2) \left( 1 - \bar{L} \bar{\Omega}_F \right) - \left( g_{\phi\phi} + 2 g_{t\phi} \bar{L} + g_{tt} \bar{\bar{L}}^2 \right) \mathcal{M}^4}{\rho_\omega^2 ((\mathcal{M}^2 - \alpha)^2).} \]  \hspace{1cm} (4.6)
Here we have reverted to using the poloidal velocity instead of the poloidal magnetic field, and have introduced $\tilde{L} \equiv L/E$. The above form is used in Takahashi et al. 1990 [101] to deduce restrictions on field aligned conserved quantities. We write it here because it will be a useful reference when we discuss the nature of the Alfvén surface where $\mathcal{M}^2 = \alpha$.

There are many different forms of integral “Bernoulli equations” that can be more or less appropriate in a given context. One of the primary difficulties in solving “the Bernoulli equation” is in identifying useful and relevant independent variables (e.g. $u_t$ and $u_\phi$ instead of $E$ and $L$, or $\eta E$ and $\tilde{L}$ instead of $E$ and $L$) that might have a corresponding ”Bernoulli Equation” that is either sufficiently devoid of pathologies to be usefully soluble or that might contain useful pathologies that can be exploited to arrive at physical constraints and conclusions.

**Differential Bernoulli Equations**

A derivation of the differential Bernoulli equation may be found in Appendix C. For the purposes of this work the primary value of a differential Bernoulli equation is in its ability to clarify the physical meaning of the various magnetohydrodynamic wave speeds, so here we will only explore the overall structure of a differential Bernoulli equation. The gradient of $u_p$ along a magnetic field line may be written in the form:

$$\left(\ln u_p\right)' = \frac{N(u_p)}{D(u_p)}.$$  \hspace{1cm} (4.7)

The exact form of the numerator $N(u_p)$ and denominator $D(u_p)$ are only relevant here in that the denominator has critical points where it vanishes:

$$D(u_p) \sim \left(u_p^2 - u_{AW}^2\right)^2 \left(u_p^2 - u_{FM}^2\right) \left(u_p^2 - u_{SM}^2\right).$$  \hspace{1cm} (4.8)
The critical magnetohydrodynamic wave speeds are given by:

\[ u_{\text{AW}}^2 = \alpha \frac{B_{\phi}^2}{4\pi \mu n}, \quad (4.9) \]

\[ u_{\text{FM}}^2 = \frac{1}{2} \left[ c_s^2 + u_{\text{AW}}^2 + \rho_\omega^2 \frac{(B_{\phi})^2}{4\pi \mu n} \right] + \sqrt{ \left( c_s^2 + u_{\text{AW}}^2 + \rho_\omega^2 \frac{(B_{\phi})^2}{4\pi \mu n} \right)^2 - 4c_s^2 u_{\text{AW}}^2 }, \quad (4.10) \]

\[ u_{\text{SM}}^2 = \frac{1}{2} \left[ c_s^2 + u_{\text{AW}}^2 + \rho_\omega^2 \frac{(B_{\phi})^2}{4\pi \mu n} \right] - \sqrt{ \left( c_s^2 + u_{\text{AW}}^2 + \rho_\omega^2 \frac{B_{\phi}^2 - 1}{4\pi \mu n \rho_\omega^2} \right)^2 - 4c_s^2 u_{\text{AW}}^2 }. \quad (4.11) \]

The relativistic sound speed \( c_s^2 \) is defined as \( c_s^2 = a_s^2/(1 - a_s^2) \) with the adiabatic sound speed \( a_s^2 \) defined as \( a_s^2 = \partial p/\partial \rho \). It is because of the critical points in the denominator \( D \) that the wave speeds are labeled Alfvénic (AW) or slow and fast magnetosonic (SM, FM); critical points in a flow may be expected to correspond to sonic points. It is from this equation that the Alfvén Mach number was selected; it was separated from the factor of \( \alpha \) because \( \alpha \) can be both negative and positive, and it is more useful to think of a Mach number as being positive-definite (physically, \( \alpha \) may be interpreted as being related to a gravitational Lorentz factor). In the limit of a magnetically dominated magnetosphere the fast magnetosonic wave speed reduces to \( 4\pi \mu n u_{\text{FM}}^2 = \alpha B_{\phi}^2 + \rho_\omega^2 (B_{\phi})^2 \); if this is to be finite on the horizon (such that the flow can become super-fast magnetosonic before crossing the event horizon) we immediately recover the horizon regularity condition that we will discuss in the next chapter.
The Transfield Equation

The stress-energy tensor is the sum of two components; an electromagnetic component and a fluid component. The transfield equation is the result of considering the poloidal component of the divergence of the combined stress energy tensor (the temporal and azimuthal components were used in Chapter 3 to arrive at conserved fluxes of energy and angular momentum). Although there are two poloidal components, ultimately they yield a restriction in only one direction, the poloidal direction that is transverse to the magnetic field line.

In order to derive the transfield equation, it is useful to first recast the stress energy tensor in terms of the magnetic field $B^A$ and plasma four velocity $u^A$; the result is (Appendix F):

$$T_A^\beta_{\tilde{\beta}} = X_A = \frac{1}{4\pi} B^C \left[ (\alpha B_C)_{,A} - (\alpha B_A)_{,C} + (g_{t\phi} + \Omega_F g_{\phi\phi}) (B_A \Omega_F, C - B_C \Omega_F, A) \right]$$

$$- \frac{1}{8\pi \rho_c^2} \left( \sqrt{-g} F^{\phi\prime} \right)_{,A}^2$$

$$+ N^C \left[ \left( \frac{\mu}{n} N_A \right)_{,C} - \left( \frac{\mu}{n} N_C \right)_{,A} \right] - N^t \left( \frac{\mu}{n} N_t \right)_{,A} - N^\phi \left( \frac{\mu}{n} N_\phi \right)_{,A}. \quad (4.12)$$

The third line is from the fluid, the first and second lines are from the electromagnetic fields. To proceed further, we need to join the two; this is done by applying the conserved particle flux per unit flux tube $\eta$ to note that:

$$N^A = -\eta B^A,$$

$$N^\phi = \Omega_F N^t - \eta B^\phi. \quad (4.13)$$

The limit of a force-free plasma may be found by taking the limit $\mu \eta \to 0$; in that limit the plasma terms will vanish from the transfield equation, and it will become a
purely electromagnetic equation (see Appendix F for additional discussion). We now define the quantity $e_c$ to be given by:

$$
\mu \eta e_c = \eta E - \eta L \Omega_F = \mu \eta (u_t + u_\phi \Omega_F).
$$  \hspace{1cm} (4.14)

In the limit of a cold flow (equivalent to a flow in which $\mu \eta$ is conserved) $e_c$ is a conserved quantity. It is ultimately generically useful to write the transfield equation in terms of $e_c$, however, even when it is not conserved. In terms of $M^2$ and $e_c$, it may be shown that the transfield equation reduces to (Appendix F):

$$
4\pi \sqrt{-g} \frac{X_A}{A_{\phi,A}} = -\frac{1}{\sin \theta} \left[ (\alpha - M^2) A_{\phi,r} \right]_r - \frac{1}{\Delta} \left[ \frac{1}{\sin \theta} (\alpha - M^2) A_{\phi,\theta} \right]_\theta \\
+ \frac{1}{\Delta \sin \theta} G_\phi \left( \Delta A^2_{\phi,r} + A^2_{\phi,\theta} \right) \Omega_{F,\psi} - \frac{\Sigma}{2\Delta \sin \theta} (4\pi \eta L)^2_{\psi} \\
- 4\pi \sqrt{-g} \left[ \frac{N_t}{\eta} \left( \mu \eta e_c \right)_{\psi} + \frac{1}{\rho^2 \omega} \mu \eta u_\phi (4\pi \eta L)_{\psi} - N^t \mu u_\phi \Omega_{F,\psi} - \frac{\mu n}{\eta} \eta_{,\psi} \right].
$$  \hspace{1cm} (4.15)

In order to simplify this further, we introduce the parameter $\chi$, defined in terms of the Alfvén Mach number as:

$$
M^2 = \frac{4\pi \mu \eta^2}{n} = 4\pi \frac{\eta}{n} (\mu \eta) = 4\pi \chi (\mu \eta).
$$  \hspace{1cm} (4.16)

The parameter $\chi$ is also related to the plasma velocity and electromagnetic fields:

$$
\chi = \sqrt{\frac{\eta^2}{n^2}} = \sqrt{-\frac{\Sigma \rho_0^2 + g_{\phi \phi} u^2_t - g_{\mu \phi} u_t u_\phi + g_{tt} u^2_\phi}{\Delta A^2_{\phi,r} + A^2_{\phi,\theta}}}. \hspace{1cm} (4.17)
$$

Technically $\chi$ is a signed quantity (as $\eta$ is a signed quantity); we have chosen to take the positive value above because for an inflow (with an outwardly directed magnetic
field) \( \eta \) is positive. Inserting \( \chi \) into the transfield equation, we find:

\[
4\pi \sqrt{-g} \frac{X_A}{A_{\phi,A}} = -\frac{1}{\sin \theta} \left[ (\alpha - 4\pi \chi \mu \eta) A_{\phi,r} \right]_r - \frac{1}{\Delta} \left[ \frac{1}{\sin \theta} (\alpha - 4\pi \chi \mu \eta) A_{\phi,\theta} \right]_\theta \\
+ \frac{G_{\phi}}{\Delta \sin \theta} \left( \Delta A_{\phi,r}^2 + A_{\phi,\theta}^2 \right) \Omega_{F,\psi} - \frac{\Sigma}{2\Delta \sin \theta} (4\pi \eta L)^2_{\psi} - \mu \eta \frac{4\pi \sqrt{-g}}{\rho_\infty^2} u_\phi (4\pi \eta L)_{\psi} \\
+ 4\pi \sqrt{-g} u^1 \chi \left[ \left( \mu \eta c \right)_{\psi} + \mu \eta u_\phi \Omega_{F,\psi} \right] + \mu \eta \frac{1}{\chi} 4\pi \sqrt{-g} (\ln \eta)_{\psi} \cdot (4.18)
\]

This form allows us to apply a more physically useful (and plausible) force-free limit than \( \mu \eta \to 0 \); the limit \( \mu \to 0 \). That limit is more useful because it allows us to decouple the particle flux as a conserved parameter from the limit of a magnetically dominated magnetosphere. It further allows us to calculate the Alfvén Mach number without having to worry about very small \( \eta \), which for a finite Alfvén Mach number would naively result in a diverging plasma density. Both limits can be mathematically valid; crudely speaking, taking the limit \( \mu \to 0 \) is taking the limit of vanishing plasma momentum while taking the limit \( \eta \to 0 \) is taking the limit of very large magnetic fields. The effect is ultimately the same, but the physical interpretation and application to the relevant equations can be different.

Solving the transfield equation is the primary difficulty of finding black hole magnetospheres; it is highly non-linear and must generally be solved numerically. That non-linearity is primarily sourced by conserved field line angular velocity \( \Omega_F \); taking the limit \( \Omega_F \to 0 \) (or at the very least taking the limit of uniform \( \Omega_F \)) can be a useful initial approach when beginning to study the transfield equation. Unfortunately energy-extracting magnetospheres generally cannot have vanishing field line angular velocity, though, so ultimately the full transfield equation must be considered if an energy-extracting magnetosphere is desired.
PROPERTIES OF BLACK HOLE MAGNETOSPHERES

In this chapter we will discuss some general features and properties of black hole magnetospheres that will be useful in interpreting our techniques and results. The nature of light surfaces, separation surfaces, and the requirement that the Alfvén surface of an inflow must be within the ergoregion if a black hole’s rotational energy is to be extracted are perhaps the most fundamental aspects of this chapter.

Light Surfaces

Rigid field line rotation immediately implies the potential existence of regions where it will be impossible for plasma to remain stationary with respect to a magnetic field line, as doing so would imply superluminal motion. Considering a classical velocity $\vec{v} = d\vec{x}/dt$, there are two obvious potential causes of superluminal motion: either $d\vec{x}$ is too large or $dt$ is too small.

Excessively large $d\vec{x}$ results in what is called the “outer light surface” and is analogous to a pulsar light cylinder. It may be understood classically by considering a rigidly rotating disk; for any non-zero angular velocity $\Omega$ there will be some cylindrical radius $s$ such that $s\Omega$ exceeds the speed of light. Excessively small $dt$ is a general-relativistic effect, resulting in what is called the “inner light surface” close to the black hole. It may be understood via time dilation; clocks will begin to run slower the closer the field line gets to the black hole, an effect that can grow faster than the shrinking value of $s\Omega$ until an observer rotating with the field line would become superluminal.

To derive the location of the light surfaces, we consider the four velocity of a stationary observer rotating with angular velocity $\Omega_F$ to be given by $u^\alpha = \ldots$
(\(u^t, 0, 0, u^t \Omega_F\)). Velocity normalization then demands that \(u_\alpha u^\alpha = 1\):

\[
(g_{tt} + 2g_{t\phi} \Omega_F + g_{\phi\phi} \Omega_F^2) \left( u^t \right)^2 = \alpha \left( u^t \right)^2 = 1.
\]

(5.1)

Here we have made the definition \(\alpha \equiv g_{tt} + 2g_{t\phi} \Omega_F + g_{\phi\phi} \Omega_F^2\). If \(\alpha < 0\) physical observers cannot remain stationary with respect to the magnetic field; the surfaces where \(\alpha = 0\) correspond to the light surfaces. Taking the limit of flat space the condition \(\alpha = 0\) reduces to the condition \(r^2 \sin^2 \theta \Omega_F^2 = 1\), leaving only the outer light surface; the inner light surface is a purely general relativistic effect. Often \(\alpha\) is referred to as the “gravitational Lorentz factor” (due to the above correspondence with \(u^t\)) and is a ubiquitous presence in the equations governing black hole magnetospheres.

In order to better elucidate the nature of the light surfaces, we may solve for \(\Omega_F\) in the equatorial plane under the condition that \(\alpha = 0\); doing so yields the condition:

\[
\Omega_F \left( \alpha = 0, \theta = \frac{\pi}{2} \right) = \frac{1}{r^3 + a^2 r - 2ma^2} \left( 2ma \pm r \sqrt{r^2 - 2mr} \right).
\]

(5.2)

This condition is plotted in Figure 5.1 to show that flat spacetimes have only one light surface for a given field line angular velocity (the pulsar light cylinder), but for both Schwarzschild and Kerr spacetimes two light surfaces can emerge for a given field line angular velocity. Field line angular velocities that never correspond to a light surface are largely unphysical; they imply that no region exists where plasma may remain stationary with respect to a field line.

In regions where \(\alpha \leq 0\) the plasma must necessarily stream either inward or outward along a magnetic field line (such travel implying concurrent toroidal motion that offsets field line rotation and allows the plasma flow to remain physical). Typically it is assumed that a field line will only cross a given light surface once, that inside the inner light surface plasma will stream towards the black hole, and that
Figure 5.1: Light surface locations for flat space, a Schwarzschild spacetime, and a Kerr spacetime. Flat spacetimes have only one light surface (analogous to a pulsar light cylinder), while black hole spacetimes can have two. The inner light surface is often very close to the horizon, and is largely a consequence of gravitational time dilation.

outside the outer light surface plasma will stream away from the black hole. There is no definitive mathematical requirement that those assumptions always hold, but it might be difficult to construct a reasonable magnetosphere where they were violated (the direction of the toroidal magnetic field would have to be adjusted in a potentially unreasonable fashion).

It is not guaranteed that the light surfaces will always be distinct. For spatially uniform field line angular velocities there exists a range of field line angular velocities $\Omega_F < \Omega_{F \text{min}}$ and $\Omega_F > \Omega_{F \text{max}}$ such that the inner and outer light surfaces merge into a single surface (we make the equivocation “spatially uniform” here because $\Omega_F$ is a function of $A_\phi$, typically making knowledge of both $\Omega_F(A_\phi)$ and $A_\phi(r, \theta)$ necessary to locate the light surfaces - but similar considerations apply to all magnetospheres). When the light surfaces merge it becomes possible for a single field line extending from the horizon to spatial infinity to disallow stationary observers along its entire length. That is typically taken to be unphysical, as it would demand that the poloidal
plasma velocity never drop to zero anywhere between the horizon and spatial infinity. For positive black hole spin $\Omega_{F_{\text{min}}}$ is always less than zero and $\Omega_{F_{\text{max}}}$ is always greater than the angular velocity of the horizon $\omega_H$. Throughout this work we will only consider field line angular velocities in the range $0 \leq \Omega_F \leq \omega_H$, so we will always assume the existence of distinct inner and outer light surfaces.

**Separation Surfaces**

There are only two primary forces capable of accelerating the plasma from rest (with respect to a field line); inward gravitational forces from the black hole and outward centripetal forces from field line rotation. Any surface on which those forces are balanced and plasma released from rest would stay at rest is called a “separation surface”.

There are several ways to demonstrate the nature of the separation surface, but one of the more physically illuminating is to consider the geodesic motion (acceleration) of an observer instantaneously at rest with respect to a field line, then find the magnitude of that acceleration in the direction of the field line. We begin by writing the observer’s acceleration as:

$$a^\alpha = -\Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma.$$  \hspace{1cm} (5.3)

We then note the definition of the Christoffel symbols:

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha \mu} (g_{\mu \beta,\gamma} + g_{\mu \gamma,\beta} - g_{\beta \gamma,\mu}).$$  \hspace{1cm} (5.4)

Due to the stationarity and axisymmetry of our metric (and noting that the metric is diagonal in the poloidal directions) the only non-vanishing Christoffel symbols
relevant to our current problem can then be seen to be:

\[ \Gamma^{\alphaYZ} = -\frac{1}{2} g^{\alphaYZ,B} g_{YZ,B}. \] (5.5)

We note that the notation convention in use here is that uppercase Latin indices near the beginning of the alphabet denote poloidal indices (e.g. \( A = r, \theta \)) and that Latin indices at the end of the alphabet denote temporal and azimuthal indices (e.g. \( Z = t, \phi \)). We then find the component of the observer’s acceleration in the direction of the magnetic field to be (recalling that \( u^\alpha = (u^t, 0, 0, u^t \Omega_F) \)):

\[
B_\alpha a^\alpha = -B_\alpha \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = -B_A \Gamma^{\alpha YZ} u^Y u^Z = (u^t)^2 B_A \left( g_{tt,A} + 2g_{t\phi,A} \Omega_F + g_{\phi\phi,A} \Omega_F^2 \right)
= \frac{1}{\alpha} B^\gamma \alpha_{\gamma}. \] (5.6)

To arrive at the above we took advantage of the four velocity normalization condition and the fact that field line angular velocity is conserved; \( B^\alpha \Omega_{F,\alpha} = 0 \). We immediately see that the only places along a field line at which centripetal and gravitational forces are balanced (in the direction of the magnetic field line) are the points where \( B^\beta \alpha_{\beta} = 0 \). From the previous section we know that \( \alpha = 0 \) on the light surfaces and that \( \alpha > 0 \) between them, so we immediately know that there will be at least one separation point between the light surfaces where plasma released at rest with respect to the field line will remain at rest. In general only one such point should be expected, though multiple separation points are not impossible.

Because velocity is not acceleration there is no guarantee that the separation point will coincide with a vanishing poloidal plasma velocity - the word “separation” only refers to forces. When considering more realistic black hole magnetospheres it can nonetheless be useful to consider a separation point as approximately delineating
the region where plasma might realistically build up and launch both an inflow and an outflow. Therefore it is common to use the term “separation surface” to mean “injection surface”, in the sense of being an outer boundary for an inflow and inner boundary for an outflow; it should be noted, however, that correspondences between a separation surface and an injection surface are typically modeling assumptions, not physical requirements.

Horizon Regularity Condition

If an ingoing observer on the horizon is to measure finite electromagnetic fields, then along the horizon we must have (cf. [127]):

\[
\sqrt{-g} F_{\theta r}^H = -\frac{(r_H^2 + a^2) (\Omega_F - \omega_H) \sin \theta_H}{\Sigma_H} A_{\phi \theta}^H. \tag{5.7}
\]

This is effectively a statement that $B_\theta^H = 0$ and $B_\phi^H \sim \Omega_F B_r^H$, which is to say that it is a somewhat tautological statement that the horizon is a horizon. This can be seen in a flat Cartesian spacetime by considering a generic Lorentz boost in the $\hat{x}$ direction from frame $S$ with electromagnetic fields $\vec{E}$ and $\vec{B}$ to frame $S'$ with electric fields $\vec{E}'$ and $\vec{B}'$; the resultant magnetic fields are given by (cf. [32]):

\[
\begin{align*}
E'_x &= E_x, & E'_y &= \gamma (E_y - v B_z), & E'_z &= \gamma (E_z + v B_y), \\
B'_x &= E_x, & B'_y &= \gamma (B_y + v E_z), & B'_z &= \gamma (B_z - v E_y). \tag{5.8}
\end{align*}
\]

If this is taken to be a lightspeed boost (i.e. $v \to 1$ and $\gamma \to \infty$) then for there to be finite electromagnetic fields in frame $S'$ after the boost, in frame $S$ prior to the boost
we must have:

\begin{align}
E_x &= E_x, & B_x &= B_x, & B_y &= -E_z, & B_z &= E_y. \quad (5.9)
\end{align}

If we make a correspondence to stationary and axisymmetric magnetohydrodynamics with \( x \to r, \ y \to \theta, \) and \( z \to \phi \) then we immediately demand that \( E_x \sim \Omega_F B_y, \ E_y \sim \Omega_F B_x, \) and \( E_z = 0 \) prior to the boost. In other words, for finite fields after the boost, prior to the boost our restrictions reduce to \( B_y = 0 \) (i.e. \( B_\theta = 0 \)) and \( B_z = -E_y \) (i.e. \( B_\phi \sim \Omega_F B_r \)). By definition the horizon is related to more distant regions in effectively the same way that frame \( S' \) is related to frame \( S \), so we immediately demand the horizon regularity condition because the horizon is a horizon.

The horizon regularity condition may be proven directly by making a coordinate transformation to ingoing coordinates (analogous to the lightspeed boost used above); one such transformation is given by:

\begin{align}
t &= t' - \int_0^{r'} \frac{r^2 + a^2}{\Delta} \, dr, \quad r = r', \quad \theta = \theta', \quad \phi = \phi' - \int_0^{r'} \frac{a}{\Delta} \, dr. \quad (5.10)
\end{align}

After somewhat lengthy but straightforward algebra, it can be shown that:

\begin{align}
F_{\theta' r'} = \frac{\Sigma}{\Delta \sin \theta} \left[ \sqrt{-g} F^{\theta r} + \frac{(r^2 + a^2) \Omega_F - a}{\Sigma} \sin \theta F_{\theta \phi} \right]. \quad (5.11)
\end{align}

In order for there to be finite electromagnetic fields on the horizon (\( \Delta = 0 \)) in this frame we immediately demand the horizon regularity condition (noting that \( \omega_H = a/(r_H^2 + a^2) \)).

Ultimately, however, while we have shown that the horizon condition is to be expected we have not shown that our governing equations are self-consistent; to accomplish that we must demonstrate that the horizon regularity condition does not
restrict the solution space implied by those equations. We will do so by examining the transfield equation on the horizon in the force-free limit (the same condition emerges without taking the force-free limit, the calculations are simply more tedious). The transfield equation in the force-free limit may be written as:

\[
\frac{1}{2} \sum \Delta \sin \theta \frac{d}{d A_\phi} \left( \sqrt{-g} F^{\theta \phi} \right)^2 = -4\pi \sum \sin \theta \frac{X_A}{A_{\phi, A}} - \frac{1}{\sin \theta} (\alpha F_{\tau \phi})_x - \frac{1}{\Delta} \left( \frac{\alpha}{\sin \theta} F_{\theta \phi} \right)_\theta \\
+ \frac{G_{\phi}}{\sin \theta} \left( F_{\tau \phi} \Omega_{F, x} + \frac{1}{\Delta} F_{\theta \phi} \Omega_{F, \theta} \right).
\] (5.12)

If this is to remain force-free on the horizon when \( \Delta \to 0 \), we immediately demand that (everything in what follows should be taken to be evaluated on the horizon; we will suppress horizon labeling for clarity):

\[
\left( \sqrt{-g} F^{\theta \phi} \right)_\theta = -2 \frac{\sin \theta}{\Sigma} F_{\theta \phi} \left( \frac{\alpha}{\sin \theta} F_{\theta \phi} \right)_\theta + \frac{2G_{\phi}}{\Sigma} F_{\theta \phi}^2 \Omega_{F, \theta}.
\] (5.13)

Along the horizon \( \alpha = G_{\phi}^2 / g_{\phi \phi} \), \( G_{\phi} = g_{\phi \phi}(\Omega_F - \omega_H) \), and \( g_{\phi \phi} = -(r^2 + a^2)^2 \sin^2 \theta / \Sigma \), and this reduces to:

\[
\left( \sqrt{-g} F^{\theta \phi} \right)_\theta = 2 \frac{F_{\theta \phi} \sin \theta}{\Sigma} \left[ (r^2 + a^2) \left( \Omega_F - \omega_H \right)^2 \frac{F_{\theta \phi} \sin \theta}{\Sigma} \right]_\theta \\
- 2 \left( r^2 + a^2 \right)^2 \left( \frac{F_{\theta \phi} \sin \theta}{\Sigma} \right)^2 \left( \Omega_F - \omega_H \right) \Omega_{F, \theta} \\
= \left[ \frac{(r^2 + a^2) (\Omega_F - \omega_H) \sin \theta}{\Sigma} F_{\theta \phi} \right]_\theta^2.
\] (5.14)

This can be immediately recognized as the square of the horizon regularity condition; it emerges as a square because the equations involved are insensitive to the sign of the toroidal magnetic field. This is only one example; the horizon regularity condition
also emerges in other areas, such as when solving for the Alfvén Mach number along the horizon or when relating the fast magnetosonic surface to the horizon (as was noted in Chapter 4). Its emergence is so ubiquitous, in fact, that it serves as a useful check that algebraic manipulations have been done correctly - if it doesn’t emerge when taking the horizon limit then a mistake has likely been made.

Within the force-free limit it is always the square of the horizon regularity condition that emerges; the force-free equations are agnostic to the direction of the toroidal magnetic field (which is to say agnostic to the inward/outward direction of any Poynting flux). It is only when enforcing physical infalling observers (such as when finding a physical plasma inflow) that the sign emerges. As such, within the force-free limit the horizon regularity condition does select the physically valid solution, but does not otherwise restrict the solution space in any way; outside of the force-free limit the horizon regularity condition provides no additional restrictions.

Due to the relatively simple form of the horizon regularity condition, it is often used as a convenient short cut in exactly the same way in which a boundary condition on the horizon might be used. As such it can easily be misinterpreted as being a boundary condition that restricts the solution space or otherwise provides additional information to the fundamental assumptions being made. That is the primary reason why we have belabored its nature; it is often used as a shortcut, and unless its role is clearly grasped misunderstandings can emerge.

**Horizon and Ergosphere**

The event horizon and ergosphere of a black hole fundamentally separate the problem space of black hole magnetospheres from magnetospheres that might be applicable in other contexts. The equations involved in black hole magnetospheres may be recast to make them reminiscent of equations more commonly used in
Newtonian contexts (e.g. via 3+1 decompositions [45] [85] [113]) or those fundamental
differences may be respun to make them analogous to flat space concepts (e.g. treating
the horizon as a resistive membrane [86] [114] or with the “light speed boost” of
the previous section), but ultimately both the horizon and ergosphere are physical
concepts largely absent from Newtonian contexts and must be treated as such. In
this section we explore some of the special properties of black hole magnetospheres
that will be relevant to our discussion.

Closed Horizon Loops

Plasma cannot flow outward from the horizon; it must flow inwards. This
immediately and directly implies that the horizon cannot support closed horizon loops
under the assumptions we have made; if there is a conserved particle flux, it cannot
flow into the horizon on one end of such a loop and outward on the other end. This
means that field lines piercing the horizon can do one of two things. First, they can
be open and extend to large distances, where “large distances” should not be taken
to imply spatial infinity but rather the point at which the assumptions made near
the horizon might have become less robust (e.g. inconsistencies with an inflow, such
as crossing the outer light surface, will typically emerge long before spatial infinity is
reached). Second, a field line may directly connect the horizon to a nearby accreting
matter structure.

In the case of an open field line extending to large distances there might not
be a clear point where the assumptions we have made break down. Instead small
deviations from our assumptions might gradually build up to the point where field-
aligned conserved quantities valid near the horizon become significantly different from
the field-aligned conserved quantities further away. In the case of field lines that
connect to nearby matter, it might generally be expected to find significant deviations
from our assumptions; for example, descriptions utilizing hydrodynamic imperfect fluids might become much more appropriate. Regardless of whether a field line is open or connects to nearby matter, however, the effect is the same: the problem is immediately separated into a “near horizon region” where our assumptions might be taken as valid and more distant regions where additional physics and assumptions must be considered.

As a note, closed horizon loops are also disallowed in the force-free limit [29, 54], even if some plasma source is present to drive an inflow down both ends. The exception is if such a loop encloses a non force-free region. Any such loop would carry no energy or angular momentum fluxes, however, as any such conserved fluxes could not both emerge from and enter the horizon. Therefore even if closed horizon loops were taken to exist in the force-free limit by appealing to an encircled non force-free region they would not extract a black hole’s rotational energy, and as such they are less interesting for the purposes of this work.

Horizon and Boundary Conditions

The horizon is not an object; it is simply spacetime curvature. As such it cannot provide any useful internal boundary conditions, source a plasma, support a magnetic field, or do many of the things that one might commonly expect a star, planet, or other astrophysical object to do. At the same time the black hole is not isolated or disconnected from the outside world, but can source significant outflows of energy and angular momentum. There are two ways of dealing with such a dichotomy of effect. The more distant matter that ultimately sources any inflow and supports a magnetic field can be considered as the primary actor, and the black hole spacetime the stage on which that matter evolves. Alternatively the black hole can be treated as the primary actor, with matter sources assumed to exist in whatever fashion might
be necessary.

Treating the black hole as a stage is advantageous whenever a specific astrophysical context is being explored, as any such context will immediately yield the structure and composition of the matter and electromagnetic fields surrounding the black hole. The limitation of such an approach is that any black hole magnetosphere so obtained cannot be decoupled from the assumed boundary conditions enforced by the astrophysical model being applied; it can be impossible to disentangle the ultimate source of any observed behaviors as being due to the black hole or due to the applied model.

If black hole magnetospheres are to be studied purely as “black hole magnetospheres” and not “the magnetosphere associated with additional astrophysical model and assumptions X, Y, and Z” the black hole must be treated as the primary actor. That is the approach that we take throughout this work, while noting that such an approach is intrinsically limited. We cannot directly apply any of the magnetospheres we obtain to any specific astrophysical object; at best we can say what distant matter structures might be more or less compatible with a given magnetosphere. The advantage of this approach is that we can more confidently state the the magnetospheres obtained are “black hole” magnetospheres - the black hole is the primary actor.

The black hole may be treated as the primary actor in at least three different ways. First, “monopolar” boundary conditions may be assumed, such that a single magnetic field line traces both the azimuthal axis and the equatorial plane. Near the horizon such boundary conditions are consistent with plasma flowing into the black hole. Away from the black hole in the equatorial plane such boundary conditions can become less robust, especially as the innermost stable circular orbit is passed; eventually a model of nearby accreting matter would need to be applied. Second, the
horizon can be taken to be the surface on which field-aligned conserved quantities are specified, then the structure of the magnetosphere can be extended to distant regions in order to discover which horizon-specified parameter configurations might be more or less compatible with different configurations of nearby matter. Third, the vector potential along the horizon can be assumed to be monotonically increasing (or decreasing). When viewing the vector potential as a flux function that traces the poloidal magnetic field, that assumption prevents the emergence of closed horizon loops, at the cost of potentially excluding magnetospheres where the presence of nearby matter might allow for alternative distributions of horizon magnetic flux. In this work we generally make all three of the above assumptions.

Ergosphere and Energy Extraction

If the Alfvén point of an inflow occurs within the ergosphere, it is possible to extract energy along that magnetic field line (but not always guaranteed). Robustly proving that statement in full generality is non-trivial [101], but for narrower cases it is relatively straightforward to show, which we will now do.

Examining the Bernoulli equation as expressed by Equation 4.6, we see that if $u^2 p = 0$ (and therefore $\mathcal{M}^2 = 0$, for non-vanishing magnetic field) we must have:

$$\left( \frac{\eta E}{\eta \mu} \right)^2 = \frac{\alpha}{(1 - \Omega_F \tilde{L})}.$$  \hfill (5.15)

The condition $u_p^2 = 0$ may only occur between the light surfaces where $\alpha > 0$. From a practical perspective, if $u_p^2 = 0$ does not occur “naturally” at an injection point where $\alpha \neq 0$ somewhere between the inner light surface and the separation surface (assuming that only one such surface occurs between the light surfaces), then $u_p^2 = 0$ will occur on the outer light surface. For most practical purposes it is therefore typical
to assume an inflow from the horizon that passes through the inner light surface before coming to rest ($u_p^2 = 0$) either somewhere interior to the separation surface or at the outer light surface. Going to rest anywhere involves divergences, however; any such injection point implies diverging plasma density in order to maintain a conserved particle flux, so in a practical model an injection point would generally be taken to coincide with finite $u_p^2$ (or additional considerations to the assumptions we have made). Caveats aside, if the point where $u_p^2 = 0$ is taken to be the injection point, then from the above we conclude that:

$$
\eta E - \Omega_F \eta L = \left( \sqrt{\alpha \mu \eta} \right) \bigg|_I.
$$

(5.16)

Here the subscript “I” denotes evaluation at the injection point; the correct sign of the root requires more detailed analysis than is appropriate here. From the Bernoulli equation as expressed by Equation 4.6 at the Alfvén point where $\mathcal{M}^2 = \alpha$ we demand that:

$$
\Omega_F \frac{\eta L}{\eta E} = -\Omega_F \frac{g_{t\phi} + g_{\phi \phi} \Omega_F}{g_{tt} + g_{t\phi} \Omega_F} \bigg|_A.
$$

(5.17)

Here the subscript “A” denotes evaluation at the Alfvén point. Inserting this condition into the injection point condition, we find:

$$
\eta E = \left( \sqrt{\alpha \mu \eta} \right) \bigg|_I \left( \frac{g_{tt} + g_{t\phi} \Omega_F}{\alpha} \right) \bigg|_A.
$$

(5.18)

Therefore $E$ becomes negative if and only if (the condition $\mathcal{M}^2 = \alpha$ implies $\alpha > 0$):

$$
(g_{tt} + g_{t\phi} \Omega_F) \bigg|_A < 0.
$$

(5.19)

The ergosphere is defined as the point where it is impossible for an observer to remain
stationary; \( u^\alpha = (u^t, 0, 0, 0) \) cannot correspond to a timelike observer because \( g_{tt} < 0 \). The metric element \( g_{t\phi} \) is always greater than zero, so we can only have a negative energy outflow (for field line angular velocity \( \Omega_F > 0 \)) when the Alfvén point of the inflow is inside the ergosphere. It can also be shown that only field line angular velocities in the range \( 0 < \Omega_F < \omega_H \) can have negative energy under the assumptions we have made, so we conclude that not only must the Alfvén point be inside the ergoregion for energy to be extracted, but also that field lines must co-rotate at less than the rate at which the black hole rotates.

In full generality the most robust statement is that the Alfvén point of a plasma inflow must be located inside the ergosphere in order for a black hole’s rotational energy to be extracted. For most practical physical flows, however, a net energy outflow requires that the energy extraction be at least partly (and generally primarily) an electromagnetic process; inflows where the particles carry a negative energy flux while the fields carry a positive energy flux are largely impractical.

Although we have tacitly assumed that any field line on which energy is extracted ultimately crosses the horizon, energy extraction is purely related to processes within the ergosphere and has almost nothing to do with the horizon. The horizon does play an indirect role in that a horizon is necessary for an ergosphere to be stable, but the horizon is nonetheless largely irrelevant when discussing black hole energy extraction (similar to the way in which distant matter is assumed to ultimately source the plasma and electromagnetic fields of an energy-extracting black hole magnetosphere, but is otherwise somewhat irrelevant).

**Concluding Remarks**

In this chapter we discussed the existence and nature of light surfaces, separation surfaces, the horizon regularity condition, and the implications of a horizon and
ergosphere in studying black hole energy extraction. We argued that black hole magnetospheres should generally be treated separately from any nearby matter model for the phrase “black hole magnetosphere” to be most appropriate. We noted that poloidal magnetic field lines must generally go “straight out” near the horizon, as a consequence of the horizon regularity condition and the absence of closed horizon loops. Interpreting the vector potential $A_\phi$ is a flux function, we can then indirectly conclude that (absent a specific matter model) $A_\phi$ should generally monotonically increase or decrease along the horizon. For a black hole magnetosphere to be energy extracting the Alfvén point of the flow must be located inside the ergoregion and have a field line angular velocity between $0 < \Omega_F < \omega_H$, noting that net energy extraction should generally be expected to be primarily an electromagnetic process, not a plasma process.

Many of the arguments and conclusions we made are at least somewhat conditional and not robustly true in full generality, but can typically be taken to be valid for most reasonable energy-extracting black hole magnetospheres. They will therefore generally form the basis for the conditions we apply when exploring and constructing energy-extracting black hole magnetospheres.
In this chapter we explore and extend the monopolar perturbation of force-free magnetospheres originally found by Blandford & Znajek [6]. We show that the monopolar magnetosphere they perturbed around is a special case of a much broader class of magnetospheres, and that the single perturbed solution they found is also a special case of a broader solution space. In so doing we find that the field line angular velocity $\Omega_F$ (which determines if a force-free magnetosphere is capable of extracting a black hole’s rotational energy) is coupled to large-scale magnetosphere structure. This chapter is significantly based on Thoelecke et al. [109], modified as appropriate for this work.

**Background**

As mentioned in the introduction, Blandford & Znajek [6] developed a mathematical procedure for perturbing force-free magnetospheres from non-rotating to rotating spacetimes. Using that procedure it was shown that a monopolar magnetic field around a slowly rotating black hole should rigidly rotate at roughly half the rate at which the horizon rotates, demonstrating that it is possible to efficiently extract a black hole’s rotational energy via an outgoing Poynting flux. The magnetosphere parameters associated with that monopolar solution are still commonly used today to describe and estimate the expected physical attributes of an energy-extracting black hole magnetosphere, to the point that any deviations from those canonical values in numerical simulations can lead to expressions of concern [79].

What is sometimes missed is that the single monopolar solution and associated magnetosphere parameters found by Blandford & Znajek [6] (and extended to higher order in later works [58 74 105]) is based on a special case of a broader class of
monopolar magnetospheres applicable to flat spacetimes found by Michel [61] (that are themselves a subset of a broader solution space). Specifically, the solution found by Blandford & Znajek [6] is the result of perturbing around an initially non-rotating magnetosphere, while the solutions found by Michel [61] allow for arbitrary uniform rotation (non-uniform rotation is also conditionally allowed).

From an analytic computational perspective there is a very practical reason for selecting an initially non-rotating magnetosphere; the perturbation techniques used are primarily useful when changes to the structure of the poloidal field are small. If the changes are not small then the non-linearity of the equations involved can lead to large corrections that reduce the utility of a perturbative approach. The non-linearity in the equations is primarily sourced by field line rotation, so assuming vanishing initial rotation diminishes the probability of the emergence of problematic effects (at least in first order corrections).

In this chapter we more generally perturb the monopolar magnetospheres of Michel [61] in a manner compatible with the approach of Blandford & Znajek [6] without demanding that the initial magnetosphere be non-rotating. The non-linear nature of the equations necessarily limits the obtained solutions’ regions of validity, and should a specific rotational profile be desired different analytic (or numeric) techniques from the somewhat ad hoc ones we employ would likely be more profitable. Nonetheless the solutions obtained are sufficient to demonstrate that more slowly rotating magnetospheres (referenced to black hole spin) should be expected to have poloidal magnetic field lines that bend upwards towards the azimuthal axis, while more rapidly rotating magnetospheres should have poloidal magnetic field lines that bend downwards towards the equatorial plane. The single monopolar (or split-monopolar) solution found by Blandford & Znajek [6] and others is a separatrix between those two behaviors, implying much smaller changes to the structure of
the poloidal magnetic field and therefore more compatibility with a perturbative approach.

From a mathematical point of view this suggests that higher-order perturbative explorations of the common monopolar solution might become problematic if deviations from the separatrix solution emerge, potentially leading to apparent inconsistencies with a perturbative approach. Should divergences or other poor behaviors emerge, they might be understood and corrected by considering how the solution deviates from the separatrix. The separatrix solution might also be more usefully mathematically interpreted as a rotation-driven selection from a group of existing solutions instead of the reaction of a single solution to the addition of spacetime rotation.

From a physical point of view this suggests that the landscape of energy-extracting black hole magnetospheres is much broader than the single monopolar solution found by Blandford & Znajek \cite{Blandford1977} and others might suggest, and by extension that different rates of black hole energy and angular momentum extraction might be coupled to global magnetosphere structure. That broader solution space allows for more flexibility in considerations of astrophysical scenarios in which black hole energy extraction might be relevant. Instead of a single solution with a fixed rate of energy extraction for a given black hole spin, the rate of energy extraction may be tuned in a way that is coupled to magnetosphere structure.

**Basic Equations and Assumptions**

The basic equations and assumptions we use in this chapter are the same as those outlined in the previous chapters. The primary difference here is that we specialize to the force-free limit, which in most instances simply reduces to taking the limit $\mu \eta \to 0$. For completeness we will briefly restate the assumptions that are relevant
here.

The first assumption is of a stationary and axisymmetric spacetime, expressed in Boyer-Lindquist coordinates as:

\[
 ds^2 = \left(1 - \frac{2mr}{\Sigma}\right)dt^2 + \frac{4mar\sin^2\theta}{\Sigma}dtd\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - \frac{A\sin^2\theta}{\Sigma}d\phi^2. \quad (6.1)
\]

We then assume a stationary and axisymmetric force-free magnetosphere, such that:

\[
 T_{\alpha\beta} = -F_{\alpha\beta}J^\beta = X_\alpha = 0. \quad (6.2)
\]

The condition that \(X^\alpha = 0\) in the force-free limit reduces to the requirement that:

\[
 4\pi \Sigma \sin \theta \frac{X_A}{A_{\phi,A}} = -\frac{1}{\sin \theta} \left(\alpha F_{r\phi}\right)_r - \frac{1}{\Delta} \left(\frac{\alpha}{\sin \theta} F_{\theta\phi}\right)_\theta + \frac{G_\phi}{\sin \theta} \left(F_{r\phi}\Omega_{F,r} + \frac{1}{\Delta} F_{\theta\phi}\Omega_{F,\theta}\right) \\
  - \frac{1}{2 \Delta \sin \theta} \frac{d}{dA_\phi} \left(\sqrt{-g} F^{\theta r}\right)^2. \quad (6.3)
\]

Stationarity, axisymmetry, and the assumption of a perfectly conducting plasma (expressed here as the vanishing contraction of the electromagnetic field strength tensor with its dual) allow us to recover the conserved field line angular velocity \(\Omega_F\);

\[
 F_{tr} = F_{r\phi}\Omega_F \quad \text{and} \quad F_{t\theta} = F_{\theta\phi}\Omega_F.
\]

Stationarity and axisymmetry demand conserved fluxes of energy \(\eta E\) and angular momentum \(\eta L\) (the concept of \(\eta\) has no meaning in a force-free context, but is used here for notational continuity); in the force-free limit \(\eta E = \eta E(A_\phi) = (1/4\pi)\sqrt{-g} F^{\theta r}\Omega_F\) and \(\eta L = \eta L(A_\phi) = (1/4\pi)\sqrt{-g} F^{\theta r}\). That in turn demands that the toroidal magnetic field \(\sqrt{-g} F^{\theta r}\) and magnetic field line angular velocity \(\Omega_F\) be conserved; \(\sqrt{-g} F^{\theta r} = \sqrt{-g} F^{\theta r}(A_\phi)\) and \(\Omega_F = \Omega_F(A_\phi)\). If those conditions are not met then \(X_t \neq 0\) and/or \(X_\phi \neq 0\) and satisfying \(X_A = 0\) in Equation 6.3 would not yield a self-consistent \(X^\alpha = 0\) solution.

Equation 6.3 is insensitive to the sign of the toroidal field, which is to say
insensitive to the inward/outward direction of Poynting flux. If an ingoing observer on the horizon is to measure finite electromagnetic fields, however, then the toroidal field must satisfy the Znajek regularity condition on the horizon ([127], Chapter 5):

\[ \sqrt{-g} F^\theta r (r_H, A_\phi) = -\frac{(r_H^2 + a^2) (\Omega - \omega_H) \sin \theta_H}{\Sigma_H} A^{\phi,\theta}_H. \] (6.4)

The angular velocity of the horizon \( \omega_H \) may be written as \( \omega_H = a/2mr_H \). This regularity condition does not restrict the solution space in any significant way, as its square is already present in Equation 6.3; the only additional information it offers is which solution for the toroidal field (ingress or upgoing Poynting flux) is physically valid on the horizon. Nonetheless it can still be a very useful simplification of Equation 6.3 when the horizon is being considered.

Schwarzschild Monopole Solution

In this section we arrive, in abbreviated form, at the monopolar solution found by Michel [61] in the context of flat space and extended by Blandford & Znajek [6] to slowly rotating spacetimes. We refer to it as a “monopolar” solution because the magnetic field in the poloidal plane is monopolar, but there can be a toroidal component of the magnetic field arising from magnetosphere rotation. A monopolar poloidal magnetic field around a Schwarzschild black hole may be described by \( A_\phi = B_0 \cos \theta \), which yields \( F_{\theta\phi} = -B_0 \sin \theta \) and a magnetic field far from the black hole \( \vec{B} \sim (B_0/r^2) \hat{r} \) in standard orthonormal spherical coordinates (Appendix C). For such a field to be force-free everywhere, far from the black hole we demand that (Equation

\footnote{Often such solutions are referred to as “split-monopolar” by adding reflection asymmetry across the equatorial plane (in order to make the magnetic field divergenceless when integrated over a volume enclosing the black hole); such distinctions are irrelevant for our current purposes, however, and can trivially be addressed by applying the restriction \( 0 \leq \theta \leq \pi/2 \) everywhere.}
Here a prime denotes a derivative with respect to $A_\phi$. Therefore the field line angular velocity as $r \to \infty$ can be specified as an arbitrary function of poloidal angle. Insertion of this condition into Equation 6.3 shows that it holds for all radii, including the horizon (as may be intuited from the coincident symmetry of the spacetime with the electromagnetic fields), yielding a general solution that is given by:

$$A_\phi = B_0 \cos \theta,$$
$$\Omega_F = \Omega(\theta),$$
$$\sqrt{-gF^{\theta r}} = B_0 \Omega(\theta) \sin^2 \theta. \quad (6.6)$$

To arrive at the above we have applied the Znajek regularity condition of Equation 6.4 to select the sign of the toroidal magnetic field, and have used $\Omega(\theta)$ to denote the field line angular velocity as an arbitrary function of poloidal angle. Far from the black hole those electromagnetic fields may be expressed in standard orthonormal spherical coordinates as:

$$\vec{B} = \frac{B_0}{r^2} \hat{r} + \frac{B_0 \Omega(\theta) \sin \theta}{r} \hat{\phi},$$
$$\vec{E} = -\frac{B_0 \Omega(\theta) \sin \theta}{r} \hat{\theta}. \quad (6.7)$$

This field configuration always yields a Poynting flux that is directed radially inward, consistent with the physical demand that energy not be extracted from a non-rotating black hole (enforced by application of the Znajek regularity condition of Equation 6.4).
As such it might be said that this solution is restricted by conditions at both the horizon and at spatial infinity. The solutions found by Michel \cite{Michel} have uniform rotation, \( \Omega(\theta) = \Omega_0 \), and Blandford \& Znajek \cite{BlandfordZnajek} perturbed around the static \( \Omega_0 = 0 \) solution.

If \( \Omega(\theta) = 1/a \sin^2 \theta \) then the solution of Equation \eqref{6.6} is also valid when applied to an arbitrarily rotating black hole with spin parameter \( a \) \cite{Bambietal1, BambiZhang,Micaela}. Unfortunately such solutions do not extract a black hole’s rotational energy and can become problematic near the poles; we note them here for the sake of completeness but will not address them further in this chapter.

\textbf{Perturbing to Kerr - Most Monopolar}

In this section we arrive at the perturbed monopolar solution of Blandford \& Znajek \cite{BlandfordZnajek} using a slightly different approach in an attempt to elucidate the nature of the solution obtained. Every solution applicable to a Schwarzschild spacetime found in the previous section (Equation \eqref{6.6}) is “monopolar” in the sense that the poloidal magnetic field traces the vacuum poloidal electric field that would be sourced by a non-rotating black hole possessing an electric charge of magnitude \( B_0 \) (we make this distinction due to magnetic monopoles being on somewhat weaker footing than electric monopoles). We now determine which solution(s) remain monopolar in that sense when extended to a rotating black hole, using the same general procedure as was used in the previous section. We begin as before by assuming a monopolar vector potential, except this time in a form appropriate to a rotating spacetime with \( a \neq 0 \) (which may be derived by taking the standard Kerr-Newman electric field and converting it to a magnetic field):

\[
A_\phi = B_0 \frac{r^2 + a^2}{r^2 + a^2 \cos^2 \theta} \cos \theta. \tag{6.8}
\]
Far from the horizon we arrive at the same condition on the toroidal field as was found in the Schwarzschild case (Equation 6.5, recast in terms of $A_\phi$):

$$\left[\left(\sqrt{-g}F^{\theta r}\right)^2\right]_{r\to\infty} = \frac{1}{B_0^2} \left[(B_0^2 - A_\phi^2)^2\Omega_F^2\right]_{r\to\infty}. \quad (6.9)$$

When the spacetime rotates the Znajek regularity condition on the horizon proves to be much more restrictive than it was in the previous section; instead of simply restricting the sign of the toroidal field, we now also require that the toroidal field satisfy:

$$\left(\sqrt{-g}F^{\theta r}\right)_H^2 = \frac{1}{B_0^2} (\Omega_F - \omega_H)^2 \left(1 - \frac{a^2 A_\phi^2}{B_0^2 m^2}\right)_H \left[\frac{m^2 B_0^2}{r_H^2} \left(1 + \sqrt{1 - \frac{a^2 A_\phi^2}{B_0^2 m^2}}\right)^2 - A_\phi^2\right]^H. \quad (6.10)$$

This will be derived in detail at the end of this section. For a slowly rotating black hole, we may proceed by considering an expansion in black hole spin to find:

$$\left(\sqrt{-g}F^{\theta r}\right)_H^2 = \frac{1}{B_0^2} (\Omega_F - \omega_H)^2 \left[\left(B_0^2 - A_\phi^2\right)^2 + \frac{1}{B_0^2} (B_0^2 - A_\phi^2)^2 \frac{a^2}{m^2} + \mathcal{O}\left(\frac{a^4}{m^4}\right)\right]^H. \quad (6.11)$$

Here we have assumed that $\Omega_F \sim \mathcal{O}(\omega_H)$; if they were of significantly different order then the inner and outer light surfaces would not be distinct and the solution would have diminished physical relevance (assuming black hole spin different enough from $a = 0$ to be interesting). Combining this with the condition at spatial infinity (Equation 6.9) yields:

$$\Omega_F^2 = (\Omega_F - \omega_H)^2 + \mathcal{O}\left(\frac{a^4}{m^4}\right). \quad (6.12)$$
Therefore for slowly rotating black holes we have \( \Omega_F \approx \omega_H / 2 \), the same conclusion Blandford & Znajek [6] and others have arrived at using slightly different approaches. Dropping the \( a^4 \) terms to arrive at that result effectively reduced the vector potential of Equation 6.8 to \( A_\phi = B_0 \cos \theta \), the same as in the Schwarzschild case. Should higher order corrections be desired to explore the more strictly “monopolar” form of Equation 6.8 or other effects, the full expression for \( \Omega_F \) is given by:

\[
\Omega_F = \omega_H \frac{\chi}{\chi + (B_0^2 - A_\phi^2)},
\]

where:

\[
\chi = \sqrt{1 - \frac{a^2 A_\phi^2}{B_0^2 m^2}} \left[ \frac{m^2 B_0^2}{r_H^2} \left( 1 + \sqrt{1 - \frac{a^2 A_\phi^2}{B_0^2 m^2}} \right) - A_\phi^2 \right].
\] (6.14)

Taking an expansion in spin on the horizon, the field line angular velocity is then given by:

\[
\Omega_F(r_H) = \omega_H \left[ \frac{1}{2} + \frac{a^2 \sin^2 \theta}{8m^2} + \frac{3a^4 \sin^2 \theta}{64m^4} + \mathcal{O}\left( \frac{a^6}{m^6} \right) \right].
\] (6.15)

The reason for selecting the techniques used in this section was to note that the assumption of a specific poloidal magnetic field configuration followed by solving for compatible distributions of field line angular velocity and toroidal field is exactly what most perturbation techniques do (at least to leading order). This is due to the fact that if there were significant changes to the poloidal field a perturbative approach would likely be ill-behaved, as the non-linear nature of the equations involved (most prominently sourced by field line rotation) could easily amplify even small changes to the fields and lead to higher-order corrections that are comparable in magnitude (if not larger) than the unperturbed terms.

At higher orders the solution obtained here differs slightly from the solutions obtained using other perturbative approaches, although direct comparisons can become
difficult: at higher orders other approaches often sacrifice the rigid conservation of field-aligned fluxes of energy and angular momentum and can have difficulty elegantly describing the region interior to $r = 2m$ (i.e. the ergoregion). Nonetheless such solutions are still broadly compatible with ours, finding $A_\phi \sim \cos \theta + a^2 \cos \theta \sin^2 \theta$ (compatible with the assumed monopolar geometry in Equation 6.8) and a field line angular velocity $\Omega_F \sim \omega_H/2 + a^3 \sin^2 \theta$ (compatible with the expansion in Equation 6.15). The differences are primarily due to our approach of exactly satisfying the force-free condition of Equation 6.3 on only the horizon and at spatial infinity, while perturbative approaches typically seek to approximately satisfy Equation 6.3 over all space (or at least between the perturbed horizon and spatial infinity). In practice that means that our solution has error somewhat off the horizon near the equatorial region, while perturbed solutions have error of similar magnitude but concentrated on and near the horizon.

In summary, when the spacetime is non-rotating a “monopolar” field may have an arbitrary rotational profile described by $\Omega_F = \Omega(\theta)$ (Section 6). In this section we found that the solution that most smoothly transitions to a rotating black hole while remaining largely “monopolar” is the solution with a rotational profile given by $\Omega(\theta) \approx 0.5\omega_H$. In the next section we will explore how solutions with $\Omega(\theta) \neq 0.5\omega_H$ might behave when extended to rotating spacetimes.

An Explicit Monopole’s Toroidal Magnetic Field

In the force-free limit the toroidal field should be a function of the vector potential $A_\phi$ due to its correspondence with a conserved flux of angular momentum from the assumption of axisymmetry. In this subsection we explicitly solve for the form of that function given by Equation 6.10. A monopolar vector potential is given
Figure 6.1: The percent error (Thoelecke et al. [111]) of a monopolar vector potential with black hole spin parameter $a = 0.5m$. The error is largest near the equatorial plane around $r = 2m$, but vanishes along the horizon and as $r \to \infty$. The increase near the equatorial plane is a result of the field line angular velocity there being too large ($\Omega_F \gtrsim 0.5\omega_H$) to be completely compatible with a “straight” magnetic field line. Standard perturbed monopole solutions have errors of similar magnitude, but concentrated on and near the horizon.

by (Equation 6.8):

$$A_\phi = B_0 \frac{r^2 + a^2}{\Sigma} \cos \theta.$$  

(6.16)

Here $\Sigma = r^2 + a^2 \cos^2 \theta$, and with $A_t = -B_0a \cos \theta/\Sigma$ the vector potential would describe the vacuum solution of a black hole possessing a magnetic charge of magnitude $B_0$. On the horizon the toroidal magnetic field is given by (Equation 6.4):

$$\sqrt{-g} F_{H}^{\theta \phi} = -\frac{(r_H^2 + a^2)(\Omega_F - \omega_H)}{\Sigma_H} A_{\phi H, \theta} = - (\Omega_F - \omega_H) \frac{A_\phi}{B_0} A_{\phi H, \theta}. \quad (6.17)$$
In order to express this purely in terms of $A_\phi$ we first evaluate $A_{\phi,\theta}$ to find:

$$A_{\phi,\theta} = B_0 \left( r^2 + a^2 \right) \left( \frac{2a^2}{\Sigma^2} \sin \theta \cos^2 \theta - \frac{1}{\Sigma} \sin \theta \right) = A_\phi \left( 1 - \frac{2r^2}{\Sigma} \right) \tan \theta. \quad (6.18)$$

Therefore the toroidal field on the horizon is given by:

$$\sqrt{-g} F^r_H = -\frac{1}{B_0} \left( \Omega_F - \omega_H \right) \left( 1 - \frac{2r^2}{\Sigma_H} \right) A_{\phi_H}^2 \tan^2 \theta$$

$$= -\frac{1}{B_0} \left( \Omega_F - \omega_H \right) \left( 1 - \frac{2r^2}{\Sigma_H} \right) \left[ \left( \frac{2mr_H B_0}{\Sigma_H} \right)^2 - A_{\phi_H}^2 \right]. \quad (6.19)$$

To proceed further we require an expression for $\Sigma(A_\phi)$ on the horizon; from the expression for $A_\phi$ given in Equation 6.16 we find:

$$A^2_\phi = \frac{B_0^2}{\Sigma^2} \left( r^2 + a^2 \right)^2 \frac{\Sigma - r^2}{a^2}. \quad (6.20)$$

Solving for $1/\Sigma$, we find:

$$\frac{1}{\Sigma} = \frac{1}{2r^2} \left( 1 \pm \sqrt{1 - 4 \frac{a^2 r^2 A^2_\phi}{B_0^2 \left( r^2 + a^2 \right)^2}} \right). \quad (6.21)$$

Taking the relevant positive branch and evaluating along the horizon, we find:

$$\frac{2r^2}{\Sigma_H} = 1 + \sqrt{1 - \left( \frac{a A_{\phi_H}}{B_0 m} \right)^2}. \quad (6.22)$$

We then finally insert this expression into the horizon condition on the toroidal field to conclude that:

$$\sqrt{-g} F^r_H = \frac{1}{B_0} \left( \Omega_F - \omega_H \right) \sqrt{1 - \left( \frac{a A_{\phi_H}}{B_0 m} \right)^2} \left[ \frac{m^2 B_0^2}{r^2_H} \left( 1 + \sqrt{1 - \frac{a^2 A^2_{\phi_H}}{B_0^2 m^2}} \right)^2 - A^2_{\phi_H} \right]. \quad (6.23)$$
This was used above to calculate the field line angular velocity (Equation 6.13) for a monopole by comparing the above condition to the condition on the toroidal field at spatial infinity. The resultant field line angular velocity on the horizon goes as $\Omega_F(r_H) \sim \omega_H(0.5 + \sin^2 \theta)$ (Equation 6.15). The relatively significant deviation away from $\Omega_F \approx 0.5\omega_H$ near the equatorial plane is generally incompatible with a “straight” magnetic field line in the poloidal plane, leading to error near the equatorial plane for median values of $r$ (Figure 6.1). The first step towards eliminating such error might be to explicitly add more “bunching” of field lines near the azimuthal axis; such bunching is observed when finding solutions numerically [111] and when conducting simulations [107], and its nature has been discussed in [30].

**Perturbing to Kerr - Generic Behavior**

We have found that the monopolar magnetosphere in Schwarzschild spacetimes that most closely corresponds to a monopolar magnetosphere in Kerr spacetimes (in the limit of low black hole spin) is specified by $\Omega_F = \omega_H/2$. We now explore what might happen to some of the other $\Omega_F \neq \omega_H/2$ solutions from Equation 6.6 when they are extended to slowly rotating spacetimes. We note from the outset that the results of the previous section indicate that those extensions might generally be expected to involve significant changes to the structure of the poloidal field; they are unlikely to remain “monopolar” over all space, and as such might have only limited regions where the concept of a small perturbation remains valid.

For simplicity we will only consider uniform (constant) field line angular velocities, expressed as $\Omega_F = x\omega_H$, with $x$ a unitless weighting factor on the angular velocity of the horizon.
We begin by perturbing the metric in spin\(^2\) such that:

\[
    ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 + \frac{4ma \sin^2 \theta}{r} dtd\phi - \frac{r}{r - 2m} dr^2 - d\Omega^2. \tag{6.24}
\]

Here \(d\Omega^2 = r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2\) and the metric is equivalent to Schwarzschild with the exception of an additional \(g_{t\phi}\) component. We then assume that the monopolar solutions’ vector potential gains an additional term when extended to that slowly rotating spacetime:

\[
    A_\phi = B_0 \cos \theta + a^2 B_0 R(r) \Theta(\theta). \tag{6.25}
\]

We demand an \(a^2\) correction here because (as noted in the previous section) the field line angular velocity is likely to be proportional to the horizon’s angular velocity if both the black hole spin and magnetosphere rotation are physically interesting, and as signed quantities the two sources of rotation should couple as \(a\Omega_F \sim a^2/m^2\).

We next demand that the vector potential near the horizon remain unchanged, in the sense that the correction to the vector potential must have \(R(2m) = 0\). The primary reason for this demand is that a generally monopolar field close to the horizon is typically more physically plausible than monopolar behavior further away. As we expect to find potentially large deviations from monopolarity in some regions for \(\Omega_F \neq \omega_H/2\) magnetospheres, we choose to directly enforce small changes to the vector potential compatible with our perturbative approach in the more relevant near-horizon region.

Next we determine the structure of the toroidal field; as \(R(2m) = 0\), the

---

\(^2\)The results of this section may also be obtained without this conceit through direct perturbation of the force-free condition of Equation \(6.3\). We choose a more circuitous route in this section and expand the metric and fields separately because doing so can lead to more illuminating intermediate expressions where the different sources of rotational effects are more easily separated and identified. A different approach will be taken in the next section to arrive at the same result.
appropriate form of the horizon regularity condition is straightforward to evaluate to find (in terms of the unitless weighting factor $x$):

$$
\left(\sqrt{-g}F_{tr}\right)_H^2 = \frac{g_{\phi\phi}}{g_{\theta\theta}} (\Omega_F - \omega_H)^2 A_{\phi,\theta}^2 \bigg|_H = \frac{\omega_H^2}{B_0^2} (x - 1)^2 \left( B_0^2 - A_{\phi}^2 \right)^2 \bigg|_H .
$$

(6.26)

The only remaining task is to evaluate the unknown functions $R(r)$ and $\Theta(\theta)$ in the vector potential. We do so by taking $\omega_H = a/4m^2$, then considering the appropriate form of the force-free condition of Equation 6.3. We then find that to $O(a^4/m^4)$ we must have $\Theta(\theta) = \sin \theta \cos^2 \theta$, and demand that $R(r)$ satisfy (primes denoting derivatives with respect to $r$):

$$
8m^4 (r - 2m)^2 R'' + 16m^5 (r - 2m) R' - 48m^4 (r - 2m) R = r^3 (1 - 2x) + 16m^3 x .
$$

(6.27)

Re-writing this in terms of the variable $y = r - 2m$ and the unknown function $Y(y)$, we find (primes denoting derivatives with respect to $y$):

$$
8m^4 y (y + 2m) Y'' + 16m^5 Y' - 48m^4 Y = (1 - 2x) \left( y^2 + 6my + 12m^2 \right) + \frac{8m^3}{y} .
$$

(6.28)

The $8m^3/y$ term on the right hand side sources undesirable logarithmic terms, so we drop it (dropping this term is also advantageous when excess momentum flux is considered, as discussed in the next section). Demanding that $Y(0) = R(r_H) = 0$ we find a general solution of:

$$
A_{\phi} = B_0 \cos \theta - \frac{a^2}{16m^4} B_0 \left( x - \frac{1}{2} \right) (r - 2m) \left( 44r^2 + 13mr + 14m^2 \right) \cos \theta \sin^2 \theta ,
$$

This expression assumes that $\rho_0^2 = g_{t\phi}^2 - g_{tt}g_{\phi\phi}$ vanishes when $r = r_H = 2m$. For the perturbed metric in use we actually have $\rho_0^2(r_H) = a^2 \sin^4 \theta$, so this should be understood as a perturbed expression.
\[ \Omega_F = x \omega_{Hp}, \]
\[ \sqrt{-g} F^{\theta r} = \frac{1}{B_0} \left( x \omega_{Hp} - \frac{a}{4m^2} \right) (B_0^2 - A_\phi^2). \] (6.29)

Here \( \omega_{Hp} = a/4m^2 \); although it has a correspondence to \( \omega_H \), \( \omega_{Hp} \) should not be taken to vanish as \( a \to 0 \); rather it should be taken as a convenient description of \( \Omega_F \) once an \( a \neq 0 \) spacetime has been specified. In application the above solution should be applied with both the metric and force free condition of Equation 6.3 in their full form (\( \omega_{Hp} \) should remain as \( a/4m^2 \)); the separate perturbed expressions shown in this section are only used to explain the considerations used to arrive at Equation 6.29.

The solution of Equation 6.29 makes it clear that the most monopolar \( x = 1/2 \) solution found in the previous section is a separatrix between two different types of solutions. It also makes it clear that the perturbative approach used can become problematic; the correction to the poloidal magnetic field is unbounded as \( r \to \infty \) unless \( x = 1/2 \). We discuss those outcomes and the error of the solution after first re-deriving the above from an alternative perspective.

**Alternative Perturbation of Generic Behavior**

In this section we solve for the generic perturbed solution found above by more directly considering the force-free transfield equation (Equation 6.3). We first note that the most monopolar solution may be expanded in spin (as originally arrived at by Blandford & Znajek [6]) to find:

\[ A_\phi = B_0 \cos \theta, \]
\[ \Omega_F = \frac{1}{2} \frac{a}{4m^2}, \]
\[
\sqrt{-g} F^{\theta r} = -B_0 \frac{a}{8m^2} \sin^2 \theta = -\frac{1}{B_0} \frac{a}{8m^2} (B_0^2 - A_{\phi}^2) .
\] (6.30)

In other words (at least to leading order) the reaction of the initially non-rotating magnetosphere to the addition of spacetime rotation is to develop outward fluxes of energy and angular momentum while maintaining the same poloidal magnetic field structure. This solution may be placed into the force-free transfield equation to find its corresponding excess momentum flux (defining \( \bar{X} \equiv 4\pi \Sigma \sin \theta X_A / A_{\phi, A} \) for the sake of compactness):

\[
\bar{X} = -\frac{1}{2} \Sigma \frac{d}{\Delta \sin \theta dA_{\phi}} (\sqrt{-g} F^{\theta r})^2 - \frac{1}{\Delta} \left( \frac{\alpha}{\sin \theta} A_{\phi, \theta} \right)_\theta \\
= \frac{2B_0 \Sigma}{\Delta} \left( \frac{a}{8m^2} \right)^2 \cos \theta \sin \theta + \frac{B_0}{\Delta} \left[ g_{tt, \theta} + 2g_{t\phi, \theta} \left( \frac{a}{8m^2} \right) + g_{\phi\phi, \theta} \left( \frac{a}{8m^2} \right)^2 \right].
\] (6.31)

The division by zero divergence of this expression on the horizon (\( \Delta = 0 \)) is the result of failing to satisfy the square of the Znajek regularity condition of Equation 6.4, which is intrinsic to the transfield equation. The metric elements \( g_{tt}, g_{t\phi}, \) and \( g_{\phi\phi} \) may be expanded in spin to find:

\[
g_{tt} = 1 - \frac{2m}{r} + \frac{2m \cos^2 \theta}{r^3} a^2 + \mathcal{O} \left( a^4 \right), \\
g_{t\phi} = \frac{2m \sin^2 \theta}{r} a + \mathcal{O} \left( a^3 \right), \\
g_{\phi\phi} = -r^2 \sin^2 \theta - \frac{r + 2m \sin^2 \theta}{r} a^2 \sin^2 \theta + \mathcal{O} \left( a^4 \right).
\] (6.32)

If we discard the metric elements of order \( a^2 \) and higher (taking \( \Sigma = r^2 \)), the excess momentum flux becomes:

\[
\bar{X} \approx \frac{2B_0 r^2}{\Delta} \left( \frac{a}{8m^2} \right)^2 \cos \theta \sin \theta + \frac{B_0}{\Delta} \left[ 8ma \left( \frac{a}{8m^2} \right) - 2r^2 \left( \frac{a}{8m^2} \right)^2 \right] \cos \theta \sin \theta
\]
\[
\frac{B_0}{\Delta} \frac{a^2}{mr} \cos \theta \sin \theta.
\]

This excess momentum flux is due entirely to the \( g_{tb, \theta} \Omega_F \) term. If we also include the order \( a^2 \) term in \( g_{tt} \), we arrive at (taking \( \Delta = r^2 - 2mr \)):

\[
\bar{X} = \frac{B_0}{\Delta} \left[ -\frac{4ma^2}{r^3} \cos \theta \sin \theta + \frac{a^2}{mr} \cos \theta \sin \theta \right] + \mathcal{O}(a^4)
\]

\[
= B_0 \frac{2m + r}{mr^4} a^2 \cos \theta \sin \theta + \mathcal{O}(a^4).
\] (6.34)

This excess momentum flux will be used in the next section when error is discussed.

We have belabored the above in order to suggest that perturbative approaches applied to the problem of black hole magnetospheres can have philosophical differences. For example, the fundamental perturbation might be taken to be the transfield equation, such that a somewhat inconsistent perturbation in the metric and fields (as above, where the second-order correction to \( g_{\phi\phi} \) is irrelevant) is acceptable. Alternatively, the metric might be taken as the fundamental perturbation, with (for example) only terms of order \( a^2 \) kept in the metric, and the fields adjusted in whatever manner might be useful in order to find a vanishing momentum flux under application of that perturbed metric.

In this chapter we largely follow Blandford & Znajek [6] and take the transfield equation to be the fundamental perturbation. However, when studying the equations involved it can sometimes be more illuminating to consider alternative approaches. In light of that, we took a hybrid approach in the previous section. We first perturbed the metric in spin, then selected the well-behaved electromagnetic fields corresponding to that metric. When exploring the error of that solution in the next section, however, we will take the approach used to arrive at Equation 6.34, which is to say we will treat the transfield equation as the fundamental perturbation.
We will now arrive at the same generically rotating solution found in Section 6 without applying a hybrid approach. We begin by taking the unperturbed fields around a non-rotating black hole to be given by:

\[
\begin{align*}
A_\phi &= B_0 \cos \theta, \\
\Omega_F &= x \frac{a}{4m^2}, \\
\sqrt{-g} F^\theta r &= \frac{1}{B_0} x \frac{a}{4m^2} (B_0^2 - A_\phi^2).
\end{align*}
\] (6.35)

Here \(x\) is a unitless weighting factor and \(a\) is arbitrary, though it will be taken to correspond to the spin of the rotating black hole (such that the inner and outer light surfaces can be taken to be distinct for \(x\) values of order 1). We now wish to solve for the structure of the fields under the constraint that the conserved field line angular velocity along each field line remains the same. Following identical logic to that applied in the previous section (which is insensitive to the metric used under the transformation \(2m \to r_H\)) we assume that leading order corrections to the fields are given by:

\[
\begin{align*}
A_\phi &= B_0 \cos \theta + a^2 B_0 R(r) \Theta(\theta), \\
\Omega_F &= x \frac{a}{4m^2}, \\
\sqrt{-g} F^\theta r &= \frac{1}{B_0} \left( x \frac{a}{4m^2} - \frac{a}{4m^2} \right) (B_0^2 - A_\phi^2).
\end{align*}
\] (6.36)

Inserting those fields into the transfield equation, we find (keeping only terms of order \(a^2\), and taking advantage of the calculations done above for the monopolar case):

\[
X = -\frac{1}{2} \frac{\Sigma}{\Delta \sin \theta} \frac{d}{dA_\phi} \left( \sqrt{-g} F^\theta r \right)^2 - \frac{1}{\sin \theta} (\alpha A_{\phi r}, r) - \frac{1}{\Delta} \left( \frac{\alpha}{\sin \theta} A_{\phi \theta} \right)_{\theta}
\]

\[
= 2B_0 \Sigma \frac{d}{\Delta} (1 - 2x) \left( \frac{a}{4m^2} \right)^2 \cos \theta \sin \theta + \frac{B_0}{\Delta} \frac{2xa}{mr} \cos \theta \sin \theta - \frac{4B_0 ma^2}{\Delta r^3} \cos \theta \sin \theta
\]
\[- \frac{B_0}{\sin \theta} \Theta(\theta) a^2 \left[ \left( 1 - \frac{2m}{r} \right) R'(r) \right]_{,r} - \frac{B_0}{\Delta} R(r) a^2 \left[ \left( 1 - \frac{2m}{r} \right) \frac{\Theta'(\theta)}{\sin \theta} \right]_{,\theta} \]  

(6.37)

We can now note two things. First, we would prefer that \( \Theta(\theta) \sim \cos \theta \sin^2 \theta \) such that all terms have the same \( \theta \) dependence. Second, the only \( a^2 \) correction to the metric to survive is the same as was used above, the correction from \( g_{tt} \) that is independent of \( x \). If we demand that the error (in terms of the excess momentum flux) of the general solution match the error of the monopolar solution to order \( a^2 \), then we must have:

\[
\left( \frac{r^2}{8m^4} - \frac{2m}{r^2 - 2mr} \right) - \left[ \left( 1 - \frac{2m}{r} \right) R'(r) \right]_{,r} + \frac{6R(r)}{r^2 - 2mr} \left( 1 - \frac{2m}{r} \right) = 0. \quad (6.38)
\]

Making the variable substitution \( y = r - 2m \) for a function \( Y(y) \), this becomes:

\[
\frac{y}{y + 2m} Y''(y) + \frac{2m}{(y + 2m)^2} Y'(y) - \frac{6}{(y + 2m)^2} Y(y) - \frac{y^2 + 6my + 12m^2}{8m^4} \frac{1 - 2x}{(y + 2m)^2} = 0. \quad (6.39)
\]

This is compatible with Equation 6.28 arrived at through different considerations in the previous section. The primary difference is that there we ignored the order \( a^2 \) term in \( g_{tt} \) and did not add in a term related to \( g_{t\phi} \) in order to explicitly force the errors of the solutions to match. Instead we dropped an undesirable logarithmic source term from the differential equation, ultimately finding that doing so resulted in exactly matching error.

Our basic goal is to find arbitrarily uniformly rotating magnetospheres with error comparable to the widely known first-order corrections to a monopolar magnetic field found by Blandford & Znajek [6] (from the same philosophical perspective of treating the transfield equation as the primary perturbation). Ultimately, however, we are perturbing magnetospheres that already possess rotation and angular momentum...
fluxes, so in a general perturbation in black hole spin it is not entirely obvious that any two factors of $a$ are truly comparable (or simultaneously vanish as black hole spin vanishes). As such, in the previous section we took a hybrid approach that attempted to emphasize that it might be more appropriate to consider a slowly rotating spacetime as a distinct perturbation, from which appropriate electromagnetic fields should be calculated in whatever manner might be most convenient.

As a final concluding remark, for the sake of completeness we should point out that Blandford & Znajek \[6\] did include a correction to $A_{\phi}$ in order to eliminate the remaining order $a^2$ error shown in Equation 6.34. That correction is common to our solution (essentially a particular solution of Equation 6.38 with $x = 1/2$ and the order $a^2$ error as a source term). We have suppressed that correction as being uninteresting; it is sourced by corrections to the metric that are only relevant inside the ergosphere and as such falls off at large radius, which can be seen by taking the limit $r \gg 2m$ in Equation 6.38 with the error of Equation 6.34 as a source term:

\[
R''(r) - \frac{6}{r^2} R(r) = \frac{1}{mr^3}.
\]

(6.40)

Solving, one can conclude that $R(r) \sim 1/4r$, which as expected isn’t significant outside the ergosphere. If the full correction is nonetheless desired, it may be written as:

\[
R_{\text{Corr}} = -\frac{1}{m^4} \left[ \frac{m^2 + 3mr - 6r^2}{12} \ln \left( \frac{r}{2m} \right) + \frac{11m^2}{72} + \frac{m^3}{3r} + \frac{mr}{2} - \frac{r^2}{2} \right]
- \left( \frac{2r^3 - 3mr^2}{8m^5} \right) \left[ \text{Li}_2 \left( \frac{2m}{r} \right) - \ln \left( 1 - \frac{2m}{r} \right) \ln \left( \frac{r}{2m} \right) \right].
\]

(6.41)
where \( \text{Li}_2 \) is the dilogarithm, defined as:

\[
\text{Li}_2(x) = \int_x^0 \frac{1}{t} \ln(1 - t) \, dt. \tag{6.42}
\]

The addition of the above correction to the vector potential might in some sense be more mathematically correct, but offers no real physical insight into the problem. We will therefore ignore it, as it would only serve to obfuscate the more fundamental behaviors involved and complicate the application of the solution to spatial regions interior to \( r = 2m \).

**Error Analysis and Behaviors**

In this section we will discuss the error and general behaviors of the perturbative extension of arbitrarily uniformly rotating magnetospheres found in the previous section, the role of the outer boundary surface, and the bending behaviors found.

**General Solution Error**

The general solution found in the previous sections (Equation 6.29) yields an excess momentum flux (Equation 6.3) that is given by \( \Omega_F = x\omega_{\text{Hp}} \):

\[
4\pi \sum \frac{X_A}{A_{\phi,A}} = a^2 B_0 \frac{2m + r}{mr^4} \cos \theta + \mathcal{O} \left( \frac{a^4}{m^4} \right). \tag{6.43}
\]

This form is a reason for dropping the \( 8m^3/y \) term when solving for \( Y(y) \) in Equation 6.28; to order \( a^2 \) the failure of the momentum flux \( X_A \) to vanish is then identical for all magnetospheres (including the standard “first order” perturbed monopole solution of Blandford & Znajek [6]). The \( \mathcal{O}(a^4) \) term goes as \( r^3 \) (as a function of \( x \)), so although the excess momentum flux \( X_A \) is of comparable magnitude for all solutions near the horizon, it generally grows with increasing \( r \) for \( x \neq 1/2 \) as the vector potential begins
to deviate ever more strongly from its $a \to 0$ monopolar behavior.

A representation of that error is shown in Figure 6.2 for three different $x$ values and black hole spin $a = 0.3m$. The error is expressed as the full excess momentum flux $X_A$ related as a percentage of the largest term of the force-free condition in Equation 6.3 (the exact method used may be found in Thoelecke et al. [111]). It is apparent that the percent error of $x \neq 1/2$ solutions grows in $r$. As was shown in Section 6, that is because $x = 1/2$ is the solution most compatible with monopolarity at both the horizon and spatial infinity.

![Figure 6.2: Three magnetospheres with $\Omega_0 = 0.45\omega_H$, $\Omega_0 = 0.5\omega_H$, $\Omega_0 = 0.55\omega_H$ (where $\omega_H \equiv a/4m^2$) for black hole spin $a = 0.3m$. The background shading is the percent error of the solutions (Thoelecke et al. [111]). The $\Omega_0 = 0.5\omega_H$ solution is a separatrix between two classes of solutions that can exhibit significant modifications to the structure of the poloidal field when extended from Schwarzschild to Kerr spacetimes. We have deliberately chosen to extend the domain to include topological changes to the field (i.e. divergences from monopolarity). In practice those regions are mostly indicative of a breakdown in the solution, and should be viewed with some suspicion.](image)

It is also apparent that the topology of the poloidal magnetic field lines changes for $x \neq 1/2$. The field lines that do not intersect the horizon are outside the domain of the toroidal field as a function of vector potential found in Equation 6.26, however, so the exact nature of those topological changes should be viewed with some suspicion.
For the purposes of this chapter we do not consider such topological changes to be anything more than an indication that the perturbation techniques might have been extended outside a domain of more robust validity, and we don’t insist upon their existence. Should a more extensive radial domain be desired we would suggest either consideration of solutions closer to $x = 1/2$ or the application of other solution techniques before any attempt to justify such topological changes.

The more robust structural change is the tendency for the poloidal magnetic field lines to begin bending away from monopolarity for $x \neq 1/2$. In Figure 6.2 we have chosen solutions that exhibit fairly significant bending close to the black hole; if for whatever reason an outer boundary further from the black hole were desired, those solutions might be problematic. Nonetheless, for any boundary surface located a finite distance from the black hole there will be a range of field line angular velocities described by $x = 1/2 \pm \epsilon$ that will exhibit the same bending tendencies while maintaining error comparable to the error of the standard $x = 1/2$ solution.

**Domain Boundary Surfaces**

In solving for the force-free magnetospheres around both non-rotating black holes and the most monopolar solution we applied conditions at both spatial infinity and the horizon (though spatial infinity was a mathematical convenience, not a requirement; a boundary surface located at finite radii would have led to the same overall conclusions). The generically rotating solution did not apply any conditions at spatial infinity, however, and as such can have potentially significant error for some $x$ values at larger radii, which might lead to that solution being viewed as less robust or more erroneous.

That might be true in some instances, but it should be noted what the solution procedures applied in both the non-rotating black hole and most monopolar cases are
actually doing: they’re demanding a specific poloidal field geometry over all space, then finding conserved fluxes of energy and angular momentum compatible with that assumption. While that is a mathematically useful approach, it tacitly allows the conditions on an (at least) super-Alfvénic plasma inflow close to the horizon to directly feed back and communicate with a plasma outflow extended to spatial infinity.

There is also a more fundamental problem in extending the outer boundary to spatial infinity: such magnetospheres fairly generically contain infinite amounts of energy and are not physically realizable. Significantly restricting the conditions on near horizon behaviors using spatial infinity should therefore be viewed as a potentially useful mathematical technique or simplification, with the knowledge that \( r \to \infty \) is only a stand-in for a more appropriate outer boundary.

Mathematically that boundary might be taken as \( r \gg m \) or similar, but physical considerations are potentially more restrictive. One such consideration, as suggested above, as that the ingoing near horizon magnetosphere might be expected to be at least somewhat independent of the more distant outgoing magnetosphere. The simplest example of such a magnetosphere would be a single spherical surface serving as an outer boundary for an inflow and inner boundary for an outflow, but that is not guaranteed. More generally (from a perspective of physical modeling) one might have an “inner magnetosphere” close to the horizon that is connected to a series of different outer magnetosphere regions (described by differing physical approximations and/or variables) that only loosely connect the near horizon region with spatial infinity (or \( r \gg m \)).

In Blandford & Znajek [6] “spark gaps” and other mechanisms between the inner and outer light surfaces are postulated and discussed specifically because conditions on the horizon and distant regions are intrinsically incompatible without some kind of intermediate joining mechanism or structure where additional physics (beyond the
rigid application of stationary and axisymmetric force-free magnetohydrodynamics) must be considered. The magnitude of any resultant decoupling of inner and outer magnetospheres is an open question that will necessarily vary from model to model and magnetosphere to magnetosphere, but it is clear that in general the horizon and distant regions cannot be self-consistently connected by a single, unbroken magnetic field line described solely by the core assumptions used here and in Blandford & Znajek [6].

Depending upon the problem being explored, a practical outer boundary for the inner magnetosphere might lie somewhere interior to the outer light surface [54], perhaps near the separation surface [101]. Regardless of the selection made, however, it is in general somewhat implausible to expect to directly drive near horizon magnetosphere behaviors using a single fixed model rigidly extended to spatial infinity (and vice-versa). Models that do rigidly connect both regions can nonetheless still be useful, such as in demonstrating that the single physical assumption of a force-free magnetosphere can in principle transmit energy from the horizon to spatial infinity. However, in a more general exploration of energy-extracting black hole magnetospheres it is overly restrictive to demand a rigid connection between the near horizon inflowing magnetosphere and an outflowing magnetosphere extended to spatial infinity.

**Bending Behaviors**

For \( x < 1/2 \) (where \( \Omega_F = x \omega_{H_0} \)) we found that field lines bend upwards towards the azimuthal axis; for \( x > 1/2 \) we found that field lines bend downwards towards the equatorial plane (compatible with our numerical results that will be discussed in the following chapters). That behavior is independent of any (finite) outer boundary selected, although the range of reasonable \( x \) values becomes more restricted as the
outer boundary moves radially outwards.

No matter what outer boundary is selected, however, if the boundary condition along the equatorial plane (or other “straight” boundary) is close to being “monopolar” (i.e. a single magnetic field line tracing the boundary) then the presence of upward bending field lines in force-free magnetospheres should not be surprising wherever $\Omega_F \lesssim 0.5\omega_H$ and the presence of downward bending field lines should not be surprising wherever $\Omega_F \gtrsim 0.5\omega_H$. The primary question is the exact nature of the bending.

The magnitude of the bending should be expected to increase with increases in black hole spin, distance from the horizon, and in deviations from $\Omega_F \sim 0.5\omega_H$, as such changes will change the strength of the toroidal field and/or the distance over which the toroidal field can force the poloidal field to bend. Numerical experiments, such as in Thoelecke et al. [111], confirm those tendencies. However we would hesitate to call such tendencies anything more than a potentially useful “rule of thumb”; additional complications, such as the simultaneous presence of opposing tendencies (i.e. regions containing both $x < 1/2$ and $x > 1/2$) or the presence of different boundary conditions could break the “rule”.

We are not the first to suggest a general rule coupling magnetosphere bending behaviors to field line rotation. Impedance matching arguments can be made using resistive membranes on the horizon and at spatial infinity (to include surfaces approximating spatial infinity) to suggest that field lines with diverging angular separation should have $\Omega_F \gtrsim 0.5\omega_H$ and that field lines with converging angular separation should have $\Omega_F \lesssim 0.5\omega_H$ [81]. In other words finding $\Omega_F \sim 0.5\omega_H$ as a separatrix between magnetosphere bending behaviors (with a monopolar configuration coinciding with the separatrix) is not a surprising result.

The ability to adjust the bending of magnetic field lines by adjusting field line
angular velocity can add both significant flexibility and restrictions when considering black hole energy extraction. The single “mostly monopolar” solution originally obtained by Blandford & Znajek \cite{6} provides only a single rate of energy extraction and angular momentum outflow in a single fixed direction. By changing field line angular velocity the rates of energy and angular momentum extraction can be easily modified in way that is coupled to the direction of their outflow.

The direct formation of jets, for example, might be aided by more slowly rotating magnetospheres. More rapidly rotating magnetospheres, meanwhile, might more easily connect to a nearby accretion disk. Both behaviors would diminish the overall rates of energy extraction while simultaneously enhancing or reducing the rates of angular momentum extraction, potentially limiting the timescales over which such magnetospheres might be relevant or applicable.

\textbf{Closing Remarks}

It was noted in Blandford & Znajek \cite{6} that solving for magnetospheres via a perturbation in black hole spin might be ill-advised unless the poloidal magnetic field remains essentially unchanged. That is a reason why only a single solution from the general class of monopolar solutions found by Michel \cite{61} is typically treated in analytic extensions to rotating spacetimes. Although the more general solution space is more difficult to compute it is still of physical interest, and indicates that slowly rotating magnetospheres might be expected to bend towards the azimuthal axis while more rapidly rotating magnetospheres might be expected to bend towards the equatorial plane. Attempts to refine the analytic treatment of the “mostly monopolar” $\Omega F \approx 0.5\omega_H$ solution might also benefit from the knowledge that it is a separatrix between two classes of behaviors, as initially ignorable deviations from the separatrix might lead to significant effects as the solution is refined.
The “mostly monopolar” solution has value in that it simultaneously provides both an inflow near-horizon solution and an outflow solution that can be extended to spatial infinity. However, there is no requirement that a near-horizon inflow solution be rigidly coupled to an outflow solution. In fact almost the complete opposite is true, and for internal self-consistency inflow and outflow solutions must be decoupled to at least some extent. Disregarding inflow solutions solely because they do not extend to spatial infinity therefore artificially limits understanding of black hole energy extraction. There can be good reasons for desiring such an extension, but applying such a restriction necessarily limits any solutions obtained to special cases, just as the “mostly monopolar” solution is a special case of a more general solution space.
In this chapter we extend the results of the previous chapter by numerically solving for the structure of uniformly rotating force-free magnetospheres across a full range of black hole spins. To do so we develop a fully relativistic extension of the magnetofrictional method often applied when solving for force free magnetospheres in Newtonian contexts [123]. Qualitatively, we find the same thing that was found in the previous chapter; slowly rotating magnetospheres have field lines that bend upwards towards the azimuthal axis, while more rapidly rotating magnetospheres have field lines that bend downwards towards the equatorial plane, with the strength of that bending increasing with increases in black hole spin. This chapter is significantly based on Thoelecke et al. [111], modified as appropriate for this work.

Background

The Blandford-Znajek mechanism is not the only way to extract black hole rotational energy ([49] reviews others), nor is it strictly necessary for magnetic field lines to cross the horizon in order to extract black hole rotational energy ([45] and references therein). However the simple idea of a single magnetic field line connecting the horizon to distant observers can be seductive, as it is reminiscent of the more familiar and well understood case of a magnetic field line threading and torquing a rotating body. As one might naively expect from such an analogy, in order for a black hole’s rotational energy to be extracted along a given magnetic field line that field line must rotate more slowly than the black hole. Ultimately the condition of a relatively slowly rotating magnetic field line is the force-free limit of the requirement that a plasma inflow become super-Alfvenic inside the ergosphere ([101], Chapter 5).

Despite the defining role that the location of the inner Alfvén surface plays in
determining whether or not a given magnetosphere extracts black hole rotational energy, it is unknown how the shape and location of the Alfvén surface might correspond to changes in the structure of a magnetosphere. The differences (if any) between a magnetosphere with an Alfvén surface near the horizon and a magnetosphere with an Alfvén surface near the boundary of the ergosphere are largely uncertain. This is due to the intractability of the equations governing plasma flows in rotating black hole spacetimes, most especially the transfield equation that describes the force balance transverse to magnetic field lines ([68], Chapter 4). No exact analytic solutions of magnetospheres with plasma flows satisfying the transfield equation are known, so there is no robust method of studying any potential correspondences between Alfvén surface location and magnetosphere structure.

In the force-free limit plasma inertial effects are ignored, the ingoing Alfvén surface collapses to the inner light surface, and the transfield equation simplifies to a purely electromagnetic condition. Due to those simplifications it is natural to consider the limit of force-free magnetospheres as a first step in attempting a study of the effects of Alfvén surface location. Unfortunately despite its relative simplicity the force-free limit of the transfield equation has very few analytic solutions, primarily limited to a handful of energy extracting magnetospheres found by perturbing in black hole spin (e.g. [6, 58, 74], though they were expanded by [109] as outlined in Chapter 6) and a single class of fully exact but non energy extracting (and mostly unphysical) solutions [20, 29, 60]). Therefore even in the force-free limit there is no robust analytic method of studying the effects of Alfvén surface location, as no solutions are known in which the field line angular velocity is a variable parameter (with the exception of the monopolar solutions discussed in Chapter 6).

Current numerical solutions and simulations also lack significant utility, as the nature of the Alfvén surface is an unknown function of the specific choice of boundary
conditions and related initial assumptions that are made. This means that the Alfvén surface is at best a result to be reported rather than an input to be explored, making it difficult to study how differing Alfvén surfaces might correspond to changes in magnetosphere structure.

The goal of this chapter is to study how Alfvén surface location might correspond to the structure of energy extracting black hole magnetospheres, thereby beginning to fill a gap in our understanding of black hole energy extraction. Due to the lack of useful analytic solutions it was necessary to calculate magnetospheres numerically. For simplicity we will focus on the limit of force-free magnetospheres. We will also focus on magnetospheres with a boundary condition in the equatorial plane compatible with a monopolar geometry; this avoids the confounding affects of the arbitrarily large forcings alternative boundary conditions can exert on the magnetosphere. This boundary condition is also compatible with those used to calculate magnetospheres in many other works [6, 18, 65], allowing us to more easily compare our results with previous studies.

To form an initial survey of the parameter space it is necessary to calculate a large number of magnetospheres for a wide range of potential black hole spins and field line angular velocities (ultimately \( \sim 500 \) magnetospheres are found), so we developed numerical techniques that are efficient at calculating magnetospheres with similar structures. Those numerical techniques are based on a relativistic extension of the magnetofrictional method widely used in Newtonian contexts [123], which have not previously been extended to fully relativistic contexts.

**Assumptions**

The assumptions used in this chapter are the same as those outlined in previous chapters, the primary difference here is that we specialize to the force-free limit. For
completeness we will briefly restate those assumptions in this section, noting that more complete explanations may be found in previous chapters.

In the stationary and axisymmetric limit, force-free black hole magnetospheres are naturally described by three scalar fields: the toroidal vector potential $A_\phi(r,\theta)$, the field line angular velocity $\Omega_F(A_\phi)$, and the toroidal magnetic field $B_\phi(A_\phi)$. In order to form a valid solution those scalar fields must satisfy a single differential equation, the force-free limit of the general relativistic Grad-Shafranov (transfield) equation. For black hole rotational energy to be extracted along a given field line that field line must rotate slower than the black hole; $0 < \Omega_F < \omega_H$, where $\omega_H$ is the angular velocity of the horizon. The notation used is largely compatible with that of Blandford & Znajek \[6\] and Takahashi et al. \[101\] to facilitate comparison with those works; any significant discrepancies will be noted. Throughout we use covariant 4-vector notation; useful explanations of the 3+1 formulations often found in other relevant works may be found in \[45, 85, 113\].

**Core Assumptions**

The first core assumption is that of a stationary and axisymmetric spacetime, expressed in Boyer-Lindquist coordinates as:

$$ds^2 = \left(1 - \frac{2mr}{\Sigma}\right) dt^2 + \frac{4mr \sin^2 \theta}{\Sigma} dtd\phi - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2 - \frac{A \sin^2 \theta}{\Sigma} d\phi^2. \quad (7.1)$$

The black hole is then immersed in a perfectly conducting (ideal) plasma and electromagnetic fields that together lack sufficient energy density to modify that background spacetime. The plasma and electromagnetic field configuration is also assumed to be stationary and axisymmetric, with an axis of symmetry corresponding to the rotational axis of the black hole. The electromagnetic fields must be consistent
with Maxwell’s equations in a coordinate basis:

\[
(\sqrt{-g} F_{\alpha\beta})_{,\beta} = -4\pi \sqrt{-g} J^\alpha,
\]

\[
F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0. \tag{7.2}
\]

The field strength tensor may be expressed in terms of an electromagnetic vector potential \( A_\alpha \) as:

\[
F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}. \tag{7.3}
\]

In this form it is clear that the toroidal electric field vanishes \( (F_{t\phi} = 0) \) due to the assumptions of stationarity and axisymmetry, so any references to a “toroidal field” always refer to a toroidal magnetic field. The stress energy tensor outside the horizon is then taken to be the combination of plasma and electromagnetic effects as:

\[
T_{\alpha\beta} = T_{\alpha\beta}^{\text{Plasma}} + T_{\alpha\beta}^{\text{EM}} = (\rho + p) u^\alpha u^\beta - pg^{\alpha\beta} + \frac{1}{4\pi} g^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{1}{16\pi} g^{\alpha\beta} F_{\mu\lambda} F^{\mu\lambda}. \tag{7.4}
\]

We finally assume that there is no external forcing over any region of interest such that the divergence of the stress energy tensor vanishes everywhere:

\[
T_{\alpha\beta}^{\alpha\beta} = 0. \tag{7.5}
\]

Everything that follows rests upon the above assumptions and conditions.

Field Aligned Conserved Quantities

The three scalar fields that describe a force-free magnetosphere (vector potential \( A_\phi \), field line angular velocity \( \Omega_F \), and toroidal magnetic field \( \sqrt{-g} F^{\theta r} \)) are field-aligned conserved quantities in the sense that for a scalar quantity \( f \) we have \( B^{\alpha} f_{,\alpha} = 0 \). The toroidal vector potential \( A_\phi \) directly describes the poloidal magnetic field,
making it a useful flux function to trace poloidal magnetic field lines. The conservation of $A_\phi$ rests upon stationarity and axisymmetry, so $A_\phi$ is also a valid flux function in non force-free magnetospheres.

The conservation of field line angular velocity $\Omega_F$ is the general relativistic extension of Ferraro’s law of isorotation [23]. The field line angular velocity is defined in terms of the field strength tensor as $F_{r\phi} \Omega_F = F_{tr}$ and $F_{\theta\phi} \Omega_F = F_{t\theta}$. In other words all electric fields are the result of the rotation of a purely magnetic configuration with angular velocity $\Omega_F$ (referenced to zero angular momentum frames, as noted in Appendix E). The rigid rotation of individual magnetic field lines through the conservation of $\Omega_F$ is a consequence of stationarity, axisymmetry, and a perfectly conducting plasma, so it is also valid in non force-free magnetospheres.

The conservation of the toroidal field may be understood by considering Poynting fluxes through the magnetosphere. The only electric field is the result of rotating a purely magnetic configuration, and is therefore both entirely poloidal and perpendicular to the poloidal magnetic field. Therefore all poloidal Poynting fluxes are weighted only by the toroidal magnetic field and aligned with the poloidal magnetic field. As such the toroidal field is naturally interpreted as a measure of a conserved flux of energy $\eta E$ and angular momentum $\eta L$ per unit field line; $\eta E = \frac{1}{4\pi} \sqrt{-g} F^{\theta r} \Omega_F$ and $\eta L = \frac{1}{4\pi} \sqrt{-g} F^{\theta r}$ (the concept of a conserved particle flux $\eta$ has no meaning in a force-free context, but is used here for notational continuity). The conservation of the toroidal field along magnetic field lines is a consequence of stationarity, axisymmetry, and a perfectly conducting force-free plasma; in non force-free magnetospheres the toroidal field $\sqrt{-g} F^{\theta r}$ is not conserved.

Varied derivations and discussions of these and other field-aligned conserved quantities may be found in [2, 6, 8, 101] and in Chapter 3.
Critical Points and Energy Extraction

In this subsection we explore key points along magnetic field lines and examine the conditions required for a magnetosphere to extract a black hole’s rotational energy. We begin by making two assumptions that are unnecessary in general but greatly simplify the discussion. First, we assume a single magnetic field line that extends directly from the horizon to a distant observer. Second, we insist that this field line be “simple” in the sense that the distance from the horizon decreases monotonically as one travels along the field line towards the black hole; a useful mental image is a straight line connecting the horizon with distant regions.

Along any such field line there are seven key points, the first being the separation point. Inside the separation point gravitational forces dominate and draw plasma inwards to the horizon; outside the separation point centrifugal forces in the rotating frame of the magnetic field line dominate and accelerate plasma away from the black hole. The remaining six points are the ingoing and outgoing slow magnetosonic, Alfvén, and fast magnetosonic points through which the plasma accelerates as it travels from the separation point to the horizon or distant regions. In the force-free limit the ingoing slow magnetosonic point coincides with the separation point, the ingoing Alfvén point coincides with the inner light surface, and the ingoing fast magnetosonic point coincides with the horizon.

The inner and outer light surfaces are defined by the inner and outer surfaces exterior to the horizon where $\alpha = 0$ (light surfaces are discussed in more detail in Chapter 5). The outer light surface is analogous to a pulsar light cylinder, where rigid rotation would cause a particle stuck to a magnetic field line to rotate at the speed of light. The particle would also rotate at the speed of light at the inner light surface as a consequence of gravitational time dilation, and it would be subluminal between the light surfaces.
A field line extracts black hole rotational energy when its ingoing Alfvén point is located inside the ergoregion. In the force-free limit that reduces to the requirement that the magnetic field line rotates more slowly than the black hole; \( 0 < \Omega_F < \omega_H \).

In general there exist regularity conditions at each of the slow magnetosonic, Alfvén, and fast magnetosonic points. In the force-free limit the most useful of these is that of the ingoing fast magnetosonic point, which reduces to a condition on the fields at the horizon (the horizon regularity condition discussed in Chapter 5):

\[
\sqrt{-g} F_{\theta r} = \pm \left( \frac{r^2 + a^2}{r^2} \right) \left( \Omega_F - \omega_H \right) \sin \theta \frac{\Sigma}{F_{\theta \phi}}. \tag{7.6}
\]

The equations in use are insensitive to the sign of the toroidal field, but if an ingoing observer on the horizon is to measure finite electromagnetic fields then the negative branch must be chosen [127]. This is often referred to as the “Znajek condition” and does not place any additional restrictions on the fields, as it was already present in our core assumptions. However it can still be a useful check to ensure that a solution is valid on the horizon.

Further discussion of the key points along a magnetic field line may be found in [99, 101], Chapter 4 and Appendix G.

**The Force-Free Transfield Equation**

In our core assumptions we demanded that there be no external forcing on the magnetosphere, as stated in the requirement that \( T^{\alpha \beta ; \gamma} = 0 \). If the poloidal components of this requirement are satisfied then the temporal and toroidal components are automatically satisfied (assuming energy and angular momentum are treated as conserved quantities). We therefore focus on the two poloidal components, which may be reduced to a single differential equation; this equation is often referred to as the “transfield equation” as it encapsulates the force balance transverse to
poloidal magnetic field lines. In the force-free limit the transfield equation may be written as (cf. Equation 3.14 of Blandford & Znajek [6]):

$$
0 = \frac{1}{\sin \theta} \frac{\alpha}{\Delta} \left[ \frac{\alpha}{\sin \theta} F_{r\phi} \right]_r + \frac{1}{\sin \theta} \frac{G_{\phi}}{\Delta F_{\theta\phi}} \left( F_{r\phi} \Omega_F + \frac{1}{\Delta} F_{\theta\phi} \Omega_F, \theta \right)
+ \frac{1}{2} \sum \Sigma \Delta \sin \theta \frac{d}{dA_{\phi}} \left( \sqrt{-g} F^{\theta r} \right)^2.
$$

(7.7)

The function $\alpha^{-1/2}$ may be interpreted as a “gravitational Lorentz factor” of a particle stuck to a magnetic field line, describing the effects of both rotation and gravitational time dilation, while $G_{\phi}$ may be interpreted as a measure of that particle’s rotational velocity with respect to the spacetime.

Solving the transfield equation is the core difficulty of finding black hole magnetospheres. In the stationary and axisymmetric limit of a rotating black hole, only one exact class of solutions is currently known (first published by [60] and discussed in a more general framework by [20, 29]):

$$
A_{\phi} = B_0 h(\theta),
\Omega_F = \frac{1}{a \sin^2 \theta},
\sqrt{-g} F^{\theta r} = B_0 \frac{1}{a} g(\theta).
$$

(7.8)

Here $B_0$ is a scalar measure of the strength of the poloidal magnetic field; $h$ and $g$ are arbitrary functions of $\theta$ that must satisfy $h' = -g \sin \theta$ (a prime denoting a derivative with respect to $\theta$). Solutions of this type generically contain an unphysical divergence in $\Omega_F$ in both the $\theta \to 0$ and low black hole spin $a \to 0$ limits, and are therefore mostly mathematical curiosities. They also do not extract rotational energy from the black hole along any field lines, further limiting their interest. However we still find them to be useful tools in determining the potential precision of a given numerical grid.
In the limit of a non-rotating \((a = 0)\) Schwarzschild black hole, there does exist a more reasonable variety of exact rotating “monopolar” solutions that are given by (Chapter 6):

\[
A_\phi = B_0 \cos \theta, \\
\Omega_F = \Omega_0, \\
\sqrt{-g} F^{\phi r} = B_0 \Omega_0 \sin^2 \theta. \tag{7.9}
\]

Such solutions do not extract rotational energy from the black hole (it has none to extract) but nonetheless serve as a convenient starting point for exploring black hole energy extraction. These solutions are sometimes written as “split-monopole” solutions by changing the sign of \(B_0\) below the equatorial plane and appealing to an equatorial current sheet. For simplicity we will always refer to them as “monopolar” solutions, as regardless of interpretation there is still reflection symmetry across the equatorial plane. However it should be noted that “monopolar” references only the poloidal magnetic field; there is often a significant toroidal field, as can be seen in the \(\Omega_0 \neq 0\) solutions. Setting \(\Omega_0 = 0\) for a static monopole and applying that solution to a slowly rotating black hole by perturbing in black hole spin yields [6, 74]:

\[
A_\phi = B_0 \cos \theta - \left\{ \frac{a^2}{m^2} B_0 R(r) \cos \theta \sin^2 \theta \right\}, \\
\Omega_F = \frac{a}{8m^2} + \left\{ \frac{a^3}{32m^4} \left[ 1 + \frac{1}{2} (1 - 2R_2) \sin^2 \theta \right] \right\}, \\
\sqrt{-g} F^{\phi r} = - B_0 \frac{a}{8m^2} \sin^2 \theta - \left\{ \frac{B_0 a^3 \sin^2 \theta}{32m^4} \left[ 1 + \frac{1}{2} (1 - 4R_2) \sin^2 \theta + 8R(r) \cos^2 \theta \right] \right\}. \tag{7.10}
\]
where:

\[
R(r) = \frac{1}{m^2} \left[ \frac{m^2 + 3mr - 6r^2}{12} \ln \left( \frac{r}{2m} \right) + \frac{11m^2}{72} + \frac{m^3}{3r} + \frac{mr}{2} - \frac{r^2}{2} \right] \\
+ \left( \frac{2r^3 - 3mr^2}{8m^3} \right) \left[ \text{Li}_2 \left( \frac{2m}{r} \right) - \ln \left( 1 - \frac{2m}{r} \right) \ln \left( \frac{r}{2m} \right) \right],
\]

\[R_2 = \frac{1}{72} (6\pi^2 - 49), \tag{7.11}\]

and \(\text{Li}_2\) is the dilogarithm, defined as:

\[
\text{Li}_2(x) = \int_x^0 \frac{1}{t} \ln (1 - t) \, dt. \tag{7.12}\]

The higher order corrections to the original Blandford & Znajek \cite{6} solution found by more recent authors are shown in curly brackets \cite{58, 74}. It is possible to extend the above to include \(O(a^5)\) corrections, but the solutions achieve a level of complication far surpassing the point of reason. There are two things to note from the above expressions; first, and most obviously, is that attempting to find perturbed solutions is not a trivial undertaking. Perturbing one of the simplest possible magnetic field configurations is effectively impossible past \(O(a^2)\) corrections to \(A_\phi\). The second thing to note that the leading order correction to \(\Omega_0 = 0\) is \(\Omega_F = 0.5\omega_H\) (to first order in \(a\)). This has partly contributed to the “rule of thumb” that energy extracting black hole magnetospheres should rotate at one half the rate of the black hole (\(\Omega_F \approx 0.5\omega_H\) also extracts the maximum amount of black hole energy at low spin).

Exploring the \(\Omega_0 \neq 0\) solutions was one of the initial questions that prompted the work outlined in this chapter, as they naturally encompass a full range of inner Alfvén surfaces and corresponding magnetospheres. Unfortunately analytically perturbing around such solutions is even more intractable than the already non-trivial \(\Omega_0 = 0\) case (Chapter \cite{6}), so we developed numerical techniques to solve for the structure of
black hole magnetospheres.

**Additional Assumptions**

In addition to the core assumptions already noted, we make two additional simplifying assumptions guided by our goal of studying correspondences between inner Alfvén surface location and magnetosphere structure. The first is the assumption of a monopolar geometry, in the sense that the poloidal magnetic field line threading the horizon on the equator must remain on the equator. This assumption is enforced via a boundary condition on the flux function $A_\phi$; we require that $A_\phi$ be constant in the equatorial plane. In making this assumption we were guided by a desire to extend the Schwarzschild monopole solutions of Equation 7.9 to rotating spacetimes and to avoid assuming a specific model for nearby accreting matter. Without the presence of such matter to restrict magnetic field lines, a configuration with varying vector potential in the equatorial plane could violate the condition that there can be no closed horizon loops in stationary force-free magnetospheres [29, 54]. Additionally, boundary conditions encoding an assumed model of matter in the equatorial plane or elsewhere have the potential to exert arbitrarily large forcings on the magnetosphere, confounding attempts to explore basic correlations between Alfvén surface location and magnetosphere structure. The interested reader may see [117, 124, 125] for some complications induced by horizon-disk connections and [22] for disk-jet connections.

The second assumption we made is that of uniform field line angular velocity, as that is the simplest possible method of classifying inner light surfaces (inner Alfvén surfaces in the force-free limit) and allowed us to know exactly where the inner light surface would be prior to calculating the structure of the field lines that pass through it. While it is convenient, this assumption restricts us to a region outside the horizon that is bounded by the outer light surface. This is due to the fact that
our dissipative numerical techniques cannot generically find magnetospheres that pass smoothly through both inner and outer light surfaces under the assumption of uniform field line angular velocity, even if such solutions exist. Ultimately we do not believe this to be nearly as limiting an assumption as it might initially appear, but it is still the most awkward assumption that we have made. As such we return to later with a more detailed discussion.

**Numerical Techniques**

In order to numerically solve for a large number of black hole magnetospheres, we have extended the Newtonian magnetofrictional method developed by [123] to the general relativistic regime. The magnetofrictional method makes an initial guess as to the structure of a force-free magnetosphere and then gradually relaxes the fields towards a valid force-free state. This makes it a useful tool for exploring our parameter space; once a solution for a given black hole spin and field line angular velocity is known adjacent solutions for slightly different values are readily found. In this section we discuss the underlying theory of the magnetofrictional method as well as the computational specifics of our implementation.

**The Magnetofrictional Method**

Once initial guesses have been made for a vector potential $A_\phi$, field line angular velocity $\Omega_F$, and toroidal field $\sqrt{-g} F^{\theta r}$, they may be inserted into the transfield equation to see if those guesses form a valid solution. This is equivalent to determining if the poloidal components of the divergence of the stress energy tensor vanish:

$$T^{A \beta}_{\beta} = X^A.$$  \hfill (7.13)
If the initial guesses formed a valid and self consistent force-free solution then $X^A$ would vanish, but in general $X^A \neq 0$ and there is an excess momentum flux that may be interpreted as an external forcing. The core idea of the magnetofrictional method is to take that external forcing and convert it into the coordinate velocity of a fictitious plasma via friction. In this way a physically invalid (or at least inconsistent) initial guess for a force-free magnetosphere may be converted into a valid non force-free magnetosphere, and the fields may be evolved towards a force-free state by applying the equations governing inertial plasma flows. The first step in this process is to evaluate the excess momentum flux $X^A$, which is found as a modification to the left hand side of the transfield equation (Equation 7.7):

$$-4\pi \Sigma \sin \theta \frac{X_A}{A_{\phi,A}} = \frac{1}{2} \sum \Delta \sin \theta \frac{d}{dA_{\phi}} \left( \sqrt{-g} F^{\theta r} \right)^2 + \ldots$$

(7.14)

Once the excess momentum flux has been determined, it may be converted into the coordinate velocity of a fictitious plasma $v^A$ ($v^r = \partial r / \partial t$, $v^\theta = \partial \theta / \partial t$) via friction:

$$X^A = \frac{1}{\nu} v^A.$$  

(7.15)

Here the coefficient $\nu$ measures the strength of the friction. We then assume that the fictitious plasma is a perfect conductor such that there is no electric field in its rest frame; $F_{\alpha \beta} u^\beta = 0$. Here $u^\alpha$ is the four velocity of the plasma, which we define in terms of its coordinate velocity as $v^c \equiv u^c / u^t$. We then find:

$$F_{tc} = F_{cb} u^b.$$  

(7.16)
If we insert this relationship into Maxwell’s homogeneous equations (Equation 7.2) we find the relativistic analog of the ideal induction equation ∂ₜ ⃗B = ⃗∇ × (⃗v × ⃗B):

\[ F_{ab,t} = (F_{bc}v^c)_a - (F_{ac}v^c)_b. \]  
\(7.17\)

Inserting either \(F_{\theta\phi}\) or \(F_{r\phi}\) into the above then shows that the time rate of change of the vector potential \(A_\phi\) is given by:

\[ A_{\phi,t} = -v^A A_{\phi,A}. \]  
\(7.18\)

In Appendix H we prove that application of this equation inevitably leads to a force-free configuration and expand it in Boyer-Lindquist coordinates. Although the coordinate velocity \(v^A\) is a complicated function of the metric and electromagnetic fields, at a high level the magnetofrictional method reduces to the straightforward evolution of an advection equation. A variety of numerical techniques for evolving such equations exist; for simplicity we have implemented a straightforward upwind differencing scheme \([35]\). More sophisticated techniques might converge to a valid solution more rapidly, but we found upwind differencing to be both very numerically stable and fast enough for our purposes. The relaxation scheme implemented by \([117]\) is similar to the method described above, using \(A_{\phi,t} \sim X^A\) instead of \(A_{\phi,t} \sim v^A\); because \(X^A \sim v^A\) the primary effective difference is in the weighting of \(v^A\).

**Fixing Kinks**

The magnetofrictional method described above is limited by the development of kinks in the magnetic field, which we now discuss. The transfield equation (Equation 7.7) changes character at the inner and outer light surfaces where the gravitational Lorentz factor \(\alpha\) changes sign. This change in character is transmitted
to the coordinate velocity $v^A$ of the fictitious plasma, which goes as:

$$v^A \sim \alpha (A_{\phi,rr} + A_{\phi,\theta\theta}).$$

(7.19)

On the inner and outer light surfaces $\alpha$ vanishes; it is positive between them and negative outside. To avoid having our numerical scheme become anti-diffusive and unstable, outside the light surfaces we introduce an overall multiplicative factor of $-1$ to maintain stability. This factor may be thought of as a correction for the fact that the fictitious plasma introduced by the magnetofrictional method is superluminal outside the light surfaces, and does not affect convergence.

However the change in sign of $\alpha$ does split the computational domain into three regions, and in general those regions do not join smoothly. In other words for an initial guess of field line angular velocity $\Omega_F$ and toroidal field $\sqrt{-g} F^{\theta r}$, the solutions that the magnetofrictional method finds for $A_\phi$ will in general not match across light surfaces. Treating $A_\phi$ as a flux function, this leads to discontinuous “kinks” in the magnetic field at the inner and outer light surfaces. As there are two light surfaces, both the field line angular velocity and the toroidal field could be evolved simultaneously in order to diminish both kinks, as in [18, 65].

We have chosen to use a fixed and uniform field line angular velocity so we do not have the freedom required to generically diminish both kinks using the inherently dissipative magnetofrictional method. We therefore divide the computational domain into two regions, one inside the inner light surface and one between the light surfaces, and only diminish the kink at the inner light surface by evolving the toroidal field there. The limitations and advantages of this approach will be discussed in more detail below.

In order to measure the kink at the inner light surface, the magnetofrictional
method is applied separately to the regions on either side until an empirical “zero level” in the velocity $v^A$ is reached. A vanishing plasma coordinate velocity corresponds to $X^A = 0$ and a valid force-free configuration, so this procedure simply finds a “close enough” solution in both regions. The two solutions are then compared across the inner light surface and the magnitude of any kink is measured in terms of $\Delta A_\phi$. This measurement is then used to modify the toroidal field in a method similar to that developed by [17] for light cylinders in pulsar magnetospheres:

$$\frac{d}{dA_\phi} \left( \sqrt{-g} F_{\theta r} \right)^2_{\text{New}} = \frac{d}{dA_\phi} \left( \sqrt{-g} F_{\theta r} \right)^2 \bigg|_{A_\phi^-} - \gamma \Delta A_\phi. \quad (7.20)$$

Here $\gamma$ is an empirical constant, usually much less than one, and $A_{\phi^-}$ denotes the value of $A_\phi$ just inside the inner light surface. After the above equation has been used to make a small correction to the toroidal field, the magnetofrictional method is again applied to the regions on either side of the inner light surface for a fixed number of time steps, and then (if the set “zero level” in the plasma coordinate velocity is still achieved) the kink is re-evaluated and toroidal field re-adjusted. The cycle repeats until matching solutions are found.

Finding the correct “zero level”, value for $\gamma$, and number of time steps to apply the magnetofrictional method between modifications of the toroidal field is akin to critically damping a simple harmonic oscillator. Poor choices can lead to either an underdamped, wildly oscillating kink or an overdamped kink that requires an excessive amount of computational time to diminish. We have found that a “zero level" of $10^{-4}$, a factor of $\gamma = 0.0005$, and 3000 time steps between toroidal field changes is typically a safe starting point, but optimal values can vary significantly depending upon everything from grid resolution to the current magnitude and nature of the kink and how good the initial guess for the toroidal field is.
Measuring Force-Freeness

In principle the magnetofrictional method could be applied to the limits of numerical precision. However even if this were practical the approximations and assumptions we have made do not justify that level of precision. Instead we evolve the fields until a “good enough” level is reached, which we now describe.

A force-free solution is a solution with a vanishing Lorentz force, $F_{\alpha\beta}J^\beta = 0$. Therefore it is reasonable to take a ratio of the Lorentz force to the magnitude of electric current and field strength tensor in order to measure how force-free a solution is, an approach taken in [58]:

$$\xi = \frac{|F_{\mu\nu}J^\nu F_{\mu k}J^k|}{|J^\mu F_{\kappa\lambda}F^{\kappa\lambda}|}.$$  \hspace{1cm} (7.21)

If $\xi \ll 1$ the configuration can be thought of as being force-free. Unfortunately such a technique is of limited utility in our case; many of our magnetospheres contain regions of vanishing or very small current $J_\alpha J^\alpha \ll 1$, leading to divergences in $\xi$.

In force-free magnetospheres with regions of vanishing current, one alternative to $\xi$ is to measure the ratio of the Lorentz force to the forces of magnetic pressure and tension, as in [56]:

$$\zeta = \frac{|F_L|}{|F_{mp}| + |F_{mt}|}.$$  \hspace{1cm} (7.22)

As with $\xi$, if $\zeta \ll 1$ then the configuration may be taken to be force free. Unfortunately some of our magnetospheres contain very strong monopolar components for which the forces of magnetic pressure and tension vanish separately, again leading to divergences. We have therefore developed a more mathematical and less physically motivated measure of force-freeness. We begin by noting that the transfield equation
may be broken up into seven terms:

\[ T_{\alpha,\beta} = -F_{\alpha\beta} J^\beta = A_{\phi,A} \sum_{i=1}^{7} D_i. \]  

(7.23)

The separation of the transfield equation into \( D_i \) components is done somewhat arbitrarily by grouping factors of \( A_{\phi} \), field line angular velocity, and the toroidal field; \( D_1 \sim \sqrt{-g} F^{\theta r} \), \( D_3 \sim A_{\phi,rr} \), et cetera. The exact separations we use are expanded in Appendix H. Mathematically the above puts us in the position of having an expression of the form \( \sum D_i = \delta \) and requiring that \( \delta \) be "small". As a measure of force-freeness, we therefore take the absolute maximum of the seven \( D_i \) terms and require that \( \delta \) be a factor of \( \epsilon \) smaller than it:

\[ |\delta| < \epsilon \cdot \text{Max} (|D_i|). \]  

(7.24)

While such a method is disadvantaged by not having an obvious physical interpretation, it does not diverge when applied to magnetospheres with regions of vanishing current or with magnetic fields containing strong monopolar components. For this work we have selected \( \epsilon = 1\% \) as a level of sufficient force-freeness. In practice our numerical techniques result in only a few segments of the inner light surface approaching \( \epsilon = 1\% \); over the rest of the domain \( \epsilon \) is significantly smaller.

Analytic Comparisons

Above we discussed some analytic solutions to the force-free transfield equation. Here we select four specific solutions that are useful for making comparisons with our numerical solutions; below we will compare them to a numerical solution with \( \Omega_F = 0.5\omega_H \) and \( a = 0.3m \) in order to examine the consequences of the \( \epsilon = 1\% \) error level discussed in the previous section.
The first analytic solution that provides a useful comparison is the exact Schwarzschild monopole described by Equation 7.9. The field line angular velocity \( \Omega_0 \) of this solution may be set to correspond to the field line angular velocity of a given numerical solution. The monopole weight \( B_0 \) is set to \( B_0 = 1 \) in order to match the numerical solution’s change in \( A_\phi \) between the axis and the equator (discussed in the next section). This solution provides a comparison with all field line angular velocities but is most useful for very small spins.

The second useful solution is the first order perturbed monopole solution (Equation 7.10 without the terms in curly brackets). This solution is very similar to the Schwarzschild monopole with \( \Omega_0 = 0.5\omega_H \), but differs slightly due to the fact that its field line angular velocity is only equivalent to \( 0.5\omega_H \) to first order in spin parameter \( a \). As with the Schwarzschild monopole solution we set \( B_0 = 1 \). The field line angular velocity of this solution is fixed, so it is most useful for comparison with numerical solutions that have \( \Omega_F = 0.5\omega_H \) and small black hole spins.

The third solution is the third order perturbed monopole solution described by Equation 7.10. As with the first order solution we set \( B_0 = 1 \). This solution is most useful for comparison with numerical solutions of \( \Omega_F = 0.5\omega_H \) for slightly higher spins than the first order solutions, and for determining how a higher order perturbation in spin translates to changes in our measure of error \( \epsilon \). It should be noted that the field line angular velocity of this solution is not uniform, as in the first order solution, but is a function of both \( r \) and \( \theta \).

The fourth solution is a fully exact solution, described by Equation 7.8 with \( h = \sin^3 \theta \) and \( B_0 = 1 \). There are infinitely many possible choices for \( h \). Our selection is motivated by a desire for physical plausibility, which is most clearly seen by taking the Newtonian limit of the fields described. In that limit, our choice of \( h \) corresponds
to magnetic and electric fields given by:
\[
\vec{B} = -\frac{3B_0}{r^2} \sin \theta \cos \theta \hat{r} - \frac{3B_0}{ar} \cos \theta \hat{\phi},
\]
\[
\vec{E} = \frac{3B_0}{ar} \cos \theta \hat{\theta}.
\] (7.25)

Here the vector components correspond to a standard orthonormal spherical basis. We emphasize that the factor of \(a\) above is from the definition of the fields; if the \(a \to 0\) limit appropriate to a Newtonian transition were also taken in the electromagnetic fields they would diverge. In the above form it is clear that our choice of \(h = \sin^3 \theta\) was made to correspond to physically plausible fields, in the limited sense that they don’t diverge anywhere in our domain; less careful choices generally diverge on the azimuthal axis. This solution is most useful for estimating the potential precision of a given numerical grid for arbitrary black hole spin; we will refer to it as the HS3 solution when we compare it to our numerical results.

**Computational Specifics**

In this section we detail the computational specifics of our numerical methods, covering the grid resolutions used, domain boundaries and boundary conditions, initial conditions, and performance.

The poloidal plane is divided into an \((r, \theta)\) grid for the vector potential \(A_\phi\). In the radial direction the grid extends from just inside the horizon to a radius \(r_{\text{max}}\) outside the ergosphere. The value of \(r_{\text{max}}\) is dependent upon the location of the outer light surface, which is in turn dependent upon the choice of field line angular velocity \(\Omega_F\) and black hole spin. For small values of \(\Omega_F\) (near 0) or black hole spin the outer light surface can be a very large distance from the black hole, so we artificially place a cap of \(20m\) on the maximum radius. For large values of \(\Omega_F\) (near \(\omega_H\)) and black
hole spin the outer light surface approaches the ergosphere and \( r_{\text{max}} \approx 2m \) is used.

It is critical to resolve the inner light surface, so for large values of \( \Omega_F \) where the inner light surface approaches the horizon we use very small grid spacings in the radial direction near the horizon, then gradually relax that spacing as \( r \) increases. For small values of \( \Omega_F \) the horizon and inner light surface are well separated and we use larger radial grid spacings near the horizon, then increase that spacing as \( r \) extends past the ergosphere to relatively larger \( r_{\text{max}} \) values. In general around 400 grid squares of varied spacing in the radial direction are used to resolve the inner light surface and ergoregion and an additional 450 of varied spacing are used to extend to a given \( r_{\text{max}} \). This yields a rough average of 850 grid squares in \( r \) per full magnetosphere; exactly how many are used varies from magnetosphere to magnetosphere.

In order to treat the value of \( A_\phi \) on the horizon as an evolving entity, we extend three grid squares past the horizon towards the black hole. We cannot extend any further as our implementation becomes anti-diffusive inside the horizon and more grid squares allow numerical instabilities to develop. We are using Boyer-Lindquist coordinates, which are singular on the horizon. This is not a difficulty, as we scale the magnetofrictional coefficient \( \nu \) in Equation 7.15 by the magnitude of the poloidal magnetic field (Appendix H). The coordinate singularity in that scaling cancels with the coordinate singularity in the transfield equation such that the horizon is well behaved. As a solution is found the horizon naturally settles into a configuration consistent with the Znajek horizon condition (the force-free limit of the fast magnetosonic regularity condition, Equation 7.6), justifying the usage of Boyer-Lindquist coordinates.

In the \( \theta \) direction we extend from \( \theta = 0 \) on the azimuthal axis to \( \theta = \pi/2 \) on the equatorial plane using 200 linearly spaced grid squares. The equations in use do diverge on the azimuthal axis, a consequence of axisymmetry requiring that the
magnetic field be purely radial there. By using the azimuthal axis as a fixed boundary condition we enforce axisymmetry and avoid that divergence, and find that double precision arithmetic is sufficient for the grid squares immediately adjacent to the axis. The other fixed boundary condition that we use is the equatorial plane, in keeping with the assumption of perturbing around a monopolar geometry. On the azimuthal axis we fix $A_{\phi} = 4$ and on the equatorial plane we fix $A_{\phi} = 3$, for a $\Delta A_{\phi}$ between them consistent with a monopole of unit weight ($A_{\phi} = \cos \theta$ in Equation 7.9). In principle any arbitrary range between the axis and equatorial plane could be used; we made our selection for convenience.

We do not use fixed boundary conditions for the vector potential $A_{\phi}$ at either $r_{\min}$ or $r_{\max}$. Instead, after every time step we shoot outwards to find a boundary that keeps $A_{\phi}$ smooth and use that boundary for the next iteration. The reason for this is practical; although the axis and equatorial plane are fixed by symmetry, there is no restriction on $A_{\phi}$ for a given $r_{\max}$. The inner boundary could be fixed by applying the horizon regularity condition, but we chose to have the horizon evolve as a simple check that the numerical algorithm is converging properly.

In order to calculate the derivatives in $A_{\phi}$ required to calculate the coordinate velocity $v^A$ of the fictitious plasma (Equations H.12 and H.13) we use centered finite difference approximations appropriate to the local grid spacing. One-sided finite difference derivatives in $A_{\phi}$ appropriate to an upwind differencing algorithm are then used to evolve the magnetofrictional advection equation (Equation 7.18).

Our initial conditions for each run were fairly simple. Any smooth $A_{\phi}$ that decreased monotonically from the azimuthal axis to the equatorial plane could be used; we found no dependence upon initial conditions. Lack of monotonicity caused “spikes” to develop in $A_{\phi}$ that required magnetic reconnection via numerical diffusion to dissipate, greatly extending computation time when the code remained stable. To
speed convergence we found it desirable to begin with the vector potential of the closest already calculated magnetosphere, “close” being defined by either slightly different black hole spin or field line angular velocity. We do not directly use the toroidal field $\sqrt{-g} F^{\theta r}$, instead we propose a function of $A_\phi$ corresponding to the derivative with respect to $A_\phi$ of the square of the toroidal field (the left hand side of Equation 7.20). From the transfield equation it can be shown that this function must vanish on the axis and equatorial plane (at $A_\phi = 4$ or $A_\phi = 3$ using the boundary conditions described above) but is otherwise largely unrestricted. Beyond increasing computation time we found no dependence upon initial choice of this function. In order to decrease computation time we often found it desirable to use three neighboring magnetospheres with the same spin but different field line angular velocities to “shoot” an initial guess for this function for a given value of $A_\phi$.

The overall number of time steps required to find a solution is sensitive to the accuracy of the initial guesses for both the vector potential and toroidal field, with the majority of computation time typically being spent reducing the kink at the inner light surface by finding a compatible toroidal field. In general with good initial guesses for the fields and well-tuned parameters a relatively standard desktop computer (6-core Intel Haswell architecture CPU assisted by an Nvidia Kepler architecture GPU) can find a magnetosphere at the $\epsilon = 1\%$ level (Section 7) in a matter of hours; a $\epsilon \sim 10\%$ level can often be achieved in less than an hour. We have found that it is possible to significantly reduce computation time by bracketing the functional form of the derivative of the toroidal field using a kind of two dimensional root-finding algorithm with initial guesses taken from adjacent magnetospheres, but that technique requires careful setup and tuning to avoid instabilities. The more naïve method of kink reduction described above is more robust, easier to implement, and while significantly slower is not prohibitively so for most purposes.
Results

We divide our results into four sections. First we explore the general structure of the magnetospheres as a function of black hole spin parameter and field line angular velocity. We then examine the rates of energy and angular momentum extraction. Lastly we explore the numerical error of our solutions.

Field Line Structure

In this section we explore the behavior of poloidal magnetic field lines. We are limited to regions near to the black hole by our model assumptions and numerical techniques. However even over that limited domain interesting behaviors emerged, as shown in Figures 7.1 and 7.2.

Figure 7.1 plots poloidal magnetic field lines using spacing on the horizon corresponding to the strength of the radial magnetic field there (using $|B^r| \sim \csc \theta A_{\phi,\theta}$ appropriate to ingoing coordinates); denser field lines imply greater magnetic field strength. The first and last field line do not lie on the axis or equatorial plane, as those are fixed, but instead lie one grid square inwards on the horizon. The shading measures conserved momentum flux per unit field line, which corresponds to the toroidal field $\sqrt{-g} F^{\phi r}$.

Figure 7.2 plots poloidal magnetic field lines using linear spacing in $\theta$ on the horizon. The first and last field line again lie one grid square inwards on the horizon. The shading measures conserved energy flux per unit field line.

The first thing to note is that as expected for very small black hole spin ($a \approx 0.1m$) all values of field line angular velocity yield magnetospheres that are almost indistinguishable from a Schwarzschild monopole (Equation 7.9, noting that “monopole” refers only to the poloidal plane). For slightly larger values of spin
Figure 7.1: The structure of poloidal magnetic field lines for various values of black hole spin and field line angular velocity. The magnitude of the outward conserved angular momentum flux per unit field line is colored, scaled to $\Delta A_\phi = 1$ between the axis and the equatorial plane ($B_0 = 1$ in Equation 7.9). The inner light surface (green), ergosphere (red), horizon (black), monopole separation surface (cyan), and calculated separation surface (dotted red) are overplotted. 80% of the total extracted energy flows outward between the azimuthal axis and the magnetic field line drawn in solid magenta; 95% flows between the dotted magenta field line and the axis. The eight black field lines are spaced according to the magnitude of the radial magnetic field on the horizon; if they fall outside the 80% line they are dotted.
Figure 7.2: The same structure of poloidal magnetic field lines as in Figure 7.1 but with the magnitude of the outward conserved energy flux per unit field line colored. The inner light surface (green), ergosphere (red), horizon (black), monopole separation surface (cyan), and calculated separation surface (dotted red) are overplotted. 80% of the total extracted energy flows outward between the azimuthal axis and the magnetic field line drawn in solid magenta; 95% flows between the dotted magenta field line and the axis. The eight black field lines are spaced evenly on the horizon; if they fall outside the 80% line they are dotted.
(\(a \approx 0.3m\)), however, noticeable deviations from a monopolar structure rapidly emerge for both low and high field line angular velocities. As spin increases beyond \(a \approx 0.3m\), only the \(\Omega_F \approx 0.5\omega_H\) solutions remain roughly monopolar.

Low field line angular velocity solutions bend towards the azimuthal axis, with the strength of that bending increasing as black hole spin increases. For large spin \((a \approx 0.9m)\) this bending can be severe, with the majority of poloidal magnetic field lines becoming nearly parallel to the azimuthal axis at a distance \(r = 20m\). The source of this bending is most easily seen in Figure 7.1; on the horizon the radial magnetic field is strong on the azimuthal axis and weak near the equatorial plane, while the toroidal magnetic field is strong on the equatorial plane and weak on the azimuthal axis. This discrepancy acts to naturally wind up the magnetic field and the magnetosphere self-collimates into a jet-like structure. The discrepancy becomes very large at high spin; for \(a = m\) (not shown) the horizon radial magnetic field strength on the axis can be \(\sim 10\) times that on the equator. For larger values of \(\Omega_F\) (implying larger electric fields as viewed by distant observers, but not necessarily others such as ZAMOs) the toroidal field becomes much smaller, radial magnetic field strength becomes almost uniform on the horizon, and the field lines begin to bend toward the equatorial plane.

There are two separation surfaces shown in the figures (the separation surface may be defined as the point where \(\alpha' = 0\), with the prime denoting differentiation along magnetic field lines); one for a monopolar geometry where \(A_\phi\) is purely a function of \(\theta\) and one corresponding to the numerically calculated \(A_\phi\). For low field line angular velocities and high spins the separation surface moves away from the horizon at higher latitudes, while for high field line angular velocities it moves towards the horizon.

Figure 7.2 makes it clear that significantly more energy is extracted along field
lines near the equator for all values of field line angular velocity, but low $\Omega_F$ and high spin magnetospheres redirect most of that extracted energy towards the azimuthal axis. We measure this behavior in the next section.

Energy Extraction

In this section we explore the rate of black hole energy extraction. Energy flux is conserved along magnetic field lines, so the rate of black hole energy extraction is most easily calculated on the horizon. If we define $P$ as the power (energy per unit time as measured by a distant observer) leaving the horizon, we find (cf. [49]):

$$P = \frac{1}{2} \int_{r^+} T^r_t \sqrt{-g} d\theta d\phi = \frac{1}{2} \int_{r^+} \Omega_F A_{\phi, \theta} \sqrt{-g} F^{\theta r} d\theta. \quad (7.26)$$

As this is evaluated on the horizon, we may use the Znajek horizon condition (Equation [7.6]) to find:

$$P = \frac{1}{2} Q (1 - Q) \frac{a^2}{r_+^2 + a^2} \int_0^{\pi} \frac{1}{\Sigma} A_{\phi, \theta}^2 \sin \theta d\theta. \quad (7.27)$$

Here $Q$ is a unitless scaling of the field line angular velocity to the horizon’s angular velocity; $\Omega_F = Q \omega_H$ (for energy extraction to take place we must have $0 < Q < 1$). In CGS units, the power $P$ becomes:

$$P = 6.5 \times 10^{20} \cdot \chi \cdot r_+^4 \frac{B_x^2 m^2}{G^2 M_\odot^2} \text{ erg s}^{-1}. \quad (7.28)$$

where

$$\chi = \frac{1}{2} Q (1 - Q) \frac{a_*^2}{(r_+^2 + a_*^2)} \int_0^{\pi} \frac{A_{\phi, \theta}^2 \sin \theta}{r_+^2 + a_*^2 \cos^2 \theta} d\theta. \quad (7.29)$$

Here $a_*$ and $r_*$ are dimensionless measures of black hole spin and horizon radius; $a = a_* m$ and $r_+ = r_*/m$. The quantity $\chi$ is a dimensionless measure of the rate of
black hole energy extraction that varies from magnetosphere to magnetosphere. The quantities $B_x$ and $r_{x^*}$ are measures of monopolar magnetic field strength, in a sense that we now explore by considering the Newtonian limit.

In the Newtonian limit, a vector potential $A_\phi = B_0 \cos \theta$ corresponds to a monopolar magnetic field that is given by (in a standard orthonormal spherical basis):

$$\vec{B} = \frac{B_0}{r^2} \hat{r}. \tag{7.30}$$

The weighting of $B_0$ on $A_\phi$ may therefore be interpreted as a measure of the monopolar magnetic field strength at a specified radius (magnetic charge in the case of a monopole). For convenience our numerical solutions for $A_\phi$ assume unit spacing between the azimuthal axis and the plane ($B_0 = 1$), but any arbitrary weighting would yield identical results. The quantities $B_x$ and $r_{x^*}$ are measures of that arbitrary weighting; $B_x$ is the field strength in Gauss of a Newtonian monopole at dimensionless radius $r_{x^*}$. For example, $B_x = 100 \text{G}$ and $r_{x^*} = 20$ would correspond to a monopolar magnetic field strength of 100 Gauss at a distance $r = 20 \text{m}$ from a black hole of mass $m$. Some caution should be taken in applying this interpretation, however. None of our numerical solutions are truly monopolar, and the strength of the magnetic field is an observer dependent quantity that does not have a simple translation from a flat space value to the spacetime of a rotating black hole. The above interpretation of $B_x$ and $r_{x^*}$ is made purely for simplicity and convenience, and in general should be taken to be nothing more than a rough estimate.

In Figure 7.3 we plot the value of $\chi$ as a measure of the rate of black hole energy extraction for select values of black hole spin and field line angular velocity. We suppress spins greater than $a = 0.99m$ as showing them would compress the curves of smaller spin that are of greater interest.
Figure 7.3: The rate of black hole energy extraction for select magnetospheres; $\chi$ is defined in Equation [7.29]. For each value of black hole spin 20 magnetospheres were calculated, indicated for $a_*=0.875$ with blue dots. The approximate maximum values of $\chi$ for various spin values are indicated with red stars; they corresponding to the 23 spin parameters $a_*=0.1, 0.2, 0.3, 0.4, 0.5, 0.55, 0.6, 0.65, 0.7, 0.725, 0.75, 0.775, 0.8, 0.825, 0.875, 0.9, 0.925, 0.95, 0.965, 0.975, 0.985,$ and $0.99$. For small spin the maximum value of $\chi$ occurs near $\Omega_F = 0.5\omega_H$; for $a_* = 0.99$ the maximum occurs near $\Omega_F = 0.58\omega_H$. For extremal spin ($a_* = 1$, not shown) the rate of energy extraction peaks slightly above $\chi = 0.033$.

For a given value of $\Omega_F$, increasing black hole spin always increases the rate of energy extraction, but changes in field line angular velocity can have a much larger effect than changing spin. The peak energy extracted for a given spin occurs at $\Omega_F \approx 0.5\omega_H$ for low spin and approaches $0.6\omega_H$ at high spin. The maximum energy extracted spans two orders of magnitude; $\chi_{\text{Max}} = 0.03$ for $a \approx 0.99m$, $\chi_{\text{Max}} = 0.003$.
for $a \approx 0.5 m$, and $\chi_{\text{Max}} = 0.0003$ for $a \approx 0.2 m$.

To a very good approximation, the functional form of $\chi(a)$ for fixed field line angular velocities $\Omega_F = 0.1 \omega_H$, $\Omega_F = 0.95 \omega_H$, and value for maximum energy extraction $\Omega_{F, \text{Max}}$ are given by:

\[
\begin{align*}
\chi_{0.1 \omega_H} & \approx 3.3 \times 10^{-3} \left(1.8a_s^{2.5} + 1.1a_s^{17}\right), \\
\chi_{0.95 \omega_H} & \approx 3.3 \times 10^{-3} \left(1.5a_s^{3.1} + 1.8a_s^{17}\right), \\
\chi_{\text{Max}} & \approx 1.0 \times 10^{-2} \left(1.9a_s^{2.6} + 1.3a_s^{14}\right). 
\end{align*}
\]

(7.31)

In this form it is clear that for a given spin the maximum rate of energy extraction corresponding to $\Omega_{F, \text{Max}}$ is roughly three times that of the lower values $\Omega_F \approx 0.1 \omega_H$ or $\Omega_F \approx 0.95 \omega_H$, an effect that can be much larger than changes in spin.

For low field line angular velocities the magnetic field bends towards the azimuthal axis, as shown in Figures 7.1 and 7.2. So while most energy is extracted on the horizon near the equatorial plane, a short distance from the horizon a large fraction of it ends up flowing outward along the azimuthal axis. In order to explore this behavior, we calculated the energy escaping through a spherical cone (both upper and lower hemispheres) at a radius $r = 20 m$ for $\Omega_F = 0.1 \omega_H$. The results are plotted in Figure 7.4. For high spins ($a_* \approx 0.85$ and greater), over 95% of the extracted energy escapes through a 45° cone, and 80% escapes through a cone of less than 30°. This indicates that for high spin and low field line angular velocities a black hole magnetosphere is naturally inclined to extract energy via jet-like structures aligned with the rotational axis of the black hole.
Figure 7.4: The rate of black hole energy extracted as a function of the angle of a spherical cone at a radius of 20 m (in both upper and lower hemispheres) for a field line angular velocity $\Omega_F = 0.1\omega_H$. The angles through which 50%, 80%, 90%, and 95% of the total extracted energy escapes are overplotted.

Angular Momentum Extraction

In this section we explore the rate of black hole angular momentum extraction. As with energy flux, angular momentum flux is conserved along magnetic field lines, and is most easily calculated on the horizon by exploiting the Znajek horizon condition. If we define $K$ as the rate of angular momentum extraction, we find:

$$K = -\int_{r_+} T^r_\phi \sqrt{-g} \theta d\phi = \frac{1}{2} \int_{r_+} A_{\phi,\theta} \sqrt{-g} F^{\phi r} d\theta. \quad (7.32)$$
Note that this only differs from the rate of energy extraction (Equation 7.26) by a factor of field line angular velocity $\Omega_F$, consistent with the conserved energy $E$ and angular momentum $L$ per unit flux tube differing by the same factor. In CGS units $K$ reduces to:

$$K = 3.2 \times 10^{15} \cdot \varphi \cdot r_{x*}^4 \frac{B_x^2}{G^2} \frac{m^3}{M_\odot^3} \text{erg},$$

(7.33)

where:

$$\varphi = \frac{1}{2} \left[ (1 - Q) a_+ \int_0^\pi \frac{A_{\phi,\theta}^2 \sin \theta}{r_+^2 + a_+^2 \cos^2 \theta} d\theta \right].$$

(7.34)

The primary structural difference between these expressions and those for the rate of energy extraction (Equations 7.28 and 7.29) is the fact that $\varphi$ does not vanish for $Q = 0$. This is a statement that it is possible to spin down a black hole without extracting energy. In Figure 7.5 we plot the value of $\varphi$ as a measure of the rate of black hole angular momentum extraction for the same values of black hole spin and field line angular velocity that were used to plot the rate of energy extraction.

To a very good approximation, the functional form of $\varphi(a)$ for fixed field line angular velocities $\Omega_F = 0.1\omega_H$, $\Omega_F = 0.95\omega_H$, and value for maximum energy extraction $\Omega_{F\text{ Max}}$ are given by:

$$\varphi_{0.1\omega_H} \approx 1.7 \times 10^{-2} \left( 9.7a_+^{1.1} + 2.8a_+^{8.0} \right),$$

$$\varphi_{0.95\omega_H} \approx 1.3 \times 10^{-3} \left( 9.4a_+^{1.3} + 9.7a_+^{9.3} \right),$$

$$\varphi_{\text{Max}} \approx 1.0 \times 10^{-2} \left( 9.3a_+^{1.1} + 2.8a_+^{6.6} \right).$$

(7.35)

In this form it is clear that the $\Omega_F = 0.1\omega_H$ solutions extract nearly twice as much angular momentum as the solutions that maximize the rate of energy extraction. It is also clear that the $\Omega_F = 0.95\omega_H$ solutions extract around 10% of the angular momentum of the $\Omega_F = 0.1\omega_H$ solutions, while from Equation 7.31 we note that these
Figure 7.5: The rate of black hole angular momentum extraction for the selection of magnetospheres used in Figure 7.3. $\varphi$ is defined in Equation 7.34. For each spin the angular momentum extracted at the maximum rate of energy extraction is indicated by red stars. The blue dots indicate the 20 values of field line angular velocity used for the $a_* = 0.875$ case. The red stars indicate the 23 different values of spin listed in Figure 7.3.

solutions extract roughly the same amount of energy. This indicates that high $\Omega_F$ solutions can extract energy from a black hole for a much longer period of time than lower $\Omega_F$ solutions can, as it will take longer for high $\Omega_F$ magnetospheres to spin down a black hole.
Numerical Error Estimation

Previously we discussed four analytic solutions that can provide useful comparisons with our numerical results, most especially in determining the reliability of the $\epsilon = 1\%$ error level we have set. In this section we will use those solutions as an aid in exploring the error of our numerical solutions.

The three perturbed monopole and exact HS3 solutions have varying regions of applicability. We therefore choose to examine the numerical solution for $a = 0.3m$ and $\Omega_F = 0.5\omega_H$ so that we can reasonably compare all four solutions simultaneously. For convenience we begin by comparing them along a slice of constant $\theta = 45^\circ$ from just inside the horizon to $r_{\text{max}} = 20m$ (Figure 7.6). Along that slice it is apparent that the Schwarzschild monopole and first order perturbed monopole are very similar, with both typically having errors of less than 5%. The third order perturbed solution does better in general, with an error of around 1% near the inner light surface and inside the ergosphere. Aside from the two grid squares immediately adjacent to the inner light surface the numerical solution has an error of less than 0.05% across the entire slice, and is sometimes “better” than the HS3 solution’s error of around 0.0005%. We place quotes around “better” because the error of the HS3 solution is a rough measure of the precision of our numerical grid and derivatives; significantly exceeding $\sim 0.0005\%$ likely involves unjustified precision. Slices along different $\theta$ values as well as different spin parameters and field line angular velocities yield qualitatively similar results.

In order to examine the increase in error along the inner light surface we plot the percent error of the numerical solution in the two grid squares immediately adjacent to the inner light surface in Figure 7.7. This is done for all values of $\theta$, again for $a = 0.3m$ and $\Omega_F = 0.5\omega_H$. Even though substantial portions of the inner light surface are well below the 1% level, there are spikes up to 1%. Reducing those spikes
Figure 7.6: The percent error along a $\theta = 45^\circ$ slice for $a = 0.3m$ and $\Omega_F = 0.5\omega_H$; the radial spacing corresponds to uniform spacing in the numerical grid. The horizon (H), inner light surface (ILS), ergosphere (E), and separation point (SP) are indicated by vertical lines. With the exception of the two grid squares immediately adjacent to the inner light surface the numerical solution is significantly better than the perturbed solutions. The various solutions used are listed in the text.

is what takes the largest amount of computational time. The perturbed solutions are either comparable to or greatly exceed the error in the numerical solutions along the entirety of the inner light surface. The $HS3$ solution is typically 10-100 times better than any other solution. The exception is near the azimuthal axis where the $HS3$ solution’s field line angular velocity diverges. Note that the error of the $HS3$ solution does not also diverge, as the combined equations are well behaved there; the error increase comes from amplification of the error in taking numerical derivatives. The
implications of the varying error levels are discussed in more depth below.

Figure 7.7: The percent error along the inner light surface for $a = 0.3m$ and $\Omega_F = 0.5\omega_H$. The grid squares on either side of the inner light surface (red and black) have almost identical error; reducing the spikes in that error to the 1% threshold we have set is what takes the most computational time. Despite those spikes the numerical solution still generally does as good or better than the perturbed solutions. The upticks in the third order monopole and exact $HS3$ solutions near $\theta = 0$ are from the conversion of the toroidal field to a differentiated function of $A_{\phi}$. Most of the terms in the transfield equation are very small there, so numerical errors in that conversion are amplified.

In order to reduce computation time we make a guess as to the ultimate structure of the fields to use as an initial condition. It is therefore reasonable to ask if our guesses bias our results towards solutions that are close to that initial guess and ignore other potentially valid magnetospheres. There are two unknown quantities that we make
guesses for; the toroidal component of the vector potential, $A_\phi(r, \theta)$, and the derivative of the square of the toroidal field with respect to the vector potential as an unknown function of the vector potential, $d/dA_\phi(\sqrt{-g} F^{\theta r})^2 = F(A_\phi)$. In order to examine how sensitive our solutions might be to changes in the initial form of these functions, we examine the magnetospheres obtained for black hole spin parameter $a = 0.8m$; the numerically obtained functions of the derivative of the toroidal field for those magnetospheres are shown in Figure 7.8.

To assess how initial conditions might modify our results, we first coupled the numerical solution for the derivative of the toroidal field for the $\Omega_F = 0.05 \omega_H$ magnetosphere with the vector potential $A_\phi$ obtained for every other value of field line angular velocity (extrapolating outward to $r = 20m$ for the higher field line angular velocity solutions). In every case we found that the numerical code rapidly converged to a solution essentially indistinguishable from the original $\Omega_F = 0.05 \omega_H$ solution. There were some minor deviations at large radii, as the 1% error level was achieved there last (in the original solution it was achieved last on the inner light surface), but significantly less than the deviation between adjacent $\Omega_F = 0.1 \omega_H$ and $\Omega_F = 0.0 \omega_H$ magnetospheres.

As modifying the initial vector potential seemed to have no effect, we next examined modifying our initial guess for the derivative of the toroidal field. Poor guesses for this function can result in differences ("kinks") in $A_\phi$ across the inner light surface that can easily exceed the difference in $A_\phi$ between the azimuthal axis and the equatorial plane. Diminishing inner light surface kinks takes the most computation time, so good guesses for the derivative of the toroidal field are the most critical initial condition and most likely source of any sensitivity our numerical solutions might have to initial conditions.

For clarity we focus on two alternative initial guesses for the functional form
Figure 7.8: The derivatives of the toroidal field $d/dA_\phi(\sqrt{-g}F^{\phi r})^2$ obtained numerically for black hole spin parameter $a = 0.8m$ and various values of field line angular velocity $\Omega_F$ (in increments of $0.05\omega_H$). The solid blue and red dashed lines indicate two different initial guesses for the derivative of the toroidal field in the $\Omega_F = 0.05\omega_H$ case. The blue and red dotted line is the final result in both cases; they completely overlap and obscure the original result obtained using a much better initial guess. The relative difference between the results obtained using the A and B initial conditions and the original result are shown in the bottom panel.

of the derivative of the toroidal field, shown in Figure 7.8 as initial conditions A and B. They are symmetric quadratics in $A_\phi$ that straddle the numerically obtained function, with the addition of periodic oscillations in Case A in an attempt to make a very poor initial guess without being overly ridiculous. Both cases led to very large initial kinks across the inner light surface and significantly increased the computation
time required to find a solution. However the solutions ultimately obtained in both cases were again essentially indistinguishable from the original $\Omega_F = 0.05\omega_H$ solution; the minor deviations in the derivative of the toroidal field (shown in the bottom panel of Figure 7.8) are dwarfed by the deviation between adjacent $\Omega_F = 0.1\omega_H$ and $\Omega_F = 0.0\omega_H$ magnetospheres.

**Discussion**

The most limiting assumption underlying our solutions is that of uniform field line angular velocity. In this section we discuss that assumption in more detail, consider two extremes of the magnetospheres that we found, and briefly explore the numerical error of our solutions.

**Assumption of Uniform $\Omega_F$**

We have assumed uniform field line angular velocities $\Omega_F$ in order to simplify the task of studying how the location of the inner Alfvén surface might correspond to changes in the structure of energy extracting black hole magnetospheres. This restricted us to solving for the structure of magnetospheres inside the outer light surface. With solutions in hand we return and examine that restriction in more detail. We begin by examining how limiting the assumption of uniform $\Omega_F$ might be in the two extreme classes of magnetospheres ($\Omega_F/\omega_H \approx 0$ and $\Omega_F \approx \omega_H$) that we found. We then discuss the existence and uniqueness of solutions that pass smoothly through both light surfaces. We close with a discussion of how limiting uniform $\Omega_F$ might be in interpreting our results.

For low field line angular velocities ($\Omega_F/\omega_H \approx 0$) where the magnetic field lines bend towards the azimuthal axis the limitations imposed by solutions restricted to the interior of the outer light surface should not be a significant concern. In that case
the outer light surface can be many thousands of gravitational radii away from the horizon near the azimuthal axis and formally infinitely far away exactly on the axis. Diminishing the kink on such a distant surface could slightly modify the structure of the magnetosphere near the horizon, but in reasonable application we would generally expect deviations from our core assumptions of stationarity, axisymmetry, a perfectly conducting force-free plasma over such an extended region to be far more significant. Should that not be the case, we also see a smooth transition from our $\Omega_F = 0.05\omega_H$ solutions to our $\Omega_F = 0$ solutions for which an outer light surface does not exist (formally located infinitely far away from the black hole). We see no reason to expect pathologies in the limit $\Omega_F \rightarrow 0$, implying that our low $\Omega_F$ solutions are close to ones those that pass smoothly through both light surfaces.

For high field line angular velocities ($\Omega_F \approx \omega_H$) where the magnetic field lines bend towards the equatorial plane we would expect to find a connection to nearby accreting matter (not necessarily a thin disk limited to the equatorial plane). Our boundary condition of a single magnetic field line in the equatorial plane might be reasonable close to the horizon, but moving away from the horizon it would become increasingly likely to find significant deviations depending upon the specific model of nearby accreting matter that was chosen. Finding solutions that passed smoothly through both light surfaces would therefore be somewhat ill-advised, as it would involve modifying the near horizon magnetosphere using restrictions found in unreasonably distant regions. A more realistic magnetosphere would include a description of nearby accreting matter that included a model of plasma injection into an inflow interior to or near the separation surface. Our solutions could be representative of such inflows, especially a thick disk or torus near the ergosphere. A solution that passed smoothly through an outer light surface, on the other hand, would also need to include some model of plasma injection into an outflow consistent
with the field line geometry obtained, which might be difficult.

In arguing that it is possible to usefully interpret the above limits on $\Omega_F$ despite the limitations of our domain, we have ignored the question of whether or not similar solutions that pass smoothly though both light surfaces even exist. Using essentially identical boundary conditions on the vector potential $A_\phi$ as were used here, Contopoulos et al. [18] and Nathanail & Contopoulos [65] (hereafter CKP13 and NC14) modified both the field line angular velocity and toroidal field to find solutions that pass smoothly through both light surfaces. In doing so they found a single nearly monopolar solution with varying $\Omega_F \approx 0.5\omega_H$ that is very similar in structure to our uniform $\Omega_F = 0.5\omega_H$ solutions. It is therefore natural to ask if the solutions found by CKP13 and NC14 are unique. The exact solutions of Equation 7.8, while non energy extracting and largely unphysical, indicate that they are not; there are in fact infinitely many solutions fulfilling our boundary conditions on $A_\phi$ that pass smoothly through both an inner and outer light surface. The question then becomes why we found the solutions that we did regardless of initial conditions, and why CKP13 and NC14 similarly found a single solution.

The answer to that question lies in our numerical techniques. We show in Appendix H that the magnetofrictional method works by minimizing the energy in the electromagnetic fields. In minimizing the kink across the inner light surface we are finding matched minimum energy solutions. The apparent uniqueness of our results is a suggestion of the uniqueness of a minimum energy solution, not the uniqueness of our solution in full generality. The majority of numerical codes are common in the behavior of being dissipative and seeking minimum energy states; any code that allows energy to be added at will is likely to be numerically unstable. For example there is a full range of valid rotating Schwarzschild monopole solutions (Equation 7.9), but it would be a very unique numerical code that would be capable of blindly
finding all of them. Most codes would always converge on the minimum energy $\Omega_0 = 0$

solution.

For any value of black hole spin, the minimum energy magnetosphere consistent with our boundary conditions will be as close to monopolar as possible ($\Omega_F \approx 0.5\omega_H$). The bunching of magnetic field lines towards the axis or equator seen in our low and high $\Omega_F$ solutions requires the addition of energy to spin the magnetosphere away from that monopole. It is therefore not surprising that CKP13 and NC14 found a single roughly monopolar solution that passes smoothly through both inner and outer light surfaces; without the explicit and very careful addition of energy to move to and maintain a rotating magnetosphere any stable code would likely converge on that solution. Our utilization of uniform $\Omega_F$ may be interpreted as a way of exploring the structure of arbitrarily rotating magnetospheres using a dissipative numerical code, and we believe that a large fraction of our solutions could be taken to be good approximations of solutions that pass smoothly through both light surfaces. In realistic application, however, they should largely be taken to be representative of inflow solutions. Outflow solutions would require the additional description of plasma injection mechanisms, and even if our solutions passed smoothly through an outer light surface there is no guarantee that plasma parameters would be either continuous or conserved across an extended plasma injection region (as implied by single solutions that pass smoothly through both light surfaces).

We have argued that our limited domain might not prevent useful interpretations and that our solutions might approximate solutions that pass smoothly through both light surfaces (even if such solutions might be of dubious value). It could still be asked how representative our solutions might be of energy extracting black hole magnetospheres, as completely uniform $\Omega_F$ magnetospheres are unlikely to exist. We cannot fully answer that question, but we emphasize that exploring uniform $\Omega_F$
magnetospheres was not the ultimate goal of this work. Instead, the ultimate goal of this work was to explore the effects of inner Alfvén surface location. Uniform $\Omega_F$ in the force-free limit is merely a useful tool to do so, both in its obvious simplicity as well as in allowing us to know exactly where the inner light surface will be prior to solving for the structure of the magnetosphere that passes through it. Slightly deforming the location of the Alfvén surfaces used here should not have significant effects, and in general we expect the Alfvén surface to be continuous, so the solutions obtained here could be merged to explore more complex scenarios. For example, if one desired an ingoing solution with a connection to accreting matter near the equator and the launching of a jet-like structure from the horizon along the rotational axis, an appeal could be made to an Alfvén surface that lies closer to the boundary of the ergosphere at higher latitudes and closer to the horizon near the equator. On the horizon this would correspond to lower $\Omega_F$ for small values of $\theta$ and higher $\Omega_F$ for large values of $\theta$, compatible with the simulations conducted by [58].

Two Types of Magnetospheres

Our solutions indicate two extreme methods for black hole energy to be extracted and transmitted to distant observers. For low field line angular velocities ($\Omega_F/\omega_H \approx 0$) jet-like structures naturally form to transmit extracted energy directly from the horizon to distant observers along the azimuthal axis. For high field line angular velocities ($\Omega_F \approx \omega_H$) we find magnetospheres that are consistent with the transmission of energy from the horizon to nearby accreting matter, which in turn might transmit the extracted energy to distant observers. In this section we discuss potential consequences of those two types of magnetospheres, noting at that outset that any potential distinction between the two in realistic astrophysical scenarios is likely to be fuzzy.
If we examine the rate of energy and angular momentum extraction through Equations 7.31 and 7.35, we see that the two types might have significant time-dependent distinctions due to their differing rates of angular momentum extraction and concurrent black hole spindown. Therefore transient high energy phenomena powered by black hole energy extraction might also have two distinct signatures. For specificity, consider a gamma-ray burst (GRB). Black hole energy extraction is a candidate for the central engine of a GRB [26], and the exponential decay associated with black hole energy extraction might be compatible with some GRB observations [66]. We do not demand a specific type or model of GRB, however. We only require a transient high energy event that might be powered by black hole energy extraction, and “GRB” is a convenient specific stand-in for “transient high energy event” even if our treatment is too crude to completely describe or differentiate between realistic GRBs of any specific type.

If we take $B_x = 10^{16}G$ and $r_x = 3$ as constants in Equations 7.28 and 7.33 (i.e. a magnetic field strength of roughly $10^{16}G$ just outside the ergosphere), assume that $a = 0.95m$ and $m = 4M_\odot$ at time $t = 0$, and for simplicity insist upon a state of suspended accretion (such as a magnetically arrested disk in a low accretion state), we find the rates of black hole energy extraction plotted in Figure 7.9.

We immediately note that all three cases exhibit roughly exponential decay, but the $\Omega_F = 0.95\omega_H$ case decays relatively slowly. After roughly 20-30 seconds both the $\Omega_F = 0.1\omega_H$ and $\Omega_F_{\text{Max}}$ luminosities have decreased by factors of $\sim 10^3$ due to the black hole spin dropping below $a \approx 0.05m$. The rate of angular momentum extraction for the $0.95\omega_H$ case is far lower, however, and after 30 seconds its luminosity has not significantly decreased. Note that in this model larger black hole masses would take less time to spin down due to the increase in magnetic field strength implied by $r_x$ corresponding to a larger dimensional radius. For example, a black hole with twice
the mass but otherwise identical parameters would be spun down in around half the time. Also note that the scaling on the axes is somewhat irrelevant; they were chosen for crude correspondence to a GRB, but a wide range of systems with different parameters, luminosities, and timescales exhibit qualitatively identical curves.

It takes over 350 seconds for the 0.95ω_H luminosity to drop to the same level that the 0.1ω_H luminosity fell to in around 20 seconds. We therefore expect that GRBs with magnetic field lines that directly connect the horizon to distant observers
might be at least 10-20 times shorter than GRBs with magnetic field lines connecting to a disk or other matter before connecting to distant observers. The addition of more realistic effects such as accretion actively spinning up the black hole would obviously complicate matters, and as noted at the outset the boundary between both types of magnetospheres is unlikely to be very distinct. However, if GRBs (again emphasizing the limited sense in which we are using the term) are powered by black hole energy extraction it would not be unreasonable to expect two broad classes with differing characteristic durations as well as different radiation signatures; the longer lived $0.95\omega_H$ type might also exhibit a greater degree of thermalization due to the deposition of the extracted energy into inflowing matter before that energy is transmitted to distant observers.

The analysis presented here is too crude to reasonably claim that distinctions between GRBs can arise due to differences in the location of their inner Alfvén surface, encapsulated here in differences in field line angular velocity. A more reasonable model is beyond the scope of our present work, but the possibility that such a model could allow finer classification of some GRBs or other transient high energy phenomena via correlation of timescale and degree of thermalization is interesting.

**Numerical Error Analysis**

In this section we discuss how reliable the numerical calculations underlying our solutions might be. With the possible exception of the kink at the inner light surface, we find that our numerical solutions are consistent with the precision of their numerical grid. For the case of $a = 0.3 m$ and $\Omega_F = 0.5 \omega_H$, the error of our solution is generally around 100 times smaller than the error in any of the perturbed monopole solutions that are considered to be good approximations at low spin. While the numerical error involved in implementing the exact HS3 solution (arising mostly from
taking numerical derivatives) can be another 100 times smaller still, there are large regions in which the numerical solution is actually “better” than the HS3 solution and likely exceeds the precision justified by our numerical grid.

Along the inner light surface the error in our numerical solution is generally a few times smaller than the third-order perturbed monopole solution, and exceeds the first-order solution by a factor of 10-100. Again the HS3 solution generally exceeds the numerical solution by another factor of 10-100. However significantly better solutions would be difficult to obtain using the kink reduction algorithm used here. At the 1% level the “kink” (change in $A_{\phi}$) in our solution across the inner light surface in the radial direction is around $10^{-6}$, much smaller than the required linear change in $A_{\phi}$ of $5 \times 10^{-3}$ in the $\theta$ direction implied by our usage of 200 grid squares between the azimuthal axis (where $A_{\phi} = 4$) and the equator (where $A_{\phi} = 3$). This means that it becomes very difficult to accurately quantify the kink, much less diminish it further.

Some experimentation reveals that a much higher error level of around 10% would not have changed our results in any significant way. This is consistent with the error of the perturbed solutions, which should be accurate enough for most purposes and have similarly sized errors. Reducing the error of our solutions to the 1% level increased the required computation time significantly, and achieves a precision that almost certainly far exceeds what our assumptions allow for in any reasonable application. However reducing the error to the 1% level allows us to confidently state that any deficiencies in our results rest within our assumptions and not in our numerical calculations.

Due to the importance of selecting appropriate initial conditions in diminishing computation time, it might be asked if our initial guesses in any way affect or drive our results. As shown in Figure 7.8 even with somewhat ridiculous choices of initial condition we ultimately arrive at effectively identical solutions that are
appreciably different from neighboring solutions. We cannot (and do not) claim that this implies that our solutions are unique in general; it is possible that there are many solutions that satisfy our boundary conditions and assumptions. The only potentially persuasive uniqueness in our solutions lies in unique minimum energy solutions that are matched across the inner light surface, as a consequence of the magnetofrictional method being an energy minimizing algorithm (shown in Appendix H). In developing our numerical code we conducted many more trials and experiments than have been discussed here, and we never observed any indication that it might be possible to converge on two different solutions. Therefore while we cannot prove that our solutions are unique matched minimum energy solutions consistent with our boundary conditions and assumptions, we do believe that it is a reasonable assumption to make.

Closing Remarks

The defining feature of energy extracting black hole magnetospheres is the location of their inner Alfvén surface, which coincides with the inner light surface in the force-free limit. Despite its importance, no comprehensive studies of the effects of modifying the location of the inner Alfvén surface have been accomplished. We have begun addressing that deficiency by studying how simple force-free magnetospheres can be modified as a function of uniform field line angular velocity, which together with black hole spin determines the location of the inner light surface. In order to do so we extended the Newtonian magnetofrictional method for computing force-free magnetospheres into the general relativistic regime, which allowed us to efficiently calculate hundreds of energy extracting black hole magnetospheres as functions of inner Alfvén surface location. In so doing we found that inner Alfvén surfaces near the horizon cause extracted energy to flow towards the equatorial plane, while inner
Alfvén surfaces near the boundary of the ergosphere cause extracted energy to flow outwards via jet-like structures aligned with the azimuthal axis. Applied to transient high energy phenomena, those magnetospheres imply two timescales that might differ by a factor of 20 or more. This suggests that two classes of transient high energy phenomena might exist; shorter ones that directly connect the horizon to distant observers, and longer ones that have a disk or other significant matter separating the energy flow from the horizon to distant observers that potentially thermalizes the extracted energy, creating a different radiation signature.
In this chapter we solve for the structure of variably rotating force-free magnetospheres for a black hole spin parameter $a = 0.8m$. Qualitatively, we find the same thing that was found in the previous chapters; slowly rotating magnetospheres have field lines that bend upwards towards the azimuthal axis, while more rapidly rotating magnetospheres have field lines that bend downwards towards the equatorial plane. When opposing tendencies are present (high field line angular velocity near the axis and low field line angular velocity near the equatorial plane) then either one tendency will overwhelm the other or the field lines will bend towards each other. Of special interest are magnetospheres with low field line angular velocity near the axis and high field line angular velocity near the equatorial plane. Those magnetospheres simultaneously develop jet-like structures near the azimuthal axis and structures compatible with connections to nearby accreting matter in the equatorial plane, behaviors that they might have significant astrophysical relevance. This chapter is significantly based on Thoelecke et al. [110], modified as appropriate for this work.

**Background**

In the previous chapter we found that relatively slowly rotating magnetospheres with inner light surfaces near the outer limits of the ergoregion resulted in the bending of magnetic field lines towards the azimuthal axis and the formation of jet-like structures. Relatively rapidly rotating magnetospheres with inner light surfaces near the horizon resulted in the bending of magnetic field lines towards the equatorial plane, compatible with a direct connection between the horizon and a disk or similar nearby accreting matter structure. Due to the complex nature of the equations involved those results were arrived at numerically, but such tendencies can also be
seen in a more restricted form analytically (Chapter 6).

In the previous chapter we made the assumption of a uniformly rotating magnetosphere. That assumption was convenient in that it allowed us to know a priori exactly where the inner light surface of a given magnetosphere would lie, and it allowed us to focus on zeroth-order effects of magnetosphere rotation. Despite those conveniences, however, uniformly rotating magnetospheres are likely to be fairly crude approximations of real black hole magnetospheres.

The goal of this chapter is to relax the assumption of uniform magnetosphere rotation and study rotational profiles that might more closely correspond to astrophysical black hole magnetospheres. We will primarily focus on near-horizon behaviors where the effects of a rotating spacetime are the strongest, making the event horizon a natural place to specify the rotation of a magnetosphere. Specifically we choose to study distributions of field line angular velocity $\Omega_F$ on the horizon corresponding to the first two terms of a series expansion of an arbitrary distribution:

$$\left.\Omega_F\right|_{r_H} = (A + B \sin \theta) \omega_H. \quad (8.1)$$

Here $\omega_H$ is the angular velocity of the horizon (i.e. the angular velocity of a zero angular momentum observer on the horizon), while $A$ and $B$ are unitless constants. Field line angular velocity $\Omega_F$ corresponds to a magnetosphere’s rotation in that it may be thought of as a measure of the rotational boost velocity to the plasma rest frame (Appendix E). In the previous chapter we studied uniformly rotating magnetospheres with $A = [0 \ldots 1]$ and $B = 0$ for a full range of black hole spins and corresponding horizon angular velocities $\omega_H$.

In this chapter we will study $A = [0 \ldots 1]$ and $B = (-1 \ldots 1)$, as those values most completely encompass arbitrarily rotating energy-extracting black hole magne-
tospheres. Values of $A$ less than 0 or greater than or equal to 1 are not considered as they would not generally correspond to energy-extracting magnetospheres. We will apply the condition $0 \leq A + B < 1$ so that the magnetospheres will extract rotational energy along almost every magnetic field line (field lines along the azimuthal axis or corresponding to $A + B \sin \theta = 0$ being the few exceptions). We will focus exclusively on a spacetime with black hole spin parameter $a = 0.8m$, selected as being large enough to be interesting without being overly extreme and potentially less widely representative.

From analysis of the previous chapters we expect (and will find) that the most interesting $A$ and $B$ values will fall into a fairly narrow range corresponding to low field line angular velocities near the azimuthal axis and high field line angular velocities near the equatorial plane, as those values would form magnetospheres with jet-like structures aligned with the azimuthal axis and structures reminiscent of horizon-disk connections at lower latitudes. Such magnetospheres (which we will label as “Jet-Disk” magnetospheres) are of special interest because they combine nearby matter structure compatibility with the formation of collimated jet-like structures without directly relying on anything except near-horizon magnetosphere conditions. As we will discuss in greater detail below, the horizon field line angular velocity distributions corresponding to those magnetospheres are also intrinsically compatible with conditions that might be imposed by the presence of nearby accreting matter structures, further increasing the likelihood of their astrophysical interest.

While some of the $A$ and $B$ pairs are of special interest, many are deliberately naïve and likely not very relevant to plausible astrophysical black hole magnetospheres. Nonetheless their calculation is still important. Not only do they form a more complete set when viewing Equation 8.1 as a generic expansion of arbitrary magnetosphere rotation, they also place the more astrophysically relevant
distributions of field line angular velocity on the horizon in a more complete context. So while many of the magnetospheres we will calculate are likely to be mostly mathematical curiosities, their illumination of more interesting black hole magnetospheres still gives them value.

Assumptions and Numerical Techniques

The primary difference between this work and the work described in the previous chapter is the relaxation of the condition of uniform field line angular velocity. Therefore we will primarily provide summaries of the assumptions and numerical techniques used, and direct attention to the more detailed discussion of Chapter 7 should greater depth be desired.

Core Assumptions

We assume a black hole whose surrounding spacetime is adequately described by the Kerr metric in Boyer-Lindquist coordinates, corresponding to the line element:

\[
\begin{aligned}
    ds^2 &= \left( 1 - \frac{2mr}{\Sigma} \right) dt^2 + \frac{4mar \sin^2 \theta}{\Sigma} dtd\phi - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2 - \frac{A \sin^2 \theta}{\Sigma} d\phi^2. \\
\end{aligned}
\] (8.2)

We then assume that the black hole is surrounded by a perfectly conducting plasma that is both stationary and axisymmetric, with the axis of symmetry corresponding to that of the black hole. We then take the magnetically dominated force-free limit and assume that plasma inertial effects may be discarded, such that the magnetosphere can be completely described by three parameters: the toroidal vector potential \( A_\phi \), the field line angular velocity \( \Omega_F \), and the toroidal magnetic field \( \sqrt{-g} F^{\theta r} \). The toroidal vector potential \( A_\phi \) and the field line angular velocity \( \Omega_F \) are related to the field strength tensor \( F^{\alpha \beta} \) by \( F_{\alpha \beta} = A_{\beta, \alpha} - A_{\alpha, \beta} \), \( F_{r\phi} \Omega_F = F_{tr} \), and \( F_{\theta \phi} \Omega_F = F_{t\theta} \).
The toroidal vector potential $A_\phi$ is conserved along magnetic field lines and as such is a useful flux function for the poloidal magnetic field. The field line angular velocity and toroidal magnetic field are also conserved along magnetic field lines, respectively statements that field lines rotate rigidly and that energy and angular momentum Poynting fluxes are conserved. The conservation of field line angular velocity is a consequence of stationarity, axisymmetry, and a perfectly conducting plasma (expressed as the vanishing contraction of the field strength tensor and its dual $\mathcal{F}^{\alpha\beta}F_{\alpha\beta} = 0$) and is also a conserved quantity when plasma inertial effects are considered. The conservation of the toroidal field is the force-free limit of angular momentum flux conservation. Further discussion of force-free conserved quantities may be found in Blandford & Znajek \[6\]; further discussion of conserved quantities when plasma inertial effects are present may be found in \[2, 8, 101\] and Chapter \[3\].

Boundary Conditions

We assume reflection symmetry across the equator so that we need only solve for the structure of the magnetosphere in the upper half of the poloidal plane. We then apply boundary conditions along the azimuthal axis and equatorial plane that are compatible with a “monopolar” magnetic field, in the sense that we assume fixed magnetic field lines tracing the azimuthal axis and equatorial plane (mathematically $A_\phi(\theta = 0) = A_{\phi_{\text{max}}}$ and $A_\phi(\theta = \pi/2) = A_{\phi_{\text{min}}}$ where $A_{\phi_{\text{max}}}$ and $A_{\phi_{\text{min}}}$ are fixed constants). We use the term “monopolar” to describe those boundary conditions while noting that has the potential to be misleading. There will often be a substantial toroidal component of the magnetic field, resulting in a helical magnetic field that only resembles a monopole when projected onto the poloidal plane. Additionally, non-zero magnetic flux through a closed surface surrounding the black hole demands a reversal of the direction of the magnetic field below the equatorial plane, so “split-monopolar”
would be a more appropriate term when the entire poloidal plane is considered. In short “monopolar” should only be taken to describe the boundary conditions on the upper half of the poloidal plane where we conduct our numerical calculations, not the magnetosphere as a whole.

A field line tracing the azimuthal axis is not a significant restriction as stationarity and axisymmetry already imply a single magnetic field line extending straight upward from the pole. However a single magnetic field line tracing the equatorial plane is a more severe restriction. Although it might be physically reasonable in regions close to the horizon, the further away one gets the less reasonable it is likely to become. This is because significant amounts of matter would generally be expected near the equatorial plane at least as close as the innermost stable circular orbit. That matter would likely anchor many different magnetic field lines and be better described by higher order multipoles in the equatorial plane.

Despite their potential deficiencies, monopolar boundary conditions are still very useful in explorations of basic interactions between the electromagnetic fields and the background spacetime. Other boundary conditions intrinsically assume a specific matter distribution outside the black hole and by extension a specific astrophysical context. Any such assumed context not only introduces its own assumptions but also has the potential to introduce arbitrarily large forcings on the magnetosphere that might obscure interactions between the electromagnetic fields and the background spacetime. In short the assumption of monopolar boundary conditions is a deliberate compromise; we are favoring a more fundamental exploration of black hole magnetospheres over direct applicability to any specific astrophysical context.
Limited Domain

We do not extend our magnetospheres past the outer light surface (pulsar light cylinder analog) or $r = 20m$, whichever occurs sooner. That choice is not made completely freely, as diffusive numerical techniques are generally incapable of finding magnetospheres that pass smoothly through both inner and outer light surfaces once a specific distribution of horizon field line angular velocity has been specified. This is because the character of the equations involved changes across a light surface; the prefactor on the second derivatives of the vector potential changes sign. This means that numerical schemes can fail to find many valid solutions, as for stability they must necessarily evolve the magnetosphere differently on either side of a light surface. Finding solutions numerically then generally reduces to “matching” minimum energy solutions across a light surface by adjusting field-aligned conserved quantities. Matching three regions across two light surfaces would require two different conserved quantities, but once a distribution of horizon field line angular velocity has been specified only the toroidal field remains as a free variable. This means that numerical techniques will generally be incapable of finding matched solutions across both inner and outer light surfaces once a distribution of horizon field line angular velocity has been specified, even if such solutions exist.

For example, within the monopolar boundary conditions we have set infinitely many solutions are known to exist that pass smoothly through both inner and outer light surfaces [60]. However when numerical techniques are applied by matching across light surfaces only a single highly monopolar (minimum energy) solution is found [18]. As we are primarily interested in near-horizon behaviors in this work, we have chosen to limit ourselves to regions interior to the outer light surface.

Despite the fact that the outer light surface is a numerical limitation, our selection of a limited domain interior to $r = 20m$ or the outer light surface has
physical motivations. For example, we have assumed that the field-aligned conserved quantities are rigidly conserved across the entire magnetosphere, from the horizon to the outer boundary. While the assumptions of stationarity, axisymmetry, and a force-free plasma that led to those conserved quantities might be approximately valid over any given region, as a field line grows in length small deviations from those assumptions can grow in significance. Our limited domain can be thought of as the assumption that field-aligned quantities are only approximately conserved, and that near horizon values might differ significantly from more distant values.

Additionally, when the force-free limit is viewed as the magnetically-dominated limit of an ideal plasma flow, then consideration must be made of the “separation surface” between the inner and outer light surfaces that separates a plasma inflow from a plasma outflow (by demarcating the change in dominance from inward gravitational forces to outward centripetal forces). Near the separation surface plasma effects can become significant and deviations from our assumptions can be expected, to some extent decoupling inner and outer magnetospheres near the separation surface. We note that the separation surface is a suggestion, however, and not a rule - problematic effects (mathematically sourced by a diminishing Alfvén Mach number and concurrent increase in plasma density) do not have to emerge there, but if they do not emerge at or interior to the separation surface they are guaranteed to emerge as the outer light surface is approached. As such we are not concerned with addressing the numerical difficulties that emerge at the outer light surface, as in the problem space of ideal plasma flows we necessarily demand physical changes to the problem (from a plasma inflow to a plasma outflow) before arriving there. Although in general we expect the separation surface to be a much more physically relevant indication of changing physics than the outer light surface, it is numerically trivial to extend the magnetosphere past the separation surface to the outer light surface, and we see no
harm in doing so.

It is possible that the selection of the outer boundary might influence (or drive) the structure of the magnetosphere contained within it (e.g. spherical outer boundaries at \( r = 3m \), \( r = 4m \), and \( r = 5m \) might result in different solutions). We studied that behavior in initially developing the numerical techniques applied in the previous chapter, and found that such differences could indeed emerge if the treatment of the outer boundary was poorly implemented. If done appropriately, however, the solutions obtained are identical and the outer boundary does not influence the solution. For the work described in this chapter we continued to verify the apparent invariance of the solutions with outer boundary, and (in addition to other tests) calculated every magnetosphere a minimum of two times: once with a spherical outer boundary (typically between \( r = 3m \) and \( r = 4m \)) completely interior to the outer light surface, and once with an outer boundary limited by the outer light surface (or \( r = 20m \)). The solutions obtained were always identical, regardless of outer boundary. For this reason we do not view the outer light surface as a “boundary condition”, but rather the point at which the rigid application of stationary and axisymmetric force-free magnetohydrodynamics (extended from the horizon) has definitively broken down. In a specific model that breakdown might be avoided or diminished by appealing to additional physics, which is a reason why Blandford & Znajek [6] and others have discussed “spark gaps” and other plasma injection mechanisms. A more generic exploration of the solution space of ingoing magnetospheres has no such luxury, however, and is restricted to a domain interior to the outer light surface [54].

Numerical Techniques and the Magnetofrictional Method The only significant change to our numerical techniques from the previous chapter is our method of
kink reduction across the inner light surface, which we have improved to allow for significantly reduced error levels there. That error reduction does not modify our final results in any appreciable way but does allow for more efficient computation of magnetospheres. We will only briefly review the magnetofrictional method and describe the improved kink reduction procedure; further details may be found in Chapter 7.

To calculate magnetospheres we apply a relativistic extension of the magnetofrictional method developed by [123], similar to the method used by [117]. This method takes an initial guess for the structure of a magnetosphere and then calculates the divergence of the stress energy tensor. If that divergence does not vanish the configuration is invalid (or at least inconsistent). To find a valid configuration the invalid excess momentum fluxes are converted to the velocity $\vec{v}$ of a fictitious plasma via empirically determined “friction”. The magnetic fields $\vec{B}$ are then modified by the relativistic analog of the ideal induction equation $\partial_t \vec{B} = \vec{\nabla} \times (\vec{v} \times \vec{B})$. The end result is that the vector potential $A_\phi$ is evolved via a simple advection equation until a solution is found:

$$A_{\phi,t} = -v^A A_{\phi,A}. \quad (8.3)$$

The magnetofrictional velocity $v^A$ also includes empirically determined weighting factors for convenience and stability, as outlined in Appendix H. A demonstration that application of the magnetofrictional method will always result in a valid force-free magnetosphere under fairly general assumptions may also be found there.

We have found that our numerical procedures always converge to the same solution regardless of the initial guess for the structure of the magnetosphere (initial $A_\phi$, $\Omega_F$, and $\sqrt{-g F^{\theta \phi}}$); poor initial guesses simply take longer to converge. The magnetofrictional method is an energy-minimizing algorithm, so that single solution
is at best an indication of the uniqueness of a minimum-energy state. We believe it likely that in addition to that minimum energy solution there are also infinitely many magnetospheres compatible with our boundary conditions and assumptions that contain more energy and are therefore unstable in some fashion. We therefore interpret the solutions found as being “most compatible” with the various assumptions made, but not unique.

Kink Reduction

Regions inside and outside the inner light surface are evolved using a different overall prefactor of $\pm 1$ on the magnetofrictional velocity in order to maintain numerical stability. This causes a “kink” to develop across the inner light surface as the vector potential $A_\phi$ is evolved in opposite directions on either side. To find a smooth solution we adjust the toroidal field as a function of the vector potential, $\sqrt{-g F^{\theta r}}(A_\phi)$, until that kink disappears. To make modifications to that function we first shoot across the inner light surface from the inside (near horizon region) to the outside such that the near horizon region provides an inner boundary condition for the region outside the inner light surface. We then modify the toroidal field corresponding to the shot grid squares by measuring how close to being force-free those squares are. Specifically we use the error level of the shot grid squares (calculation of that error is discussed in the next section) to gradually correct the functional form of the toroidal field:

$$\sqrt{-g F^{\theta r}}|_{\text{New}} = \sqrt{-g F^{\theta r}}|_{\text{Old}} - \lambda \cdot \text{Error.}$$ (8.4)

Here $\lambda$ is an empirically determined constant; optimal values vary widely, depending primarily upon grid resolution and current error level.

The above method of kink reduction differs slightly from the method used in the previous chapter. There we adjusted the toroidal field by evolving both sides of the
inner light surface separately, then measured the magnitude of the difference in $A_\phi$ across the light surface as an input in adjusting the toroidal field. While that method works reasonably well, as the kink in $A_\phi$ becomes smaller it can become very difficult to accurately quantify and therefore reduce the error level of the final solution.

By directly using the error level to modify the toroidal field we are able to reduce the error along the inner light surface significantly from what was obtainable in the previous chapter. Error levels of at least 0.001% are now fairly easy to obtain along the entire extent of the inner light surface (computation time being the primary limiting factor) while our previous method would sometimes struggle to significantly exceed 0.1% as a worst case.

Despite the advantages in error reduction, the primary motivation for the change in kink reduction method was to enable more rapid convergence to a solution. Enhanced error reduction was merely a side effect that did not change our results in any appreciable fashion.

Both our previous and current methods for kink reduction are identical in basic principle to the method developed by [17] for pulsar magnetospheres and applied more recently by others such as [18, 55, 65, 76] to black hole magnetospheres.

Measuring Error

Our magnetospheres can contain both strongly monopolar regions as well as regions with very small current, so most commonly used measures of force-freeness that rely directly on some physical attribute of the fields can yield unreliable (or at least potentially inconsistent) results due to one or more of the measured physical attributes becoming vanishingly small. We have therefore developed a more mathematical technique for measuring the error of our solutions. A valid and self-consistent solution will have a stress energy tensor with vanishing divergence, so we
measure how close to zero the divergence of a magnetosphere’s stress energy tensor is in order to determine the magnetosphere’s error level. Specifically, we first separate the divergence into seven terms:

\[ T^{A\beta,\beta} \sim \sum_{i=1}^{7} D_i = \delta. \]  \hfill (8.5)

The exact form of \( D_i \) we use is detailed in Appendix [H]; their sum is not completely equivalent to the divergence of the stress energy tensor because we apply overall weighting factors for convenience. When \( \delta \) is close enough to zero a solution has been found, close enough being determined by comparing \( \delta \) to the largest of the \( D_i \) terms:

\[ |\delta| < \epsilon \cdot \text{Max} (|D_i|). \]  \hfill (8.6)

We have set \( \epsilon = 1\% \) over the entire domain as an adequate error level, but in practice most of the domain will be significantly less. Typically the largest \( r \) values are the last regions to achieve the 1\% level, at which time averages of 0.0001\% inside the ergosphere are common. Magnetospheres with an error level of 10\% are generally not substantively different from those at 1\%, which are in turn effectively indistinguishable from those at 0.1\% and below. We chose 1\% as an error level in order to remove as much numerical uncertainty as possible while avoiding excessive amounts of computation time.

**Computational Specifics**

The vector potential \( A_\phi \) is calculated over a rectangular \((r, \theta)\) grid with 200 evenly spaced grid squares in \( \theta \) and on average around 1000 variably spaced grid squares in \( r \). The radial spacing varies from magnetosphere to magnetosphere; magnetospheres with inner light surfaces near the horizon have tighter spacing there
in order to adequately resolve the inner light surface. The radial grid extends from just inside the horizon to \( r = 20m \) or the outer light surface, whichever is smaller, with radial spacing of around 0.1m near \( r = 20m \).

The toroidal field is implemented as a function of \( A_\phi \) with over 1000 points of varied spacing between \( A_{\phi\text{min}} \) on the equatorial plane and \( A_{\phi\text{max}} \) on the azimuthal axis. That spacing is determined by convenience, as convergence can be optimized by using very fine sampling in regions where the toroidal field as a function of \( A_\phi \) is steep.

To evolve the advection equation for \( A_\phi \) we use an upwind differencing algorithm similar to the one described in [35]. One-sided finite difference approximations appropriate to that algorithm are made to evolve \( A_\phi \), but centered finite difference approximations appropriate to the local grid spacing are used to determine the magnetofrictional velocity \( v^A \).

The azimuthal axis and equatorial plane are taken as fixed boundaries. At all other boundaries (\( r_{\text{min}}, r_{\text{max}}, \) and/or along the outer light surface) we shoot outwards using a quadratic fit after every time step, an approach that is largely equivalent to using one-sided derivatives on those boundaries.

Performance

We computed all magnetospheres on a single desktop computer with a 6-core Intel Haswell CPU assisted by Nvidia Kepler GPUs, which can generally find a magnetosphere at the 1% error level within a few hours. Exact time to completion can vary widely, however, from well under an hour to days in extreme cases. Computation time is highly dependent upon how good the initial guess was, how tight the grid spacing is, and how optimal various empirical tunings (strength of friction, modification to the toroidal field, etc.) are. At a high level our algorithm
is conceptually similar to finding the root of a computationally expensive function by crawling along that function, and as such can be susceptible to large inefficiencies similar to those found when over or under evaluating an expensive function and from taking steps that are either too big or too small. Significantly improving the speed of our code and its algorithmic inefficiencies should be possible, but we found the current performance level to be adequate for our current purposes.

Results

We divide our results into three sections. First we explore the general structure of the magnetospheres obtained as a function of field line angular velocity, measured by the \( A \) and \( B \) parameters in \( \Omega_f(r_H, \theta) = (A + B \sin \theta) \omega_H \). We then explore the rates of energy and angular momentum extraction from the black hole. Lastly we explore the behavior of magnetospheres containing both jet-like regions and structures resembling horizon-disk connections in more detail, as those are likely to be the magnetospheres of greatest astrophysical interest.

General Structure

We calculated 400 distinct magnetospheres with different horizon field line angular velocities \( \Omega_f = (A + B \sin \theta) \omega_H \) using a spacing of 0.05 in both \( A \) and \( B \) over the ranges \( A = [0 \ldots 1) \) and \( B = (-1 \ldots 1) \) under the condition that \( 0 \leq A + B < 1 \). It would be impractical to show all 400 in detail, so instead we classify magnetospheres based upon their general structure and show some representative types.

In the poloidal plane there are only three things that a magnetic field line can do: bend upwards toward the azimuthal axis, remain straight, or bend downwards towards the equatorial plane. We classify each of those three tendencies as being “jet-like”, “monopole-like”, or “disk-like”, respectively, and then classify magnetospheres by the
\[ \Omega_F = (0.20 + 0.65 \sin \theta) \omega_H \]

Figure 8.1: The structure of a magnetosphere with horizon field line angular velocity \( \Omega_F = (0.20+0.65 \sin \theta)\omega_H \). The black poloidal magnetic field lines are spaced evenly on the horizon. The green lines trace the inner and outer light surfaces, the red line traces the boundary of the ergosphere, and the cyan line traces the separation surface (the point at which gravitational and centripetal forces are balanced). The dotted magenta line traces the monopolar separatrix between field lines bending towards the axis and field lines bending towards the equatorial plane. This magnetosphere is classified as a Jet-Disk magnetosphere, denoted by the “J-D” text inside the horizon. The background shading denotes the magnitude of the conserved field-aligned Poynting flux; \( E = (1/4\pi)\sqrt{-gF^{\phi\theta}} \Omega_F \). A plot of this magnetosphere’s ergoregion is shown in Figure 8.2, and an additional 12 magnetospheres are displayed in similar fashion in Figure 8.3.
Figure 8.2: The structure of a magnetosphere with horizon field line angular velocity \( \Omega_F = (0.20 + 0.65 \sin \theta) \omega_H \), the same as shown in Figure 8.1. The inner light surface is shown in green, the boundary of the ergosphere is marked in red, and the monopolar separatrix between Jet and Disk behaviors is shown as a dotted magenta line. The black magnetic field lines are spaced evenly on the horizon. The three blue magnetic field lines rotate with field line angular velocities \( \Omega_F = 0.4 \omega_H \), \( \Omega_F = 0.5 \omega_H \), and \( \Omega_F = 0.6 \omega_H \). The shading to the left of the horizon \( (r_H = 1.6m) \) is a measure of the poloidal magnetic field strength on the horizon; \( |B^H_p| \sim A_{\phi,\rho} \csc \theta \). The shading outside the horizon is a measure of both the toroidal magnetic field and conserved angular momentum Poynting flux; \( L = (1/4\pi) \sqrt{-g} F^{\phi\rho} \). The three dotted blue lines correspond to the inner light surfaces of uniformly rotating magnetospheres with \( \Omega_F = 0.6 \omega_H \) (closest to the horizon), \( \Omega_F = 0.5 \omega_H \), and \( \Omega_F = 0.4 \omega_H \) (furthest from the horizon). An additional 12 magnetospheres are displayed in similar fashion in Figure 8.3.
Figure 8.3: Twelve magnetospheres shown using the same conventions as in Figures 8.1 and 8.2. The main plot shows the structure of the magnetosphere on a Cartesian grid colored by outward energy flux. The inset plot shows the structure of the ergoregion on an \((r, \theta)\) grid colored by outward momentum flux. The colorbar to the left of the horizon denotes poloidal magnetic field strength on the horizon using \(|B_H^p| \sim A_{\phi, \theta} \csc \theta\). The inner and outer light surfaces are shown in green, the boundary of the ergosphere is marked in red, and the separation surface is shown in cyan. The black field lines are spaced evenly on the horizon; the most monopolar field line, if relevant, is marked in dotted magenta. The magnetosphere classification type (Jet, Monopole, Disk) of high and low latitude regions is denoted by text inside the horizon. The inset plots show the locations of the inner light surfaces of magnetospheres with uniform field line angular velocities \(0.6 \omega_H\), \(0.5 \omega_H\), and \(0.4 \omega_H\) as dotted blue lines. The inset plots on the middle two rows and middle of the bottom row (d, e, f, g, h, i, and k) mark the field lines rotating at \(0.6 \omega_H\), \(0.5 \omega_H\), and \(0.4 \omega_H\) in blue.
typical behaviors of their field lines in high latitudes and in low latitudes. For example a purely monopolar magnetosphere is classified as “Monopole-Monopole” while a magnetosphere with field lines that bend upwards in high latitudes and downwards in low latitudes is classified as a “Jet-Disk” magnetosphere. The classification of a magnetosphere is accomplished by subjective inspection; as such the boundary between high and low latitudes and what is more “monopolar” than not varies from magnetosphere to magnetosphere.

In Figure 8.1 we plot a Jet-Disk magnetosphere with horizon field line angular velocity distribution $\Omega_F = (0.20 + 0.65 \sin \theta)\omega_H$. The magnetosphere is limited to the region interior to the outer light surface, shown as a green line. The separation surface is shown as a cyan line and might be considered as a more realistic outer boundary, as it delineates the region where the forces on the plasma shift from being dominated by outward centripetal forces to being dominated by inward gravitational forces. As such a large accumulation of plasma might be expected near the separation surface, breaking the assumption of a force-free plasma and the rigid conservation of field-aligned quantities. In Figure 8.2 we plot the same magnetosphere using an $(r, \theta)$ grid focused on the ergoregion. The shading in Figure 8.2 corresponds to the outward momentum flux (or toroidal magnetic field); the strength of the poloidal field on the horizon is shown as a colorbar immediately inside the horizon, allowing for a comparison of the relative strengths of the two magnetic fields. The inner light surface is shown as a green line while the inner light surfaces of uniformly rotating magnetospheres with field line angular velocities of $0.4\omega_H$, $0.5\omega_H$, and $0.6\omega_H$ are shown as dotted dark blue lines, allowing for a rough determination of correlation of inner light surface location with magnetosphere behavior. Such correlations and other effects are discussed in more detail below, and an additional twelve magnetospheres of various different types are plotted in Figure 8.3 using methods identical to Figures
Monopole-Jet magnetospheres were the only type not found. The other eight types were present, although as shown in Figure 8.4 some types were much more common than others. By far the most common type was Jet-Jet, followed by Jet-Disk and Disk-Disk. The boundaries and transitions between different types of magnetospheres are slightly fuzzy due to the subjective nature of their classification, but within a given classification region the behaviors are robust.

Figure 8.4: Classification of magnetospheres according to their high and low latitude behaviors as a function of field line angular velocity on the horizon. The boundaries between regions are moderately susceptible to subjective interpretation, but the classification within a region is robust. The cyan shading denotes magnetospheres with at least 90% of the maximum luminosity (cf. Figure 8.5); the magenta shading denotes magnetospheres with at least 90% of the maximum rate of angular momentum extraction (cf. Figure 8.6). The green numbers denote the Jet-Disk magnetosphere numbering scheme described in the text. The blue circles mark the magnetospheres shown in Figure 8.3; the red circle marks the magnetosphere shown in Figures 8.1 and 8.2.

As a general rule the structure of a magnetosphere is predictable by considering the average field line angular velocity of small collections of field lines. If the average
is less than half of the horizon’s angular velocity, $\langle \Omega_F \rangle < 0.5 \omega_H$, then the field lines will bend upwards toward the azimuthal axis. If the average is greater than half of the horizon’s angular velocity, $\langle \Omega_F \rangle > 0.5 \omega_H$, then the field lines will bend towards the equatorial plane. The strength of either bending increases the further away from $0.5 \omega_H$ the average field line angular velocity becomes.

The exception to that rule is when contradictory preferences are present, such as when high latitude groups want to bend downwards and low latitude groups want to bend upwards. In that case one group will generally dominate over the other and cause the entire magnetosphere to be either completely Jet-Jet or Disk-Disk. However there are a few transitional magnetospheres where neither behavior dominates, resulting in Disk-Jet magnetospheres where field lines converge along an approximately 45° line through the poloidal plane (as in the middle of the bottom row of Figure 8.3).

The structure of the magnetospheres near the horizon and inside the ergoregion are shown using an $(r, \theta)$ grid in Figure 8.2 and in the inset plots of Figure 8.3. As might be expected from monopolar boundary conditions and a minimum energy solution, no significant bending of field lines occurs within the ergoregion; this means that the distribution of horizon field line angular velocity is highly predictive of the shape and location of the inner light surface. This in turn means that the bending of field lines and structure of a magnetosphere can just as easily be attributed to inner light surface location as to average field line angular velocity. In other words it can be said that field lines want to bend upwards when they cross the inner light surface closer to the outer limits of the ergoregion and want to bend downwards when they cross the inner light surface near the horizon.

There is a third potential indicator of magnetosphere structure in addition to average field line angular velocity and the location of the inner light surface. The colorbars to the left of the horizon in Figure 8.2 and the inset plots of Figure 8.3 are
measures of poloidal magnetic field strength, while the shading outside the horizon is a measure of the toroidal magnetic field (as the conserved angular momentum Poynting flux). Comparison of the two indicates that large-scale field line bending could also be predicted via the relative strengths of poloidal and toroidal magnetic fields on the horizon. A strong toroidal field relative to the poloidal field generally causes bending towards the azimuthal axis, while a weak toroidal field and stronger poloidal field results in bending towards the equatorial plane.

In Jet-Disk and Disk-Jet magnetospheres a single monopolar field line can be defined as the separatrix between the two regions of opposite bending. We determined that separatrix by doing a Cartesian \((x, z)\) transformation from the \((r, \theta)\) computational grid near the outer boundary of the magnetosphere followed by finding the absolute minimum of the second derivative in \(x\) along different field lines. The resulting monopolar field line was then visually verified in comparison with the entire magnetosphere to ensure reasonableness. In general the field line angular velocity of that field line falls between \(0.5\omega_H\) and \(0.6\omega_H\), compatible with the notion that \(0.5\omega_H\) field lines “want” to be straight. Not much more than compatibility should be concluded, however, as a careful inspection of the monopolar field lines drawn in magenta in Figure 8.3 makes it clear that the determination of monopolarity can be somewhat arbitrary and dependent upon the region of the magnetosphere chosen for analysis.

Jet-Disk magnetospheres are perhaps the most interesting in terms of astrophysical relevance and are generally predicted by two features. First, the field line angular velocity on the azimuthal axis must be less than or equal to \(0.4\omega_H\). Second, the field line angular velocity must increase from the azimuthal axis to the equatorial plane with an ultimate value greater than or equal to \(0.6\omega_H\). This is again compatible with the general rule of low/high \(\Omega_F\) jet/disk bending, and the more extreme the
difference between azimuthal axis and equatorial plane field line angular velocities the more obvious the “jet-disk” behavior becomes. There is significant variation in the amount of energy and angular momentum flowing into either the “jet” or to the “disk”, as shown by the shading in the middle two rows of Figure 8.3, we explore that variation in more detail below.

Energy and Angular Momentum Extraction

We measure the net rate of black hole energy extraction via the dimensionless parameter $\chi$, calculated as an integral over the horizon (cf. [49]):

$$\chi = \frac{1}{2} \frac{a^2}{(r^2_{+s} + a^2_s)} \int_0^{\pi} Q (1 - Q) \frac{A^2_{\phi,\theta} \sin \theta}{r^2_{+s} + a^2_s \cos^2 \theta} d\theta. \quad (8.7)$$

Here $Q$ is a unitless scaling of the field line angular velocity on the horizon; $Q = A + B \sin \theta$. In terms of $\chi$, the net luminosity is given by:

$$P = \int_{r^+} T_r t \sqrt{-g} d\theta d\phi = 6.5 \times 10^{20} \cdot \chi \cdot r^4_{x*} \frac{B^2_x m^2}{G^2 M^2_\odot} \text{ erg} \cdot s. \quad (8.8)$$

Here $a_*$ and $r_{+*}$ are dimensionless measures of black hole spin and horizon radius; $a = a_* m$ and $r_H = r_{H*} m$. The quantity $B_x$ corresponds to monopolar magnetic field strength at dimensionless radius $r_{x*}$, in the sense that in the Newtonian limit of a monopole we would have a magnetic field that in spherical orthonormal coordinates is given by:

$$\vec{B} = \frac{B_x r^2_x}{r^2} \hat{r}. \quad (8.9)$$

We emphasize that the definitions of $B_x$ and $r_{x*}$ are made purely for convenient compatibility with our monopolar boundary conditions and should be taken to be nothing more than a rough average of magnetic field strength as our magnetospheres
are neither Newtonian nor generally truly monopolar. The rate at which a given magnetosphere extracts energy in terms of $\chi$ is shown in Figure 8.5.

![Image of Figure 8.5](image_url)

**Figure 8.5:** The net rate of black hole energy extraction for all magnetospheres in terms of the dimensionless parameter $\chi$. The top panel plots lines of constant $A$; the bottom panel plots $\chi$ as a function of average field line angular velocity on the horizon. The cyan shading denotes the region within 90% of the maximum luminosity. That region includes every type of observed magnetosphere, as shown by the compatible cyan shading in Figure 8.4.

We measure the net rate of angular momentum extraction via the dimensionless parameter $\varphi$ in almost identical fashion to the measurement of $\chi$:

$$\varphi = \frac{1}{2} a_* \int_0^\pi (1 - Q) \frac{A^2_{\phi, \theta} \sin \theta}{r_{+*}^2 + a_*^2 \cos^2 \theta} d\theta. \quad (8.10)$$
In terms of $\varphi$, the net rate of black hole angular momentum extraction is given by:

$$K = - \int_{r^+} T^\varphi \sqrt{-g} d\varphi d\phi = 3.2 \times 10^{15} \cdot \varphi \cdot r_{\gamma}^4 \frac{B^2}{G^2} \frac{m^3}{M^3} \text{erg.} \quad (8.11)$$

The rate at which a given magnetosphere extracts momentum in terms of $\varphi$ is shown in Figure 8.6.

![Figure 8.6](image)

Figure 8.6: The net rate of black hole angular momentum extraction for all magnetospheres in terms of the dimensionless parameter $\varphi$. The top panel plots lines of constant $A$; the bottom panel plots $\varphi$ as a function of average field line angular velocity on the horizon. The magenta shading denotes the region within 90% of the maximum rate of angular momentum extraction. That region is entirely composed of Jet-Jet magnetospheres, as shown by the compatible magenta shading in Figure 8.4.

Perhaps the most striking feature of the net rate of energy extraction in Figure
is the very broad peak. The maximum luminosity corresponds to a horizon field line angular velocity of $\Omega_F = (0.5 + 0.05 \sin \theta) \omega_H$, but there are a large number of magnetospheres that have effectively equivalent luminosities within 90% of that maximum. Those magnetospheres encompass every single type of observed magnetosphere, as shown by the cyan shading in Figure 8.4. The bottom panel of Figure 8.5 indicates that average field line angular velocity is not very predictive of a maximally luminous magnetosphere; anything from $\langle \Omega_F \rangle = 0.3 \omega_H$ to $\langle \Omega_F \rangle = 0.8 \omega_H$ can yield a very close to maximum luminosity magnetosphere. Averages near $0.5 \omega_H$ could be assumed to be relatively luminous and averages closer to the outer limits of energy extracting magnetospheres ($\langle \Omega_F \rangle = 0$ or $\langle \Omega_F \rangle = \omega_H$) could be assumed to be relatively dim, but anything else would require closer analysis.

The rate of energy extraction calculated by Blandford & Znajek [6] for a monopolar geometry corresponds to a $\chi$ parameter of:

$$\chi_{BZ77} = \frac{1}{8} \frac{a_*^2}{r_+^2 + a_*^2} \int_0^{\pi} \frac{\sin^3 \theta}{r_+^2 + a_*^2 \cos^2 \theta} d\theta. \quad (8.12)$$

In this work we used a dimensionless black hole spin parameter $a_* = 0.8$ and corresponding horizon radius $r_+ = 1.6$, yielding $\chi_{BZ77} = 0.0124$. Despite being well outside the “low spin” assumption used to derive that value, it is still only about 10% over our maximum value of $\chi$. This is primarily a result of our solutions concentrating more horizon poloidal magnetic flux near the azimuthal axis than a monopole would, as shown by the middle of the top row of Figure 8.3.

The closeness of such a crude approximation indicates that any estimate of the net luminosity of an energy-extracting black hole magnetosphere based in some way on the assumptions of a Blandford and Znajek monopole (such as made in [58, 74, 107], among others) will in general be successful. There are a wide range of magnetospheres
within 10% or so of the near energy maximum such an estimate would yield, including essentially every magnetosphere with an average horizon field line angular velocity \( \langle \Omega_F \rangle = 0.5 \omega_H \) typically used in such estimates.

The net rate of angular momentum extraction is more discriminating in its behavior than the net rate of energy extraction, as shown in Figure 8.6. Very low field line angular velocity magnetospheres always extract more angular momentum than others. Those magnetospheres also always correspond to Jet-Jet magnetospheres, as shown by the magenta shading in Figure 8.4. The spread in the rate of angular momentum extraction for a given average horizon field line angular velocity is typically not very large, so it is also generally safe to assume that if average field line angular velocity is increased then the net rate of angular momentum extraction will be decreased.

So far we have only considered net rates of energy and angular momentum extraction. However the direction of those flows could in many instances be of far greater importance than the net values, as indicated by comparing the luminosities of Figure 8.5 with magnetosphere type in Figure 8.4 and magnetosphere structure in Figure 8.3. Magnetospheres near the luminosity maximum are those that are closest to exhibiting monopolar behaviors; magnetospheres that bend most tightly towards either the azimuthal axis or the equatorial plane are the dimmest. When compared with the rates of angular momentum extraction, we find that magnetospheres that would most rapidly spin down a black hole will be fairly underluminous in a global sense, but that most of that energy will be very tightly directed along the azimuthal axis. Magnetospheres that would take the longest amount of time to spin down a black hole will be similarly underluminous in a global sense, but most of that energy will be transmitted into a small nearby region in the equatorial plane. So while some magnetospheres might be dimmer overall than others, they might direct that energy
in a far more efficient fashion to specific regions of interest while at the same time correlating with other behaviors due to their different rates of angular momentum extraction.

We examine the importance of the direction of energy and angular momentum flows in more depth in the next section within the context of Jet-Disk magnetospheres.

Structure of Jet-Disk Magnetospheres

Jet-Disk magnetospheres are probably the most interesting type of magnetosphere we found, as they most closely resemble what might often be expected in an astrophysical environment: open field lines aligned with the azimuthal axis and lower latitude structures compatible with a connection to nearby accreting matter. In light of that interest, in this section we divide the fluxes of energy and angular momentum in Jet-Disk magnetospheres into their “jet” and “disk” components and compare their magnitudes.

Of the 400 calculated magnetospheres, 69 are classified as Jet-Disk, with \( A \) parameters in the distribution of horizon field line angular velocity \( \Omega_F = (A + B \sin \theta) \omega_H \) ranging from 0.0 to 0.4. In order to discuss the general tendencies of Jet-Disk magnetospheres we first number each of the 69 magnetospheres by their different \( A \) parameters as shown in Figure 8.4: magnetospheres with \( A = 0.0 \) and \( B = [0.60 \ldots 0.95] \) are numbered by increasing \( B \) from 1 to 8, those with \( A = 0.05 \) and \( B = [0.55 \ldots 0.90] \) are numbered by increasing \( B \) from 9 to 16, and so on.

We separate magnetospheres into jet and disk regions by using the most monopolar field line (i.e. the line with minimal bending in the poloidal plane) as the separatrix between the two regions. The determination of the most monopolar field line is slightly arbitrary in that it is dependent upon the parameters and regions used to measure bending, which leads to some variability in the field line angular velocity
of the most monopolar field line. In this case that variability is not significant, as both the ratios of energy and angular momentum flow to jet and disk regions as well as the trends in their magnitudes from magnetosphere to magnetosphere do not significantly change over the spread of what could reasonably be called a valid separatrix between jet and disk regions.

Figure 8.7: The rates of energy and angular momentum flow into the jet and disk regions for all 69 Jet-Disk magnetospheres. The magnetosphere numbering scheme is shown in Figure 8.4; the vertical lines separate magnetospheres into $A$ parameter sections, while $B$ parameter increases across a section $(\Omega_F = (A + B \sin \theta)\omega_H)$. The total rates of energy and angular momentum extracted are fairly constant but the ratio between jet and disk regions can be large. The maximum rate of energy flow into the jet is $\chi_{\text{Jet Max}} \approx 3 \times 10^{-3}$ while the minimum is $\chi_{\text{Jet Min}} \approx 2 \times 10^{-5}$, around 130 times smaller. The blue circles mark the magnetospheres shown in Figure 8.3; the red circle marks the magnetosphere shown in Figures 8.1 and 8.2.
Figure 8.7 shows the rates of energy and angular momentum flux into both jet and disk regions for all 69 Jet-Disk magnetospheres, grouped by the $A$ parameter of their horizon field line angular velocities. In general much more energy flows into the disk region than into the jet region. Similarly much more angular momentum flows into the disk region than into the jet region, although the relative difference is generally smaller. The total rates of energy and angular momentum extraction don’t vary that much across all of the Jet-Disk magnetospheres, but as the average field line angular velocity goes up very little of either gets sent into the jet region.

The amounts of energy and angular momentum flowing into the disk region per unit of energy flowing into the jet region are shown in Figure 8.8. For an assumed jet energy up to 200 times more energy could be flowing into the disk region, with a median of around 10 times more energy. For that same assumed jet energy there is also a large range of possible momentum fluxes into the disk region; the maximum is almost 60 times larger than the minimum amount. Those ranges are coupled to the strength of the jet in terms of how sharply field lines bend towards the axis, as illustrated by the two middle rows of Figure 8.3. The tightest jet-like structures have the smallest ratios between jet energy and momentum and disk energy and momentum, while the loosest jet-like structures have the largest ratios between jet and disk energies and momenta.

Putting the ratio of jet energy and disk energy and momentum flows into more concrete terms, for every erg of energy flowing into the jet region there will be between 2 and 200 ergs of energy flowing into the disk region and between $1 \times 10^{-4} m/M_\odot$ and $5 \times 10^{-3} m/M_\odot$ erg-seconds of angular momentum flowing into the disk region, with median values given by:

$$1 \text{ Jet erg} \sim 10 \text{ Disk erg} \sim 3 \times 10^{-4} \frac{m}{M_\odot} \text{ Disk erg-s.} \quad (8.13)$$
In this section we discuss three topics in more depth. First we compare our results with our previous work in calculating uniform field line angular velocity
magnetospheres. Second we discuss the original conceit that led to that work, namely a desire to determine if the inner Alfvén surface of a magnetosphere might convey any additional useful information beyond the potential for an individual field line crossing it to extract a black hole’s rotational energy. Lastly we explore how reasonable the Jet-Disk magnetospheres we found might be, and what potential implications they might have for astrophysical objects.

Comparison with Uniform Field Line Angular Velocity

In the previous chapter we calculated the structure of energy extracting force-free black hole magnetospheres as a function of uniform field line angular velocity using the same monopolar boundary conditions along the azimuthal axis and equatorial plane that were used here. We found that rapidly rotating magnetospheres (referenced to the horizon’s angular velocity) with inner light surfaces near the horizon had poloidal magnetic field lines that bent towards the equatorial plane, while slowly rotating magnetospheres with inner light surfaces near the outer boundary of the ergoregion had poloidal magnetic field lines that bent upwards towards the azimuthal axis. Such behavior may also be seen analytically (Chapter 6), though in a more limited form.

In this chapter we largely found the same behaviors. If a collection of field lines had a relatively small average field line angular velocity they bent upwards towards the azimuthal axis; if they had a relatively large average field line angular velocity they bent downwards towards the equatorial plane. The only exceptions to that behavior were in cases where adjacent groups had incompatible bending preferences; in those cases the group with most extreme field line angular velocity would generally “win” and bend the other group.

In the previous chapter we speculated that consideration of the direction of energy and angular momentum flows might be critical to any consideration of black
hole energy extraction as a plausible central engine driving astrophysical phenomena. In that speculation we were hampered by the crudity of our assumptions, perhaps most notably that of uniform field line angular velocity, but nonetheless were able to use the timescale of a transient object as an example of how the direction of energy and angular momentum flows could potentially modify observed behaviors.

Having now solved for more realistic distributions of field line angular velocity we are more strongly convinced that the direction of energy flow should be a primary consideration in determining the applicability of black hole energy extraction to any given astrophysical object. Within the context of Jet-Disk magnetospheres, for example, any assumed amount of jet energy would need to be coupled to a consideration of the effects of the concurrent flows of energy and angular momentum into nearby accreting matter, something we discuss in more detail below.

Lastly, we previously speculated that changes in magnetosphere structure could be more significant in varying luminosity than changing either black hole spin or magnetic field strength. That is what we found here; within Jet-Disk magnetospheres we found that jet energies varied by a factor of 130 (Figure 8.7). As a general rule the rate of energy extraction varies with black hole spin as $a^2$ (e.g. Equation 8.12 or 8.7), although for very high spin better approximations can be made (e.g. [107]). If we select what might be a reasonable range of black hole spins for active galactic nuclei, 0.3$m$ to 0.95$m$ (e.g. [120]), then an $a^2$ estimate only yields a factor of 10 variation in luminosity due to changing black hole spin. The results of the previous chapter suggest that a factor of 30 variation might be a better estimate, but that’s still subdominant to the factor of 130 variation within Jet-Disk magnetospheres found above. When both effects are combined, jet luminosity variations in excess of a factor of 1000 could be expected for the jets of Jet-Disk magnetospheres over a reasonable range of black hole spins, a variation that would be correlated with both the degree
of jet collimation and the amount of energy and angular momentum concurrently flowing into the disk region.

**Alfvén Surface Location**

In the introduction and elsewhere we have made reference to the inner light (Alfvén) surface as being an important element of black hole energy extraction that might influence magnetosphere structure. In this subsection we discuss in what sense we make those statements and how robust they might be.

As noted in the introduction and in Chapter 5, under fairly general assumptions if a given magnetic field line of an ideal plasma flow has an ingoing Alfvén point within the ergoregion then it is possible for black hole rotational energy to be extracted along that field line [101]. Further, the exact position of the Alfvén point within the ergoregion restricts the allowed relationships between the conserved plasma parameters associated with that field line [99]. Those restrictions necessarily couple Alfvén point location to both global magnetosphere structure and the rate of black hole energy extraction.

Quantifying any coupling between Alfvén point location and global magnetosphere structure could be a useful tool in exploring black hole magnetospheres, as the general relativistic Grad-Shafranov (transfield) equation is almost completely analytically intractable [68]. As such analytic explorations of black hole magnetospheres almost universally assume some poloidal magnetic field line structure and/or focus on explorations of the interactions between plasma parameters along a single magnetic field line (e.g. [27, 87, 104]). If a given Alfvén point location and its concurrent restrictions on plasma parameters were known to be most compatible with an assumed poloidal magnetic field line structure, that knowledge could provide a valuable constraint on analytic explorations of black hole magnetospheres. It is
from this perspective that Alfvén surface location coupled to the structure of black hole magnetospheres could be most valuable.

However, it should be understood that when we make statements about Alfvén (or light) surface location coupling to magnetosphere structure that we are making those statements within a specific context. When considering a single magnetic field line, for example, there are infinitely many relationships between the plasma parameters passing through a given Alfvén point, and as such infinitely many magnetospheres that might be associated with that single magnetic field line. A collection of magnetic field lines is similar; there can be infinitely many magnetospheres associated with any given Alfvén surface. In other words Alfvén surface location cannot possibly encode complete information as to the structure of a magnetosphere - additional restrictions or information must be given for context. In this work that context should largely be taken to be force-free magnetospheres in a minimum energy state that are compatible with monopolar boundary conditions near the horizon.

The magnetospheres we calculated in this work did not exhibit any significant bending of field lines within the ergoregion, making it impossible to decouple inner light surface location from horizon field line angular velocity distribution. Additionally, large scale structure could just as easily be attributed to the ratio of poloidal to toroidal magnetic field strength, which is coupled to field line angular velocity on the horizon via the Znajek regularity condition ([127] and Chapter 5). The 3-way equivalence between inner light surface location, horizon field line angular velocity, and horizon toroidal and poloidal magnetic field strength allows for many interpretations as to what “causes” global magnetosphere behavior. Our view is that inner light surface location is potentially the most fundamentally interesting and that horizon field line angular velocity is the most useful descriptor, but we would in no
way suggest that as being a universally correct or useful perspective.

It is possible that the addition of plasma inertia effects might help decouple Alfvén surface location from field line angular velocity or otherwise break the 3-way equivalence in interpretation of the source of global magnetosphere structure. However, the relationships between plasma parameters at the Alfvén point intimately involve field line angular velocity and as such it might ultimately be impossible to disentangle their respective effects within the context of reasonable magnetospheres. Our belief (or more accurately hope, biased by our perspective and background in examining 1D plasma flows along single magnetic field lines) is that some decoupling will occur and that Alfvén surface location will prove to be useful, but additional study is required to determine if that is the case.

**Jet-Disk Magnetospheres**

In this section we explore some potential implications of the Jet-Disk magnetospheres that were found. We first examine whether or not the distributions of horizon field line angular velocities leading to those magnetospheres are reasonable, then examine what restrictions Jet-Disk magnetospheres might place on black hole energy extraction in astrophysical contexts.

To determine how reasonable the distributions of Jet-Disk horizon field line angular velocities might be, we consider what might be expected of the black hole’s nearby environment. An isolated black hole cannot support a magnetic field, so the magnetic flux that we assume exists on the horizon must be maintained by nearby matter. A likely configuration of such matter compatible with our assumption of stationarity is matter rotating near the equatorial plane with an angular velocity distribution corresponding to centripetal forces roughly balancing gravitational forces. For convenience we will call that configuration of matter a “disk” (a choice made to
aid discussion, not to imply preference for a thin/thick disk, torus, or other structure). The disk is likely to be highly conductive, meaning that the magnetic field will rotate with the disk and possess a field line angular velocity compatible with the disk’s angular velocity. This means that near the equatorial plane the angular velocity of magnetic field lines should be largest near the black hole and decrease as the distance from the black hole increases, formally vanishing infinitely far away.

The field lines on the horizon near the equatorial plane should connect to the disk in nearby regions, and by virtue of rotating with the disk should have $\Omega_F \approx \omega_H$. Proceeding up the horizon field lines will connect with the disk further and further away, resulting in a gradual diminishing of field line angular velocity towards the azimuthal axis. That is exactly the type of horizon field line angular velocity distribution that results in Jet-Disk magnetospheres, and has been used by [124] to calculate magnetospheres that directly connect the horizon to a nearby disk (the problem setup used there prohibits the emergence of open field lines connected to the horizon, however, making the emergence of a Jet-Disk magnetosphere impossible).

In other words the simplest of assumptions one could make about the black hole’s environment are both compatible with and intrinsically imply the presence of jet-like structures aligned with the azimuthal axis and disk-like connections near the equatorial plane. It indicates that at a basic level asking how a jet didn’t form in a given black hole magnetosphere could be a far more difficult question to answer than how it did in another.

The last feature implied by our assumptions is shown in cartoon form in Figure 8.9. We previously noted that for a given amount of energy flowing upward into the jet region, a large amount of energy and angular momentum will also flow into the disk region. That energy will be absorbed by nearby disk anchoring those field lines, providing a natural energy reservoir to fuel a jet and help convert the “jet-like”
Figure 8.9: Cartoon depiction of Jet-Disk magnetospheres. The field line angular velocities of magnetic field lines in the disk decrease from $\Omega_F \approx 0.0 \omega_H$ far from the black hole to $\Omega_F \approx \omega_H$ at the inner edge of the disk. With rigid field line rotation the simplest assumption for horizon field line angular velocity distributions mirrors the disk distribution, starting with $\Omega_F \approx 0.0 \omega_F$ on the pole and ending with $\Omega_F \approx \omega_H$ near the equator. That assumption leads directly to Jet-Disk magnetospheres, with the separatrix between jet and disk connected field lines being a roughly monopolar field line with $\Omega_F \approx 0.5 \omega_H$. The large amounts of energy and angular momentum flowing outward along field lines between $0.5 \omega_H \lesssim \Omega_F \lesssim \omega_H$ will be deposited into a relatively small disk region, creating a natural reservoir from which to launch an energetic jet along upward bending field lines.
structure into a bona fide jet. In simple terms the terminal Lorentz factor of such jet would be given by (derived in Appendix I):

\[ u_{t\text{ Final}} = u_{t\text{ Initial}} + \kappa \frac{\Omega_F}{4 \pi \mu |\eta|} \sqrt{-g} F_{\text{Initial}} \, . \] (8.14)

Here \( u_t \) is the temporal component of the plasma’s four velocity (\( \sim \) Lorentz factor), \( \kappa \) a measure of the efficiency of the conversion of electromagnetic Poynting flux to plasma energy, \( \mu \) is the relativistic enthalpy of the plasma (particle mass in the limit of a cold flow), \( \eta \) is the particle flux per unit flux tube, and we have taken absolute values for ease of interpretation. So long as the toroidal field at the base of the jet is large relative to plasma inertia (\( \sim \mu \eta \)) a highly relativistic jet should be expected.

The physical acceleration mechanism is largely the same as the acceleration of a bead along a rotating wire, as used in Blandford & Payne [5] to describe disk-launched jets. This can be noted from the fact that conversion of electromagnetic Poynting flux into plasma energy flux simultaneously demands transference of angular momentum to the plasma (cf. Equation I.12). A more detailed analytic treatment of how magnetic energy can be transformed into plasma kinetic energy and launch a jet may be found in [116], numerical simulations using similarly “monopolar” field lines as those used here including more discussion of acceleration mechanisms may be found in [48, 106], and observational evidence for a highly magnetized jet base using the event horizon telescope may be found in [41].

For convenience we have used the word “disk” to describe the structure of nearby accreting matter, not intending to imply a demand for a geometrically thin disk close to the black hole. Nonetheless the assumption of a thin disk is useful to gauge how compatible a Jet-Disk horizon field line angular velocity distribution might be with a “disk” by converting the specific energies and angular momenta of a thin disk into
Figure 8.10: Top Panel: Plots of the specific energy and angular momentum of a particle orbiting in a circular orbit in the equatorial plane for a black hole spin parameter $a = 0.8m$. Bottom Panel: The angular velocity of a magnetic field line anchored by matter in circular orbits in the equatorial plane. The radii corresponding to the horizon, co-rotating photon orbit, ergosphere, and innermost stable circular orbit (ISCO) are marked as vertical lines. The presumed approximate horizon-disk connection region between $\Omega_F = 0.95\omega_H$ and $\Omega_F = 0.50\omega_H$ is shaded (corresponding to the $E$ and $L$ deposition region of Figure 8.9); a substantial fraction of extracted energy and angular momentum flows to that region. Derivations of the relationships plotted here may be found in Appendix 1.

a field line angular velocity distribution. The results for the same black hole spin parameter $a = 0.8m$ we used to calculate our magnetospheres are shown in Figure 8.10.

The radius of the disk corresponding to $\Omega_F = 0.95\omega_H$ (the maximum value we
used in calculating magnetospheres) corresponds to what might reasonably be called
the inner edge of the disk. It is interior to the innermost stable circular orbit (ISCO),
but a hard disk cutoff there is not necessarily to be expected. A more reasonable
inner edge would be the point at which the specific angular momentum required to
maintain an orbit begins to noticeably increase before diverging at the radius of a
co-rotating photon orbit. Subjectively that point might be taken to be near the point
where $\Omega_F = 0.95\omega_H$.

In the absence of direct observational evidence of field line angular velocity
in real Jet-Disk type magnetospheres we can also consider the results of numerical
simulations. Within that context it can be difficult to quantify field line angular
velocity (it is most relevant to stationary and axisymmetric plasmas) and as such
is unfortunately often not reported in detail in the literature. However [58] is an
exception, finding a Jet-Disk type magnetosphere and reporting a horizon field line
angular velocity distribution that increased from $0.4\omega_H$ on the azimuthal axis to $0.8\omega_H$
near the equatorial plane, compatible with our Jet-Disk distributions.

The radius of the disk corresponding to $\Omega_F = 0.5\omega_H$ is roughly only a single
gravitational radii away from the radius corresponding to $\Omega_F = 0.95\omega_H$, meaning that
only a very small portion of the disk would have to absorb the potentially enormous
amounts of energy and angular momentum required for a given amount of energy
directed into the jet. This could have at least two useful effects. First, it might
assist in jet launching by heating the plasma near the base of the jet. Second, it
might provide the increase in specific angular momentum required for disk material
to continue orbiting in relatively stable fashion inside the ISCO. However, disk energy
and angular momentum deposition could also be harmful to jet formation, in that
the rate of angular momentum deposition could become too large for a stationary
disk to absorb. In that case the disk could be blown away and the jet halted until
accretion was able to resume, implying that the most luminous jets might necessarily be intrinsically variable.

Quantifying the behaviors discussed above in more depth is beyond the scope of this work, and are not fundamentally new considerations in any event. For example, magnetic fields torquing the inner portions of a black hole’s accretion disk and modifying its behavior were being considered years before Blandford & Znajek [6] was published [112], and such effects have been examined before within the context of energy extracting black hole magnetospheres [52, 118, 119]. What we wish to suggest with this section is not the creation of a new model of energy-extracting black hole magnetospheres, but rather the potentially intrinsic compatibility of near-horizon electromagnetic field structure (expressed by Jet-Disk distributions of field line angular velocity) with both a compact connection to nearby accreting matter and structures reminiscent of jets.

In qualitative form, that intrinsic compatibility might best be summarized by the statement that the term “Blandford-Znajek mechanism” could potentially be more accurately described as the “Blandford-Znajek/Blandford-Payne mechanism” - the launching of a self-collimated jet could be intrinsically implied by some of the most basic assumptions for the structure of a realistic energy-extracting black hole magnetosphere.

Closing Remarks

In this chapter we calculated 400 energy-extracting black hole magnetospheres with varying horizon field line angular velocity distributions given by $\Omega_F = (A + B \sin \theta)\omega_H$, corresponding to the first two terms of a series expansion of an arbitrary horizon field line angular velocity distribution.

We found that horizon field line angular velocity and the location of the inner
light surface are equally predictive of large scale magnetosphere structure. Groups of field lines with field line angular velocity $\Omega_F \lesssim 0.5\omega_H$ (inner light surfaces closer to the outer limits of the ergoregion) tend to bend towards the azimuthal axis. Groups of field lines with field line angular velocity $\Omega_F \gtrsim 0.5\omega_H$ (inner light surfaces closer to the horizon) tend to bend towards the equatorial plane. The strength of either bending increases the further away from $0.5\omega_H$ the field line angular velocity becomes (the closer the inner light surface gets to the horizon or outer limit of the ergoregion).

We also found that the horizon field line angular velocity distribution perhaps most compatible with conditions introduced by nearby accreting matter naturally correspond to magnetospheres that both connect the horizon to that matter and might easily launch a jet. This implies that near-horizon jet launching might be expected as a general feature of energy-extracting black hole magnetospheres. The structure of those magnetospheres further implies that temporal jet variability might be expected as a necessary and intrinsic feature of high luminosity jets. Finally, varying black hole spin from $a = 0.3m$ to $a = 0.95m$ coupled to variations in magnetosphere structure could easily lead to a factor of 1000 or more difference in the luminosity of black hole jets. Much of that variation would be due to changes in magnetosphere structure, both in terms of the degree of jet collimation and in terms of highly variable rates of energy and angular momentum deposition into small equatorial regions near the black hole’s innermost stable circular orbit.
In this chapter we discuss numerical techniques for solving for the structure of energy-extracting black hole magnetospheres in the case of an ideal plasma flow without taking the force-free limit. For simplicity only the limit of a cold flow will be considered, although in principle the numerical techniques we describe could be extended to include the more generic case of a hot flow. A single solution describing uniform accretion will be given, and its properties explored to illustrate some of the fundamental properties of energy-extracting black hole magnetospheres in the case of an ideal plasma flow. This chapter is based on original work that is being prepared for publication.

Background

In the previous three chapters, we solved for the structure of energy-extracting black hole magnetospheres in the limit of a force-free plasma. Taking that limit effectively removes all references to plasma parameters from the problem; the only role the plasma plays is in ensuring that there exists a frame in which the electric field vanishes. That condition may be encapsulated by restrictions on the invariant contractions of the field strength tensor, specifically that the electric and magnetic fields be perpendicular and that the field configuration be magnetically dominated. As such the problem can be cast in purely electromagnetic terms, with absolutely no reference to a plasma whatsoever.

It should generally be expected that practical black-hole energy extraction via an appropriately configured magnetosphere will largely be an electromagnetic process. A significantly negative-energy plasma would in general be extremely rapidly counter-rotating near the event horizon (from the perspective of zero angular momentum
observers), a configuration that might be difficult to form naturally from an inflow originating outside the ergosphere. As such, the purely electromagnetic force-free solutions arrived at in the previous chapters might be considered to correspond to zeroth-order approximations of energy-extracting black hole magnetospheres, while the introduction of plasma effects might be considered a first-order correction.

The simplest plasma approximation for such a first-order correction that might be made is that of an ideal plasma comprised of a perfect fluid in the limit of a cold flow. Under those conditions finite temperature effects encoded by a pressure and equation of state are discarded and effects such as viscosity and heat conduction are ignored. Within the limit of a stationary and axisymmetric configuration, the plasma is then effectively reduced to beads sliding down a wire traced by the magnetic field, and the primary interaction of the plasma with the fields might generally be taken to be an exchange of angular momentum (and therefore energy) as a consequence of the rigid rotation of magnetic field lines.

In this chapter we will assume such a cold flow, and develop numerical techniques capable of solving for the structure of energy-extracting black hole magnetospheres under that assumption. The assumption of a cold flow is advantageous in that, despite its simplicity, it might still generally be expected to yield a very good description of magnetically dominated magnetospheres that would be similar in structure to the completely force-free magnetospheres we examined previously. As already noted effective black hole energy extraction should generally be expected to be a predominantly electromagnetic process, so magnetically dominated magnetospheres are a natural initial assumption to make. The limit of a cold flow also directly encodes accretion rate, one of the most fundamental attributes of astrophysical phenomena that might be related to black hole energy extraction.

The primary deficiency of a cold flow is that it discards any dependencies
on temperature and an equation of state that might be relevant when treating a
given magnetosphere solution as a background for additional studies, such as the
propagation of magnetohydrodynamic waves and radiative and neutrino processes.
With that in mind, we have deliberately focused on developing numerical techniques
that might be amenable to extension to the more general case of a hot flow, and have
avoided applying techniques that might only be relevant to a cold flow. An extension
to a hot flow would nonetheless still be a non-trivial challenge and is beyond the
scope of our current work, but as appropriate we will comment on how it might be
accomplished.

Assumptions

The primary difference between this work and the work described in the previous
chapters is the relaxation of the force-free condition to the limit of an ideal (cold)
plasma flow. Therefore we will only provide reminders of the assumptions in use;
more detailed discussion of those assumptions and resultant consequences may be
found in Chapters 2, 3 and 4.

Core Assumptions

We first assume an isolated, uncharged, rotating black hole whose surrounding
spacetime is adequately described by Boyer-Lindquist coordinates. We then assume
that the black hole is immersed in a stationary and axisymmetric plasma (with axis of
symmetry corresponding to that of the black hole) that lacks sufficient energy density
to modify the surrounding spacetime. We then take that plasma to be described by
ideal magnetohydrodynamics and the limit of a cold flow under approximation of the
plasma as a perfect fluid.

Within those assumptions there will be a conserved field line angular velocity
Ω, particle flux η, energy flux ηE, and angular momentum flux ηL that together with a flux function for the poloidal field (the toroidal component of the electromagnetic vector potential Aφ) will completely describe a magnetosphere around a black hole of mass m and spin a. Descriptions and derivations of those parameters may be found in Chapter 3.

Cold Flow

The limit of a cold flow is equivalent to taking the limit of vanishing pressure \( p \to 0 \). The effect of that limit is to make the relativistic enthalpy \( \mu \) a field-aligned conserved quantity related to the average mass per particle of the plasma:

\[
\lim_{p \to 0} \mu = \lim_{p \to 0} \left( \frac{\rho + p}{n} \right) = m_p. \tag{9.1}
\]

The Bernoulli and transfield equations (Chapter 4 Appendices F and G) described in this work are ultimately always stated in terms of enthalpy \( \mu \), not pressure \( p \), so the limit of a cold flow does not significantly modify the structure of the equations, only the interpretation of the enthalpy \( \mu \) of the flow. The lone exception is that the sound four velocity and by extension slow magnetosonic wave speed vanish, such that the flow is always super-slow magnetosonic. That means that the assumption of a cold flow requires a consideration of the Alfvén and fast magnetosonic points (or surfaces), as opposed to the force-free case that only required consideration of an Alfvén surface (coinciding with the inner light surface - the fast magnetosonic surface coincides with the horizon in the force-free limit) and the case of a hot flow in which all three magnetosonic points need to be considered.
Boundary Conditions and Domain

The only significant boundary condition that we impose is that of a single monopolar field line tracing the equatorial plane. A single magnetic field line also traces the azimuthal axis, but that is primarily a consequence of stationarity and axisymmetry and not an additional restriction. The computational domain will extend from the horizon to the point at which the plasma inflow velocity (as measured by the Alfvén Mach number) begins to vanish and the plasma number density begins to diverge (by virtue of conserved particle flux $\eta$). That point cannot be known a priori, but must be numerically determined as a solution is found. Therefore the outer boundary is largely a quantity to be reported after a solution has been found, not a condition that may be initially imposed. The initial configuration of the electromagnetic fields will be taken to correspond to a known force-free solution already calculated in the previous chapters, and plasma parameters will be selected that might be compatible with treating plasma inertia effects as a small correction to the structure of those magnetospheres.

Solving the Transfield Equation

In the limit of a force-free plasma only the transfield equation needed to be solved. Outside of the force-free limit, both the transfield equation (in a more complicated form) and the Bernoulli equation need to be simultaneously solved. Therefore there are three basic elements to finding a self-consistent magnetosphere when plasma inertial effects are considered. First, the transfield equation must be solved. Second, the Bernoulli equation must be solved. Third, techniques must be developed to productively join a transfield solution to the restrictions imposed by the Bernoulli equation such that a completely self-consistent solution may be found. In
this section we discuss the first of those elements, finding a solution to the transfield equation outside of the force-free limit.

We are already in possession of useful numerical techniques for solving the force-free transfield equation via a relativistic extension of the magnetofrictional method, so we choose to solve the more general case by adapting those magnetofrictional techniques. In the force-free limit, the essential elements of the excess momentum flux $X^A$ that underpins the magnetofrictional method may be written as (these are similar but not equivalent to the magnetofrictional $C_{ij}$ coefficients outlined in Appendix H):

$$X^A_{FF} = C_{rr}A_{\phi,rr} + C_{\theta\theta}A_{\phi,\theta\theta} + C_rA_{\phi,r} + C_\theta A_{\phi,\theta} + C_L \frac{d}{dA_{\phi}} \left( \sqrt{-g} F^{\theta r} \right)^2 + C_x F(x). \quad (9.2)$$

There are various coefficients on the derivatives of $A_\phi$, a single term involving an unknown function of conserved angular momentum (with independent variable $A_{\phi}$), and an additional “constant” term (which is not actually constant, but may be crudely taken to be so over small spatial domains). The conceit of the magnetofrictional method is to transform the above momentum flux into the velocity of a fictitious plasma such that an advection equation for the vector potential is found. Evolving that equation using standard numerical techniques ultimately results in convergence to a solution of the transfield equation via $X^A_{FF} \rightarrow 0$.

In solving for the transfield equation outside of the force-free limit, we convert the additional terms introduced by plasma inertial effects into a form that results in an equation for excess momentum flux $X^A$ that is structurally equivalent to the force-free case. This may be thought of as effectively applying the force-free method to a modified spacetime (such that the coefficients on the derivatives of $A_\phi$ are modified) with some external forcing added to the trailing term. As such it should be expected
that the magnetofrictional method can remain useful outside of the force-free limit, and we have found that to be the case. The primary difficulty is in finding a useful translation of plasma inertia effects into such a modified force-free context.

The first terms in $X^A$ that must be translated are the $C_{rr}$ and $C_{\theta\theta}$ terms, which in the force-free limit are both proportional to $\alpha$. That led to a “kink” developing along the inner light surface where $\alpha$ changes sign; to prevent the equations from becoming anti-diffusive ($A_{\phi,t} \sim -A_{\phi,rr}$) and numerically unstable, an overall prefactor of $\pm 1$ must be added to the magnetofrictional velocity on either side of the inner light surface. That addition causes $A_\phi$ to evolve in opposite directions on either side of the inner light surface, resulting in a “kink” that must be minimized to find a valid solution. The inner light surface is the force-free limit of the ingoing Alfvén surface, so in the more general case we take $C_{rr}$ and $C_{\theta\theta}$ to be proportional to $\alpha - M^2$, such that a “kink” will still develop on either side of the Alfvén surface.

In the force-free limit we adjusted a function of the conserved angular momentum flux (in that limit $\eta L \sim \sqrt{-g} F^{\theta r}$) in order to minimize the kink on either side of the inner light surface. That function was taken to be the derivative of the square of the toroidal field, which had three significant built-in advantages. First, the prefactor for that function was purely a function of the spacetime metric, which meant it could effectively be considered to be constant. Second, the value of that function on both the azimuthal axis and equatorial plane was known (it vanishes there), which provided useful compatibility with our applied boundary conditions on the axis and the equatorial plane. As those boundaries were initially known, we could easily implement the unknown function of conserved angular momentum as being bounded from above and below after assuming a monotonically increasing/decreasing distribution of $A_\phi$ along the horizon (from demanding no closed horizon loops). Third, because the term involving that unknown function was a linear addition to the excess
momentum flux, it could easily be adjusted along the Alfvén surface in order to minimize any kinks that developed there.

In the more general case we desire similar behavior for our unknown function of $A_\phi$, so we manipulate the excess momentum flux $X^A$ such that a conserved function of $A_\phi$ emerges that corresponds to the square of the conserved angular momentum flux $\eta L$ with a “constant” (purely metric-dependent) coefficient. That causes the more general case to be qualitatively identical to the force-free case such that the techniques developed there may still be productively applied. There is one additional complication that must be considered, however. In the force-free limit only the derivative of the square of $\eta L$ is present when solving for $X^A$; in the general case there is also a derivative of $\eta L$, with non-constant (i.e. not purely metric) coefficients. Therefore as the unknown function for $(\eta L)^2_\psi$ is evolved in the more general case, the value of $(\eta L)_\psi$ must be simultaneously found and adjusted as appropriate. Ultimately knowledge of $\eta L$ is required to solve the Bernoulli equation, however, so that complication was already unavoidable.

The final term in $X^A$ that must be considered is the “external forcing” $C_x F(x)$ term; in the force-free limit this was a simple function of derivatives of $\Omega_F$ (and its nature as an “external forcing” term was the result of the selected mathematical representation, not a genuine physical forcing). In the general case the “external forcing” term becomes a complicated function of the metric, Alfvén Mach number $\mathcal{M}^2$, plasma four velocity, and field-aligned conserved quantities. Finding those quantities self-consistently requires solving the Bernoulli equation.

Once $\mathcal{M}^2(r, \theta)$, $u_Z(r, \theta)$ (i.e. the temporal and azimuthal components of the four velocity), $\Omega_F(A_\phi)$, $\eta(A_\phi)$, and $\eta E(A_\phi)$ are specified, then an external forcing term may be found. The resultant excess momentum flux $X^A$ can then be minimized by application of a modified magnetofrictional method, such that a function $A_\phi(r, \theta)$
and $\eta L(A_\phi)$ that satisfy the transfield equation may be found. We will outline the specific form we use in the next subsection.

Solving the transfield equation alone will not find a self-consistent solution, as if $A_\phi$ and $\eta L$ are modified then any plasma input values (e.g. $M^2$) that initially satisfied the Bernoulli equation will no longer do so. However if a useful method can be developed to simultaneously solve the Bernoulli equation such that the changes to $A_\phi$ and $\eta L$ gradually diminish, then eventually an acceptable solution will be found. We will discuss our method of solving the Bernoulli equation and the coupling of the Bernoulli equation to the transfield equation in the next sections. Before doing so, however, we will outline the specific techniques we use to solve the transfield equation.

Transfield Equation Specifics Finding a solution to the transfield equation via a modified magnetofrictional method reduces to evolving an advection equation for the vector potential $A_\phi$; $A_{\phi,t} = -v^A A_{\phi,A} = -\nu X^A A_{\phi,A}$. As with the force-free case, we weight the magnetofrictional coefficient $\nu$ by the magnitude of the local poloidal field, such that:

$$\nu = \nu_0 \frac{\Sigma \Delta \sin^2 \theta}{A_{\phi,\theta}^2 + \Delta A_{\phi,r}^2}.$$  \hspace{1cm} (9.3)

The appropriate value of $\nu_0$ is determined empirically; if it is too large instabilities will rapidly emerge, and if it is too small excessive amounts of computation time can be required (related to a Courant condition that is most simply found experimentally). The magnetofrictional velocities $v^A$ are then given by $v^r = \nu_0 CP \Delta A_{\phi,r}$ and $v^\theta = \nu_0 CP A_{\phi,\theta}$, where “CP” denotes a common prefactor that is given by:

$$CP = \frac{1}{A_{\phi,\theta}^2 + \Delta A_{\phi,r}^2} \left[ C_{rr} A_{\phi,rr} + C_{\phi,\theta A_{\phi,\theta} + C_r A_{\phi,r} + C_\theta A_{\phi,\theta} + C_4 \pi \eta L F(A_\phi) + \text{PFT} \right].$$  \hspace{1cm} (9.4)
Here the unknown function of \( F(A_\phi) \) is the derivative of \((\eta L)^2\) with respect to \( A_\phi \).

The leading coefficients are given by:

\[
4\pi C_{rr} = -\frac{\Delta}{\Sigma} \left( \alpha - M^2 \right), \quad 4\pi C_{\theta \theta} = -\frac{1}{\Sigma} \left( \alpha - M^2 \right),
\]
\[
4\pi C_r = -\Delta \left( \alpha - M^2 \right)_r + G_\phi \Delta \Omega_{F,r}, \quad 4\pi C_\theta = -\sin \theta \left( \frac{\alpha - M^2}{\sin \theta} \right)_\theta + G_\phi \Omega_{F,\theta}.
\]

The \( C_{4\pi\eta L} \) coefficient is given by \( C_{4\pi\eta L} = -1/8\pi \), and the plasma forcing term \( \text{PFT} \) is given by:

\[
\frac{1}{4\pi} \text{PFT} = -u_\phi \mu \eta (4\pi \eta L)_\Psi + \Delta \sin^2 \theta (\mu n) (\ln \mu \eta)_\Psi
\]
\[
+ \mu n (g_{\phi \phi} u_t - g_{\phi \Psi} u_\phi) \left[ e_{c,\Psi} + e_c (\ln \mu \eta)_\Psi - u_\phi \Omega_{F,\Psi} \right].
\]

Here a derivative of a conserved quantity with respect to \( A_\phi \) is denoted as a derivative with respect to \( \Psi \), and the conserved quantity \( e_c \) will be discussed in the next section on solving the Bernoulli equation. The intervals over which to update the plasma forcing term, the spatial distributions of the conserved quantities, and the angular momentum flux in order to diminish the kink are subject to empirical tuning. In order, we have typically found it useful to adjust the spatial distribution of \(((\eta L)^2)_\Psi\) the most frequently, then the spatial distribution of \(\Omega_F\), then the functional form of \(((\eta L)^2)_\Psi(A_\phi)\) to diminish the Alfvén surface kink, then finally adjusting the plasma forcing term the most infrequently. The relative frequencies of updating those quantities primarily affects the rate of convergence; very different ratios can be more or less appropriate in different contexts.
Solving the transfield equation as outlined above reduces to finding a vector potential $A_\phi (r, \theta)$ and a distribution of conserved angular momentum flux $\eta L$. There are three remaining conserved quantities that are necessary to completely specify a magnetosphere: the conserved field line angular velocity $\Omega_F$, conserved particle flux $\eta$, and conserved energy flux $\eta E$. Specifying those three parameters in a self-consistent way is equivalent to solving for an energy-extracting black hole magnetosphere. The number that are freely specifiable will be a function of how many must be adjusted in order to solve the Bernoulli equation; we will ultimately discover that typically only one requires adjustment.

The Bernoulli equation is typically taken to describe plasma acceleration along a single magnetic field line. Starting from some “injection point” (where the flow is assumed to be sub-Alfvénic), the plasma will accelerate through an Alfvén point, a fast magnetosonic point, and then pass through the horizon (that simple sequence is not strictly guaranteed from first principles, but for a roughly monopolar field line may be generally assumed). Finding conserved flow parameters that yield a smooth (continuous or unshocked) flow across the Alfvén and fast magnetosonic points is the entire difficulty of solving the Bernoulli equation; once “smooth” parameters are known, finding any relevant plasma parameters along the remainder of a field line becomes a relatively trivial and straightforward exercise.

Due to its interpretation as a field-aligned plasma acceleration equation, the Bernoulli equation is most naturally considered in the one-dimensional context of a single field line. Finding smooth plasma parameters within that context then becomes a “non-local” exercise that must match plasma flow parameters across critical points by simultaneously considering the entire length of a field line, with
no prior knowledge of where those critical points might occur along that field line. The magnetofrictional method for solving the transfield equation, on the other hand, does not utilize field lines but rather a continuous function $A_\phi(r, \theta)$ whose contours correspond to magnetic fields lines. Such a mismatch between the Bernoulli and transfield equations might in principle be addressed by tracing contours of $A_\phi$ provided by the magnetofrictional method followed by applying the Bernoulli equation to those contours. After implementing such a numerical scheme, however, we discovered that the often extreme sensitivity of the Bernoulli equation to variations in plasma parameters coupled to the error inherent in tracing contours, leading to unsatisfactory results when transitioning between transfield and Bernoulli equation contexts. The problems found were not primarily due to tracing $A_\phi$, but rather due to the fact that both $A_\phi$ and its derivatives along a contour are required by the Bernoulli equation, and that field-line perpendicular derivatives of any resultant plasma parameters are required by the transfield equation. Therefore we found it desirable to develop a method of solving the Bernoulli equation in a largely local manner that relies primarily on $A_\phi(r, \theta)$ rather than a collection of magnetic field lines. We will discuss that method later, after first exploring the behavior of the Bernoulli equation within the context of a single field line.

There are two critical points through with a smooth flow is desired, so at first glance it might appear that the Bernoulli equation would restrict two of our conserved parameters. This is not the case. Ultimately a smooth transition through the Alfvén point is largely guaranteed; if the flow parameters are such that the Alfvén Mach number $M^2$ approaches $\alpha$, then it will smoothly pass through an Alfvén point. In other words from the perspective of solving the Bernoulli equation the Alfvén point is more of a binary question (are the conserved parameters physical or not) than a question of fine tuning. The fast magnetosonic point, on the other hand, generally
requires very fine tuning; the difference between a perfectly tuned set of conserved
parameters and an obviously unacceptable set can often be measured in thousandths
of a percent. That means that in practice the Bernoulli equation generally only
restricts a single conserved parameter.

As already mentioned, there are four conserved parameters involved in the
problem; solving the transfield equation via the modified magnetofrictional method
restricts one, and solving the Bernoulli equation restricts another. The remaining
two parameters may then be freely specified. Following our previous results, we
choose to specify the field line angular velocity $\Omega_F$ and the accretion rate $\mu \eta$ (in
the limit of a cold flow $\mu$ is conserved). Solving the transfield equation specifies
conserved angular momentum flux $\eta L$, and the Bernoulli equation specifies the final
parameter via restrictions found at the fast magnetosonic surface. We find it most
useful to specify that conserved parameter as $e_c$, with $\mu \eta e_c = \eta E - \eta L \Omega_F$ such that
$e_c = u_r + u_\phi \Omega_F$. As with the accretion rate $\mu \eta$, $e_c$ is only conserved within the limit
of a cold flow, but due to the way $\mu \eta$ and $e_c$ combine within the equations involved
an extension to a hot flow could rely on the same parameters; determining $\mu$ in the
limit of a hot flow would be equivalent to determining the non-conserved scaling of
$\mu \eta$ and $e_c$.

In the following subsections we will address the points raised above in more
detail. We will begin by examining the selection of $e_c$ as a useful conserved parameter
to adjust in order to solve the Bernoulli equation. Next, we will explore the nature of
both the Alfvén and fast magnetosonic points, demonstrating that only adjustments
to $e_c$ are necessary to solve the Bernoulli equation. The method we have developed for
solving the Bernoulli equation in a primarily local (i.e. intrinsic to each grid square)
manner will then be discussed. Finally we will examine the behavior of the Bernoulli
equation along both the horizon and the azimuthal axis in greater detail, as they are
the most potentially problematic regions.

**Bernoulli Equation Conserved Parameter**

Solving the Bernoulli equation typically involves finding a single conserved parameter (as a function of $A_\phi$) that is compatible with a smooth fast-magnetosonic solution once a function $A_\phi(r, \theta)$ and three other conserved quantities have been specified. There is ambiguity, however, as to which conserved parameters should be specified and which should be solved for. We have found that for our purposes the most appropriate conserved parameter to solve for is given by $e_c$, where $\mu\eta e_c = \eta E - \eta L \Omega_F$ and $e_c = u_t + u_\phi \Omega_F$; we will now discuss why we have found that to be the case.

In Chapter 5 we found that at the plasma injection point (where $u^2_p = \mathcal{M}^2 = 0$) we must have:

$$\eta E - \Omega_F \eta L = (\sqrt{\alpha \mu \eta})_{|I}. \quad (9.7)$$

This in turn implies that at the injection point $e_c = \sqrt{\alpha}$. That further implies that specifying $e_c$ by solving the Bernoulli equation is roughly equivalent to specifying the initial plasma velocity at an outer boundary internal to the injection point that is required for a plasma flow of specified mass flux and angular momentum to smoothly accelerate through an Alfvén and fast magnetosonic point before passing through the horizon. Due to the fact that the angular momentum of the flow is an initially unknown output of the transfield equation, that initial velocity is a natural quantity to adjust for a successful inflow. If it weren’t able to be freely adjusted, then outward centripetal forces could potentially overwhelm inward gravitational forces and a valid flow solution might not exist for a given outer boundary. We will explore how $e_c$ relates to an injection point in more detail in later subsections.

It should be noted that $e_c$ is only conserved within the limit of a cold flow.
(the reason for its subscript) due to the fact that the relativistic enthalpy $\mu$ is only a conserved quantity within that limit. We do not see this as being especially problematic if the procedures we outline in this chapter were to be extended to include a hot flow. The extension to a hot flow could be made by maintaining the principle of $\eta\mu$ as a conserved parameter via the translation $\eta\mu \rightarrow \eta m_p$; the conserved entropy parameter $h_{\text{inj}}$ (Chapter 3 which would be determined by the slow magnetosonic point) would then be applied with an appropriate equation of state to find compatible $M^2$, $\mu$, and $e_c$ pairs everywhere in the domain such that the techniques we outline would remain valid. The primary additional difficulty (which might be substantial) would not be in treating $e_c$ as a quasi-conserved parameter, but rather the potential need to simultaneously solve the Bernoulli equation for both $e_c$ and $h_{\text{inj}}$.

### Bernoulli Equation Critical Points

In the limit of a cold flow, the Bernoulli equation may be expressed as a quartic in terms of the Alfvén Mach number $M^2$ (Appendix G) as:

$$C_4 (M^2)^4 + C_3 (M^2)^3 + C_2 (M^2)^2 + C_1 (M^2) + C_0 = 0. \quad (9.8)$$

In this subsection we will explore the nature of the four roots of that quartic as functions of $r$ along a field line tracing the equatorial plane for a black hole mass $m = 1$, black hole spin $a = 0.8m = 0.8$, field line angular velocity $\Omega_F = 0.5\omega_H = 0.125$, and (unless otherwise explicitly noted) a plasma accretion rate $\mu\eta = 10^{-4}$ (normalized to a black hole mass $m = 1$ and magnetic field strength described by a change in $A_\phi$ of 1 between the azimuthal axis and the equatorial plane). The relevant derivatives of $A_\phi$ will be taken from the force-free solution applicable to uniform field line angular velocity $\Omega_F = 0.5\omega_H$ and black hole spin $a = 0.8m$, which is to say the field line structure roughly corresponding to a rotating monopole. The value of $\eta L$ will be
expressed as a percentage of the value of the toroidal field in the equatorial plane found in that solution; $0.8\eta L$ should be understood as corresponding to $0.8(1/4\pi)\sqrt{-gF^{\theta r}_{FF}}$, i.e. 80% of the rate of the outward flux of angular momentum found in the force-free case.

Using that field line and conventions, we will in turn examine how the Alfvén point is ultimately not restrictive in most cases, the nature of smooth fast magnetosonic crossings, and the nature of the injection point as it relates to the separation surface. We in no way claim that the conclusions we draw in this subsection should be expected to hold in full generality (most, if not all, have counterexamples of varying degree of contrivance). Rather they are conclusions that we have found to be typically applicable across physically relevant magnetospheres that might more closely correspond to the force-free magnetospheres we have solved for in the preceding chapters - which is to say magnetically dominated energy-extracting magnetospheres with a plasma inflow injected between the light surfaces.

Alfvén Point Behavior Although a plasma inflow must accelerate through two critical points (the Alfvén and fast magnetosonic points), ultimately the Bernoulli equation only restricts one conserved parameter, not two. The reason is that a smooth solution that passes through the Alfvén point is in some sense guaranteed, as we will now explore. We will consider two cases with identical plasma parameters except one has $e_c = 0.58$ while the other has $e_c = 29.25$, as shown in Figure 9.1. In the case of $e_c = 0.58$, which has an injection point where $\mathcal{M}^2 = 0$ slightly interior to the separation point where $\alpha' = 0$ at $r = 3.73$, the plasma does not experience acceleration sufficient to become super Alfvénic; rather it returns to $\mathcal{M}^2 = 0$ at the inner light surface. In order for the plasma to become super-Alfvénic and smoothly pass through the fast magnetosonic point, a significantly larger value of $e_c = 29.25$
Figure 9.1: The effects of the Alfvén point along an equatorial field line for black hole spin $a = 0.8$ and field line angular velocity $\Omega_F = 0.5\omega_H = 0.125$ are shown. The real and imaginary (dashed) components of the four roots of $M^2$ are plotted, along with the value of $\alpha$. In the top panel, an unphysical parameter set (specifically, an unphysical value of $e_c$) is shown; the physical parameter set is shown in the bottom panel. If the value of $e_c$ is too small then the flow never becomes super Alfvénic, as in the top panel. Once the value of $e_c$ becomes large enough, however, a smooth Alfvén crossing from Root 1 to Root 4 is guaranteed; the value of $e_c$ then must be more finely tuned to allow for a smooth fast magnetosonic crossing (Figure 9.2).
Figure 9.2: The effects of the fast magnetosonic point along an equatorial field line for black hole spin $a = 0.8$ and field line angular velocity $\Omega_F = 0.5\omega_H = 0.125$ are shown. The real and imaginary (dashed) components of the four roots of $M^2$ are plotted, along with the value of $\alpha$. In the top panel, the value of $e_c$ is too large; after the transition from Root 1 to Root 4 at the Alfvén point, Root 4 never meets Root 3, the physical horizon root. In the bottom panel, the value of $e_c$ is too small; Root 4 meets Root 3, but after meeting they become complex conjugates. Tuning the value of $e_c$ for a smooth fast magnetosonic crossing is effectively equivalent to solving the Bernoulli equation.
Figure 9.3: The relationship between the separation point and the value of $e_c$ along an equatorial field line for black hole spin $a = 0.8$ and field line angular velocity $\Omega_F = 0.5\omega_H = 0.125$ is shown. A larger accretion rate ($\mu\eta = 10^{-3}$) than is used elsewhere is shown here in order to increase the clarity of the figures. The separation point (where $\alpha' = 0$) occurs at $r = 3.73$, where it has a value of $\sqrt{\alpha} = 0.5816$. When $e_c$ is larger than $\sqrt{\alpha}$ at the separation point, the plasma will not go to rest ($M^2 = 0$) until the outer light surface, as shown in the top panel. When $e_c$ is less than $\sqrt{\alpha}$ at the separation point the plasma will come to rest ($M^2 = 0$, where Root 1 and Root 2 meet) interior to the separation point, as shown in the bottom panel.
is required. Values of $e_c \gtrsim 18$ will pass smoothly through the Alfvén point, but they will not pass smoothly through the fast magnetosonic point (as we will discuss in the next subsection).

There is a minimum value of $e_c$ that will allow a smooth super-Alfvénic transition; once that value is exceeded, the Alfvén surface offers no additional restrictions. The minimum value of $e_c$ that will allow for a smooth super-Alfvénic transition is always less than the value of $e_c$ required for a smooth super-fast magnetosonic transition, so ultimately the Alfvén point places no restrictions on $e_c$.

The physical reason that a minimum value of $e_c$ is required for a smooth super-Alfvénic transition may be seen by noting that $e_c = u_t + u_\phi \Omega_F$, and that for our current purposes $u_t > 0$ and $u_\phi < 0$. As $e_c$ increases, therefore, the energy of the plasma as related to its poloidal motion begins to dominate that related to its azimuthal motion. Viewing the magnetic field as a rotating helical wire along which a “bead” of plasma travels, the minimum value of $e_c$ necessary to become super-Alfvénic may be thought of as the minimum initial “push” required to successfully slide down to the bottom of the wire.

As already stated above, the conclusions drawn here are not generic to all magnetosphere configurations, but we have found them to hold within the ranges of plasma values and magnetosphere structures we have studied, which is to say magnetically dominated magnetospheres that may be approximated to a reasonable degree by a force-free energy extracting magnetosphere.

**Fast Magnetosonic Point Behavior** Once a minimum value of $e_c$ that passes through the Alfvén point is known, it must be fine-tuned to allow for a smooth fast magnetosonic solution. Two cases with identical plasma parameters with the exception of the first having $e_c = 4.1$ and the second having $e_c = 3.9$ are shown in
Figure 9.2. Within the quartic root-taking conventions we discuss in Appendix G, Root 3 is the only possible physical horizon root; the other roots either vanish at the horizon or are negative there. At the Alfvén point Root 1 always transitions to Root 4 (assuming sufficiently large $e_c$), and within the appropriate domain Root 2 is always negative and irrelevant. Therefore for the flow to smoothly accelerate to the horizon in a physical fashion Root 4 must transition to Root 3 at the fast magnetosonic point.

Unlike the case of the Alfvén point, where all that is required is a minimum initial value of $e_c$, at the fast magnetosonic point there is only a single value of $e_c$ that allows for a physical transition from Root 4 to Root 3. Values of $e_c$ that are too large will result in both Root 3 and Root 4 being real-valued, but they will never meet. Values of $e_c$ that are too small will result Root 3 meeting Root 4, but they will become complex (conjugates) after meeting. Absent an appeal to a shocked flow, there is therefore only a single value of $e_c$ that will allow for a physical and smooth super-fast magnetosonic transition. Finding that value as a function of $A_\phi$ across the entire magnetosphere is what solving the Bernoulli equation largely entails. Due to the fact that smooth first derivatives of $M^2$ are required in order to solve the transfield equation, the range of acceptable values of $e_c$ can be very small; deviations of thousandths of a percent or less from the correct value can often be obviously unacceptable.

An additional problem in solving for $e_c$ at the fast magnetosonic point is that when rotational effects are not present the fast magnetosonic point no longer constrains $e_c$. This means that $e_c$ is completely unconstrained along the azimuthal axis, and only very weakly constrained near the axis. In regions where $e_c$ is weakly constrained, tiny variations in the parameters (such as from numerical noise) can become significant. We will address those issues in more detail later when we discuss the horizon and the azimuthal axis.
As a first approximation the correct value of $e_c$ will decrease as $\mu \eta$ increases and will increase as $\eta L$ increases (for an energy-extracting magnetosphere where $\eta L < 0$ this means $\eta L$ closer to 0). As with everything else we have mentioned those are less fundamental rules than observations relevant to the current context we are considering. Significantly increasing $\mu \eta$ will simultaneously modify both $\eta L$ and the magnetic field as solutions to the transfield equation, which can in turn have significant effects upon the correct value of $e_c$.

**Separation Surface Behavior** A fundamental feature of black hole magnetospheres is the separation point (or points, though that is much less common) along a given magnetic field line. Residing between the inner and outer light surfaces, a separation point is a point where inward gravitational forces and outward centripetal forces balance one another; a plasma released from rest at the separation point would remain at rest. Interior to the separation point the plasma accelerates inward, exterior to the separation point the plasma accelerates outward. The separation point is therefore often conflated with an “injection point” (or “injection region”) where a plasma inflow and outflow might be sourced.

The value of $e_c$ is intimately related to the separation point. As shown in Figure 9.3, if the value of $e_c$ is less than the value of $\sqrt{\alpha}$ at the separation point then the plasma flow will have $\mathcal{M}^2 = 0$ somewhere interior to the separation point (where Root 1 vanishes). If the value of $e_c$ is greater than $\sqrt{\alpha}$ at the separation point then Root 1 will remain valid until it vanishes at the outer light surface.

With respect to an outer boundary condition, this means that if the distribution of $e_c$ is appropriately larger than $\alpha$ everywhere then the outer light surface demarcates the boundary where plasma number density diverges. If $e_c$ is not appropriately large then the plasma number density will diverge somewhere interior to the separation
surface. That divergence cannot be part of the computational domain (it is the
unphysical result of demanding that particle flux $\eta$ be rigidly conserved as plasma
velocity vanishes), so as the distribution of $e_c$ is solved for it must be checked to make
sure that it remains compatible with whatever outer boundary has been set.

It is often the case that the correct value of $e_c$ will be appropriately larger
than $\alpha$ such that the physical inflow solution goes to rest at the outer light surface.
This means that a plasma could in principle be injected from regions exterior to the
separation surface so long as its injection velocity was sufficiently large. That is a
reason why in the previous chapter we extended the outer boundary of our domain to
the outer light surface; there is no fundamental requirement that an inflow be injected
at or interior to the separation point, it is merely a suggestive location to do so.

Solving the Bernoulli Equation Locally

In the previous subsections we considered the behavior of the Bernoulli equation
in terms of the roots of a quartic in $M^2$, solving those roots along a single equatorial
field line. In principle a collection of field lines could be traced via contours of $A_\phi(r, \theta)$
and the correct value of $e_c$ solved for along each of them. We created a numerical code
that did so, but ultimately discarded it as being undesirably limited. The ultimate
problem is that $A_\phi$, $A_{\phi, r}$, and $A_{\phi, \theta}$ are all required to solve the Bernoulli equation
along a field line, and adequately minimizing the error in tracing field line contours
by interpolating nearby grid squares can be problematic. Additionally, once the
Bernoulli equation has been solved along a given field line then $M^2$, $M_{r, \theta}^2$, and $M_{\phi}^2$
must all be determined as functions of the $(r, \theta)$ grid used to solve the transfield
equation, enhancing the difficulty of finding appropriate interpolation techniques
between $A_\phi(r, \theta)$ and a given contour of $A_\phi$.

We therefore developed a method of solving the Bernoulli equation that does
not require field lines to be traced. The Bernoulli equation (in integral form) can be
generically reduced to a function of $\mathcal{M}^2$ that vanishes for potentially valid values of $\mathcal{M}^2$:

$$SBZ = F(\mathcal{M}^2). \quad (9.9)$$

Here “SBZ” stands for “Should Be Zero”. If the SBZ function were the quartic
equation used in the previous subsection, then its (real) roots would correspond to zero
crossings of SBZ as a function of $\mathcal{M}^2$. There is no requirement that SBZ correspond
to the quartic formulation used above, however; there are infinitely many formulations
of such an SBZ function that might be more or less useful for different reasons and
in different contexts. One we have found to be particularly useful is given by the
following procedure:

$$\frac{1}{4\pi} \sqrt{-g} F^{\theta r} = -\frac{\mu \eta}{\mathcal{M}^2} \sqrt{\rho_\omega^2 (\epsilon_c^2 - \alpha) - \alpha \left( \frac{\mathcal{M}^2}{4\pi \mu \eta} \right)^2 \frac{\Delta A_{\phi, r}^2 + A_{\phi, \theta}^2}{\Sigma}},$$ $$u_\phi = \frac{1}{\mu \eta} \left( \frac{1}{4\pi} \sqrt{-g} F^{\theta r} - \eta L \right),$$ $$u_t = \epsilon_c - u_\phi \Omega_F,$$

$$SBZ = \rho_\omega^2 + g_{\phi \phi} u_t^2 - 2 g_{\theta \phi} u_t u_\phi + g_{\mu} u_\phi^2 + \left( \frac{\mathcal{M}^2}{4\pi \mu \eta} \right)^2 \frac{\Delta A_{\phi, r}^2 + A_{\phi, \theta}^2}{\Sigma}. \quad (9.10)$$

The evaluation of the plasma parameters should be understood to take place in a
logically sequential fashion from top to bottom and then inserted into the form of SBZ
shown. There are two primary reasons we have found the above form to be useful.
First, its evaluation specifically involves prior evaluation of the dependent plasma
variables that the transfield equation relies upon. Treating those variables as primary
considerations when solving the Bernoulli equation as opposed to secondary outputs
diminishes the error that secondary function evaluations can introduce. Second, the
function is both well behaved and usefully predictable across the range of plasma
parameters that we have found to be most typically relevant. In Figure 9.4 we diagram
that behavior.

Figure 9.4: The behavior of the “Should Be Zero” function described in the text at a
typical grid square. Analyzing that function at every grid square independently results
in an \((r, \theta)\) grid of minimum \(e_c\) values corresponding the parabola’s minima coinciding
with zero (yielding a single valid value of \(M^2\)). Tracing the peak of those minimum
values across the computational domain immediately outlines the fast magnetosonic
surface, and finding a smooth distribution of \(M^2(r, \theta)\) as a function of \(e_c(A_\phi)\) across
that outline adequately solves the Bernoulli equation.

At a typical grid point (and for reasonable plasma values) the SBZ function given
above behaves essentially as a simple parabola in \(M^2\) with a minima below zero and
two roots. Increasing the value of \(e_c\) at that grid point will cause the parabola to
move downward, while increasing the value of \(e_c\) at that grid point will cause the
parabola to move upward. The left (smaller) root corresponds to the correct solution for $M^2$ exterior to the fast magnetosonic point (Root 1 or Root 4 in the language of the previous subsection) while the correct solution for $M^2$ interior to the fast magnetosonic point corresponds to the right (larger) root (Root 3 in the language of the previous section). That means that if the parabola’s minima is above zero then no physical solution is possible at that grid square. Therefore every relevant grid square (which is to say every grid square that might correspond to a super-fast magnetosonic flow) can report the minimum value of $e_c$ that will allow for a physical flow at that point.

Once every relevant grid square has reported its minimum valid value of $e_c$, the peak of those minimum values may be immediately taken to trace the fast magnetosonic surface. This is due to the fact that a fast magnetosonic point corresponds to the point where the parabola has only one root, such that a smooth transition from the left root to the right root is possible. If $e_c$ is too large, then a smooth transition will be impossible (corresponding to the upper panel of Figure 9.2); if $e_c$ is too small, then there will be grid squares with no valid roots (corresponding to the bottom panel of Figure 9.2). Therefore identifying the peak in minimum values of $e_c$ over the domain is equivalent to identifying the grid squares that must be on either side of the fast magnetosonic surface.

Once the grid squares on either side of the fast magnetosonic surface have been identified, it is relatively straightforward to solve for a function $e_c(A_\phi)$ that minimizes “kinks” in $M^2(r, \theta)$, thereby finding a smooth fast magnetosonic solution. We have found that minimizing variations in the second derivative of $M^2$ using second order finite differencing approximations is a useful method of minimizing those kinks, but other methods exist.

The primary deficiency of the overall approach we have outlined is that we never
directly appeal to a single magnetic field line, and when such field lines are traced significant failures of a smooth fast magnetosonic transition might be observed. It is then that the often extreme sensitivity of a smooth solution to variations in $e_c$ becomes useful. It is because of that sensitivity that the distribution of $e_c(A_\phi)$ that might be more “correctly” arrived at via a field-aligned treatment of the Bernoulli equation cannot significantly differ from the treatment we have outlined (i.e. physical conclusions will not change). It is only if the Bernoulli equation were largely insensitive to the value of $e_c$ that significantly different physical conclusions might be drawn.

We have glossed over many complications in the above description. For example, the SBZ function we outlined above can have vastly different behavior near the “parabola” region, there are regions where SBZ becomes complex, the correct sign of the toroidal field must generally be deduced, and there exist parameter space regions where $e_c$ does not cause the parabola’s minima to monotonically decrease (the relevant values of $e_c$ typically reside on the right side of a downward parabola describing those minima, but regions near and to the left of the peak can be observed in some cases). Properly handling such complications increases the complexity of a practical implementation of the SBZ function as we have discussed it, but the fundamental techniques do not change. We have also found other representations of SBZ functions that can be more numerically useful in certain contexts (e.g. catastrophic cancellation can sometimes be a concern in implementing the above) but those representations often have significantly increased logical complexity without being fundamentally different in approach.

When a hot flow is considered, we expect that a treatment conceptually identical to the above may be productively applied to find a solution that also makes a smooth super-slow magnetosonic transition. Evaluating an appropriate “Should Be Zero”
function will be significantly more complicated, as \( \mu \) will become a function of \( \mathcal{M}^2 \), but the qualitative behavior near the fast magnetosonic point should be identical.

### The Event Horizon and Azimuthal Axis

Along the event horizon, the Alfvén Mach number is given by (Appendix G):

\[
\mathcal{M}_H^2 = -\frac{2mr_H}{\Sigma} \left[ 2mr_H \sin^2 \theta (\Omega_F - \omega_H)^2 - 4\pi \Sigma \sin \theta \left( \frac{\mu \eta e_c + \eta L (\Omega_F - \omega_H)}{F_{\theta\phi}} \right) \right].
\]

(9.11)

This has immediate value in that it may be solved for the minimum value of \( e_c \) that might possibly result in a horizon crossing solution. Applied to the example outlined above in Figure 9.1 \((m = 1, a = 0.8, \, A_\phi \approx \cos \theta, \, \omega_F = 0.125, \, \eta L = -0.002, \text{and} \, \mu \eta = 10^{-4} \text{ applied to } \theta = \pi/2)\) we immediately find a crude estimate of \( e_c \gtrsim 13 \) if there is to be a valid solution on the horizon \((\mathcal{M}_H^2 > 0)\). The value of \( e_c \) obtained by demanding a smooth Alfvén point crossing is not significantly different \((e_c \gtrsim 18)\), meaning that the above horizon condition can immediately yield a useful initial floor for \( e_c \) from which an analysis can begin. When exploring the roots of an SBZ function for a given value of \( e_c \) the horizon condition also immediately gives a rough scale of the magnitude of the potentially valid roots, making an initial analysis of an SBZ function much simpler.

The horizon condition also illustrates a potential problem, however. Along the azimuthal axis (where \( \eta L \to 0 \) and \( F_{\theta\phi} \sim -\sin \theta \)) we immediately find:

\[
\mathcal{M}_H^2 (\theta = 0) \approx 8\pi mr_H (\mu \eta) e_c.
\]

(9.12)

This is always positive and valid for any value of \( e_c \), implying that there might be no constraint upon \( e_c \) along the azimuthal axis. Taking a naïve \( \theta \to 0 \) limit of the SBZ function described above immediately (but incorrectly) results in \( \text{SBZ} \sim \mathcal{M}^4 \),
indicating that even if the axis is not unconstrained the formulation of SBZ that we outlined in the previous subsection might be poorly behaved there.

The solution to that dilemma is to recast SBZ in terms of parameters that do not vanish on the azimuthal axis. Specifically, we define:

\[
\frac{1}{4\pi} \sqrt{-gF^{\theta r}} \sin \theta = -\frac{\mu \eta}{\mathcal{M}^2} \sqrt{\Delta (e_c^2 - \alpha) - \alpha \left( \frac{\mathcal{M}^2}{4\pi \mu \eta} \right)^2 \frac{1}{\Sigma} \left[ \Delta \left( \frac{A_{\phi,r}}{\sin \theta} \right)^2 + \left( \frac{A_{\phi,\theta}}{\sin \theta} \right)^2 \right]},
\]

\[
\frac{u_\phi}{\sin \theta} = \frac{1}{\mu \eta} \left( \frac{1}{4\pi} \sqrt{-gF^{\theta r}} \sin \theta - \eta L \right),
\]

\[
u_t = e_c - u_\phi \Omega_F,
\]

\[
\text{SBZ} = \Delta + \frac{g_{\phi \phi}}{\sin^2 \theta} u_t^2 - 2 \frac{g_{t \phi}}{\sin \theta} u_\phi \frac{u_\phi}{\sin \theta} + g_{tt} \left( \frac{u_\phi}{\sin \theta} \right)^2 + \left( \frac{\mathcal{M}^2}{4\pi \mu \eta} \right)^2 \frac{1}{\Sigma} \left[ \Delta \left( \frac{A_{\phi,r}}{\sin \theta} \right)^2 + \left( \frac{A_{\phi,\theta}}{\sin \theta} \right)^2 \right].
\] (9.13)

In other words we have effectively divided the form of SBZ given in the previous subsection by $1/\sin^2 \theta$. This may be simplified to find:

\[
\text{SBZ} = \Delta + \left( \mathcal{M}^4 - \alpha^2 \right) \left( \frac{1}{4\pi \mu \eta} \right)^2 \frac{1}{\Sigma} \left[ \Delta \left( \frac{A_{\phi,r}}{\sin \theta} \right)^2 + \left( \frac{A_{\phi,\theta}}{\sin \theta} \right)^2 \right]
\]

\[
+ \frac{\alpha}{(\mu \eta)^2} \left( \frac{\eta L}{\sin \theta} \right)^2 + e_c^2 \frac{g_{\phi \phi}}{\sin^2 \theta} + 2 \frac{e_c}{\mu \eta} \frac{\eta L}{\sin \theta} \frac{G_{\phi}}{\sin \theta} + \frac{\Delta \alpha}{\mathcal{M}^4} \left( e_c^2 - \alpha \right)
\]

\[
- 2 \left( e_c \frac{G_{\phi}}{\sin \theta} + \frac{\alpha}{\mu \eta} \frac{\eta L}{\sin \theta} \right) \sqrt{\Delta (e_c^2 - \alpha) - \alpha \left( \frac{\mathcal{M}^2}{4\pi \mu \eta} \right)^2 \frac{1}{\Sigma} \left[ \Delta \left( \frac{A_{\phi,r}}{\sin \theta} \right)^2 + \left( \frac{A_{\phi,\theta}}{\sin \theta} \right)^2 \right]}.
\] (9.14)

In the limit $\theta \to 0$ we must have $\eta L/\sin \theta \to 0$ and $A_{\phi,r}/\sin \theta \to 0$, but $A_{\phi,\theta}/\sin \theta$ will remain finite. Taking SBZ = 0 and solving for $\mathcal{M}^4$, we then immediately find:

\[
\mathcal{M}^4|_{\theta=0} = (4\pi \mu \eta)^2 \left( \Sigma^2 e_c^2 - \Sigma \Delta \right) \left( \frac{\sin \theta}{A_{\phi,\theta}} \right)^2.
\] (9.15)
Therefore any value of $e_c$ will result in a valid flow, and the values of $e_c$ obtained near the axis should be expected to have more error than the values of $e_c$ obtained near the equatorial plane. Additionally, the term involving $e_c$ in the above expression corresponds to the $\theta \to 0$ limit of $e_c^2 - \alpha$. On the axis $\alpha$ goes from 0 on the horizon to 1 at spatial infinity, so if the value of $e_c$ on the axis is taken such that the distribution of $e_c(A_\phi)$ is smooth in $A_\phi$, then any value of $e_c$ on the axis less than 1 immediately indicates an effective injection point (where $M^2 \to 0$) at finite radii along the azimuthal axis.

Due to the fact that $e_c$ on the azimuthal axis is unconstrained, we simply impose a value of $e_c$ there that is compatible with the distribution of $e_c$ off the axis (such that $\partial_\phi e_c$ remains smooth near the axis). Near the axis we also typically apply the SBZ function in the form found in this subsection, as the form found in the previous subsection can be numerically problematic near the axis.

Equation Coupling

When solving the transfield equation, we multiply the magnetofrictional velocity on either side of the Alfvén surface by a factor of $\pm 1$ in order to maintain numerical stability. If we did not, then the equations would become anti-diffusive and unstable. In Equation [9.5] it can be seen that the coefficients on $M^2_{r}$ and $M^2_{\theta}$ will therefore necessarily change sign across the Alfvén surface. If $M^2$ were purely a function of $A_\phi$ that would be not be an issue, but $M^2$ is a function of both $A_{\phi,r}$ and $A_{\phi,\theta}$; taking a derivative of $M^2$ is effectively equivalent to introducing additional second derivatives of $A_\phi$. Those additional derivatives can then source anti-diffusive behavior, leading to numerical instability.

We therefore must find a way to couple the Bernoulli equation to the transfield equation that does not allow such numerical instabilities to develop. One possible
way of doing so is to apply smoothing to either $A_\phi$ or its derivatives before they are passed from the transfield equation context to the Bernoulli equation context, and/or applying the same smoothing to either $M^2$ or its derivatives before they are passed from the Bernoulli equation to the transfield equation. We chose not to implement such an approach, as accurately quantifying the effects of any such smoothing might be difficult, even if it was successful. Another possible way of preventing numerical instabilities would be to selectively apply derivative approximations that are less sensitive to amplifying numerical noise than the second-order finite differencing approximations that we typically apply. Savitsky-Golay filtering derivatives are one potential example (i.e. differentiating a local least squares polynomial fit). However, even though such derivative approximations can prevent anti-diffusive instabilities from being amplified for a longer period of time than finite-differencing derivatives can (such that a solution might be found before numerical error gets amplified to the point of being problematic), there is no guarantee that they will always prove satisfactory.

We ultimately chose to fit both $A_\phi$ and $M^2$ to an analytic function, then differentiate that function when transitioning between transfield and Bernoulli contexts. If the poloidal magnetic field structure were largely uncertain, such an approach might have limited value. However, because we are solving for the structure of magnetically-dominated magnetospheres with known force-free solutions, we may determine before calculations begin what functions might be best applied to approximate $A_\phi(r, \theta)$ and $M^2(r, \theta)$. For the purposes of this work we have applied the same description to both $A_\phi(r, \theta)$ (and similarly for $M^2$):

$$A_\phi(r, \theta) \approx R_0 + R_1 \cos \theta + R_2 \cos^2 \theta + R_3 \cos^3 \theta + R_4 \cos^4 \theta.$$  \hspace{1cm} (9.16)
The coefficients $R_i$ are first identified by fitting them (in a least-squares sense) across each $\theta$ row, under the condition that $R_0 = 3$ and that $R_1 + R_2 + R_3 + R_4 = 1$. The $R_i$ coefficients are then fit (again in a least-squares sense) to a polynomial in $r$. Once a function of $A_\phi(r, \theta)$ has been obtained in that manner, its derivatives are evaluated analytically and passed to the Bernoulli equation. The primary deficiencies of such a procedure are found in cases where the deviation of $A_\phi(r, \theta)$ evaluated by the transfield equation from the approximation are significant. Fortunately that significance may be directly measured by the output of the Bernoulli equation. The “true” value of $A_\phi(r, \theta)$ and approximate values of $\mathcal{M}^2$ produced by the Bernoulli equation may be used to evaluate the velocity normalization condition $u_\alpha u^\alpha = 1$; the poorer the approximation of $\mathcal{M}^2$ (as related to a poor approximation of $A_\phi(r, \theta)$ used to solve the Bernoulli equation), the further from unity the result will be.

The remaining issue is how often the Bernoulli equation should be evaluated as the transfield equation is solved. For this work we have settled on finding a solution to the transfield equation under a given set of of plasma parameters before calling the Bernoulli equation to update those parameters. Such a process is extremely computationally inefficient, but allows us to get a better sense of how the solutions converge, such that a more efficient but still robust frequency with which to update plasma parameters might be identified for use in future work.

Uniform Accretion Example

In developing the numerical techniques described in the previous sections, as a background we used a force-free magnetosphere with $\Omega_F = 0.5\omega_H$ applied to a black hole spacetime with spin $a = 0.8m$. Using an accretion rate of $\mu \eta = 10^{-4}$ (normalized to a black hole spin $m = 1$ and magnetic field strength $A_\phi \sim \cos \theta$), we found a solution with both the transfield equation and the four velocity normalization
Figure 9.5: The distributions of $e_c$ and $\eta L$ for the magnetosphere with $\mu \eta = 10^{-4}$ and $\Omega_F = 0.5 \omega_H$ described in the text. The specific values are evaluated on the horizon, but due to the monopolar structure of the poloidal magnetic field the distributions are representative of all radii. The values of $e_c$ (dashed line) are clearly larger than $\sqrt{\alpha}$ over the entire domain, so the plasma does not go to rest anywhere in the domain. The force-free distribution of $\eta L$ from the initial background shows less net outward angular momentum flux than in the plasma case due to the effective addition of an angular momentum inflow when adding an accreting plasma.

The conserved angular momentum flux did not substantially change from the force-free case (Figure 9.5), as was to be expected; the value of $\mu \eta$ was deliberately chosen such that the magnetosphere would be significantly magnetically dominated and approximately force-free. We also found that $e_c(A_\phi) \sim 2.5$ over the entire domain (Figure 9.5). In the force-free case $\eta E - \eta L \Omega_F = 0$, and along the equatorial plane we have $\eta L_{FF} \sim -0.0105$, such that $\eta L_{FF} \Omega_F \sim -0.00131$. In the plasma case along the equatorial plane we have $\eta L_{\text{Plasma}} \sim -0.00989$, such that $\eta L_{\text{Plasma}} \Omega_F \sim -0.00124$. The difference in those values is just below the $10^{-4}$ level, compatible with the
Figure 9.6: The location of the Alfvén surface is shown for the magnetosphere with $\mu \eta = 10^{-4}$ and $\Omega_F = 0.5 \omega_H$ described in the text. The addition of plasma has moved the Alfvén surface outward from the inner light surface. Although the majority of the Alfvén surface resides within the ergosphere, energy is only extracted when the Alfvén surface is within the “negative energy surface” described by the condition $g_{tt} + g_{t\phi} \Omega_F = 0$ (see the discussion in Chapter 5). In order to achieve a net extraction of energy closer to the azimuthal axis, either the accretion rate or the field line angular velocity would have to go down there. There is still a substantial outward Poynting flux near the axis, but the rest mass energy associated with the plasma inflow has overwhelmed it.

variation from zero in $\eta E - \eta L \Omega_F$ of $\mu \eta c \sim 2.5 \times 10^{-4}$.

As was expected, the outward angular momentum flux was very slightly decreased from the force-free case (by about 5%), and the outward conserved energy flux was similarly decreased, but far more significantly (roughly a 25% decrease). The reason for such a relatively substantial decrease is that the energy flux includes the inward flux of the plasma’s rest mass energy; the outward Poynting flux was effectively unchanged. The relationship of the Alfvén surface of the solution to the ergosphere is shown in Figure 9.6, although the majority of the Alfvén surface remains inside the ergoregion, energy is not always extracted. This is due to the fact that an Alfvén point inside the ergosphere is a necessary but not sufficient condition for black hole
energy extraction.

We hesitate to draw significant physical conclusions from the single result described above. While nothing about it is unreasonable and its error level appears to be more than adequate, without additional solutions covering different accretion rates (to include non-uniform accretion), force-free backgrounds over broader spatial domains (to especially include the effects introduced by approaching regions where $M^2 \to 0$ and the number density $n$ diverges), and different magnetic field line structures as related to different distributions of field line angular velocity, it is difficult to place that single result in context or have a sense of how robust or general it might be.

Solving for broader classes of magnetospheres that might provide fuller context is the focus of currently ongoing work. No solutions for stationary and axisymmetric energy-extracting black hole magnetospheres where plasma inertial effects are present were known prior to the development of the numerical techniques described by this chapter. There is therefore no prior work with which we might directly compare our results. As such we choose to defer a more detailed analysis beyond what was already discussed until more solutions have been found and compared with one another.

Error

There are two primary sources of error: the failure of the transfield equation to be exactly satisfied, and the failure of the velocity normalization condition to be exactly satisfied (which is to say a failure of the Bernoulli equation to be exactly satisfied). The failure of the transfield equation to be satisfied may be measured in a manner qualitatively identical to that already applied in the force-free case. By dividing the excess momentum flux $X^A$ into different terms, any failure of $X^A$ to vanish may be weighted to the largest term in its sum for an estimation of a percent
error. That is the approach we have currently taken; specifically, we have considered $X^A$ in the form:

$$\frac{-4\pi\sqrt{-g}\rho_0}{B_0^2} X_A = D_{rr} A_{\phi,r} + D_r A_{\phi,r} + D_{\theta\theta} A_{\phi,\theta} + D_\theta A_{\phi,\theta} + D_{\Omega_F} \Omega_{F,\psi} + D_{4\pi\eta L^2} (4\pi\eta L)^2_{,\psi} + D_{4\pi\eta L} (4\pi\eta L)_{,\psi} + D_{\mu\eta c} (\mu\eta c)_{,\psi} + D_{\Omega_{F,u}} \Omega_{F,\psi} + D_\eta (\ln \eta)_{,\psi},$$

where the coefficients are given by:

$$D_{rr} = \Delta \sin \theta (\alpha - M^2), \quad D_{\Omega_F} = -\sin \theta (g_{t\phi} + g_{\phi\phi} \Omega_F) (\Delta A_{\phi,r}^2 + A_{\phi,\theta}^2),$$

$$D_r = \Delta \sin \theta (\alpha - M^2)_{,r}, \quad D_{4\pi\eta L} = 4\pi \Sigma \mu \eta u_{t,\phi} \sin \theta,$$

$$D_{\theta\theta} = \sin \theta (\alpha - M^2), \quad D_{\mu\eta c} = 4\pi \Sigma \left(4\pi \mu \eta M^2\right) \sin \theta (g_{t\phi} u_{t,\phi} - g_{\phi\phi} u_t),$$

$$D_\theta = \sin^2 \theta \left(\frac{\alpha - M^2}{\sin \theta}\right), \quad D_{\Omega_{F,u}} = -4\pi \Sigma \left(4\pi \mu^2 \eta^2 M^2\right) \sin \theta (g_{t\phi} u_{t,\phi} - g_{\phi\phi} u_t), u_{\phi},$$

$$D_{4\pi\eta L^2} = \frac{1}{2} \Sigma \sin \theta, \quad D_\eta = -4\pi \Sigma \Delta \sin^3 \theta \left(4\pi \mu^2 \eta^2 M^2\right).$$

In the force-free case, we observed the convergence of hundreds of solutions as related to different error metrics, and were able to confidently state that a 1% error level as evaluated with the force-free expressions shown in Appendix H was a useful one. We are unable to state with the same level of confidence that a 1% error level obtained from the above description is useful, as we have not applied it to a significant number of magnetospheres. Rather we can only state can state that we have not yet observed any failures; a force-free background with a small accretion rate $\mu \eta = 10^{-5}$ applied will typically not satisfy the above to the 1% level and a solution solved to that level appears to have strongly converged. More experimentation might reveal a more useful form or inadequacies of the above description, so while we have found it to be a good
starting point it might prove advantageous to modify it in the future.

In order to measure the failure of the Bernoulli equation to be satisfied, we measure the failure of the plasma four velocity \( u_\alpha u^\alpha \) to be properly normalized to 1. There are infinitely many possibilities for such a normalization to be calculated from the plasma parameters. We have chosen one of the representations shown in Chapter 4:

\[
1 = -\frac{1}{\rho^2} (g_{\phi\phi} u_t^2 - 2g_{\phi\theta} u_t u_\phi + g_{tt} u_\phi^2) - \frac{1}{16\pi^2 (\mu\eta)^2} \mathcal{M}^4 \frac{\Delta A^2_{\phi,\phi} + A^2_{\phi,\theta}}{\Sigma \Delta \sin^2 \theta}.
\]

(9.19)

The reason for selecting this representation is that it directly relies on the plasma parameters used to solve the transfield equation. We have found that less than 1% deviation from the true value appears to indicate a generally very good solution to the Bernoulli equation. As already discussed above we have not applied this error metric to a wide enough range of magnetospheres and situations to be able to state that this is a broadly useful metric, but it has sufficed for the situations we have considered.

Conclusions

We described a system for numerically solving for the structure of energy extracting black hole magnetospheres, to include plasma inertia effects. Those techniques have been applied and shown to work in a test case of a strongly magnetically dominated flow where plasma inertial effects do not significantly modify the structure of the electromagnetic fields from the force-free case. This is the first time that such a solution has been reported; previously, only force-free solutions were known. As such we hesitate to draw significant physical conclusions without first solving for more magnetospheres that might together better inform any physical
conclusions and possibly expose limitations of the numerical techniques we have described in this chapter. However, merely demonstrating that it is possible in principle to reasonably calculate energy extracting black hole magnetospheres outside of the force-free limit is an important first step in obtaining better understanding of the role of an ideal plasma inflow in energy-extracting black hole magnetospheres.
Prior to this work there existed very few solutions for the structure of energy-extracting black hole magnetospheres. Both the paucity and nature of those solutions prevented explorations of correlations between magnetosphere structure and the fundamental parameters that describe an energy-extracting black hole magnetosphere.

In this work we generalized the single most widely used analytic solution in the force-free limit to a more general context, developed numerical techniques that allowed us to solve for thousands of energy-extracting force-free magnetospheres, and then extended those numerical techniques to the more general case of an ideal plasma flow, for the first time finding an energy-extracting black hole magnetosphere solution with self-consistent plasma parameters. The most fundamental attribute of energy-extracting black hole magnetospheres that we discovered was that slowly rotating magnetospheres naturally form jet-like structures aligned with the azimuthal axis and that more rapidly rotating magnetospheres are naturally compatible with connections to nearby accreting matter. When both tendencies are present, Jet-Disk magnetosphere structures naturally emerge that might directly correspond to physical conditions relevant to astrophysical phenomena. Analysis of those Jet-Disk magnetosphere structures suggests that variations in jet luminosity in excess of a factor of 1000 might naturally emerge within a description of such phenomena, with the more highly luminous cases becoming more intrinsically variable than the lower luminosity cases.

There exist three broad avenues for future extensions of the work we have described, not necessarily exclusive. First, in the force-free limit, more realistic boundary conditions (i.e. more compatible with nearby accreting matter) might
be applied in the equatorial plane. It is now known which field lines connected to that matter will most naturally connect to the black hole and which will not, so magnetospheres that might more directly apply to astrophysical scenarios can be solved for without appealing to potentially artificial forcings on the magnetic fields. Outflow solutions (in the sense of passing through the outer light surface) might also be added by anchoring “outgoing” field lines to a matter structure, allowing for explorations of correlations between an inflow and an outflow. If “outgoing” field lines do not connect to a specific matter structure, but instead directly connect to “ingoing” field lines, then a study could also be conducted of the distribution of the toroidal field necessary for an outgoing solution to smoothly pass through the outer light surface. Comparing the distribution of that toroidal field with the toroidal field of the inflow would allow for measurement of the necessary failure of field-aligned parameters to be conserved in order for a solution to extend from a near horizon inflow to a distant outflow.

Second, additional magnetospheres in the presence of plasma inertia effects might be solved for. Explorations of the effects of accretion rate, to include variations in that rate appropriate to specific astrophysical contexts (such as extending a Jet-Disk magnetosphere to include plasma accretion primarily along the “Disk” field lines) could be explored as a function of both black hole spin and field line angular velocity. The numerical techniques developed for a cold flow could also be extended to a hot flow, allowing for an exploration of the effects of equation of state on black hole energy extraction. Plasma inertia effects might also be applied to a more complete force-free solution containing outgoing field lines, allowing for a more direct discussion of jet launching as coupled to black hole energy extraction.

Third, the magnetosphere solutions obtained could be used as backgrounds in a study of the propagation of magnetohydrodynamic waves within energy-extracting
black hole magnetospheres. It is already known that a flow must become super-Alfvénic within the ergosphere in order for a black hole’s energy to be extracted, a fact that intimately couples the study of magnetohydrodynamic waves with black hole energy extraction. Studying the propagation of different perturbations through different magnetospheres might therefore allow for deeper fundamental insights into the nature of black hole energy extraction via an appropriately configured magnetosphere. The backgrounds necessary for such a study might also be applied to other studies that might similarly find a library of stationary backgrounds useful, such as a study of neutrino production and propagation or a study of the emission, absorption, and propagation of radiation within energy-extracting black hole magnetospheres.
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APPENDIX A

PENROSE PROCESS ANGULAR VELOCITIES
The Penrose process [83, 84] typically refers to the generation of negative energy particles within the ergosphere. In this appendix we explore the nature of such particles, demonstrating that negative energy particles typically travel at a significant fraction of the speed of light and that they cannot escape the ergosphere. For simplicity we will restrict ourselves to particles moving purely in the azimuthal direction in the equatorial plane; the general case is much more complicated without significant additional explicative utility. The specific energy of a particle $E_p$ is defined as the contraction of that particle’s four velocity $u^\alpha$ with the spacetime’s temporal Killing vector $k^\alpha$:

$$E_p \equiv k_\alpha u^\alpha = u^t (g_{tt} + g_{t\phi} \Omega) . \quad (A.1)$$

Here we have defined the particle’s four velocity as $u^\alpha = (u^t, 0, 0, u^t \Omega)$ such that $\Omega$ is the particle’s angular velocity. The angular velocity that results in vanishing energy is therefore given by $\Omega = -g_{tt}/g_{t\phi}$; in the equatorial plane this reduces to $\Omega = (2m - r)/2ma$. The speed of light is reached when the particle’s four velocity becomes null, $u_\alpha u^\alpha = 0$, or:

$$(u^t)^2 (g_{tt} + 2g_{t\phi} \Omega + g_{\phi\phi} \Omega^2) = 0. \quad (A.2)$$

This is satisfied when $\Omega = (1/g_{\phi\phi})(-g_{t\phi} \pm \sqrt{\rho^2})$, which in the equatorial plane reduces to:

$$\Omega_{\text{Light}} = \frac{1}{r^3 + a^2 r + 2ma^2} \left( 2ma \pm \sqrt{r^4 + a^2 r^2 - 2mr^3} \right) . \quad (A.3)$$

In Figure A.1 we plot the angular velocities corresponding to co-rotating and counter-rotating light, zero angular momentum observers, and particles with vanishing specific energy. The outer limits of the ergosphere ($g_{tt} = 0$ or $r = 2m$) are the point at which particles can have vanishing energy, although to do so they must rotate at
the speed of light. Outside the ergosphere a particle with vanishing energy would be moving faster than the speed of light; negative energy particles can only exist inside the ergosphere. Within the ergosphere it is possible for particles to have negative energy if they have an angular velocity within the shaded region. It is immediately apparent that for a particle to have significant negative energy it must rotate at a substantial fraction of the speed of light (referenced to zero angular momentum observers). If a particle collision is to result in both a negative energy particle and a positive energy particle that escapes the ergosphere to distant regions, the positive energy particle must generally also be endowed with a significant outward radial velocity post-collision. From conservation of momentum, the negative energy particle must have a correspondingly significant inward radial velocity, further boosting its speed towards the speed of light. The additional “penalty” of that inward radial velocity increases as the particle collision takes place closer to the horizon, which works to diminish the advantage of the observation that negative energy particles can rotate more slowly (as a fraction of the speed of light, according to zero angular momentum observers) closer to the horizon.

![Angular velocities in the equatorial plane for a black hole spin parameter $a = 0.9m$. The shaded region denotes the angular velocities of negative energy particles, which can only exist inside the ergosphere ($r \leq 2m$).](image_url)
The ultimate conclusion is that a particle collision that results in a negative energy particle and a positive energy particle that escapes to distant regions generally requires that the negative energy particle be moving at a substantial fraction of the speed of light after the collision (according to zero angular momentum observers). That in turn suggests that any such collisions are typically likely to be either highly contrived or very rare, and that such collisional processes are unlikely to exist naturally in astrophysical contexts as efficient or significant extractors of a black hole’s rotational energy.
APPENDIX B

MAGNETIC FIELD FOUR VECTOR
In this appendix we discuss the magnetic field expressed as a four vector. The poloidal magnetic field strength $B_p^2$ is often defined as:

$$B_p^2 \equiv -\frac{B^A B_A}{(g_{tt} + g_{t\phi} \Omega_F)^2},$$  \hspace{1cm} (B.1)

where:

$$B^\alpha = \frac{1}{2\sqrt{-g}} \epsilon^{\alpha\beta\gamma\delta} k_\beta F_{\gamma\delta},$$

$$B_\alpha = -\frac{\sqrt{-g}}{2} \epsilon_{\alpha\beta\gamma\delta} k^\beta F^{\gamma\delta}. \hspace{1cm} (B.2)$$

Here $\epsilon^{\alpha\beta\gamma\delta}$ corresponds to the fully antisymmetric symbol; we have explicitly written the tensor density components associated with Levi-Civita symbols for clarity. The contraction $B_A B^A$ then yields:

$$B_A B^A = - (g_{tt} + g_{t\phi} \Omega_F) \left( F_{r\phi} F^{r\phi} + F_{\theta\phi} F^{\theta\phi} \right) = \left[ \frac{n}{\eta} (g_{tt} + g_{t\phi} \Omega_F) \right]^2 \left( u_r u_r + u_\theta u_\theta \right). \hspace{1cm} (B.3)$$

If the poloidal plasma velocity is defined as $u_p^2 \equiv -u_A u^A$, then the particle flux per unit flux tube reduces to the simple expression $\eta = -n u_p / B_p$. This expression has significant utility in simplifying (and interpreting) various expressions that emerge when studying black hole energy extraction under the assumptions made in this work. Unfortunately its simplicity comes at a cost; the vector $B^\alpha$ is not divergenceless. In component form, $B^\alpha$ may be written as:

$$B^\alpha = \frac{1}{\sqrt{-g}} \left( -g_{t\phi} F_{\theta r} - g_{tt} F_{\theta\phi} - g_{t\phi} F_{r\phi} + g_{tt} F_{rr} + g_{t\phi} F_{tr} \right). \hspace{1cm} (B.4)$$

By inspection $B^\alpha_{;\alpha} = (1/\sqrt{-g})(\sqrt{-g}B^\alpha),_{\alpha} \neq 0$, even in the limit of stationarity and axisymmetry. This limits the usefulness of $B^\alpha$ as a magnetic field four vector outside
of the definition of $B^2$. The issue with $B^\alpha$ ultimately comes from the temporal Killing vector, which picks up metric components when its indices are lowered. Fixing that problem leads to the definition of the magnetic field four vector used in this work:

$$B^\alpha = \frac{1}{2\sqrt{-g}}\epsilon^{\alpha\beta\gamma\delta}t_\beta F_{\gamma\delta}. \quad (B.5)$$

Under this definition the components of $B^\alpha$ are given by:

$$B^\alpha = \left(0, -\frac{1}{\sqrt{-g}}F_{\theta\phi}, \frac{1}{\sqrt{-g}}F_{r\phi}, \frac{1}{\sqrt{-g}}F_{\theta r}\right) = \left(0, -\frac{1}{\sqrt{-g}}A_{\phi,\theta}, \frac{1}{\sqrt{-g}}A_{\phi,r}, \frac{1}{\rho_\omega^2}\sqrt{-g}F_{\theta r}\right). \quad (B.6)$$

It is now obvious that $B^{\alpha;\alpha} = 0$ (at least in the limit of axisymmetry). In the non-asymmetric limit, the divergence of $B^\alpha$ reduces to:

$$B^{\alpha;\alpha} = \frac{1}{\sqrt{-g}}\left(-F_{\theta\phi,r} + F_{r\phi,\theta} + F_{\theta r,\phi}\right). \quad (B.7)$$

This necessarily vanishes even in the non axisymmetric limit by application of Maxwell’s equations (Equation [C.22]), which demand that $F_{\phi\theta,r} + F_{r\phi,\theta} + F_{\theta r,\phi} = 0$. Therefore $B^\alpha$ can be a much more generically useful definition of a magnetic field four vector than $B^\alpha$. Nonetheless, far from the black hole both definitions identically yield $B^r \sim B^\phi$, $B^\theta \sim B^\phi/r$, and $B^\phi \sim B^\phi/r\sin\theta$ (Equation [C.18]), offering equivalent correspondences to the standard Newtonian orthonormal spherical coordinate representation of the magnetic field.

It should be clearly noted that any magnetic field four vector is a mathematical convenience, likely to be primarily useful or applicable in a specific coordinate system or context. A robust description of the magnetic fields necessarily comes from the field strength tensor, not any such four vector. In this work we primarily use the magnetic field four vector as a useful way of expressing the stress energy tensor;
as a divergenceless vector, such a description greatly simplifies the task of taking
divergences of the stress energy tensor. The name “magnetic field four vector” is
taken from its similarity to Newtonian fields far from the black hole, and should not
to be taken to necessarily imply a direct correspondence to any magnetic fields or
that the vector usefully transforms to other coordinate systems or contexts.
APPENDIX C

THE FIELD STRENGTH TENSOR AND MAXWELL'S EQUATIONS
In fully relativistic contexts, the electromagnetic field strength tensor $F^{\alpha\beta}$ can be the most appropriate way to specify and define electromagnetic fields. The components of $F^{\alpha\beta}$ (which is to say the electromagnetic fields) will vary from observer to observer. However, the “metric” perspective (which is to say the perspective of observers far away from the black hole) is widely used to define the electromagnetic fields; the vector potential $A_{\phi}$ and toroidal magnetic field $\sqrt{-g}F^{\theta r}$ used in this work are referenced to that perspective.

It can therefore be useful to solve for the components of the field strength tensor in flat space using spherical coordinates as functions of Newtonian electric and magnetic fields. Although any correspondences will not be exact, comparing those field strength tensor components to the components found in fully relativistic contexts can allow for physical intuition by facilitating comparisons to more well-known Newtonian contexts.

In this appendix we therefore provide a brief review of the field strength tensor, its explicit form in spherical coordinates, and its relation to Maxwell’s equations. Throughout we assume a $(+,−,−,−)$ metric signature. The field strength tensor $F_{\alpha\beta}$ may be defined in terms of the vector potential $A_{\alpha}$ as $F_{\alpha\beta} \equiv \partial_{\alpha}A_{\beta} - \partial_{\beta}A_{\alpha}$; the partial derivatives here imply a coordinate basis [63]. If we then write $A_{\alpha} = (\Phi, -\vec{A})$ and $\partial_{\alpha} = (\partial_t, \vec{\nabla})$ with the conventions for the electric and magnetic fields $\vec{E} = -\partial_t\vec{A} - \vec{\nabla}\Phi$ and $\vec{B} = \vec{\nabla} \times \vec{A}$, along with the convention $(t, x, y, z) \leftrightarrow (0, 1, 2, 3)$, we find that the field strength tensor in Cartesian coordinates may be written as:

$$F_{\alpha\beta} = \begin{pmatrix} F_{00} & F_{01} & F_{02} & F_{03} \\ F_{10} & F_{11} & F_{12} & F_{13} \\ F_{20} & F_{21} & F_{22} & F_{23} \\ F_{30} & F_{31} & F_{32} & F_{33} \end{pmatrix}$$
\[
\begin{pmatrix}
0 & \partial_t A_x - \partial_x A_t & \partial_t A_y - \partial_y A_t & \partial_t A_z - \partial_z A_t \\
-\partial_t A_x + \partial_x A_t & 0 & \partial_x A_y - \partial_y A_x & \partial_x A_z - \partial_z A_x \\
-\partial_t A_y + \partial_y A_t & -\partial_x A_y + \partial_y A_x & 0 & \partial_y A_z - \partial_z A_y \\
-\partial_t A_z + \partial_z A_t & -\partial_x A_z + \partial_z A_x & -\partial_y A_z + \partial_z A_y & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & E^x & E^y & E^z \\
-E^x & 0 & -B^z & B^y \\
-E^y & B^z & 0 & -B^x \\
-E^z & -B^y & B^x & 0 \\
\end{pmatrix}
\]

The raised indices are important here; \( \vec{E} \equiv (E^x, E^y, E^z) \). Lowering those indices would result in a minus sign due to our choice of metric. For example the \( F_{tx} = \partial_t A_x - \partial_x A_t \) term results in \( E^x \) by virtue of \( E^i \equiv -\partial^i A^i + \partial^i A^i \); the \( F_{xy} = \partial_x A_y - \partial_y A_x \) term results in \( -B^z \) by virtue of \( B^k \equiv \epsilon^{ijk} \partial_i A_j \) and \( B^k \equiv \epsilon^{ijk} \partial^j A^i \). The above representation of \( F_{\alpha\beta} \) is equivalent to that given in standard texts [37], though often lowered indices are used to denote vector components. With some care as to definitions the same procedure as above could be made to find the components of the field strength tensor in spherical coordinates, but we find it more instructive to simply apply a coordinate transformation. The transformation between Cartesian coordinates and standard spherical coordinates is given by:

\[
x = r \sin \theta \cos \phi, \quad r = \sqrt{x^2 + y^2 + z^2}, \quad (C.2)
\]

\[
y = r \sin \theta \sin \phi, \quad \theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z}, \quad (C.3)
\]

\[
z = r \cos \theta, \quad \phi = \tan^{-1} \frac{y}{x}. \quad (C.4)
\]

In order to transform from a Cartesian basis to a spherical basis, we apply the transformation \( \Lambda_\beta^\alpha = \frac{\partial x_\beta}{\partial x^\alpha} \) to the basis vectors such that \( \vec{e}_\alpha' = \Lambda_\beta^\alpha \vec{e}_\beta \). The relevant
transformation elements from Cartesian to spherical coordinates may be written as:

\[ \Lambda^\mu_t = (1, 0, 0, 0), \quad (C.5) \]

\[ \Lambda^\mu_x = (0, \sin \theta \cos \phi, \frac{1}{r} \cos \phi \cos \theta, -\frac{1}{r} \sin \phi), \quad (C.6) \]

\[ \Lambda^\mu_y = (0, \sin \theta \sin \phi, \frac{1}{r} \sin \phi \cos \theta, \frac{1}{r} \cos \phi), \quad (C.7) \]

\[ \Lambda^\mu_z = (0, \cos \theta, -\frac{1}{r} \sin \theta, 0). \quad (C.8) \]

Applied to the flat space metric tensor \( \eta^{\alpha\beta} \), for example, we find the standard spherical metric (in a coordinate basis) to be:

\[
\eta^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta \eta^{\alpha\beta} = \Lambda^\mu_t \Lambda^\nu_t - \Lambda^\mu_x \Lambda^\nu_x - \Lambda^\mu_y \Lambda^\nu_y - \Lambda^\mu_z \Lambda^\nu_z
\]

\[
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -\frac{1}{r^2} & 0 \\
0 & 0 & 0 & -\frac{1}{r^2 \sin^2 \theta}
\end{pmatrix}.
\quad (C.9)
\]

This corresponds to the expected flat space line element in spherical coordinates, \( ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \). We may apply the same transformation elements to the six independent elements of the field strength tensor to find:

\[
F^{\mu t} = E^x \Lambda^t_x \Lambda^t_t + E^y \Lambda^t_y \Lambda^t_t + E^z \Lambda^t_z \Lambda^t_t
\]

\[
= E^x \sin \theta \cos \phi + E^y \sin \theta \sin \phi + E^z \cos \theta, \quad (C.10)
\]

\[
F^{\phi t} = E^x \Lambda^t_x \Lambda^t_\phi + E^y \Lambda^t_y \Lambda^t_\phi + E^z \Lambda^t_z \Lambda^t_\phi
\]

\[
= \frac{E^x}{r} \cos \phi \cos \theta + \frac{E^y}{r} \sin \phi \cos \theta - \frac{E^z}{r} \sin \theta, \quad (C.11)
\]

\[
F^{\phi \phi} = E^x \Lambda^\phi_x \Lambda^\phi_t + E^y \Lambda^\phi_y \Lambda^\phi_t + E^z \Lambda^\phi_z \Lambda^\phi_t
\]
\[
F^\phi_\theta = B_x \left( \Lambda^r_x \Lambda^y_y - \Lambda^r_y \Lambda^y_z \right) + B_y \left( \Lambda^r_x \Lambda^y_z - \Lambda^r_z \Lambda^y_x \right) + B_z \left( \Lambda^r_y \Lambda^x_x - \Lambda^r_x \Lambda^y_y \right) \\
= B_x \frac{\cos \phi}{r^2} + B_y \frac{\sin \phi}{r^2} + B_z \frac{\cot \theta}{r^2}, \quad \text{(C.13)}
\]

\[
F^{r\phi} = B_x \left( \Lambda^r_y \Lambda^\phi_y - \Lambda^r_y \Lambda^\phi_z \right) + B_y \left( \Lambda^r_x \Lambda^\phi_z - \Lambda^r_z \Lambda^\phi_x \right) + B_z \left( \Lambda^r_y \Lambda^\phi_x - \Lambda^r_x \Lambda^\phi_y \right) \\
= B_x \cot \theta \cos \phi + B_y \cot \theta \sin \phi - B_z \frac{1}{r}, \quad \text{(C.14)}
\]

\[
F^{\theta r} = B_x \left( \Lambda^\theta_x \Lambda^r_y - \Lambda^\theta_y \Lambda^r_z \right) + B_y \left( \Lambda^\theta_x \Lambda^r_z - \Lambda^\theta_z \Lambda^r_x \right) + B_z \left( \Lambda^\theta_y \Lambda^r_x - \Lambda^\theta_x \Lambda^r_y \right) \\
= -B_x \frac{\sin \phi}{r} + B_y \frac{\cos \phi}{r}. \quad \text{(C.15)}
\]

Typically orthonormal spherical coordinates are used in Newtonian contexts, such that:

\[
B^\hat{r} = E^x \sin \theta \cos \phi + E^y \sin \theta \sin \phi + E^z \cos \theta, \\
B^\hat{\theta} = E^x \cos \theta \cos \phi + E^y \cos \theta \sin \phi - E^z \sin \theta, \\
B^\hat{\phi} = -E^x \sin \phi + E^y \cos \phi. \quad \text{(C.16)}
\]

Writing the field strength tensor using spherical coordinates in a coordinate basis, but expressed in terms of standard orthonormal spherical vectors, we then find:

\[
F^{\mu \nu}_C = \begin{pmatrix}
F^{tt} & F^{tr} & F^{t\theta} & F^{t\phi} \\
F^{rt} & F^{rr} & F^{r\theta} & F^{r\phi} \\
F^{\theta t} & F^{\theta r} & F^{\theta \theta} & F^{\theta \phi} \\
F^{\phi t} & F^{\phi r} & F^{\phi \theta} & F^{\phi \phi}
\end{pmatrix} = \begin{pmatrix}
0 & -E^\hat{r} & -E^{\theta \frac{1}{r}} & -E^{\phi \frac{1}{r \sin \theta}} \\
E^\hat{r} & 0 & -B^{\phi \frac{1}{r}} & B^{\theta \frac{1}{r \sin \theta}} \\
E^{\theta \frac{1}{r}} & B^{\phi \frac{1}{r}} & 0 & -B^{\hat{r} \frac{1}{r^2 \sin \theta}} \\
E^{\phi \frac{1}{r \sin \theta}} & -B^{\theta \frac{1}{r \sin \theta}} & B^{\hat{r} \frac{1}{r^2 \sin \theta}} & 0
\end{pmatrix},
\]

\text{(C.17)}
and:

\[
F_{\mu\nu} = \begin{pmatrix}
0 & E^\hat{r} & E^\hat{\theta} r & E^\hat{\phi} r \sin \theta \\
-E^\hat{r} & 0 & -B^\hat{\phi} r & B^\hat{\theta} r \sin \theta \\
-E^\hat{\theta} r & B^\hat{\phi} r & 0 & -B^\hat{\phi} r^2 \sin \theta \\
-E^\hat{\phi} r \sin \theta & -B^\hat{\theta} r \sin \theta & B^\hat{\phi} r^2 \sin \theta & 0
\end{pmatrix}.
\]  
(C.18)

In an orthonormal basis the field strength tensor would simply be written as:

\[
F^\alpha\beta = \begin{pmatrix}
0 & -E^\hat{r} & -E^\hat{\theta} & -E^\hat{\phi} \\
E^\hat{r} & 0 & -B^\hat{\phi} & B^\hat{\theta} \\
E^\hat{\theta} & B^\hat{\phi} & 0 & -B^\hat{\phi} \\
E^\hat{\phi} & -B^\hat{\theta} & B^\hat{\phi} & 0
\end{pmatrix}.
\]  
(C.19)

The Kerr and Schwarzschild metrics are typically written in a coordinate basis, so ultimately such an orthonormal representation has minimal utility here. However the coordinate basis representation (together with the metric determinant \(\sqrt{-g} = r^2 \sin \theta\)) can often prove useful in interpreting the significance of different terms in fully relativistic expressions by allowing rough correspondences to more familiar Newtonian contexts.

We next expand Maxwell’s equations, and arrive at an alternate proof of the conservation of field line angular velocity. Maxwell’s equations may be written as:

\[
F_{\alpha\beta}^{\gamma} = -\frac{4\pi}{c} J^\alpha
\]

\[
\mathcal{F}^{\alpha\beta} = F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0.
\]  
(C.20)

In the second equation the transformation from a covariant derivative to a partial derivative is valid in a coordinate basis \[63\]. Expanded, Maxwell’s inhomogeneous
equations become:

\[
(\sqrt{-g} F^{rt})_r + (\sqrt{-g} F^{\theta t})_\theta + (\sqrt{-g} F^{\phi t})_\phi = \frac{4\pi}{c} \sqrt{-g} J_t \\
\rightarrow \hat{\nabla} \cdot \vec{E} = 4\pi \rho,
\]

\[
(\sqrt{-g} F^{\theta r})_\theta - (\sqrt{-g} F^{\phi r})_\phi = \frac{4\pi}{c} \sqrt{-g} J^r + (\sqrt{-g} F^{r t})_t \\
\rightarrow [\hat{\nabla} \times \vec{B}] \cdot \hat{r} = \left[ \frac{4\pi}{c} J + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right] \cdot \hat{r},
\]

\[
(\sqrt{-g} F^{r\phi})_r - (\sqrt{-g} F^{r\theta})_\theta = \frac{4\pi}{c} \sqrt{-g} J^\phi + (\sqrt{-g} F^{\phi t})_t \\
\rightarrow [\hat{\nabla} \times \vec{E}] \cdot \hat{\theta} = \left[ \frac{4\pi}{c} J + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right] \cdot \hat{\theta},
\]

\[
(\sqrt{-g} F^{\theta r})_\phi - (\sqrt{-g} F^{\phi t})_\theta = \frac{4\pi}{c} \sqrt{-g} J^\phi + (\sqrt{-g} F^{\phi r})_r \\
\rightarrow [\hat{\nabla} \times \vec{E}] \cdot \hat{\phi} = \left[ \frac{4\pi}{c} J + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right] \cdot \hat{\phi}.
\]

Maxwell’s homogeneous equations become:

\[
F_{\phi \theta, r} + F_{r \phi, \theta} + F_{\theta r, \phi} = 0 \quad \rightarrow \quad \hat{\nabla} \cdot \vec{B} = 0,
\]

\[
F_{t \phi, \theta} - F_{t \theta, \phi} = -F_{\phi \theta, t} \quad \rightarrow \quad [\hat{\nabla} \times \vec{E}] \cdot \hat{r} = -\frac{1}{c} \frac{\partial}{\partial t} \vec{B} \cdot \hat{r},
\]

\[
F_{t r, \phi} - F_{t \phi, r} = -F_{r \phi, t} \quad \rightarrow \quad [\hat{\nabla} \times \vec{E}] \cdot \hat{\theta} = -\frac{1}{c} \frac{\partial}{\partial t} \vec{B} \cdot \hat{\theta},
\]

\[
F_{t \theta, r} - F_{t r, \theta} = -F_{\theta r, t} \quad \rightarrow \quad [\hat{\nabla} \times \vec{E}] \cdot \hat{\phi} = -\frac{1}{c} \frac{\partial}{\partial t} \vec{B} \cdot \hat{\phi}.
\]

(C.21)

In a stationary and axisymmetric configuration the partial derivatives with respect
to $t$ and $\phi$ vanish. Excluding the possibility of a current of magnetic monopoles we also have $F_{t\phi} = F^{t\phi} = 0$ (assuming a metric that is diagonal in $r$ and $\theta$). Under these conditions Maxwell’s inhomogeneous reduce to:

\begin{align*}
\left(\sqrt{-g}F_{rt}^{r}\right)_{,r} + \left(\sqrt{-g}F_{\theta t}^{\theta}\right)_{,\theta} = \frac{4\pi}{c}\sqrt{-g}J_{t} \quad \rightarrow \quad \vec{\nabla} \cdot \vec{E} = 4\pi \rho,
\end{align*}

\begin{align*}
\left(\sqrt{-g}F_{r\theta}^{\theta}\right)_{,\theta} = \frac{4\pi}{c}\sqrt{-g}J_{r} \quad \rightarrow \quad \left[\vec{\nabla} \times \vec{B}\right] \cdot \hat{r} = \left[\frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}\right] \cdot \hat{r},
\end{align*}

\begin{align*}
-\left(\sqrt{-g}F_{\theta r}^{\theta}\right)_{,r} = \frac{4\pi}{c}\sqrt{-g}J_{\theta} \quad \rightarrow \quad \left[\vec{\nabla} \times \vec{B}\right] \cdot \hat{\theta} = \left[\frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}\right] \cdot \hat{\theta},
\end{align*}

\begin{align*}
\left(\sqrt{-g}F_{r\phi}^{\phi}\right)_{,r} - \left(\sqrt{-g}F_{\phi r}^{\phi}\right)_{,\theta} = \frac{4\pi}{c}\sqrt{-g}J_{\phi} \quad \rightarrow \quad \left[\vec{\nabla} \times \vec{B}\right] \cdot \hat{\phi} = \left[\frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}\right] \cdot \hat{\phi}.
\end{align*}

This form leads many authors to interpret the conservation of the toroidal field $\sqrt{-g}F_{\theta r}^{\theta}$ in the force-free limit as the conservation of electric current. Ultimately we find this view less helpful, as it is the conservation of energy and angular momentum fluxes (related to Poynting fluxes) that are typically more physically relevant. Maxwell’s homogeneous equations reduce to:

\begin{align*}
F_{\phi \theta, r} + F_{r \phi, \theta} = 0 \quad \rightarrow \quad \vec{\nabla} \cdot \vec{B} = 0,
\end{align*}

\begin{align*}
F_{t \theta, r} - F_{t r, \theta} = 0 \quad \rightarrow \quad \left[\vec{\nabla} \times \vec{E}\right] \cdot \hat{\phi} = -\frac{1}{c} \frac{\partial}{\partial t} \vec{B} \cdot \hat{\phi}. \quad (C.24)
\end{align*}

The first of Maxwell’s homogeneous equations is trivially satisfied if we use a vector potential $A_{\phi}$ to represent the poloidal magnetic fields; $F_{\phi \theta, r} + F_{r \phi, \theta} = -A_{\phi, \theta r} + A_{\phi, r \theta} = 0$. This is a relativistic statement of the vanishing divergence of a curl; $\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$. If we require the electric field be perpendicular to the magnetic
field, such that \( \mathcal{F}^{\alpha\beta} F_{\alpha\beta} = 4 E_i B^i = 0 \), we demand that:

\[
\mathcal{F}^{\alpha\beta} F_{\alpha\beta} = \frac{1}{2\sqrt{-g}} e^{\alpha\beta\mu\nu} F_{\mu\nu} F_{\alpha\beta} = \frac{4}{\sqrt{-g}} (F_{tr} F_{\theta\phi} - F_{t\theta} F_{r\phi}) = 0.
\]  

(C.25)

Therefore if the electric and magnetic fields are perpendicular we generally demand that \( F_{tr}/F_{r\phi} = F_{t\theta}/F_{\theta\phi} = \Omega_F \). Using this definition of the field line angular velocity, the second homogeneous equation becomes:

\[
F_{t\theta,r} - F_{tr,\theta} = (\Omega_F F_{\theta\phi})_r - (\Omega_F F_{r\phi})_\theta = A_{\phi,\theta} \Omega_F,r - A_{\phi,r} \Omega_F,\theta.
\]  

(C.26)

If we take \( \Omega_F \) be a function of \( A_{\phi} \), this immediately yields:

\[
A_{\phi,\theta} \frac{d\Omega_F}{dA_{\phi}} A_{\phi,r} = A_{\phi,r} \frac{d\Omega_F}{dA_{\phi}} A_{\phi,\theta}
\]  

(C.27)

Therefore we take \( \Omega_F \) to be a function of \( A_{\phi} \) as a generic consequence of the homogeneous Maxwell equations and the assumption of a perfectly conducting plasma, in that such an assumption requires \( \vec{E} \cdot \vec{B} = 0 \) so that there can exist a frame in which the electric field might vanish. To be fully complete that logic requires a second condition, however; \( F^{\alpha\beta} F_{\alpha\beta} = 2(B_i B^i - E_i E^i) > 0 \). Solutions where the field line angular velocity is conserved and the force-free condition \( T^{\alpha\beta;\gamma} = F^{\alpha\gamma} J_\gamma = 0 \) are satisfied can be found where the configuration is not magnetically dominated and that second condition is not fulfilled (i.e. \( F^{\alpha\beta} F_{\alpha\beta} = 2(B_i B^i - E_i E^i) \leq 0 \)). Such solutions can generally be taken to violate the assumption of a perfectly conducting plasma, as they contain a frame in which only an electric field exists, so their utility can be somewhat limited.
APPENDIX D

SELECTED NEWTONIAN LIMITS
In this appendix we take the Newtonian (non-relativistic) limit of the divergence of the stress energy tensor in order to arrive at the standard equations governing non-relativistic magnetohydrodynamics. We then take the non-relativistic limit of the conserved energy flux \( E \) and angular momentum flux \( L \). The overall goal is to provide better understanding of the variables and equations that we use, while simultaneously outlining limiting techniques that could be applied to additional quantities as desired.

### Magnetohydrodynamics from the Stress Energy Tensor

The stress energy tensor used throughout this work may be written as:

\[
T^{\alpha\beta} = \left( \rho_0 + \frac{\epsilon}{c^2} + \frac{p}{c^2} \right) u^{\alpha} u^{\beta} - pg^{\alpha\beta} + \frac{1}{4\pi} g^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{1}{16\pi} g^{\alpha\beta} F_{\mu\lambda} F^{\mu\lambda} \tag{D.1}
\]

Here \( \epsilon \) is the internal energy density (e.g. thermal energy) of the fluid and \( \rho_0 \) is its rest mass density, which are typically combined into the relativistic energy density \( \rho = \rho_0 + \epsilon/c^2 \). The divergence of the stress energy tensor vanishes in the absence of external forcings; \( T^{\alpha\beta;\alpha} = 0 \). In the non-relativistic limit the four equations described by that restriction should reduce to the standard magnetohydrodynamic differential equations governing momentum and energy conservation. To see if this is the case, we first consider the purely hydrodynamic limit.

### The Hydrodynamic Limit

We now see if we can arrive at the standard equations of hydrodynamics by only considering the fluid portion of the stress energy tensor. Using \( u^{\alpha} \approx (c + |\vec{u}|^2/2c, \vec{u} + \vec{u}|\vec{u}|^2/2c^2) \), \( g^{\alpha\beta} = \eta^{\alpha\beta} \) (flat space), and \( \partial_{\alpha} = (\frac{1}{\epsilon} \frac{\partial}{\partial t}, \vec{\nabla}) \), the spatial fluid components
are given by:

\[
T^{\alpha i}_{F ; \alpha} = \partial_\alpha \left[ \left( \rho_0 + \frac{\epsilon}{c^2} + \frac{p}{c^2} \right) u^\alpha u^i + p \delta^\alpha_i \right]
\]

\[
= \frac{\partial}{\partial t} \left[ \left( \rho_0 + \frac{\epsilon}{c^2} + \frac{p}{c^2} \right) \ddot{u} \right] + \left( \ddot{u} \cdot \nabla \right) \left( \rho_0 + \frac{\epsilon}{c^2} + \frac{p}{c^2} \right) \ddot{u}
\]

\[
+ \left( \ddot{u} \cdot \nabla \right) \left[ \left( \rho_0 + \frac{\epsilon}{c^2} + \frac{p}{c^2} \right) \ddot{u} \right] + \nabla p
\]

\[
= \frac{\partial}{\partial t} \left( \rho_0 \ddot{u} \right) + \left( \ddot{u} \cdot \nabla \right) \rho_0 \ddot{u} + \left( \ddot{u} \cdot \nabla \right) \rho_0 \ddot{u} + \nabla p
\]

\[
= \rho_0 \frac{\partial \ddot{u}}{\partial t} + \rho_0 \left( \ddot{u} \cdot \nabla \right) \ddot{u} + \nabla p + \ddot{u} \left[ \frac{\partial \rho_0}{\partial t} + \rho_0 \ddot{\nabla} \cdot \ddot{u} + \ddot{u} \cdot \nabla \rho_0 \right]
\]

\[
= \rho_0 \frac{\partial \ddot{u}}{\partial t} + \rho_0 \left( \ddot{u} \cdot \nabla \right) \ddot{u} + \nabla p. \quad \text{(D.2)}
\]

In the above we discarded all but leading order terms. We also used applied mass conservation:

\[
\frac{\partial \rho_0}{\partial t} + \nabla \cdot \left( \rho_0 \ddot{u} \right) = 0. \quad \text{(D.3)}
\]

As we will now show, mass conservation comes from the considering the temporal component of the divergence of the stress energy tensor and comparing terms of equivalent order in \( c \) (we take one value of \( u^t \) to be given by \( c \), the other by the form noted previously - effectively factoring out one \( u^t \) into an overall prefactor, then multiplying it back in using only its leading order \( c \) component):

\[
T^{\alpha t}_{F ; \alpha} = \partial_\alpha \left[ \left( \rho_0 + \frac{\epsilon}{c^2} + \frac{p}{c^2} \right) u^\alpha u^t - p g^{\alpha t} \right]
\]

\[
\approx \frac{1}{c} \frac{\partial}{\partial t} \left( c^2 \rho_0 + \epsilon + p + \frac{\rho_0}{2} |\ddot{u}|^2 \right) - \frac{1}{c} \frac{\partial p}{\partial t} + \ddot{\nabla} \cdot \left[ \left( \rho_0 + \frac{\epsilon}{c^2} + \frac{p}{c^2} \right) \left( c + \frac{1}{2c} |\ddot{u}|^2 \right) \ddot{u} \right]
\]

\[
= c \left[ \frac{\partial \rho_0}{\partial t} + \ddot{\nabla} \cdot \left( \rho_0 \ddot{u} \right) \right] + \frac{1}{c} \left[ \frac{\partial}{\partial t} \left( \frac{\rho_0}{2} |\ddot{u}|^2 \right) + \ddot{\nabla} \cdot (\epsilon \ddot{u}) + \ddot{\nabla} \cdot (p \ddot{u}) + \ddot{\nabla} \cdot \left( \frac{\rho_0}{2} |\ddot{u}|^2 \ddot{u} \right) \right]
\]

\[
= \frac{1}{c} \left[ \frac{\partial}{\partial t} E_K + \ddot{\nabla} \cdot (E_K \ddot{u}) + \ddot{u} \cdot \nabla p \right] + \frac{1}{c} \left[ \frac{\partial}{\partial t} E_T + \ddot{\nabla} \cdot (E_T \ddot{u}) + p \ddot{\nabla} \cdot \ddot{u} \right]. \quad \text{(D.4)}
\]
The order $c$ terms yielded the mass conservation equation; they were discarded (satisfied separately) as being much larger than the order $1/c$ terms. $E_K$ and $E_T$ are the kinetic and thermal energy densities, respectively; we have assumed that $E_K = (1/2)\rho_0|\vec{u}|^2$ and that $E_T = \epsilon$. The derivatives of the kinetic energy density $E_K$ are given by (using the mass conservation equation to eliminate the derivatives of $\rho_0$):

$$\frac{\partial}{\partial t} E_K + \nabla \cdot (E_K \vec{u}) = \left[ \frac{1}{2} \frac{\partial}{\partial t} \rho_0 |\vec{u}|^2 + \frac{1}{2} \nabla \cdot (\rho_0 |\vec{u}|^2 \vec{u}) \right]$$

$$= \frac{1}{2} |\vec{u}|^2 \left[ \frac{\partial}{\partial t} \rho_0 + \rho_0 \vec{\nabla} \cdot \vec{u} + \vec{u} \cdot \nabla \rho_0 \right] + \frac{1}{2} \rho_0 \left[ \frac{\partial}{\partial t} |\vec{u}|^2 + \vec{u} \cdot \nabla |\vec{u}|^2 \right]$$

$$= \rho_0 \left[ \frac{1}{2} \frac{\partial}{\partial t} (\vec{u} \cdot \vec{u}) + \frac{1}{2} \vec{u} \cdot \nabla (\vec{u} \cdot \vec{u}) \right]$$

$$= \vec{u} \cdot \left[ \rho_0 \frac{\partial \vec{u}}{\partial t} + \rho_0 (\vec{u} \cdot \nabla) \vec{u} \right]$$

$$= -\vec{u} \cdot \nabla p. \quad (D.5)$$

The second to last line used $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$, with $\vec{v}$ as the curl of $\vec{u}$. The last line used the spatial components of the divergence of the stress energy tensor (the momentum conservation equation). Therefore the first term in the temporal component of the stress energy tensor vanishes, and we are left with:

$$T_{\alpha i}^{\text{tot}} = \frac{1}{c} \left[ \frac{\partial}{\partial t} E_K + \vec{\nabla} \cdot (E_K \vec{u}) + \vec{u} \cdot \nabla p \right] + \frac{1}{c} \left[ \frac{\partial}{\partial t} E_T + \vec{\nabla} \cdot (E_T \vec{u}) + p\vec{\nabla} \cdot \vec{u} \right]$$

$$= \frac{1}{c} \left[ \frac{\partial}{\partial t} E_T + \vec{\nabla} \cdot (E_T \vec{u}) + p\vec{\nabla} \cdot \vec{u} \right]. \quad (D.6)$$

We next take the thermal energy density to be given as:

$$E_T = \frac{1}{\gamma - 1} p. \quad (D.7)$$
The temporal component of the divergence of the stress energy tensor then reduces to:

\[ T_{\alpha t}^{\alpha} = \frac{1}{c} \left[ \frac{\partial}{\partial t} E_T + \vec{\nabla} \cdot (E_T \vec{u}) + p \vec{\nabla} \cdot \vec{u} \right] \]

\[ = \frac{1}{c} \frac{1}{\gamma - 1} \left[ \frac{\partial p}{\partial t} + \vec{u} \cdot \nabla p + p \vec{\nabla} \cdot \vec{u} + (\gamma - 1) p \vec{\nabla} \cdot \vec{u} \right] \]

\[ = \frac{1}{c} \frac{1}{\gamma - 1} \left[ \frac{\partial p}{\partial t} + \vec{u} \cdot \nabla p + \gamma p \vec{\nabla} \cdot \vec{u} \right]. \quad (D.8) \]

Therefore the four components of the divergence of the fluid portion of the stress energy tensor yield the standard hydrodynamic mass, momentum, and energy conservation equations in the form:

\[ \frac{\partial \rho_0}{\partial t} + \vec{\nabla} \cdot (\rho_0 \vec{u}) = 0, \quad (D.9) \]

\[ \rho_0 \frac{\partial \vec{u}}{\partial t} + \rho_0 (\vec{u} \cdot \nabla) \vec{u} = -\nabla p + \vec{f}, \quad (D.10) \]

\[ \frac{\partial p}{\partial t} + \vec{u} \cdot \nabla p = -\gamma p \vec{\nabla} \cdot \vec{u} + (\gamma - 1) \dot{Q}. \quad (D.11) \]

The external forces \( \vec{f} \) are taken to vanish because we have presumed flat space; if we had used the weak field metric and associated connection coefficients we would have recovered a gravitational force. Because the stress energy tensor for a perfect fluid presumes adiabatic flow we also have no heating; \( \dot{Q} = 0 \).

**Gravitational Forces**

We now recover the gravitational forcings from the connection coefficients (Christoffel symbols). In considering the spatial components of the divergence of the fluid stress tensor we assumed flat space, and as such ignored the connection
coefficients present when taking the derivative of the plasma’s four velocity:

\[ \rho_0 \partial_\alpha (u^\alpha u^i) = \rho_0 \left[ \frac{\partial \vec{u}}{\partial t} + \left( \vec{\nabla} \cdot \vec{u} \right) \vec{u} + \left( \vec{u} \cdot \vec{\nabla} \right) \vec{u} \right] + \rho_0 \left[ \Gamma^\alpha_{\mu \alpha} u^\mu u^i + \Gamma^i_{\mu \alpha} u^\mu u^\alpha \right]. \quad (D.12) \]

We now see how the gravitational force emerges from the second term, by considering a static weak field metric given by:

\[ ds^2 = \left( 1 + \frac{2\phi}{c^2} \right) c^2 dt^2 - \left( 1 - \frac{2\phi}{c^2} \right) d\vec{x}^2. \quad (D.13) \]

Here \( d\vec{x}^2 \) represents the spatial line element; in Cartesian coordinates it would be given by \( d\vec{x}^2 = dx^2 + dy^2 + dz^2 \). The gravitational potential \( \phi \) is defined such that for a spherical body of mass \( m \) we have \( \phi \equiv -Gm/r \). Connection coefficients may be defined as:

\[ \Gamma^\alpha_{\mu \nu} = \frac{1}{2} g^{\alpha \beta} (g_{\beta \mu,\nu} + g_{\beta \nu,\mu} - g_{\mu \nu,\beta}). \quad (D.14) \]

If we assume typical coordinate systems, such that both the metric \( g_{\alpha \beta} \) and its inverse \( g^{\alpha \beta} \) may be taken to be diagonal, and also assume that \( \phi \) is not a function of time then there are only three relevant types of non-vanishing connection coefficients:

\[ \Gamma^i_{tt} = \frac{1}{2} g^{00} g_{00,i} = \frac{1}{c^2} g^{00} \phi, \]
\[ \Gamma^i_{tt} = -\frac{1}{2} g^{ij} g_{00,j} = -\frac{1}{c^2} g^{ij} \phi, \]
\[ \Gamma^i_{ii} = \frac{1}{2} g^{ii} g_{ii,i} = \frac{1}{c^2} g^{ii} \phi. \quad (D.15) \]

Although \( \Gamma^i_{jj} \) and \( \Gamma^i_{ij} \) might not vanish, to leading order they do not contribute to the divergence of the stress energy tensor. To first order in \( \phi \), the inverse metric
is given by:

\[ g^{tt} = \frac{1}{c^2 + 2\phi} = \frac{1}{c^2} - \frac{2\phi}{c^4} + \mathcal{O}(\phi^2), \]
\[ g^{ii} = -\frac{c^2}{c^2 - 2\phi} = -1 - \frac{2\phi}{c^2} + \mathcal{O}(\phi^2). \]  
(D.16)

Therefore the relevant connection coefficients are given by:

\[ \Gamma^0_{ti} = \frac{1}{c^2} \phi,i - \frac{2\phi}{c^4} \phi,i, \]
\[ \Gamma^i_{tt} = \frac{1}{c^2} \phi,i + \frac{2\phi}{c^4} \phi,i, \]
\[ \Gamma^i_{ii} = -\frac{1}{c^2} \phi,i - \frac{2\phi}{c^4} \phi,i. \]  
(D.17)

Therefore to leading order the geometric correction to the spatial component of the divergence of the stress energy tensor is given by:

\[ \rho_0 \left( \Gamma^\alpha_{\mu\alpha} u^\mu u^i + \Gamma^i_{\mu\alpha} u^\mu u^\alpha \right) = \rho_0 \Gamma^i_{tt} u^t u^t = \rho_0 \nabla \phi. \]  
(D.18)

Therefore the weak field metric gives rise to an external gravitational force \( \vec{f} \) in the momentum conservation equation given by \( \vec{f} = -\rho_0 \nabla \phi \), and as expected spacetime curvature corresponds to gravity.

**Adding Electromagnetism**

We now consider the electromagnetic portion of the stress energy tensor:

\[ T^{\alpha\beta}_{EM} = \frac{1}{4\pi} g^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{1}{16\pi} g^{\alpha\beta} F_{\mu\lambda} F^{\mu\lambda}. \]  
(D.19)
In terms of the electromagnetic fields, the stress energy tensor may be re-written as (returning to a Cartesian flat space metric):

\[
T_{EM}^{tt} = \frac{1}{8\pi} \left( |\vec{E}|^2 + |\vec{B}|^2 \right),
\]

\[
T_{EM}^{ti} = \frac{1}{4\pi} \left( \vec{E} \times \vec{B} \right)^i,
\]

\[
T_{EM}^{ij} = -\frac{1}{4\pi} \left[ E^i E^j + B^i B^j - \frac{1}{2} \delta^{ij} \left( |\vec{E}|^2 + |\vec{B}|^2 \right) \right].
\]

(D.20)

Therefore the spatial components of the divergence of the stress energy tensor may be written as:

\[
4\pi T_{EM;\alpha}^{ai} = \frac{1}{c} \partial_t \left( \vec{E} \times \vec{B} \right)^i - \partial_j \left[ E^i E^j + B^i B^j - \frac{1}{2} \delta^{ij} \left( |\vec{E}|^2 + |\vec{B}|^2 \right) \right]
\]

\[
= \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \times \vec{B} + \vec{E} \times \frac{1}{c} \frac{\partial \vec{B}}{\partial t} - \left( \vec{\nabla} \cdot \vec{E} \right) \vec{E} - \left( \vec{E} \cdot \vec{\nabla} \right) \vec{E} - \left( \vec{B} \cdot \vec{\nabla} \right) \vec{B}
\]

\[
+ \frac{1}{2} \nabla \left( \vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B} \right)
\]

\[
= \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \times \vec{B} - \vec{E} \times \left( \vec{\nabla} \times \vec{E} \right) - \left( \vec{\nabla} \cdot \vec{E} \right) \vec{E} + \vec{E} \times \left( \vec{\nabla} \times \vec{E} \right) + \vec{B} \times \left( \vec{\nabla} \times \vec{B} \right)
\]

\[
= \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \times \vec{B} - \left( \vec{\nabla} \cdot \vec{E} \right) \vec{E} + \vec{B} \times \left( \vec{\nabla} \times \vec{B} \right).
\]

(D.21)

In typical Newtonian magnetohydrodynamics the displacement current \( \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \) is typically very small relative to \( \vec{J} \) and is ignored; the inhomogeneous Maxwell equations become \( \vec{\nabla} \cdot \vec{E} = 4\pi \rho \) and \( \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} \). Using those modified equations, we recover the standard charged particle force law:

\[
T_{EM;\alpha}^{ai} = -\rho \vec{E} - \frac{1}{c} \vec{J} \times \vec{B}.
\]

(D.22)
Typically $\rho \vec{E}$ is very small and is also dropped; if the electric field is related to a boost from the plasma rest frame then $|\vec{E}| \sim \frac{|\vec{u}| |\vec{B}|}{c}$, so the $(\vec{\nabla} \cdot \vec{E}) \vec{E}$ term from the divergence of the stress energy tensor is of order $1/c^2$. We now consider the temporal component of the divergence of the stress energy tensor:

$$T_{\text{EM};t}^{\text{tot}} = \frac{1}{8\pi c} \partial_t \left(|\vec{E}|^2 + |\vec{B}|^2\right) + \frac{1}{4\pi} \partial_t \left(\vec{E} \times \vec{B}\right)$$

$$= \frac{1}{4\pi} \left[\vec{E} \cdot \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \vec{B} \cdot \frac{1}{c} \frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \cdot \left(\vec{E} \times \vec{B}\right)\right]$$

$$= \frac{1}{4\pi} \left[\vec{E} \cdot \left(\vec{\nabla} \times \vec{B} - \frac{4\pi}{c} \vec{J}\right) - \vec{B} \cdot \left(\vec{\nabla} \times \vec{E}\right) + \vec{B} \cdot \left(\vec{\nabla} \times \vec{E}\right) - \vec{E} \cdot \left(\vec{\nabla} \times \vec{B}\right)\right]$$

$$= -\frac{1}{c} \vec{E} \cdot \vec{J}. \quad (D.23)$$

To order $1/c$ the electric field is related to a boost from the plasma rest frame in combination with Ohm’s Law:

$$\vec{E} = -\frac{1}{c} \vec{u} \times \vec{B} + \frac{1}{\sigma} \vec{J}.$$ \quad (D.24)

Therefore the temporal component of the divergence of the stress energy tensor becomes (to order $1/c$):

$$T_{\text{EM};t}^{\text{tot}} = -\frac{1}{\sigma c} \vec{J} \cdot \vec{J} = -\frac{1}{c^2} \frac{c^2}{4\pi \sigma} |\vec{\nabla} \times \vec{B}|^2 = -\frac{1}{c} \frac{\eta}{4\pi} |\vec{\nabla} \times \vec{B}|^2. \quad (D.25)$$

Here the magnetic diffusivity (resistivity) $\eta$ is related to the conductivity $\sigma$ via $\eta = c^2 / 4\pi \sigma$. Combining the above results with the hydrodynamic results, we then find the equations of magnetohydrodynamics to be given by:

$$\frac{\partial \rho_0}{\partial t} + \vec{\nabla} \cdot (\rho_0 \vec{u}) = 0,$$
\[ \rho_0 \frac{\partial \vec{u}}{\partial t} + \rho_0 (\vec{u} \cdot \nabla) \vec{u} = -\nabla p + \vec{f}_G + \frac{1}{4\pi} \left( \vec{\nabla} \times \vec{B} \right) \times \vec{B} + \vec{f}, \]

\[ \frac{\partial p}{\partial t} + \vec{u} \cdot \nabla p = -\gamma p \vec{\nabla} \cdot \vec{u} + (\gamma - 1) \frac{\eta}{4\pi} |\vec{\nabla} \times \vec{B}|^2 + (\gamma - 1) \dot{Q}. \] (D.26)

In this work we do not consider any external forcings, so \( \vec{f} = 0 \) (gravity emerges from connection coefficients in curved spacetimes). The perfect fluid assumption of the stress energy tensor requires adiabatic flow, so \( \dot{Q} = 0 \). We also assume ideal magnetohydrodynamics and infinite conductivity, so \( \eta \to 0 \). The above results may alternatively be arrived at by directly assuming the form of the stress energy tensor used in this work (in terms of \( B^\alpha \) and \( u^\alpha \)) as opposed to the method above of assuming its form in terms of electric and magnetic fields, but many of the intermediate simplifications can become much more complex without significant additional value.

**Conserved Energy Flux**

In this section we will consider the Newtonian limit of the conserved energy flux \( E \), which is given by:

\[ E = \mu u_t + \frac{1}{4\pi\eta} \sqrt{-g} F^{\theta r} \Omega_F. \] (D.27)

Here \( \mu \) is the enthalpy of the plasma, \( u_t \) is the temporal component of the plasma’s 4-velocity, \( \eta \) is the particle flux per unit flux tube, \( \sqrt{-g} F^{\theta r} \) is associated with the toroidal magnetic field, and the field line angular velocity \( \Omega_F \) is a ratio of field strength tensor components.

The first term in the conserved energy flux contains the enthalpy of the flow \( \mu \), which combines the internal energy of the fluid with the \( pdV \) work done by it. It is written as the sum of the fluid’s energy density \( \rho \) and its pressure \( p \) divided by \( n \);
\[ \mu = (\rho + p)/n. \] To find \( u_t \) we will assume a static weak field metric:

\[
    ds^2 = \left(1 + \frac{2\phi}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2\phi}{c^2}\right) d\bar{x}^2. \tag{D.28}
\]

Here \( \phi \) is the gravitational potential. The temporal component of the four velocity \( u^\alpha = (u^t, u^i \tilde{v}) \) (where \( \tilde{v} \) is the standard three-velocity of the plasma) may then be found from the normalization condition \( u_\alpha u^\alpha = c^2 \) (generically taking \( \tilde{v} \) to be oriented in direction \( x \)):

\[
    u_\alpha u^\alpha = g_{tt} (u^t)^2 + g_{xx} (u^\bar{x} \tilde{v})^2 = (u^t)^2 (g_{tt} + g_{xx} v^2) = c^2. \tag{D.29}
\]

Solving for \( u^t \), we find:

\[
    u^t = \frac{c}{\sqrt{g_{tt} + g_{xx} v^2}}. \tag{D.30}
\]

In flat space this corresponds to the well known result \( u^t = c\gamma \). We may now solve for \( u_t \) to find:

\[
    u_t = g_{tt} u^t = c^2 + \phi + \frac{1}{2} v^2 + O \left( \frac{1}{c^4} \right). \tag{D.31}
\]

We may now expand the first term in \( E \) to find:

\[
    \mu u_t \approx \frac{1}{n} \left( \rho_0 + \frac{\epsilon}{c^2} + \frac{p}{c^2} \right) \left( c^2 + \phi + \frac{1}{2} v^2 \right)
    \approx \frac{1}{n} \left( \rho_0 c^2 + \frac{1}{2} \rho_0 v^2 + \frac{\Gamma}{\Gamma - 1} p \right) \tag{D.32}.
\]

To arrive at the above we assumed that the internal energy \( \epsilon \) was entirely thermal, with \( E_T = p/(\gamma - 1) \). The energy \( E \) is conserved in the sense that \( (EN^\alpha)_{\alpha} = 0 \). Expanding that statement into a continuity equation (and considering only the fluid
component of \( E \), we find:

\[
\frac{d}{dt} (\mu u_t) = -\tilde{\nabla} \cdot \left[ \left( \rho_0 c^2 + \frac{1}{2} \rho_0 v^2 + \frac{\gamma}{\gamma - 1} p + \rho_0 \phi \right) \tilde{u} \right]. \tag{D.33}
\]

From standard Newtonian hydrodynamics the rate of change of the energy \( E_R \) contained within a stationary volume \( R \) is given by:

\[
\frac{dE_R}{dt} = -\oint_{\partial R} \left( \frac{1}{2} \rho_0 |\tilde{u}|^2 + \frac{\Gamma}{\Gamma - 1} p \right) \tilde{u} \cdot d\tilde{a} + \int_R \tilde{u} \cdot \tilde{f} d^3x + \int_R \dot{Q} d^3x. \tag{D.34}
\]

For a perfect fluid \( \dot{Q} = 0 \). In our case the only force density \( \tilde{f} \) is given by \( \tilde{f} = -\rho_0 \nabla \phi \); integrating the force term by parts and applying the mass continuity equation immediately yields:

\[
\frac{d}{dt} (E_K + E_T + E_G)_R = -\oint_{\partial R} \left( \frac{1}{2} \rho_0 |\tilde{u}|^2 + \frac{\Gamma}{\Gamma - 1} p + \rho_0 \phi \right) \tilde{u} \cdot d\tilde{a} \tag{D.35}
\]

Upon comparison of the above with the equation for \( \mu u_t \), we immediately see that the first term in \( E \) is the average energy per particle of the plasma, including rest mass energy.

It should be expected that the second term in \( E \) will correspond to an electromagnetic energy in terms of a Poynting flux, somehow weighted to the particle density. To see how such a term might arise, we first propose a static magnetic field in an orthonormal basis:

\[
\begin{align*}
B_R &= B_R = F^{\phi \theta}, & E_R &= 0 = F^{r t}, \\
B_\theta &= B_\theta = F^{r \phi}, & E_\theta &= 0 = F^{\theta t}, \\
B_\phi &= B_\phi = F^{\theta r}, & E_\phi &= 0 = F^{\phi t}. \tag{D.36}
\end{align*}
\]
Here the subscripts on $B$ and $E$ are not indices, but labels. If we then boost the magnetic field about the $\hat{z}$ axis, the electric and magnetic fields as observed by a stationary observer become:

$$B'_R = \Lambda^{\phi'}_{\phi} B_R = F^{\phi\theta}, \quad E'_R = -\Lambda^{\phi'}_{\phi} F^{\phi\theta} = \Lambda^{\phi'}_{\phi} B_\Theta = -\frac{\Lambda^{\phi'}_{\phi}}{\Lambda^{\phi'}_{\phi}} B'_R,$$

$$B'_\Theta = \Lambda^{\phi'}_{\phi} B_\Theta = F^{\theta\phi}, \quad E'_\Theta = -\Lambda^{\phi'}_{\phi} F^{\phi\theta} = -\Lambda^{\phi'}_{\phi} B_R = -\frac{\Lambda^{\phi'}_{\phi}}{\Lambda^{\phi'}_{\phi}} B'_R,$$

$$B'_\Phi = B_\Phi = F^{\theta\phi}. \quad E'_\Phi = 0. \quad \text{(D.37)}$$

If we define the ratio of transformation elements in terms of a field line angular velocity, such that $\Lambda^{\phi'}_{\phi}/\Lambda^{\phi'}_{\phi} = -\Omega_F$ then we have:

$$B'_R = B'_R, \quad E'_R = -\Omega_F B'_\Theta,$$

$$B'_\Theta = B'_\Theta, \quad E'_\Theta = \Omega_F B'_R,$$

$$B'_\Phi = B'_\Phi, \quad E'_\Phi = 0. \quad \text{(D.38)}$$

In terms of these fields, the Poynting flux vector $\vec{S}$ is given by:

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}$$

$$= \frac{c}{4\pi} \left( E'_R B'_\Theta \hat{\phi} - E'_R B'_\Phi \hat{\theta} - E'_\Theta B'_R \hat{\phi} + E'_\Theta B'_\Phi \hat{\theta} \right)$$

$$= \frac{c}{4\pi} \Omega_F \left[ B'_R B'_\Phi \hat{\phi} + B'_\Theta B'_\Phi \hat{\theta} + \left( B'_\Theta^2 + B'_R^2 \right) \hat{\phi} \right]. \quad \text{(D.39)}$$

In terms of a Poynting flux vector, the rate of change of the electromagnetic energy density $E_{EM}$ in a volume $V$ may be written as:

$$\frac{\partial E_{EM}}{\partial t} + \nabla \cdot \vec{S} = 0. \quad \text{(D.40)}$$
If the configuration is static then the first term vanishes. If the configuration is axially symmetric, then the toroidal component of the Poynting vector $\vec{S}$ may be discarded in the divergence; it contributes nothing to $\nabla \cdot \vec{S}$. If we therefore discard the toroidal component of $\vec{S}$ above, we find the toroidal component $\vec{S}_p$ to be:

$$\vec{S}_p = \frac{c}{4\pi} \Omega F B' \left( B'_R \hat{r} + B'_{\Phi} \hat{\theta} \right).$$

(D.41)

We now seek to rewrite this vector in terms of the plasma’s velocity. To do so we recall that the plasma flow is in the direction of the magnetic field lines, which led to the various definition of the particle flux per unit flux tube $\eta$ as the ratio of plasma velocity to magnetic field. We now consider the two definitions in the poloidal plane to be:

$$\eta = \sqrt{-g n u^r F_{\theta\phi}} = -\frac{\sqrt{-g n u^r}}{B'_R},$$

$$\eta = \sqrt{-g n u^\theta F_{\phi r}} = -\frac{\sqrt{-g n u^\theta}}{B'_{\theta}}.$$  

(D.42)

We may therefore rewrite the vector $\vec{S}_p$ in terms of the plasma four velocity as:

$$\vec{S}_p = -\frac{c}{4\pi \eta} \sqrt{-g} B'_{\phi} \Omega_F \left( n u^r \hat{r} + n u^\theta \hat{\theta} \right).$$  

(D.43)

Exploiting time independence and axial symmetry, the vanishing divergence of this vector may be combined with particle number conservation to find:

$$\nabla \cdot \vec{S}_p = \left( -\frac{c}{4\pi \eta} \sqrt{-g} B'_{\phi} \Omega_F N^\alpha \right)_{,\alpha} = N^\alpha \left( -\frac{c}{4\pi \eta} \frac{\sqrt{-g} B'_{\phi} \Omega_F}{,\alpha} \right) = 0.$$  

(D.44)

Therefore the Poynting flux transport of electromagnetic energy may be mathematically regarded as the advection of electromagnetic energy by a plasma flow. With
\[ B_\phi = -F^{\theta r} \] the exact form of the conserved energy \( E \) is recovered. Generally neither the fluid energy component nor the Poynting flux component are separately conserved, but the above analysis serves to illustrate why the conserved energy \( E \) takes the form that it does.

**Conserved Angular Momentum Flux**

The conserved angular momentum \( L \) is given by:

\[
L = -\mu u_\phi + \frac{1}{4\pi \eta} \sqrt{-g} F^{\theta r} \quad (D.45)
\]

We will examine the two terms in \( L \) separately. Expanding the first (fluid) term yields:

\[
-\mu u_\phi = -\left( \frac{\rho + p}{n} \right) u_\phi = -\frac{1}{n} \left( \rho_0 + \frac{\epsilon}{c^2} + \frac{p}{c^2} \right) u_\phi. \quad (D.46)
\]

At best \( u^{\phi} \sim c \); therefore at best \( u_\phi \sim g_{\phi\phi} c \). The static weak field metric adds to flat space a component that goes as \( \frac{\phi}{c^2} \); therefore we have:

\[
-\mu u_\phi \sim -\frac{1}{n} \left( \rho_0 + \frac{\epsilon}{c^2} + \frac{p}{c^2} \right) \left( c + \frac{\phi}{c} \right). \quad (D.47)
\]

To leading order we can then assume that \( -\mu u_\phi = -(\rho_0/n) u_\phi \). In flat space Cartesian coordinates, the four velocity of the plasma is given by \( u^\alpha = \gamma(c, v^x, v^y, v^z) \).

Transforming this to spherical coordinates yields (in coordinate and orthonormal bases):

\[
\begin{align*}
u_C^0 &= \gamma(c, v^r, \frac{1}{r} v^\theta, \frac{1}{r \sin \theta} v^{\phi}), & u_O^0 &= \gamma(c, v^r, v^\theta, v^{\phi}), \\
u_C^r &= \gamma(c, -v^r, -r v^\theta, -r \sin \theta v^{\phi}), & u_O^r &= \gamma(c, -v^r, -v^\theta, -v^{\phi}).
\end{align*} \quad (D.48)
\]
Here \( v^r, v^\theta, \) and \( v^\phi \) are the standard three velocity components as measured in an orthonormal basis. In this context it is most appropriate to use the coordinate basis representation of \( u_\alpha \), as that corresponds to the spacetime metrics in use. Using the cylindrical radius \( s = r \sin \theta \), to leading order the first term in \( L \) then becomes:

\[
-\mu u_\phi = -\frac{\rho_0}{n} (-sv^\phi) = \frac{1}{n} (\rho_0 s^2 \Omega). \tag{D.49}
\]

Here \( \Omega = \frac{v^\phi}{s} \) is the angular velocity of the plasma. As with the energy \( E \), the angular momentum \( L \) satisfies the continuity equation \( (LN^\alpha)_{,\alpha} = 0 \); expanding this into a continuity equation, we find:

\[
\frac{d}{dt} (-\mu u_\phi) = -\nabla \cdot \left[ (\rho_0 s^2 \Omega) \bar{u} \right]. \tag{D.50}
\]

From standard Newtonian hydrodynamics we know that the angular momentum continuity equation for a stationary volume \( R \) is given by:

\[
\frac{d}{dt} \left( \vec{r} \times \vec{P}_R \right) = -\oint_{\partial R} \rho_0 (\vec{r} \times \bar{u}) (\bar{u} \cdot d\vec{a}) - \oint_{\partial R} p (\vec{r} \times d\vec{a}) + \int_R (\vec{r} \times \vec{f}) \, d^3x. \tag{D.51}
\]

Reducing this to the vertical \( \hat{z} \) component yields (in terms of the cylindrical radius \( s \) and using \((s, \theta, z)\) cylindrical coordinates):

\[
\frac{dL_z}{dt} = -\oint_{\partial R} \rho_0 \vec{r} s u^\theta (\bar{u} \cdot d\vec{a}) - \oint_{\partial R} ps d\sigma^\theta + \int_R sf^\theta d^3x. \tag{D.52}
\]

For an axisymmetric system the second term vanishes. For the static weak field metric there is no \( f^\theta \) force, so the the third term vanishes as well. Therefore we are left with:

\[
\frac{dL_z}{dt} = -\oint_{\partial R} \rho_0 su^\phi (\bar{u} \cdot d\vec{a}) = -\oint_{\partial R} \rho_0 s^2 \Omega (\bar{u} \cdot d\vec{a}). \tag{D.53}
\]
We therefore identify the first (fluid) term in $L$ as the average angular momentum per particle.

In order to explore the second electromagnetic term we will calculate the electromagnetic angular momentum flux in terms of the divergence of the stress tensor and then show that this quantity may be considered to be an angular momentum that is advected by the plasma flow. To begin, we consider the electromagnetic momentum continuity equation:

$$\frac{\partial}{\partial t} \vec{P}_{EM} - \vec{\nabla} \cdot \vec{T}_{EM} = 0.$$  \hspace{1cm} (D.54)

Here the momentum density $\vec{P}_{EM}$ is the momentum stored in an electromagnetic field, and the spatial Maxwell stress tensor $\vec{T}_{EM}$ is given by:

$$T_{EM}^{ij} = \frac{1}{4 \pi} \left[ E^i E^j + B^i B^j - \frac{1}{2} \left( \vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B} \right) \delta^{ij} \right].$$  \hspace{1cm} (D.55)

We are interested in the $\hat{\phi}$ (angular momentum) component; therefore we proceed in an orthonormal basis to find:

$$\frac{\partial}{\partial t} \vec{P}_{EM}^{\phi} - \partial_i T_{EM}^{i \phi} = -\partial_r T_{EM}^{r \phi} - \partial_\theta T_{EM}^{\theta \phi} = -\vec{\nabla} \cdot \vec{T} = 0.$$  \hspace{1cm} (D.56)

Here time independence and axial symmetry were used, and we have defined a divergenceless vector $\vec{T}$ as $\vec{T} \equiv T_{EM}^{r \phi} \hat{r} + T_{EM}^{\theta \phi} \hat{\theta}$. A rotating magnetic field configuration in an orthonormal basis was calculated in the previous section when exploring the second electromagnetic term in $E$:

$$B'_R = B'_R, \hspace{1cm} E'_R = -\Omega_F B'_{\Theta},$$

$$B'_\Theta = B'_\Theta, \hspace{1cm} E'_\Theta = \Omega_F B'_R,$$

$$B'_\Phi = B'_\Phi, \hspace{1cm} E'_\Phi = 0.$$  \hspace{1cm} (D.57)
Using these fields and the definitions of the particle flux per unit flux tube $\eta$, the vector $\vec{T}$ may be expanded to find:

\[
\vec{T} = T^r \hat{r} + T^\theta \hat{\theta}
\]

\[
= \frac{1}{4\pi} \left[ (B_R' B_\phi') \hat{r} + (B_\phi B_\phi') \hat{\theta} \right]
\]

\[
= \frac{1}{4\pi} B_\phi' \left[ -\sqrt{-g} \eta u^r \hat{r} - \sqrt{-g} \eta u^\theta \hat{\theta} \right]
\]

\[
= - \frac{1}{4\pi} \frac{\sqrt{-g} \eta}{\eta} B_\phi' \left[ u^r \hat{r} + u^\theta \hat{\theta} \right].
\]

(D.58)

Exploiting the vanishing divergence of $\vec{T}$ along with stationarity and axisymmetry, we then have:

\[-\vec{\nabla} \cdot \vec{T} = - \left( - \frac{1}{4\pi} \frac{\sqrt{-g}}{\eta} B_\phi' N^\alpha \right) = N^\alpha \left( \frac{1}{4\pi} \frac{\sqrt{-g}}{\eta} B_\phi' \right).\]

(D.59)

Therefore we identify the second term in $L$ as the average electromagnetic angular momentum per particle, expressed mathematically as a quantity that is advected by the plasma flow. As with the energy $E$, the fluid and electromagnetic components are not conserved separately, and we have only shown a crude correspondence between the form of $L$ and a Newtonian limit. Nonetheless it is sufficient to demonstrate why $L$ takes the form that it does.
APPENDIX E

FIELD LINE ANGULAR VELOCITY AND ZERO ANGULAR MOMENTUM
OBSERVERS
In this appendix we demonstrate that the field line angular velocity is indeed an angular velocity, referenced to zero angular momentum observers. In flat space, a stationary observer is an observer with vanishing three velocity. In the ergoregion of a Kerr black hole, such an observer is impossible; all timelike observers must rotate with the spacetime. The natural extension of a stationary observer from flat space to a Kerr metric is therefore more appropriately taken to be an observer with vanishing specific angular momentum, called a zero angular momentum observer or ZAMO. The specific angular velocity of an observer may be defined as the contraction of that observer’s four velocity with the Killing vector associated with axisymmetry, \( l^\alpha \):

\[
l \equiv -l^\alpha u_\alpha. \tag{E.1}
\]

Far from the black hole, a rotating observer with a four velocity \( u^\alpha = (u^t, 0, 0, u^t \Omega) \) would have an angular velocity given by \( l \sim r^2 \sin^2 \theta \Omega^2 \), the standard Newtonian definition of a specific angular velocity. Near the black hole, the contraction would be given by:

\[
l = -l_\alpha u^\alpha = -u^t (g_{t\phi} + g_{\phi\phi} \Omega). \tag{E.2}
\]

If this is to vanish, then a ZAMO must rotate with angular velocity \( \omega_Z = -g_{t\phi}/g_{\phi\phi} \).

On the horizon this reduces to the angular velocity of the black hole, \( \omega_H = a/2mr_H \).

By virtue of the condition of ideal magnetohydrodynamics \( F^{\alpha\beta} u_\beta = 0 \), a non-rotating stationary plasma (in the sense of co-rotating with a ZAMO such that \( u_\phi = 0 \)) necessarily sees a vanishing electric field; \( F^{tr} = F^{t\theta} = 0 \). A more generically rotating observer with angular velocity \( \Omega \) and four velocity \( u^\alpha = (u^t, 0, 0, u^t \Omega) \) will see electric fields given by:

\[
F^{rt} u_t + F^{r\phi} u_\phi = 0 \quad \longrightarrow \quad F^{tr} = F^{r\phi} g_{t\phi} + g_{\phi\phi} \Omega \over g_{tt} + g_{t\phi} \Omega.
\]
\[ F^{\theta t} u_t + F^{\phi \phi} u_\phi = 0 \quad \rightarrow \quad F^{t \theta} = \frac{F^{\theta \phi} g_{t \phi} + g_{\phi \phi} \Omega}{g_{tt} + g_{t \phi} \Omega} \tag{E.3} \]

If \( \Omega = -\frac{g_{t \phi}}{g_{\phi \phi}} = \omega_Z \) these vanish, as expected. If we now lower the indices to find the field strength tensor components \( F_{tr} \) and \( F_{r \phi} \), we find:

\[
F_{tr} = g_{tt} g_{rr} F^{tr} + g_{t \phi} g_{r \phi} F^{\phi r} \quad \rightarrow \quad F_{tr} = g_{rr} F^{r \phi} \Omega \frac{g_{tt} g_{\phi \phi} - g_{t \phi}^2}{g_{tt} + g_{t \phi} \Omega},
\]

\[
F_{r \phi} = g_{rr} g_{\phi \phi} F^{r \phi} + g_{r \phi} g_{t \phi} F^{rt} \quad \rightarrow \quad F_{r \phi} = g_{rr} F^{r \phi} \frac{g_{tt} g_{\phi \phi} - g_{t \phi}^2}{g_{tt} + g_{t \phi} \Omega}. \tag{E.4}
\]

Identical results hold for \( F_{t \theta} \) and \( F_{\theta \phi} \) with \( r \rightarrow \theta \). Taking the ratio of the field strength tensor elements, we find the field line angular velocity to be \( \Omega_F = F_{tr}/F_{r \phi} = \Omega \). This is a reason why the “field line angular velocity” is called an angular velocity; it corresponds to the angular velocity of a rotating observer that sees vanishing electric fields according to the constraint that \( F^{\alpha \beta} u_\beta = 0 \). The electric fields as “seen” by a distant or metric observer are the result of a rotational boost from the rest from of that rotating observer. There is, however, a caveat - if \( F^{tr} \) vanishes (as in the case of a ZAMO) then \( F_{tr} \) can be non-vanishing as a result of spacetime rotation. It is for this reason that the field line angular velocity is said to be referenced to ZAMO frames; ZAMO frames provide a distinction between fields sourced by a rotating field (corresponding to a plasma rest frame with non-vanishing angular momentum) and being sourced by spacetime rotation.
APPENDIX F

THE TRANSFIELD EQUATION
In this appendix we derive the various forms of the transfield equation used throughout this work. The overall approach is to first re-write the stress energy tensor in terms of divergenceless vectors, then take the divergence of the stress energy tensor and manipulate the result into a useful form.

**Electromagnetic Components**

The standard form of the electromagnetic stress energy tensor is:

\[
T_{EM}^{\alpha\beta} = \frac{1}{4\pi} g^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{1}{16\pi} g^{\alpha\beta} F_{\mu\lambda} F^{\mu\lambda}. \quad (F.1)
\]

We now recall the definition of the magnetic field four-vector \( B^\alpha \):

\[
F_{\alpha\beta} = -\sqrt{-g} \epsilon_{\alpha\beta\mu\nu} (k^\mu + \Omega F l^\mu) B^\nu,  \\
B^\alpha = \frac{1}{2\sqrt{-g}} \epsilon^{\alpha\beta\gamma\delta} t_\beta F_{\gamma\delta}. \quad (F.2)
\]

We may use this definition of \( B^\alpha \) to re-write the first term of the stress energy tensor as:

\[
F_{\mu\lambda} F^{\lambda\beta} = \left[ -\sqrt{-g} \epsilon_{\mu\lambda\omega\rho} (k^\omega + \Omega F l^\omega) B^\rho \right] \left[ \frac{1}{\sqrt{-g}} \epsilon^{\lambda\beta\sigma\delta} (k_\sigma + \Omega F l_\sigma) B_\delta \right]  \\
= -\epsilon_{\mu\lambda\omega\rho} \epsilon^{\lambda\beta\sigma\delta} r_\sigma B^\rho B_\delta. \quad (F.3)
\]

Here we have defined the rotation vector \( r^\alpha \) as \( r^\alpha \equiv k^\alpha + \Omega F l^\alpha \), and note that by virtue of stationarity and axisymmetry \( r^\alpha \) is divergenceless; \( r^{\alpha\,\alpha} = 0 \). The contraction of the antisymmetric symbols is given by:

\[
\epsilon_{\mu\lambda\omega\rho} \epsilon^{\lambda\beta\sigma\delta} = \epsilon_{\mu\rho\sigma\lambda} \epsilon^{\beta\delta\sigma\lambda}
\]
\[ \begin{align*}
&= \delta^\delta \mu \delta^\rho \delta^\sigma \omega + \delta^\delta \rho \delta^\beta \omega \delta^\sigma \mu + \delta^\delta \omega \delta^\beta \mu \delta^\rho \\
&- \delta^\delta \rho \delta^\mu \delta^\sigma \omega - \delta^\delta \mu \delta^\beta \omega \delta^\rho - \delta^\delta \omega \delta^\beta \rho \delta^\sigma \mu. \tag{F.4}
\end{align*} \]

Contracting this with the rotation and magnetic field vectors, we find:

\[ F_{\mu \lambda} F^{\lambda \beta} = -\alpha B_\mu B^\beta - B_\gamma B^\gamma r_\mu r^\beta - (r^\gamma B_\gamma)^2 \delta^\beta \mu + \alpha B_\gamma B^\gamma \delta^\beta \mu + r^\gamma B_\gamma B_\mu r^\beta + r^\gamma B_\gamma r_\mu B^\beta. \tag{F.5} \]

Therefore the first term of the electromagnetic stress energy tensor may be written as:

\[ \frac{1}{4\pi} g^{\alpha \mu} F_{\mu \lambda} F^{\lambda \beta} = -\frac{\alpha}{4\pi} B^\alpha B^\beta - \frac{B_\gamma B^\gamma}{4\pi} r^\alpha r^\beta + \frac{g_{\mu \phi} + g_{\phi \phi} \Omega_F}{4\pi \rho_\omega^2} \sqrt{-g} F^{\theta \mu} (B^\alpha r^\beta + r^\alpha B^\beta) \\
- \frac{(r^\gamma B_\gamma)^2}{4\pi} g^{\alpha \beta} + \frac{\alpha B_\gamma B^\gamma}{4\pi} g^{\alpha \beta}. \tag{F.6} \]

The full contraction of the field strength tensor in the second term of the electromagnetic stress energy tensor in terms of \( B^\alpha \) is then found to be:

\[ F_{\mu \lambda} F^{\mu \lambda} = \left[ -\sqrt{-g} \epsilon_{\mu \omega \rho \sigma} (k^\omega + \Omega_F l^\omega) B^\rho \right] \left[ \frac{1}{\sqrt{-g}} \epsilon^{\mu \lambda \sigma \delta} (k_\sigma + \Omega_F l_\sigma) B_\delta \right] \\
= -\epsilon_{\mu \omega \rho \sigma} \epsilon^{\mu \lambda \sigma \delta} r^\omega r_\sigma B^\rho B_\delta. \tag{F.7} \]

The contraction of the antisymmetric symbols is given by:

\[ \epsilon_{\mu \omega \rho} \epsilon^{\mu \lambda \sigma \delta} = 2 \delta^\sigma \omega \delta^\delta \rho - 2 \delta^\sigma \rho \delta^\delta \omega. \tag{F.8} \]

Therefore the full contraction of the field strength tensor is given by:

\[ F_{\mu \lambda} F^{\mu \lambda} = -2\alpha B^\gamma B_\gamma + 2 (r^\gamma B_\gamma)^2. \tag{F.9} \]
We therefore conclude that the second term in the electromagnetic stress energy tensor may be written as:

$$\frac{1}{16\pi} g^{\alpha\beta} F_{\mu\lambda} F^{\mu\lambda} = -\frac{\alpha}{8\pi} B_\gamma B_\gamma g^{\alpha\beta} + \frac{1}{8\pi} (r_\gamma B_\gamma)^2 g^{\alpha\beta}. \quad (F.10)$$

Combining this with the first term already found above, we find that the electromagnetic stress energy tensor may be written in terms of the magnetic field $B^\alpha$ as:

$$T^\alpha{}_{\beta}^{\text{EM}} = -\frac{\alpha}{4\pi} B^\alpha B^\beta - \frac{B_\gamma B_\gamma}{4\pi} r^\alpha r^\beta + \frac{g_{t\phi} + g_{\phi\phi} \Omega_F}{4\pi \rho_\omega^2} \sqrt{-g} F^{\theta r} (B^\alpha r^\beta + r^\alpha B^\beta) - \frac{(r_\gamma B_\gamma)^2}{8\pi} g^{\alpha\beta}$$

$$+ \frac{\alpha B_\gamma B_\gamma}{8\pi} g^{\alpha\beta}.$$  \quad (F.11)

It will ultimately prove to be convenient to fully separate the $B^\phi$ portions of the last two terms; doing so, we find:

$$T^\alpha{}_{\beta}^{\text{EM}} = -\frac{\alpha}{4\pi} B^\alpha B^\beta - \frac{B_\gamma B_\gamma}{4\pi} r^\alpha r^\beta + \frac{g_{t\phi} + g_{\phi\phi} \Omega_F}{4\pi \rho_\omega^2} \sqrt{-g} F^{\theta r} (B^\alpha r^\beta + r^\alpha B^\beta) + \frac{\alpha B^A B^A}{8\pi} g^{\alpha\beta}$$

$$- \frac{(\sqrt{-g} F^{\theta r})^2}{8\pi \rho_\omega^2} g^{\alpha\beta}. \quad (F.12)$$

To arrive at this result we used $r_\phi = g_{t\phi} + \Omega_F g_{\phi\phi}$, $B^\phi = \sqrt{-g} F^{\theta r}/\rho_\omega^2$, and $\alpha g_{\phi\phi} - (g_{t\phi} + g_{\phi\phi} \Omega_F)^2 = g_{tt} g_{\phi\phi} - g_{\phi\phi}^2 = -\rho_\omega^2$.

We may now take the divergence of the electromagnetic stress energy tensor, focusing on the poloidal components in order to arrive at the transfield equation. To begin, we first note several derivatives that will prove to be relevant:

$$k_{A;\alpha} = -\frac{1}{2} g_{\alpha,A},$$

$$l_{A;\alpha} = -\frac{1}{2} g_{\phi\alpha,A},$$

$$B_{A;\alpha} = g_{\beta A} B^{\beta;\alpha} + \frac{1}{2} B^\mu (g_{A\mu;\alpha} + g_{A\alpha;\mu} - g_{\mu A;\alpha}). \quad (F.13)$$
These are straightforward to show. The first is given by:

\[
\begin{align*}
k_{A;\alpha} &= g_{\beta A} k_{\beta;\alpha} = g_{\beta A} \left[ k_{\beta,\alpha} + \Gamma_{\mu\alpha}^{\beta} k^{\mu} \right] \\
&= g_{\beta A} \left[ \frac{1}{2} k^{\mu} g^{\beta\gamma} (g_{\gamma\mu,\alpha} + g_{\gamma\alpha,\mu} - g_{\mu\alpha,\gamma}) \right] \\
&= -\frac{1}{2} g_{\beta A} k^{\mu} g^{\beta\gamma} g_{\mu\alpha,\gamma} = -\frac{1}{2} g_{t\alpha,A}.
\end{align*}
\] (F.14)

The second is effectively identical, with \( t \to \phi \):

\[
\begin{align*}
l_{A;\alpha} &= g_{\beta A} l_{\beta;\alpha} = g_{\beta A} \left[ l_{\beta,\alpha} + \Gamma_{\mu\alpha}^{\beta} l^{\mu} \right] \\
&= g_{\beta A} \left[ \frac{1}{2} l^{\mu} g^{\beta\gamma} (g_{\gamma\mu,\alpha} + g_{\gamma\alpha,\mu} - g_{\mu\alpha,\gamma}) \right] \\
&= -\frac{1}{2} g_{\beta A} l^{\mu} g^{\beta\gamma} g_{\mu\alpha,\gamma} = -\frac{1}{2} g_{\phi\alpha,A}.
\end{align*}
\]

The third is evaluated in similar fashion:

\[
\begin{align*}
B_{A;\alpha} &= g_{\beta A} B_{\beta;\alpha} = g_{\beta A} \left[ B_{\beta,\alpha} + \Gamma_{\mu\alpha}^{\beta} B^{\mu} \right] \\
&= g_{\beta A} \left[ B_{\beta,\alpha} + \frac{1}{2} B^{\mu} g^{\beta\gamma} (g_{\gamma\mu,\alpha} + g_{\gamma\alpha,\mu} - g_{\mu\alpha,\gamma}) \right] \\
&= g_{\beta A} B_{\beta,\alpha} + \frac{1}{2} B^{\mu} (g_{A\mu,\alpha} + g_{A\alpha,\mu} - g_{\mu\alpha,A}).
\end{align*}
\] (F.15)

Combining the expressions for \( k^{\alpha} \) and \( l^{\alpha} \), we find:

\[
\begin{align*}
r_{A;\alpha} &= k_{A;\alpha} + \Omega F l_{A;\alpha} + \Omega_{F,\alpha} l_{A} \\
&= -\frac{1}{2} (g_{t\alpha,A} + \Omega_{F} g_{\phi\alpha,A}) + \Omega_{F,\alpha} l_{A}.
\end{align*}
\] (F.16)

In application the final \( \Omega_{F,\alpha} \) term will generally immediately drop out of consideration by virtue of stationarity and axisymmetry. We are now ready to evaluate the poloidal
components of the divergence of the stress energy tensor. Exploiting the fact that the
electromagnetic stress energy tensor is made up of vectors with vanishing divergences,
the poloidal components of its divergence are found to be (exploiting stationarity,
axisymmetry, the field-aligned conservation of $\Omega_F$, and $r_A = 0$):

$$T_A^\beta_{;\beta} = \left[ -\frac{\alpha}{4\pi} B_AB^\beta - \frac{B_\gamma B^\gamma}{4\pi} r_A r^\beta + \frac{g_{t\phi} + g_{\phi\phi} \Omega_F}{4\pi \rho_\omega^2} \sqrt{-g} F^\theta r (B_A r^\beta + r_A B^\beta) + \frac{\alpha B_C B^C}{8\pi} g_{A}^\beta \right]_{;\beta}$$

$$= \left[ -\frac{\alpha}{4\pi} B_A \right]_{;\beta} B^\beta - \left[ \frac{B_\gamma B^\gamma}{4\pi} r_A \right]_{;\beta} r^\beta + \left[ \frac{g_{t\phi} + g_{\phi\phi} \Omega_F}{4\pi \rho_\omega^2} \sqrt{-g} F^\theta r (B_A r^\beta + r_A B^\beta) \right]_{;\beta}$$

$$+ \left[ \frac{\alpha B_C B^C}{8\pi} \right]_{,A} - \left[ \frac{(\sqrt{-g} F^\theta r)^2}{8\pi \rho_\omega^2} \right]_{,A}$$

$$= -\frac{1}{4\pi} \left[ g_{\mu A} (\alpha B^\mu)_{,\beta} + \frac{1}{2} \alpha B^\mu (g_{A\mu,\beta} + g_{A\beta,\mu} - g_{\mu\beta,A}) \right] B^\beta$$

$$- \frac{1}{4\pi} B_\gamma B^\gamma \left[ -\frac{1}{2} (g_{t\beta,A} + \Omega_F g_{\phi\beta,A}) \right] r^\beta$$

$$+ \frac{g_{t\phi} + g_{\phi\phi} \Omega_F}{4\pi \rho_\omega^2} \sqrt{-g} F^\theta r \left\{ \left[ \frac{1}{2} B^\mu (g_{A\mu,\beta} + g_{A\beta,\mu} - g_{\mu\beta,A}) \right] r^\beta - \frac{1}{2} (g_{t\beta,A} + \Omega_F g_{\phi\beta,A}) B^\beta \right\}$$

$$+ \frac{1}{8\pi} (\alpha B_C B^C)_{,A} - \frac{1}{8\pi} \left( \frac{(\sqrt{-g} F^\theta r)^2}{\rho_\omega^2} \right)_{,A}$$

$$= -\frac{1}{4\pi} \left[ g_{CA} (\alpha B^C)_{,D} B^D + \frac{1}{2} \alpha B^C g_{AC,D} D^D + \frac{1}{2} \alpha B^C B^D g_{AD,C} - \frac{1}{2} \alpha B^\mu B^\beta g_{\mu\beta,A} \right]$$

$$+ \frac{1}{8\pi} B_\gamma B^\gamma \left[ g_{t\beta,A} + \Omega_F g_{\phi\beta,A} + \Omega_F g_{t\beta,A} + \Omega_F^2 g_{\phi\phi,A} \right]$$

$$+ \frac{g_{t\phi} + g_{\phi\phi} \Omega_F}{8\pi \rho_\omega^2} \sqrt{-g} F^\theta r \left[ B^\phi (g_{t\phi,A} - \Omega_F g_{\phi\phi,A}) - B^\phi (g_{t\phi,A} + \Omega_F g_{\phi\phi,A}) \right]$$

$$+ \frac{1}{8\pi} (\alpha B_C B^C)_{,A} - \frac{1}{8\pi} \left( \frac{(\sqrt{-g} F^\theta r)^2}{\rho_\omega^2} \right)_{,A}$$
\[= -\frac{1}{4\pi} \left[ (\alpha B_A)_{,D} B^D - \alpha B^C B^D g_{C,A,D} + \alpha B^C B^D g_{AC,D} - \frac{1}{2} \alpha B^\mu B^\beta g_{\mu\beta,A} \right] \]

\[+ \frac{1}{8\pi} B_\gamma \gamma \left[ \alpha, A - (2g_{t\phi} + 2\Omega_F g_{\phi\phi}) \Omega_{F,A} \right] \]

\[= -\frac{1}{4\pi} \left[ (\alpha B_A)_{,D} B^D \right] + \frac{1}{8\pi} \alpha B^\phi B^\phi g_{\phi\phi,A} \]

\[+ \frac{1}{8\pi} B_C B^C \left[ \alpha, A - (2g_{t\phi} + 2\Omega_F g_{\phi\phi}) \Omega_{F,A} \right] + \frac{1}{8\pi} g_{\phi\phi} B^\phi B^\phi \left[ \alpha, A - (2g_{t\phi} + 2\Omega_F g_{\phi\phi}) \Omega_{F,A} \right] \]

\[-\frac{1}{4\pi} \left( g_{t\phi} + g_{\phi\phi} \Omega_F \right) B^\phi B^\phi \left( g_{t\phi,A} + \Omega_F g_{\phi\phi,A} \right) \]

\[+ \frac{1}{8\pi} \left[ 2 (\alpha B_C)_{,A} B^C - \alpha, A B_C B^C \right] - \frac{1}{8\pi\rho_\omega^2} \left( \sqrt{-gF^{\theta r}} \right)_A \]

\[+ \frac{1}{8\pi} B^\phi B^\phi \left( 2g_{t\phi g_{t\phi,A} - g_{t\phi A} g_{\phi\phi} - g_{t\phi} g_{\phi\phi,A} \right) \]

\[= \frac{1}{4\pi} B^C \left[ (\alpha B_C)_{,A} - (\alpha B_A)_{,C} \right] - \frac{g_{t\phi} + \Omega_F g_{\phi\phi}}{4\pi} B_C B^C \Omega_{F,A} - \frac{1}{8\pi\rho_\omega^2} \left( \sqrt{-gF^{\theta r}} \right)_A \]

\[+ \frac{1}{8\pi} B^\phi B^\phi \left[ \alpha g_{\phi\phi,A} + g_{\phi\phi} \alpha, A - 2g_{\phi\phi} (g_{t\phi} + g_{\phi\phi} \Omega_F) \Omega_{F,A} \right] \]

\[- 2 (g_{t\phi} + g_{\phi\phi} \Omega_F) (g_{t\phi,A} + \Omega_F g_{\phi\phi,A}) + 2g_{t\phi} g_{t\phi,A} - g_{t\phi A} g_{\phi\phi} - g_{t\phi} g_{\phi\phi,A} \right] \]

\[= \frac{1}{4\pi} B^C \left[ (\alpha B_C)_{,A} - (\alpha B_A)_{,C} + (g_{t\phi} + \Omega_F g_{\phi\phi}) (B_A \Omega_{F,C} - B_C \Omega_{F,A}) \right] \]

\[+ \frac{1}{8\pi} B^\phi B^\phi \left[ \alpha g_{\phi\phi,A} - (g_{t\phi} g_{\phi\phi})_{,A} - 2 (g_{t\phi} g_{\phi\phi} \Omega_F)_{,A} - (g_{\phi\phi} \Omega_F^2)_{,A} \right] \]

\[- \frac{1}{8\pi\rho_\omega^2} \left( \sqrt{-gF^{\theta r}} \right)^2_1 \]
\[
= \frac{1}{4\pi} B^C \left[ (\alpha B_C)_A - (\alpha B_A)_C + (g_{\phi} + \Omega_F g_{\phi}) (B_A \Omega_{F,C} - B_C \Omega_{F,A}) \right]
- \frac{1}{8\pi \rho_w^2} \left( \sqrt{-g} F^{\theta r} \right)^2. \tag{F.17}
\]

We express the divergence in this way in order to emphasize its symmetry. The standard form of the force-free transfield equation as expressed in [111] is readily found from the above expression, as the antisymmetry in the first term selects \(B^C\)_[\(\ldots\)] in \(T_{r^\beta;}^\beta\) and \(B^r[\ldots] \sim A_{\phi,r}[\ldots]\) in \(T_{r^\beta;}^\beta\); while \(\sqrt{-g} F^{\theta r}\) in the last term is known to be a function of \(A_\phi\) from \(T_{\phi}^{\beta;}^\beta = 0\). The above form is more useful than the force-free transfield equation, however, as when plasma inertial effects are added the \(B^C\) prefactor can be trivially transformed into an \(N^C\) prefactor and the \(\sqrt{-g} F^{\theta r}\) term can be easily combined with derivatives of plasma parameters to form terms like \(E_{,A}, L_{,A}\), and \(\eta_{,A}\).

### Fluid Components

We now consider the fluid component of the stress energy tensor, which is given by:

\[
T_F^{\alpha \beta} = (\rho + p) u^\alpha u^\beta - pg^{\alpha \beta}. \tag{F.18}
\]

This is for a perfect fluid (no heat conduction or viscosity); \(\rho\) is relativistic energy density (rest mass, thermal energy, etc.) and \(p\) is proper pressure. As with the electromagnetic stress energy tensor, it is convenient to express this in terms of divergenceless vectors:

\[
T_F^{\alpha \beta} = \frac{\mu}{n} N^\alpha N^\beta - pg^{\alpha \beta}. \tag{F.19}
\]

Here \(n\) is particle number density in the fluid rest frame and \(\mu\) is the relativistic enthalpy; \(\mu \equiv (\rho + p)/n\). Particle number conservation requires that \(N^\alpha_{;\alpha} = 0\), such
that the fluid stress energy tensor is now composed of divergenceless vectors. In
order to calculate the poloidal components of the divergence of the fluid stress energy
tensor, we need to evaluate $N_{A;\alpha}$:

\[ N_{A;\alpha} = g_{\beta A} N_{;\alpha}^\beta = g_{\beta A} \left[ N_{;\alpha}^\beta + \Gamma_{\mu\alpha}^\beta N^\mu \right] 
= g_{\beta A} \left[ N_{;\alpha}^\beta + \frac{1}{2} N^\mu g^{\beta\gamma} (g_{\gamma\mu;\alpha} + g_{\gamma\alpha;\mu} - g_{\mu\alpha;\gamma}) \right] 
= g_{\beta A} N_{;\alpha}^\beta + \frac{1}{2} N^\mu (g_{A\mu;\alpha} + g_{A\alpha;\mu} - g_{\mu\alpha;A}). \quad (F.20) \]

We may now calculate the poloidal components of the divergence of the fluid stress
tensor:

\[ T_A^{;\beta} = \left[ \frac{\mu}{n} N_A N_{^\beta} - pg_A^{;\beta} \right]^{;\beta} = \left( \frac{\mu}{n} N_A \right)^{;\beta} N^\beta - p_A \]
\[ = N_A N^C \left( \frac{\mu}{n} N_A \right)_{,C} - p_A + \frac{\mu}{n} N^\beta \left[ g_{A\mu} N_{;\beta}^{\mu} A + \frac{1}{2} N^\mu (g_{A\mu;\beta} + g_{A\beta;\mu} - g_{\mu\beta;A}) \right] \]
\[ = N_A N^C \left( \frac{\mu}{n} N_A \right)_{,C} - p_A + \frac{\mu}{n} N^C g_{A\mu} N_{;\beta}^{\mu} A + \frac{\mu}{2n} N^C N^D g_{A\beta,D} \]
\[ + \frac{\mu}{2n} N^C N^D g_{AC,D} - \frac{\mu}{2n} N^\beta N^\mu g_{\mu\beta;A} \]
\[ = N_A N^C \left( \frac{\mu}{n} N_A \right)_{,C} - p_A + \frac{\mu}{n} N^C N_{A,C} - \frac{\mu}{2n} N^\beta N^\mu g_{\mu\beta;A} \]
\[ = N^C \left[ \left( \frac{\mu}{n} N_A \right)_{,C} - \left( \frac{\mu}{n} N^C \right)_{,A} \right] + N^C \left( \frac{\mu}{n} N_C \right)_{,A} - p_A + \frac{\mu}{2n} (N_{\beta} N^\beta)_{,A} - \frac{\mu}{n} N^\beta N_{\beta;A} \]
\[ = N^C \left( \frac{\mu}{n} N_A \right)_{,C} - \frac{\mu}{n} N^C \left( \frac{\mu}{n} N_A \right)_{,A} + N^C N_{C} \left( \frac{\mu}{n} N_C \right)_{,A} - p_A + \mu n_{,A} - \frac{\mu}{n} N^A N_{t;A} - \frac{\mu}{n} N^\phi N_{\phi;A} \]
\[ = N^C \left( \frac{\mu}{n} N_A \right)_{,C} - \frac{\mu}{n} N^C \left( \frac{\mu}{n} N_A \right)_{,A} + \left( n^2 - N_t N^t - N_{\phi} N^\phi \right) \left( \frac{\mu}{n} N_A \right)_{,A} - p_A + \mu n_{,A} \]
\[ - \frac{\mu}{n} N^t N_{t;A} - \frac{\mu}{n} N^\phi N_{\phi;A} \]
\[ N^C \left[ \left( \frac{\mu}{n} N_A \right)_{,C} - \left( \frac{\mu}{n} N_C \right)_{,A} \right] + n \mu,_{A} - p,_{A} - N^t \left( \frac{\mu}{n} N_t \right)_{,A} - N^\phi \left( \frac{\mu}{n} N_\phi \right)_{,A} = \]

\[ N^C \left[ \left( \frac{\mu}{n} N_A \right)_{,C} - \left( \frac{\mu}{n} N_C \right)_{,A} \right] - N^t \left( \frac{\mu}{n} N_t \right)_{,A} - N^\phi \left( \frac{\mu}{n} N_\phi \right)_{,A}. \] (F.21)

In the last line we used the condition of an adiabatic flow to eliminate the derivatives of pressure and enthalpy; \( n \mu,_{\alpha} - p,_{\alpha} = 0 \), as we will now show. From thermodynamics we have \( dU = dQ - pdV \). In terms of the energy density \( \rho \), particle number density \( n \), and entropy per particle \( S \), this yields [94]:

\[ nTdS = d\rho - \mu dn. \] (F.22)

Here \( T \) refers to a temperature. We have an adiabatic flow with \( dS = 0 \), so we immediately recover:

\[ 0 = \rho,_{A} - \mu n,_{A} = n \left( \frac{\rho,_{A} + p,_{A}}{n} \right) - p,_{A} - \mu n,_{A} = n \mu,_{A} - p,_{A}. \] (F.23)

Now that we have both fluid and electromagnetic stress energy tensor divergences, we must couple them and find a unified expression.

**Coupling Fields and Fluid**

The fluid and electromagnetic stress energy tensors may be coupled via the condition of ideal magnetohydrodynamics, expressed as \( u_{\alpha}F^{\alpha\beta} = 0 \). From that condition we arrive at a conserved particle flux per unit flux tube \( \eta \), which may be written as:

\[ N^A = -\eta B^A, \]

\[ N^\phi = \Omega^F N^t - \eta B^\phi. \] (F.24)
By exploiting those definitions we may couple the fluid to the electromagnetic fields. In order to do so, we first combine the electromagnetic and fluid stress energy divergences onto a single equation:

\[ T_{A}^{\gamma;\beta} = \frac{1}{4\pi} B^{C} \left[ (\alpha B_{C})_{,A} - (\alpha B_{A})_{,C} + (g_{t\phi} + \Omega_{F} g_{\phi\phi}) (B_{A} \Omega_{F,C} - B_{C} \Omega_{F,A}) \right] 
+ N^{C} \left[ \left( \frac{\mu}{n} N_{A} \right)_{,C} - \left( \frac{\mu}{n} N_{C} \right)_{,A} \right] 
- \frac{1}{8\pi \rho_{\omega}^{2}} \left( \sqrt{-g} F^{\theta r} \right)_{,A} - N^{t} \left( \frac{\mu}{n} N_{t} \right)_{,A} - N^{\phi} \left( \frac{\mu}{n} N_{\phi} \right)_{,A}. \]  

(F.25)

In order to proceed further, we define the Alfvén Mach number \( M^{2} = 4\pi \mu \eta^{2}/n \) and the quantity \( e_{c} \) as \( \mu \eta e_{c} = \eta E - \eta L \Omega_{F} = \mu \eta (u_{t} + u_{\phi} \Omega_{F}) \). In the limit of a cold flow, where \( \mu \) is conserved, \( e_{c} \) is also a conserved quantity. Using \( M^{2} \), \( e_{c} \), and \( \eta \), we now combine the similar parts of the transfield equation to find:

\[ T_{A}^{\gamma;\beta} = \frac{1}{4\pi} B^{C} \left\{ (\alpha B_{C})_{,A} - (\alpha B_{A})_{,C} + (g_{t\phi} + \Omega_{F} g_{\phi\phi}) (B_{A} \Omega_{F,C} - B_{C} \Omega_{F,A}) \right\} 
- 4\pi \eta \left[ \left( \frac{\mu}{n} N_{A} \right)_{,C} - \left( \frac{\mu}{n} N_{C} \right)_{,A} \right] 
- \frac{1}{8\pi \rho_{\omega}^{2}} \sqrt{-g} F^{\theta r} \left( \sqrt{-g} F^{\theta r} \right)_{,A} - \frac{N^{t}}{\eta} \left( \eta E - \frac{1}{4\pi} \sqrt{-g} F^{\theta r} \Omega_{F} \right)_{,A} 
+ \frac{N^{\phi}}{\eta} \left( \eta L - \frac{1}{4\pi} \sqrt{-g} F^{\theta r} \right)_{,A} + \frac{\mu}{n\eta} \left( N^{t} N_{t} + N^{\phi} N_{\phi} \right) \eta_{,A}. \]  

(F.26)

Examining the last two lines, we find:

\[ L^{2} L = -\frac{1}{4\pi \rho_{\omega}^{2}} \sqrt{-g} F^{\theta r} \left( \sqrt{-g} F^{\theta r} \right)_{,A} - \frac{N^{t}}{\eta} \left( \eta E - \frac{1}{4\pi} \sqrt{-g} F^{\theta r} \Omega_{F} \right)_{,A} 
+ \frac{N^{\phi}}{\eta} \left( \eta L - \frac{1}{4\pi} \sqrt{-g} F^{\theta r} \right)_{,A} + \frac{\mu}{n\eta} \left( N^{t} N_{t} + N^{\phi} N_{\phi} \right) \eta_{,A} \]

\[ = -\frac{N^{t}}{\eta} \left( \eta E \right)_{,A} + \frac{N^{\phi}}{\eta} \left( \eta L \right)_{,A} + \frac{1}{4\pi} \frac{N^{t}}{\eta} \sqrt{-g} F^{\theta r} \left( \Omega_{F} \right)_{,A} + \frac{\mu}{n\eta} \left( N^{t} N_{t} + N^{\phi} N_{\phi} \right) \eta_{,A} \]
Because $B^\phi = \sqrt{-g} F^{\theta r}/\rho_\omega^2$, the quantity in the square brackets on the last line vanishes by virtue of the definition of $\eta$. The remaining terms on the upper line all involve derivatives of field aligned conserved quantities, so we may reduce the two $T_{\alpha\beta} = X_A$ equations into a single condition, the transfield equation (where $f,\psi$ denotes a derivative with respect to $A_\phi$):

$$X_A A_{\phi,A} = \frac{1}{4\pi\sqrt{-g}} \left\{ (\alpha B_\theta)_{,r} - (\alpha B_r)_{,\theta} + (g_{l\phi} + \Omega_F g_{\phi\phi}) (B_r \Omega_{F,\theta} - B_\theta \Omega_{F,r}) \right. \\
- 4\pi \eta \left[ \left( \frac{\mu}{n} N_r \right)_{,\theta} - \left( \frac{\mu}{n} N_\theta \right)_{,r} \right] \left\} - \frac{N^t}{\eta} (\eta E)_{,\psi} + \frac{N^\phi}{\eta} (\eta L)_{,\psi} + \frac{1}{4\pi} \frac{N^t}{\eta} \sqrt{-g} F^{\theta r} (\Omega_F)_{,\psi} + \frac{\mu}{n \eta} (N^t N_t + N^\phi N_\phi) \eta,\psi. \right.$$

(F.28)

We now collapse the four-vector terms in the curly brackets to find:

$$4V T = \frac{1}{4\pi\sqrt{-g}} \left\{ (\alpha B_\theta)_{,r} - (\alpha B_r)_{,\theta} - 4\pi \eta \left[ \left( \frac{\mu}{n} N_r \right)_{,\theta} - \left( \frac{\mu}{n} N_\theta \right)_{,r} \right] \right. \\
= \frac{1}{4\pi\sqrt{-g}} \left\{ (\alpha g_{\theta\theta} B^\theta)_{,r} - (\alpha g_{r\theta} B^r)_{,\theta} - 4\pi \eta \left[ \left( -\frac{\mu}{n} g_{rr} \eta B^r \right)_{,\theta} + \left( \frac{\mu}{n} g_{r\theta} \eta B^\theta \right)_{,r} \right] \right\} \\
= \frac{1}{4\pi\sqrt{-g}} \left\{ \frac{g_{\theta\theta}}{\sqrt{-g}} \left( \alpha - 4\pi \frac{\mu \eta^2}{n} \right) A_{\phi,r} \right\}_{,r} + \left\{ \frac{g_{rr}}{\sqrt{-g}} \left( \alpha - 4\pi \frac{\mu \eta^2}{n} \right) A_{\phi,\theta} \right\}_{,\theta} \left\} + \frac{1}{4\pi\sqrt{-g}} \left[ \frac{g_{rr}}{\sqrt{-g}} A_{\phi,\eta,\theta} + \frac{g_{\theta\theta}}{\sqrt{-g}} A_{\phi,\eta,r} \right]. \right. \right.$$

(F.29)

To proceed further, we next note that for a conserved quantity $f$:

$$f^t = \frac{df}{dA_\phi} = f_{,\phi} A_{\phi,\alpha} A_\phi^{,\alpha} = \frac{1}{\rho_\omega^2 B_p^2} \left[ g^{rr} F_{r\phi} f_{,r} + g^{\theta\theta} F_{\theta\phi} f_{,\theta} \right] = -\frac{1}{\sqrt{-g} B_p^2} \left[ B_\theta f_{,r} - B_r f_{,\theta} \right].$$

(F.30)
Here $B_p^2$ is given by:

$$B_p^2 = -\frac{1}{\rho^2} \left[ g^{rr} F_{r\phi}^2 + g^{\theta\theta} F_{\phi\phi}^2 \right], \quad (F.31)$$

such that:

$$f' = \frac{g^{rr} A_{\phi,r} X_r + g^{\theta\theta} A_{\phi,\theta} X_\theta}{g^{rr} A_{\phi,r}^2 + g^{\theta\theta} A_{\phi,\theta}^2}. \quad (F.32)$$

In Boyer-Lindquist coordinates, we have $g^{rr} = -\Delta/\Sigma$ and $g^\theta = -1/\Sigma$, such that:

$$X' = \frac{\Delta A_{\phi,r} X_r + A_{\phi,\theta} X_\theta}{\Delta A_{\phi,r}^2 + A_{\phi,\theta}^2}. \quad (F.33)$$

Therefore the last line of 4VT (the four vector term of the transfield equation) found above reduces to:

$$4VT_{LL} = \frac{1}{4\pi\sqrt{-g}} \left[ \frac{4\pi \mu \eta}{n} \left( \frac{g^{rr} A_{\phi,\theta} \eta,\theta + g_{\theta\theta} A_{\phi,r} \eta,r}{\sqrt{-g}} \right) \right]$$

$$= -\frac{1}{4\pi \Sigma \Delta \sin^2 \theta} \frac{4\pi \mu \eta}{n} [\Delta A_{\phi,r} \eta,r + A_{\phi,\theta} \eta,\theta]$$

$$= -\frac{1}{\Sigma \Delta \sin^2 \theta} \frac{\mu \eta}{n} (\Delta A_{\phi,r}^2 + A_{\phi,\theta}^2) \eta,\psi. \quad (F.34)$$

The term involving $\Omega_F$ in the transfield equation reduces to a similar form:

$$OFT = \frac{1}{4\pi\sqrt{-g}} \left\{ (g_{\theta\phi} + \Omega_F g_{\phi\phi}) (B_r \Omega_{F,\theta} - B_\theta \Omega_{F,r}) \right\}$$

$$= \frac{1}{4\pi \Sigma \sin \theta} G_\phi \left\{ -\frac{g^{rr}}{\sqrt{-g}} A_{\phi,\theta} \Omega_{F,\theta} - \frac{g_{\theta\theta}}{\sqrt{-g}} A_{\phi,r} \Omega_{F,r} \right\}$$

$$= \frac{1}{4\pi \Sigma \Delta \sin^2 \theta} G_\phi \left( A_{\phi,r} \Omega_{F,r} + A_{\phi,\theta} \Omega_{F,\theta} \right)$$

$$= \frac{1}{4\pi \Sigma \Delta \sin^2 \theta} G_\phi \left( \Delta A_{\phi,r}^2 + A_{\phi,\theta}^2 \right) \Omega_{F,\psi}. \quad (F.35)$$
Combining all of the above results, we may write the transfield equation as:

\[
\frac{X_A}{A_{\phi,A}} = \frac{1}{4\pi \sqrt{-g}} \left\{ \left[ \frac{g_{\theta \theta}}{\sqrt{-g}} (\alpha - \mathcal{M}^2) A_{\phi,r} \right] + \left[ \frac{g_{\tau \tau}}{\sqrt{-g}} (\alpha - \mathcal{M}^2) A_{\phi,\theta} \right] \right\},
\]

\[
+ \frac{1}{4\pi \Delta \sin^2 \theta} G_\phi (\Delta A^2_{\phi,r} + A^2_{\phi,\theta}) \Omega_{F,\psi} - \frac{1}{\Delta \sin^2 \theta} \frac{\mu \eta}{n} (\Delta A^2_{\phi,r} + A^2_{\phi,\theta}) \eta,\psi
\]

\[
- \frac{N^t}{\eta} (\eta E)_{\psi} + \frac{N^t}{\eta} (\eta L)_{\psi} + \frac{1}{4\pi} \frac{N^t}{\eta} \sqrt{-g} F^\theta r (\Omega_F)_{\psi} + \frac{\mu}{\eta} (N^t N_t + N^\phi N_\phi) \eta,\psi.
\] (F.36)

In order to make this more useful, we wish to express the last line in terms of \(e_c\); we therefore find (ignoring the trailing \(\eta,\psi\) term):

\[
LL = -\frac{N^t}{\eta} (\eta E)_{\psi} + \frac{N^t}{\eta} (\eta L)_{\psi} + \frac{1}{4\pi} \frac{N^t}{\eta} \sqrt{-g} F^\theta r (\Omega_F)_{\psi}
\]

\[
= -\frac{N^t}{\eta} (\mu \eta e_c + \eta L \Omega_F)_{\psi} + \frac{N^t}{\eta} (\eta L)_{\psi} + \frac{1}{4\pi} \frac{N^t}{\eta} \sqrt{-g} F^\theta r (\Omega_F)_{\psi}
\]

\[
= -\frac{N^t}{\eta} (\mu \eta e_c)_{\psi} + \frac{1}{\eta} (N^\phi - N^t \Omega_F) (\eta L)_{\psi} + \frac{N^t}{\eta} \left( -\eta L + \frac{1}{4\pi} \sqrt{-g} F^\theta r \right) \Omega_{F,\psi}
\]

\[
= -\frac{N^t}{\eta} (\mu \eta e_c)_{\psi} - B^\phi (\eta L)_{\psi} + \frac{N^t}{\eta} (\mu \eta u_\phi) \Omega_{F,\psi}
\]

\[
= -\frac{N^t}{\eta} (\mu \eta e_c)_{\psi} - \frac{4\pi}{\rho^2_\omega} (\eta L + \mu \eta u_\phi) (\eta L)_{\psi} + N^t \mu \eta u_\phi \Omega_{F,\psi}
\]

\[
= -\frac{N^t}{\eta} (\mu \eta e_c)_{\psi} - \frac{1}{8\pi \rho^2_\omega} (4\pi \eta L)_{\psi}^2 - \frac{4\pi}{\rho^2_\omega} \mu \eta u_\phi (\eta L)_{\psi} + N^t \mu \eta u_\phi \Omega_{F,\psi}. \] (F.37)

Therefore we can write the transfield equation as:

\[
4\pi \sqrt{-g} \frac{X_A}{A_{\phi,A}} = \left[ \frac{g_{\theta \theta}}{\sqrt{-g}} (\alpha - \mathcal{M}^2) A_{\phi,r} \right] + \left[ \frac{g_{\tau \tau}}{\sqrt{-g}} (\alpha - \mathcal{M}^2) A_{\phi,\theta} \right] \eta,\psi
\]

\[
+ \frac{1}{\Delta \sin \theta} G_\phi (\Delta A^2_{\phi,r} + A^2_{\phi,\theta}) \Omega_{F,\psi} - \frac{1}{\Delta \sin \theta} \frac{4\pi \mu \eta}{n} (\Delta A^2_{\phi,r} + A^2_{\phi,\theta}) \eta,\psi
\]

\[
- \frac{N^t}{\eta} (\mu \eta e_c)_{\psi} - \frac{\sqrt{-g}}{2\rho^2_\omega} (4\pi \eta L)_{\psi}^2 - \frac{4\pi \sqrt{-g}}{\rho^2_\omega} \mu \eta u_\phi (4\pi \eta L)_{\psi}
\]

\[
+ N^t \frac{4\pi \sqrt{-g} \mu \eta u_\phi \Omega_{F,\psi}}{\frac{N^2_0}{B^2_0} \sqrt{-g} \frac{4\pi \mu}{n \eta} (N^t N_t + N^\phi N_\phi) \eta,\psi. \] (F.38)
The last $\eta,\psi$ term may now be simplified to find:

\[
LT = \sqrt{-g} \frac{4\pi\mu}{n\eta} \left( N^t N_t + N^\phi N_\phi \right) \eta,\psi
\]

\[
= \sqrt{-g} \frac{4\pi\mu}{n\eta} \left[ n^2 - g_{rr} (N^r)^2 - g_{\theta\theta} (N^\theta)^2 \right] \eta,\psi
\]

\[
= \sqrt{-g} \frac{4\pi\mu n}{\eta} \eta,\psi + \frac{1}{\sqrt{-g} n} \left( -g_{rr} A_{\phi,\theta}^2 - g_{\theta\theta} A_{\phi,\phi}^2 \right) \eta,\psi
\]

\[
= \sqrt{-g} \frac{4\pi\mu n}{\eta} \eta,\psi + \frac{1}{\Delta \sin \theta} \left( \Delta A_{\phi,r}^2 + A_{\phi,\phi}^2 \right) \eta,\psi, \quad (F.39)
\]

The last term here cancels with the second term on the second line of the transfield equation written above, so we finally conclude that:

\[
4\pi\sqrt{-g} \frac{X_A}{A_{\phi,A}} = -\frac{1}{\sin \theta} \left[ \left( \alpha - \mathcal{M}^2 \right) A_{\phi,r} \right]_r - \frac{1}{\Delta} \left[ \frac{1}{\sin \theta} \left( \alpha - \mathcal{M}^2 \right) A_{\phi,\theta} \right]_\theta
\]

\[
+ \frac{1}{\Delta \sin \theta} G_\phi \left( \Delta A_{\phi,r}^2 + A_{\phi,\phi}^2 \right) \Omega_{F,\psi} - \frac{\Sigma}{2\Delta \sin \theta} (4\pi\eta L)_{\psi}^2
\]

\[
- \frac{4\pi\sqrt{-g} N^t}{\eta} \left( \mu \eta e_c \right)_{\psi} - \frac{4\pi\sqrt{-g}}{\rho_{\psi}^2} \mu \eta u_{\phi} (4\pi\eta L)_{\psi}
\]

\[
+ 4\pi\sqrt{-g} N^t \mu u_{\phi} \Omega_{F,\psi} + \sqrt{-g} \frac{4\pi\mu n}{\eta} \eta,\psi. \quad (F.40)
\]

Solving the above equation with self-consistent field aligned conserved quantities is the core difficulty of studying energy-extracting black hole magnetospheres in the limit of ideal magnetohydrodynamics. It is highly non-linear, and finding self-consistent field-aligned conserved quantities is directly coupled to any solution process, as those quantities can be highly sensitive to the structure of the magnetic field.

**The Force-Free Limit**

While the form of the transfield equation found above is useful within the context of ideal magnetohydrodynamics, it is not completely obvious how it might smoothly
transition to force-free magnetospheres. The force-free limit is the magnetically dominated limit, so we expect some limit related to $\eta \to 0$ to be appropriate. In order to make such a transition robust, we introduce the parameter $\chi$, defined in terms of the Alfvén Mach number as:

$$\mathcal{M}^2 = \frac{4\pi \mu \eta^2}{n} = 4\pi \frac{\eta}{n} (\mu \eta) = 4\pi \chi (\mu \eta).$$  \hspace{1cm} (F.41)$$

To find an additional definition of $\chi$, we examine the poloidal components of the magnetic field four vector and exploit the definition of conserved particle flux $\eta$ to find:

$$B_A B^A = g_{rr} (B^r)^2 + g_{\theta \theta} (B^\theta)^2$$

$$= g_{rr} \left( -\frac{n}{\eta} u^r \right)^2 + g_{\theta \theta} \left( -\frac{n}{\eta} u^\theta \right)^2$$

$$= \frac{n^2}{\eta^2} \left( 1 - u_t u^t - u_\phi u^\phi \right)$$

$$= \frac{n^2}{\eta^2} \frac{1}{\rho_\omega^2} \left( \rho_\omega^2 + g_{\phi \phi} u_t^2 - g_{t \phi} u_t u_\phi + g_{u \phi} u_\phi^2 \right).$$  \hspace{1cm} (F.42)$$

In terms of the vector potential, this contraction is given by:

$$B_A B^A = g_{rr} \left( -\frac{1}{\sqrt{-g}} A_{\phi, r} \right)^2 + g_{\theta \theta} \left( \frac{1}{\sqrt{-g}} A_{\phi, \theta} \right)^2$$

$$= -\frac{1}{\Sigma \Delta \sin^2 \theta} \left( \Delta A_{\phi, r}^2 + A_{\phi, \theta}^2 \right).$$  \hspace{1cm} (F.43)$$

Comparing the two expressions for $B_A B^A$, we then deduce that:

$$\chi = \sqrt{\frac{n^2}{\eta^2}}$$

$$= \sqrt{-\frac{\rho_\omega^2 + g_{\phi \phi} u_t^2 - g_{t \phi} u_t u_\phi + g_{u \phi} u_\phi^2}{\Delta A_{\phi, r}^2 + A_{\phi, \theta}^2}}.$$  \hspace{1cm} (F.44)$$
Technically $\chi$ is a signed quantity (as $\eta$ is a signed quantity), but we have chosen to take the positive value as for an inflow (with an outward directed magnetic field) $\eta$ is positive. Using this expression in the transfield equation, we conclude that:

$$4\pi\sqrt{-g}\frac{X_A}{A_{\phi,A}} = -\frac{1}{\sin \theta} \left[ (\alpha - 4\pi \chi \mu \eta) A_{\phi,r} \right]_r - \frac{1}{\Delta} \left[ \frac{1}{\sin \theta} (\alpha - 4\pi \chi \mu \eta) A_{\phi,\theta} \right]_\theta$$

$$+ \frac{G_\phi}{\Delta \sin \theta} \left( \Delta A_{\phi,r}^2 + A_{\phi,\theta}^2 \right) \Omega_{\psi,\psi} - \frac{\Sigma}{2\Delta \sin \theta} (4\pi \eta L)^2_{,\psi} - \frac{4\pi \sqrt{-g}}{\rho_\omega^2} \mu \eta u_\phi \left( 4\pi \eta L \right)_{,\psi}$$

$$+ \frac{4\pi \sqrt{-g} u^t}{\chi} \left[ - \left( \mu \eta e_c \right)_{,\psi} + \mu \eta u_\phi \Omega_{\psi,\psi} \right] + \frac{4\pi \sqrt{-g}}{\chi} \mu \eta (\ln \eta)_{,\psi} \quad (F.45)$$

This allows us to identify a useful force-free limit: the limit $\mu \to 0$. This is useful because it allows us to avoid explicitly taking the limit $\eta \to 0$, thereby maintaining the coupling effects (i.e. the coupling of the electromagnetic fields with the plasma) driven by $\eta$. It also allows us to calculate the Alfvén Mach number without having to (directly) worry about what happens as $\eta \to 0$. 
APPENDIX G

THE BERNOULLI EQUATION
The Bernoulli equation in classical hydrodynamics is typically used to find the density, pressure, and fluid velocity of a hydrodynamic flow along a streamline by applying the conservation of energy. In our case there are four conserved quantities along a streamline (poloidal magnetic field line): the particle flux per unit flux tube $\eta$, field line angular velocity $\Omega_F$, average energy flux per particle $E$, and average angular momentum flux per particle $L$.

When those quantities are combined with the restriction that the plasma’s four velocity be properly normalized ($u_{\alpha}u^\alpha = 1$), a “Bernoulli equation” emerges that relates the velocity of the plasma to the various conserved parameters and flux tube structure. There are many different “Bernoulli equations” in the literature with different focuses; in this appendix we will derive a few of the more useful representations (that utility arising both from their final form and the procedures used to arrive at the given representations).

### Canonical Integral Form

In this section we will arrive at the integral form of the Bernoulli equation used in Takahashi et al. 1990 [101]. When appropriate the variables used in that work will also be applied here, in an attempt to provide a kind of “Rosetta Stone” between their arguments and conclusions and our current work. The overall procedure in this section is to find useful representations of the various components of the plasma’s four velocity, then combine them into a single Bernoulli equation by applying the velocity normalization condition. We begin with the conserved energy $E$ and angular momentum $L$:

$$E = \mu u_t + \frac{1}{4\pi \eta} \sqrt{-gF^{\theta r} \Omega_F},$$
\[ L = -\mu u_\phi + \frac{1}{4\pi \eta} \sqrt{-g} F^{0r}. \]  

We may then immediately write down expressions for \( u_t \) and \( u_\phi \):

\[ \mu u_t = E - \frac{1}{4\pi \eta} \sqrt{-g} F^{0r} \Omega_F, \]
\[ \mu u_\phi = -L + \frac{1}{4\pi \eta} \sqrt{-g} F^{0r}. \]  

In their current form these expressions are not terribly useful due to their dependency upon the toroidal magnetic field \( \sqrt{-g} F^{0r} \). We therefore apply the definition of the conserved particle flux per unit flux tube \( \eta \) to find:

\[ \sqrt{-g} F^{0r} = \frac{n}{\eta} \rho_\omega^2 \left( -u_\phi + \Omega_F u^t \right) \]
\[ = \frac{n}{\eta} \rho_\omega^2 \left[ - \left( g^{0\phi} u_t + g^{\phi\phi} u_\phi \right) + \Omega_F \left( g^{tt} u_t + g^{t\phi} u_\phi \right) \right] \]
\[ = \frac{n}{\eta} \left[ \left( g_{tt} + g_{t\phi} \Omega_F \right) u_\phi - \left( g_{t\phi} + g_{\phi\phi} \Omega_F \right) \right] \]
\[ = (G_t u_\phi - G_\phi u_t). \]  

Using this result in our expressions for \( E \), we find:

\[ E = \mu u_t + \frac{1}{4\pi \eta} \Omega_F \left[ \frac{n}{\eta} (u_\phi G_t - u_t G_\phi) \right] \]
\[ = \mu u_t - \Omega_F \left( \frac{n}{4\pi \mu_\eta^2} \right) (\mu u_t G_\phi - \mu u_\phi G_t) \]
\[ = \mu u_t - \Omega_F \frac{1}{M^2} (\mu u_t G_\phi - \mu u_\phi G_t). \]  

\[ \text{(G.1)} \]

\[ \text{(G.2)} \]

\[ \text{(G.3)} \]

\[ \text{(G.4)} \]
Here $\mathcal{M}$ is the relativistic Alfvén Mach number. An identical procedure follows for $L$, and we are left with:

$$E = \mu u_t - \Omega_F \frac{1}{\mathcal{M}} (\mu u_t G_\phi - \mu u_\phi G_t),$$

(G.5)

$$L = -\mu u_\phi - \frac{1}{\mathcal{M}^2} (\mu u_t G_\phi - \mu u_\phi G_t).$$

(G.6)

We now have two linear equations for $u_t$ and $u_\phi$ which may be immediately solved to find:

$$\mu u_t = \frac{EM^2 - G_t (E - L\Omega_F)}{M^2 - (G_t + \Omega_F G_\phi)} = \frac{EM^2 - G_t e}{M^2 - \alpha},$$

$$\mu u_\phi = -\frac{LM^2 + G_\phi (E - L\Omega_F)}{M^2 - (G_t + \Omega_F G_\phi)} = -\frac{LM^2 + G_\phi e}{M^2 - \alpha}.$$ (G.7)

Here the conserved quantity $e$ is defined as $e \equiv E - L\Omega_F$. We now need to find expressions for the poloidal components of the plasma’s four velocity. In principle we could do so by exploiting the definition of the particle flux $\eta$ to relate them to the poloidal magnetic field via the Alfvén Mach number, but for historical reasons we simply leave them as the unknown variable $u_p^2$ via $u_p^2 \equiv -u_A u^4$. Applying the four velocity normalization condition, we then find:

$$\mu^2 (1 + u_p^2) = \mu^2 (1 - u_r u^r - u_\theta u^\theta)$$

$$= \mu^2 (u_t u^t + u_\phi u^\phi)$$

$$= \mu^2 [u_t (g^{tt} u_t + g^{t\phi} u_\phi) + u_\phi (g^{\phi\phi} u_\phi + g^{t\phi} u_t)]$$

$$= (\mu u_t)^2 g^{tt} + 2 (\mu u_t u_\phi) g^{t\phi} + (\mu u_\phi)^2 g^{\phi\phi}$$

$$= \frac{(EM^2 - G_t e)^2 g^{tt} - 2 (EM^2 - G_t e) (LM^2 + G_\phi e) g^{t\phi} + (LM^2 + G_\phi e)^2 g^{\phi\phi}}{(M^2 - \alpha)^2}.$$ (G.8)
This is sometimes left as-is, but can also be re-written in terms of lowered metric components to find:

\[ \mu^2 (1 + u_p^2) = \frac{-(E \mathcal{M}^2 - G_t e)^2 g_{\phi\phi} - 2 (E \mathcal{M}^2 - G_t e) (L \mathcal{M}^2 + G_\phi e) g_{t\phi} - (L \mathcal{M}^2 + G_\phi e)^2 g_{tt}}{\rho_\omega^2 (\mathcal{M}^2 - \alpha)^2} \]

\[ = \frac{C_0 + C_2 \mathcal{M}^2 + C_4 \mathcal{M}^4}{\rho_\omega^2 (\mathcal{M}^2 - \alpha)^2}. \tag{G.9} \]

Solving for \( C_0 \), we find:

\[ C_0 = -C_2^2 e^2 g_{\phi\phi} + 2 G_t G_\phi e^2 g_{t\phi} - G_\phi e^2 g_{tt} \]

\[ = e^2 [(-g_{\phi\phi} (g_{tt} + g_{t\phi} \Omega_F)^2) + 2 g_{t\phi} (g_{tt} + g_{t\phi} \Omega_F) (g_{t\phi} + g_{\phi\phi} \Omega_F)] - g_{tt} (g_{t\phi} + g_{\phi\phi} \Omega_F)^2 \]

\[ = e^2 [(g_{t\phi} g_{tt} + g_{t\phi} g_{tt}) + (g_{t\phi} g_{t\phi} + 2 g_{t\phi} g_{t\phi}) \Omega_F + (g_{t\phi} g_{\phi\phi} - g_{tt} g_{\phi\phi}) \Omega_F^2] \]

\[ = e^2 [g_{t\phi}^2 - g_{tt} g_{\phi\phi}] [g_{tt} + 2 g_{t\phi} \Omega_F + g_{\phi\phi} \Omega_F^2] \]

\[ = e^2 \rho_\omega^2 \alpha. \tag{G.10} \]

Solving for \( C_2 \), we find:

\[ C_2 = 2 G_t e E g_{\phi\phi} + 2 G_t e L g_{t\phi} - 2 G_\phi e E g_{t\phi} - 2 G_\phi L g_{tt} \]

\[ = 2 e [(g_{tt} + g_{t\phi} \Omega_F) (E g_{\phi\phi} + L g_{t\phi}) - (g_{t\phi} + g_{\phi\phi} \Omega_F) (E g_{t\phi} + L g_{tt})] \]

\[ = 2 e [(g_{tt} g_{\phi\phi} - g_{tt}^2) E - (g_{tt} g_{\phi\phi} - g_{t\phi}^2) L \Omega_F] \]

\[ = -2 e^2 \rho_\omega^2. \tag{G.11} \]

Solving for \( C_4 \), we find:

\[ C_4 = -E^2 g_{\phi\phi} - 2 L E g_{t\phi} - L^2 g_{tt}. \tag{G.12} \]
Inserting $C_0$, $C_2$, and $C_4$ into the above, we then find:

$$
\mu^2 (1 + u_p^2) = \frac{e^2 \rho_\omega^2 \alpha - 2e^2 \rho_\omega^2 \mathcal{M}^2 - (E^2 g_{\phi\phi} + 2LE g_{t\phi} + L^2 g_{tt}) \mathcal{M}^4}{\rho_\omega^2 \left( \mathcal{M}^2 - \alpha \right)^2}.
$$

(G.13)

In Takahashi et al. 1990 [101] this expressed in terms of the variables $k_0$, $k_2$, and $k_4$ (corresponding to variables used by Camenzind 1986b [9]) to arrive at a “poloidal” (Bernoulli) equation that is given by:

$$
u_p^2 + 1 = \left( \frac{E}{\mu} \right)^2 \frac{k_0 k_2 - 2k_2 \mathcal{M}^2 - k_4 \mathcal{M}^4}{(\mathcal{M}^2 - k_0)^2},
$$

(G.14)

where:

$$
k_0 = \alpha = G_t + \Omega_F G_\phi = g_{tt} + 2g_{t\phi} \Omega_F + g_{\phi\phi} \Omega_F^2,
$$

$$
k_2 = \frac{e^2}{E^2} = \left( 1 - \frac{L}{E} \Omega_F \right)^2 = \left( 1 - \tilde{L} \Omega_F \right)^2,
$$

$$
k_4 = \frac{1}{\rho_\omega^2} \frac{1}{E^2} \left( E^2 g_{\phi\phi} + 2LE g_{t\phi} + L^2 g_{tt} \right) = \frac{1}{\rho_\omega^2} \left( g_{\phi\phi} + 2g_{t\phi} \tilde{L} + g_{tt} \tilde{L}^2 \right).
$$

(G.15)

The above form can be useful when proving that negative energy inflows must have an Alfvén point inside the ergoregion.

**Canonical Differential Form**

In this section we arrive at the canonical differential form of the Bernoulli equation, often referred to as the “poloidal acceleration equation”. This form is primarily useful for elucidating the critical points of the Bernoulli equation and their correspondences to magnetohydrodynamic wave speeds.

The plasma flows along magnetic field lines, so the poloidal acceleration of the plasma may be calculated by differentiating $u_p$ along field lines; such a field-aligned
derivative is denoted with a prime. From the previous section, the poloidal velocity is given by:

\[ 1 + u_p^2 = \left( \frac{E}{\mu} \right)^2 \frac{k_0 k_2 - 2 k_2 M^2 - k_4 M^4}{(M^2 - k_0)^2} = \left( \frac{E}{\mu} \right)^2 F_M. \] (G.16)

The variable \( F_M \) is referred to as the “characteristic flow function”. Differentiating both sides along field lines yields:

\[ 2u_p^2 (\ln u_p)' = -2 \left( \frac{E}{\mu} \right)^2 (\ln \mu)' F_M + \left( \frac{E}{\mu} \right)^2 F_M'. \] (G.17)

Re-arranging and simplifying yields:

\[ (\ln u_p)' = \frac{1}{u_p^2} \left( \frac{E}{\mu} \right)^2 \left[ -\frac{1}{\mu} \mu' F_M + \frac{1}{2} F'_M \right] \]

\[ = \frac{1}{u_p^2} \left( \frac{E}{\mu} \right)^2 \left[ -\left( \frac{\partial \ln \mu}{\partial \ln n} \right)_{ad} (\ln n)' F_M + \frac{1}{2} F'_M \right]. \] (G.18)

Here the subscript “ad” refers to an adiabatic flow. We now find the derivative of \( \ln n \) along field lines (recalling that \( \eta \) is a conserved quantity along field lines, and using the definition \( \eta = n u_p/B_p \) from Appendix B):

\[ (\ln n)' = \frac{1}{n} n' = \frac{u_p}{\eta B_p} \left( \frac{\eta B_p}{u_p} \right)' = \frac{u_p}{B_p} \left[ \frac{B_p'}{u_p} - \frac{B_p}{u_p} (\ln u_p)' \right] = (\ln B_p)' - (\ln u_p)'. \] (G.19)

We now find the derivative of \( M^2 \) along field lines:

\[ (M^2)' = \left( \frac{4\pi \mu \eta^2}{n} \right)' = 4\pi \eta^2 \left( \frac{\mu}{n} \left( \frac{\partial \ln \mu}{\partial \ln n} \right)_{ad} (\ln n)' - \frac{\mu}{n} (\ln n)' \right) \]

\[ = M^2 \left( \left( \frac{\partial \ln n}{\partial \ln n} \right)_{ad} - 1 \right) (\ln n)'. \] (G.20)

Identifying \( \left( \frac{\partial \ln \mu}{\partial \ln n} \right)_{ad} \) as the adiabatic sound speed squared \( a_s^2 \) we may now fully
evaluate the poloidal acceleration along field lines (recalling that $k_2$ is purely a function of $E$, $L$, and $\Omega_F$ and as such is a field-aligned conserved quantity): 

\[
\left(\ln u_p\right)' \frac{u_p^2}{\mu E} = -a_s^2 \left(\ln n\right)' F_M + \frac{1}{2} F_M' \\
= -a_s^2 \left(\ln n\right)' \left[\frac{k_0 k_2 - 2k_2 M^2 - k_4 M^4}{(M^2 - k_0)^2}\right] + \frac{1}{2} \left[\frac{k_0 k_2 - 2k_2 M^2 - k_4 M^4}{(M^2 - k_0)^2}\right]' \\
= -a_s^2 \left(\ln n\right)' \left[\frac{k_0 k_2 - 2k_2 M^2 - k_4 M^4}{(M^2 - k_0)^2}\right] \\
+ \frac{1}{2} \left[\frac{k_0 k_2 - 2k_2 (M^2)' - k_4 M^4 - 2k_4 M^2 (M^2)'}{(M^2 - k_0)^2}\right] \\
- \frac{k_0 k_2 - 2k_2 M^2 - k_4 M^4}{(M^2 - k_0)^3} \left[(M^2)' - k_0'\right] \\
= -a_s^2 \left(\ln n\right)' \frac{1}{2} \frac{12 \left(k_0 k_2 - 2k_2 M^2 - k_4 M^4\right) (M^2 - k_0)}{(M^2 - k_0)^3} \\
- \frac{1}{2} \frac{M^4 (M^2 - k_0)}{(M^2 - k_0)^3} k_4' \\
+ \frac{1}{2} \frac{k_2 (M^2 - k_0) + 2 \left(k_0 k_2 - 2k_2 M^2 - k_4 M^4\right) k_0'}{(M^2 - k_0)^3} \\
+ \frac{1}{2} \frac{1 \left(-2k_2 - 2k_4 M^2\right) (M^2 - k_0) - 2 \left(k_0 k_2 - 2k_2 M^2 - k_4 M^4\right) k_0'}{(M^2 - k_0)^3} \\
\cdot [M^2 \left(a_s^2 - 1\right) (\ln n)'] \\
= \frac{M^4 \left(k_0 - M^2\right) k_4' + \left(k_0 k_2 - 3k_2 M^2 - 2k_4 M^4\right) k_0'}{2 (M^2 - k_0)^3} \\
+ \frac{-2a_s^2 F_M (M^2 - k_0)^3 + \left(a_s^2 - 1\right) M^4 \left(2k_0 k_4 + 2k_2\right)}{2 (M^2 - k_0)^3} (\ln n)' .
\] 

\[(G.21)\]

Noting that $(\ln n)'$ contains a factor of $(\ln u_p)'$, this may be expanded and re-arranged
to find:

\[
\begin{align*}
\left( \ln u_p \right)' & \left[ u_p^2 \left( \frac{\mu}{\sqrt{E}} \right)^2 + a_s^2 F_M (k_0 - \mathcal{M}^2)^3 + (a_s^2 - 1) \mathcal{M}^4 (k_0 k_4 + k_2) \right] \\
= & \frac{\mathcal{M}^4 (k_0 - \mathcal{M}^2) k_4' + (k_0 k_2 - 3k_2 \mathcal{M}^2 - 2k_4 \mathcal{M}^4) k_0'}{2 (\mathcal{M}^2 - k_0)^3} \\
+ & \frac{a_s^2 F_M (k_0 - \mathcal{M}^2)^3 + (a_s^2 - 1) \mathcal{M}^4 (k_0 k_4 + k_2)}{\left( \mathcal{M}^2 - k_0 \right)^3} (\ln \mathcal{B}_p)' .
\end{align*}
\]

This can be re-written and compacted to find:

\[
\left( \ln u_p \right)' = \frac{\frac{1}{2} \left( \frac{E}{\mu} \right)^2 (F_1 k_4' + F_0 k_0') + \left( \frac{E}{\mu} \right)^2 F_B (\ln \mathcal{B}_p)'}{u_p^2 (\mathcal{M}^2 - k_0)^3 + \left( \frac{E}{\mu} \right)^2 (a_s^2 F_M (k_0 - \mathcal{M}^2)^3 + (a_s^2 - 1) \mathcal{M}^4 (k_0 k_4 + k_2))} ,
\]

where:

\[
\begin{align*}
F_0 & = k_0 k_2 - 3k_2 \mathcal{M}^2 - 2k_4 \mathcal{M}^4 \\
F_4 & = \mathcal{M}^4 (k_0 - \mathcal{M}^2) \\
F_B & = a_s^2 F_M (k_0 - \mathcal{M}^2)^3 + (a_s^2 - 1) \mathcal{M}^4 (k_0 k_4 + k_2) .
\end{align*}
\]

After dividing both the numerator and denominator by \((a_s^2 - 1)\), the numerator \(N\) becomes:

\[
\begin{align*}
N = & -\frac{1}{1 - a_s^2} \left[ \left( \frac{E}{\mu} \right)^2 \frac{1}{2} [F_1 k_4' + F_0 k_0'] + \left( \frac{E}{\mu} \right)^2 F_B (\ln \mathcal{B}_p)' \right] \\
= & -\frac{1}{2 (1 - a_s^2)} \left( \frac{E}{\mu} \right)^2 \left[ \mathcal{M}^4 (k_0 - \mathcal{M}^2) k_4' + (k_0 k_2 - 3k_2 \mathcal{M}^2 - 2k_4 \mathcal{M}^4) k_0' \right] \\
- & \frac{1}{1 - a_s^2} \left( \frac{E}{\mu} \right)^2 (a_s^2 F_M (k_0 - \mathcal{M}^2)^3 + (a_s^2 - 1) \mathcal{M}^4 (k_0 k_4 + k_2)) (\ln \mathcal{B}_p)'.
\end{align*}
\]
Here the Alfvén wave speed $u_{AW}$, fast magnetosonic wave speed $u_{FM}$, and slow
magnetosonic wave speed $u_{SM}^2$ are given by:

$$u_{AW}^2 = \frac{k_0}{k_u},$$

$$u_{FM}^2 = \frac{1}{2} \left\{ \left[ c_s^2 + u_{AW}^2 + \left( \frac{E}{\mu} \right)^2 \mathcal{M}^2 \frac{k_0 k_4 + k_2}{(k_0 - \mathcal{M}^2)^2} \right] + \sqrt{\left[ c_s^2 + u_{AW}^2 + \left( \frac{E}{\mu} \right)^2 \mathcal{M}^2 \frac{k_0 k_4 + k_2}{(k_0 - \mathcal{M}^2)^2} \right]^2 - 4 c_s^2 u_{AW}^2} \right\},$$

$$u_{SM}^2 = \frac{1}{2} \left\{ \left[ c_s^2 + u_{AW}^2 + \left( \frac{E}{\mu} \right)^2 \mathcal{M}^2 \frac{k_0 k_4 + k_2}{(k_0 - \mathcal{M}^2)^2} \right] - \sqrt{\left[ c_s^2 + u_{AW}^2 + \left( \frac{E}{\mu} \right)^2 \mathcal{M}^2 \frac{k_0 k_4 + k_2}{(k_0 - \mathcal{M}^2)^2} \right]^2 - 4 c_s^2 u_{AW}^2} \right\}. \quad (G.28)$$

From the definition of the Alfvén Mach number $\mathcal{M}^2$, the constant $k_u$ is given by $k_u = (4\pi\mu n)/B_p^2$, so the Alfvén wave speed reduces to $u_{AW}^2 = B_p^2/(4\pi\mu n)$.

The fast and slow magnetosonic wave speeds have a correction from their standard non-relativistic form due to the fact that they are an effective “poloidal” wave speed; that correction is seen in the $E^2$ term. If there is no toroidal magnetic field there should be no correction, so for transparency it is preferable to rewrite the $E^2$ term in terms of the toroidal magnetic field $\sqrt{-g F^{0r}}$. This may be done by examining the expressions for $E$ and $\mu u_t$ found in the previous section:

$$E = \mu u_t + \frac{1}{4\pi\eta} \sqrt{-g F^{0r}} \Omega_F,$$

$$\mu u_t = \frac{E \mathcal{M}^2 - G_{te}}{\mathcal{M}^2 - k_0} = \frac{E \mathcal{M}^2 - (g_{tt} + g_{t\phi} \Omega_F) (E - L \Omega_F)}{\mathcal{M}^2 - k_0}. \quad (G.29)$$
Combining these expressions, we conclude that:

\[
\sqrt{-g} F^{\theta r} = -E \frac{4\pi \eta}{\Omega_F} \left[ \frac{k_0 - (g_{tt} + g_{t\phi}\Omega_F) \left( 1 - \bar{L}\Omega_F \right)}{\mathcal{M}^2 - k_0} \right].
\]  

(G.30)

The \( E^2 \) term in the fast and slow magnetosonic wave speed expressions may then be re-written in terms of the toroidal field \( \sqrt{-g} F^{\theta r} \) as:

\[
k_E = \left( E \frac{\mu}{\kappa} \right)^2 \mathcal{M}^2 \frac{k_0 k_4 + k_2}{(k_0 - \mathcal{M}^2)^2}
\]

\[
= \left( \sqrt{-g} F^{\theta r} \right)^2 \left( \frac{\Omega_F}{4\pi \eta \mu} \right)^2 \mathcal{M}^2 \left[ \frac{\mathcal{M}^2 - k_0}{k_0 - (g_{tt} + g_{t\phi}\Omega_F) \left( 1 - \bar{L}\Omega_F \right)} \right]^2 \frac{k_0 k_4 + k_2}{(k_0 - \mathcal{M}^2)^2}
\]

\[
= \left( \sqrt{-g} F^{\theta r} \right)^2 \left( \frac{\Omega_F}{4\pi \eta \mu} \right)^2 \left( \frac{4\pi \eta^2}{n} \right) \frac{k_0 k_4 + k_2}{k_0 - (g_{tt} + g_{t\phi}\Omega_F) \left( 1 - \bar{L}\Omega_F \right)}
\]

\[
= \left( \sqrt{-g} F^{\theta r} \right)^2 \frac{k_0}{4\pi \eta n}.
\]

(G.31)

The \( k_E \) correction is now identical in form to \( u_{AW}^2 \) with \( B_p \rightarrow \sqrt{-g} F^{\theta r} \) and the geometrical correction \( k_0 \) replaced with \( k_\phi \). Explicitly, \( k_\phi \) takes the form:

\[
k_\phi = \Omega_F^2 \frac{k_0 k_4 + k_2}{\left[ k_0 - (g_{tt} + g_{t\phi}\Omega_F) \left( 1 - \bar{L}\Omega_F \right) \right]^2}
\]

\[
= \Omega_F^2 \frac{k_0 \left( g_{\phi\phi} + 2g_{t\phi} \bar{L} + g_{tt} \bar{L}^2 \right)}{k_0^2 - 2k_0 \left( g_{tt} + g_{t\phi}\Omega_F \right) \left( 1 - \bar{L}\Omega_F \right)} + k_2
\]

\[
= \Omega_F^2 \frac{k_N}{\rho_{\phi}^2} \frac{\rho_D}{k_D}.
\]

(G.32)

Expanding the numerator \( k_N \) yields:

\[
k_N = (g_{tt} + 2g_{t\phi}\Omega_F + g_{\phi\phi}\Omega_F^2) \left( g_{\phi\phi} + 2g_{t\phi} \bar{L} + g_{tt} \bar{L}^2 \right) + (g_{t\phi}^2 - g_{tt}g_{\phi\phi} + g_{tt} \bar{L}^2) k_2
\]
Expanding the denominator \( k_D \) yields:

\[
\begin{align*}
k_D &= (g_{tt} + 2 g_{t\phi} \Omega_F + g_{\phi\phi} \Omega_F^2) (g_{\phi\phi} + 2 g_{t\phi} \tilde{L} + g_{tt} \tilde{L}^2) - g_{tt} g_{\phi\phi} \left(1 - \Omega_F \tilde{L}\right)^2 + g_{t\phi}^2 k_2 \\
&= \left(g_{\phi\phi}^2 + 2 g_{t\phi} g_{t\phi} \Omega_F \right) \Omega_F^2 + \left(2 g_{t\phi} g_{\phi\phi} + 4 g_{t\phi}^2 \tilde{L} + 2 g_{tt} g_{\phi\phi} \tilde{L}^2 + 2 g_{tt} g_{t\phi} \tilde{L}^2 \right) \Omega_F \\
&+ \left(2 g_{tt} g_{t\phi} \tilde{L} + g_{tt}^2 \tilde{L}^2 \right) + g_{t\phi}^2 k_2 \end{align*}
\]

(G.33)

Therefore \( k_\phi \) greatly simplifies to \( k_\phi = \frac{1}{\rho_\phi} \), and the Alfvén wave speed, fast magnetosonic wave speed, and slow magnetosonic wave speed similarly reduce to:

\[
\begin{align*}
u_{AW}^2 &= \frac{B_\phi^2}{4\pi\mu m} k_0 \\
u_{FM}^2 &= \frac{1}{2} \left[ \left(c_s^2 + u_{AW}^2 + \frac{B_\phi^2}{4\pi\mu m \rho_\phi^2} \right) + \sqrt{\left(c_s^2 + u_{AW}^2 + \frac{B_\phi^2}{4\pi\mu m \rho_\phi^2} \right)^2 - 4c_s^2 u_{AW}^2} \right] \quad \text{(G.35)} \\
u_{SM}^2 &= \frac{1}{2} \left[ \left(c_s^2 + u_{AW}^2 + \frac{B_\phi^2}{4\pi\mu m \rho_\phi^2} \right) - \sqrt{\left(c_s^2 + u_{AW}^2 + \frac{B_\phi^2}{4\pi\mu m \rho_\phi^2} \right)^2 - 4c_s^2 u_{AW}^2} \right] \quad \text{(G.36)}
\end{align*}
\]

(G.37)

Here we have written the square of toroidal field \((\sqrt{-g} F^{\theta r})^2\) as \(B_\phi^2\) for compactness. If there is no toroidal field then the poloidal flow velocity is the actual flow velocity.
(\(u_p^2 \rightarrow v^2\)) and the MHD wave speeds reduce to expressions qualitatively identical to their non-relativistic forms.

The most relevant point of this section is that the denominator of the differential Bernoulli equation has three critical points corresponding to the appropriate magnetohydrodynamic wave speeds (or effective poloidal components of the same). When \(u_p^2\) is equal to \(u_{AW}^2\), \(u_{FM}^2\), or \(u_{SM}^2\) the denominator \(D\) vanishes and a physical flow must also have vanishing numerator \(N\) such that the poloidal acceleration remains finite. Analysis of the restrictions imposed by \(N = 0\) when \(D = 0\) can yield restrictions on the flow, similar to the way in which the classical case of spherically uniform accretion (i.e. Bondi accretion [7]) results in an \(x\)-type solution space that restricts flow parameters when the flow transitions from subsonic to supersonic (or supersonic to subsonic for the outflow solution).

**Cold Flow Quartic Form**

In this section we will explore the Bernoulli equation in the limit of a cold flow (vanishing pressure such that \(\mu\) is a conserved quantity corresponding to particle mass), reducing it to a quartic in \(M^2\). We then solve for the various roots using the quartic equation. To start with, we consider the integral form of the Bernoulli equation found above:

\[
\mu^2 (1 + u_p^2) = \frac{e^2 \rho_\omega^2 \alpha - 2e^2 \rho_\omega^2 M^2 - (E^2 g_{\phi\phi} + 2LEg_{\phi\phi} + L^2 g_{tt}) M^4}{\rho_\omega^2 (M^2 - \alpha)^2}. \tag{G.38}
\]

Here \(e^2\) is given by \(e^2 = (E - L\Omega_F)^2\), and \(u_p^2\) is given by \(u_p^2 = -g_{rr} (u^r)^2 - g_{\theta\theta} (u^\theta)^2\). Because \(u_p^2\) is generally unknown, we must replace it with something; the most useful quantity is the Alfvén Mach number. From the definition of particle flux per unit flux tube \(\eta\) and the metric as expressed in Boyer-Lindquist coordinates, it is
straightforward to show that:

\[ u_p^2 = \frac{M^4}{16\pi^2 (\mu \eta)^2} \left( \frac{F_{\Phi \Phi}^2 + \Delta F_{\phi \phi}^2}{\Sigma \Delta \sin^2 \theta} \right). \]  

Inserting this into the Bernoulli Equation, we find:

\[ 0 = \mu^2 \rho^2 (1 + u_p^2) (M^2 - \alpha)^2 - e^2 \rho^2 \alpha + 2e^2 \rho^2 \mathcal{M}^2 + (E^2 g_{\phi \phi} + 2LE g_{t \phi} + L^2 g_{tt}) \mathcal{M}^4 \]

\[ = \frac{1}{\eta^2} \left\{ (\mu \eta)^2 \rho^2 \left[ 1 + \frac{M^4}{16\pi^2 (\mu \eta)^2} \left( \frac{F_{\Phi \Phi}^2 + \Delta F_{\phi \phi}^2}{\Sigma \Delta \sin^2 \theta} \right) \right] (M^2 - \alpha)^2 \right. 
\]

\[ - \eta^2 e^2 \rho^2 \alpha + 2\eta^2 e^2 \rho^2 \mathcal{M}^2 + \alpha_{EL} \mathcal{M}^4 \} \].

We have defined \( \alpha_{EL} \) as \( \alpha_{EL} \equiv \eta^2 L^2 g_{tt} + 2\eta L \eta E g_{t \phi} + \eta^2 E^2 g_{\phi \phi} \) due to its similarity in form with \( \alpha \). Discarding the \( \mu^2/\eta^2 \) prefactor the above may be simplified into a quartic of the form:

\[ C_4 (\mathcal{M}^2)^4 + C_3 (\mathcal{M}^2)^3 + C_2 (\mathcal{M}^2)^2 + C_1 (\mathcal{M}^2) + C_0 = 0, \]  

where:

\[ C_4 = \frac{F_{\Phi \Phi}^2 + \Delta F_{\phi \phi}^2}{16\pi^2 \Sigma}, \quad C_1 = 2\rho^2 (\eta E - \eta \Omega_F)^2 - 2\alpha \rho^2 (\mu \eta)^2, \]

\[ C_3 = -2\alpha \frac{F_{\Phi \Phi}^2 + \Delta F_{\phi \phi}^2}{16\pi^2 \Sigma}, \quad C_0 = \alpha^2 \rho^2 (\mu \eta)^2 - \alpha \rho^2 (\eta E - \eta \Omega_F)^2. \]

\[ C_2 = (\mu \eta)^2 \rho^2 + \alpha \frac{F_{\Phi \Phi}^2 + \Delta F_{\phi \phi}^2}{16\pi^2 \Sigma} + \alpha_{EL}. \]  

Because a quartic has a closed form analytic solution, from this form we may in principle calculate \( \mathcal{M}^2 \). However no single root is valid for all space; a valid physical solution will switch between roots. Therefore multiple roots must be solved for and
then compared in order to find a physical solution. Those roots are given by:

\[
\begin{align*}
\mathcal{M}_1^2 &= -\frac{C_3}{4C_4} - S + \frac{1}{2} \sqrt{-4S^2 - 2p + \frac{q}{S}}, \\
\mathcal{M}_2^2 &= -\frac{C_3}{4C_4} - S - \frac{1}{2} \sqrt{-4S^2 - 2p + \frac{q}{S}}, \\
\mathcal{M}_3^2 &= -\frac{C_3}{4C_4} + S + \frac{1}{2} \sqrt{-4S^2 - 2p - \frac{q}{S}}, \\
\mathcal{M}_4^2 &= -\frac{C_3}{4C_4} + S - \frac{1}{2} \sqrt{-4S^2 - 2p - \frac{q}{S}}, \\
\end{align*}
\]

(G.43)

where:

\[
\begin{align*}
p &= \frac{8C_2C_4 - 3C_3^2}{8C_4^2}, \\
q &= \frac{C_3^3 - 4C_2C_3C_4 + 8C_1C_4^2}{8C_3^2}, \\
\Delta_0 &= C_2^2 - 3C_1C_3 + 12C_0C_4, \\
\Delta_1 &= 2C_2^3 - 9C_1C_2C_3 + 27C_0C_3^2 + 27C_0^2C_4^2 - 72C_0C_2C_4, \\
\Delta &= \frac{1}{27} \left( 4\Delta_0^3 - \Delta_1^2 \right), \\
Q^3 &= \frac{1}{2} \left( \Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3} \right), \\
S &= \frac{1}{2} \sqrt{-\frac{2p}{3} + \frac{1}{3C_4} \left( Q + \frac{\Delta_0}{Q} \right)}. \\
\end{align*}
\]

(G.44)

In the above \( \Delta \) is the discriminant of the polynomial and should not be confused with the metric element; for the remainder of this section the variables \( p, q, S, Q, \Delta_0, \Delta_1, \) and \( \Delta \) should be taken to correspond to the above standard expressions for solving a quartic equation, and not any values or quantities they might have previously specified. Although the full form of the roots is generally less than illuminating, various properties of a plasma inflow may be determined by evaluating
their expressions in special cases. The above coefficients may be simplified to find:

\[ p = \frac{2\rho^2_\omega (\mu\eta)^2 + 2\alpha_{EL} - \alpha^2 C_4}{2C_4}, \]
\[ q = \frac{2\rho^2_\omega e^2 + \alpha_{EL} - \alpha \rho^2_\omega (\mu\eta)^2}{C_4}, \]
\[ \Delta_0 = [\rho^2_\omega (\mu\eta)^2 + \alpha^2 C_4 + \alpha_{EL}]^2, \]
\[ \Delta_1 = 2 [\rho^2_\omega (\mu\eta)^2 + \alpha^2 C_4 + \alpha_{EL}]^3 + 108C_4\rho^2_\omega [e^2 - \alpha (\mu\eta)^2] (\rho^2_\omega e^2 + \alpha_{EL}), \]
\[ \Delta = -16C_4\rho^2_\omega [e^2 - \alpha (\mu\eta)^2] (\rho^2_\omega e^2 + \alpha_{EL}) \cdot \left\{ [\rho^2_\omega (\mu\eta)^2 + \alpha^2 C_4 + \alpha_{EL}]^3 + 27C_4\rho^2_\omega [e^2 - \alpha (\mu\eta)^2] (\rho^2_\omega e^2 + \alpha_{EL}) \right\}. \quad \text{(G.45)} \]

The above can be usefully applied to solve for the appropriate horizon root and the points at which one root can switch to another root, as we will now show.

**Horizon Root**

In order to find the correct horizon root, we need to simplify the quartic variables in the limit that \( \rho^2_\omega \to 0 \). We begin with \( Q \) to find:

\[ Q^3 = \frac{1}{2} \left( \Delta_1 + \sqrt{\Delta^2_1 - 4\Delta^3_0} \right) \]
\[ = \frac{1}{2} \left\{ [2(\alpha^2 C_4 + \alpha_{EL})]^3 + \sqrt{2(\alpha^2 C_4 + \alpha_{EL})^2} - 4[(\alpha^2 C_4 + \alpha_{EL})]^3 \right\} \]
\[ = (\alpha^2 C_4 + \alpha_{EL})^3. \quad \text{(G.46)} \]

We then solve for \( S \) on the horizon:

\[ S = \frac{1}{2} \sqrt{-\frac{2p}{3} + \frac{1}{3B_p} \left( Q + \Delta_0 \right)} \]
\[ = \frac{1}{2} \left[ -\frac{2\alpha_{EL} - \alpha^2 C_4}{3C_4} + \frac{1}{3C_4} \left[ \frac{(\alpha^2 C_4 + \alpha_{EL}) + \frac{(\alpha^2 C_4 + \alpha_{EL})}{\alpha^2 C_4 + \alpha_{EL}}}{\alpha^2 C_4 + \alpha_{EL}} \right] \right] \]
\[
\frac{1}{2} \sqrt{\frac{1}{3C_4} [-2\alpha_{EL} + \alpha^2 C_4 + 2\alpha^2 C_4 + 2\alpha_{EL}]} = \frac{\alpha}{2}.
\]  
(G.47)

We can now solve for the four roots on the horizon, starting with \( M_1^2 \):

\[
M_1^2 = \frac{\alpha}{2} - S + \frac{1}{2} \sqrt{-4S^2 - 2p + \frac{q}{S}} = \frac{\alpha}{2} - \frac{\alpha}{2} + \frac{1}{2} \sqrt{-4 \left( \frac{\alpha}{2} \right)^2 - 2 \left( \frac{2\alpha_{EL} - \alpha^2 C_4}{2C_4} \right) + \frac{2\alpha\alpha_{EL}}{C_4\alpha}} = 0.
\]  
(G.48)

Then second \( M_2^2 \) root also vanishes due to its similarity with \( M_1^2 \):

\[
M_2^2 = \frac{\alpha}{2} - S - \frac{1}{2} \sqrt{-4S^2 - 2p + \frac{q}{S}} = 0.
\]  
(G.49)

We then find \( M_3^2 \) to be:

\[
M_3^2 = \frac{\alpha}{2} + S + \frac{1}{2} \sqrt{-4S^2 - 2p - \frac{q}{S}} = \alpha + \frac{1}{2} \sqrt{-4 \left( \frac{\alpha}{2} \right)^2 - 2 \left( \frac{2\alpha_{EL} - \alpha^2 C_4}{2C_4} \right) - \frac{2\alpha\alpha_{EL}}{C_4\alpha}} = \alpha + \sqrt{-\frac{\alpha_{EL}}{C_4}}.
\]  
(G.50)

We then find \( M_4^2 \) to be:

\[
M_4^2 = \frac{\alpha}{2} + S - \frac{1}{2} \sqrt{-4S^2 - 2p - \frac{q}{S}} = \alpha - \sqrt{-\frac{\alpha_{EL}}{C_4}}.
\]  
(G.51)
On the horizon $\alpha = g_{\phi\phi} (\Omega_F - \omega_H)^2 \leq 0$. We demand that $\mathcal{M}^2 > 0$ on the horizon for a physical flow (i.e. finite proper particle number density $n$, as the poloidal velocity cannot vanish there). Therefore the only physical root on the horizon is $\mathcal{M}_3^2$, which may be simplified to find that (using the horizon equivalences $g_{tt} = g_{tt} g_{\phi\phi}$ and $\omega_H = -g_{t\phi}/g_{\phi\phi}$):

$$
\mathcal{M}_3^2 = g_{\phi\phi} (\Omega_F - \omega_H)^2 + \sqrt{-\left(\eta^2 L^2 g_{tt} + 2\eta L \eta E g_{t\phi} + \eta^2 E^2 g_{\phi\phi}\right) \left(\frac{16\pi^2 \Sigma}{F_{\theta\phi}^2}\right)}
$$

$$
= g_{\phi\phi} (\Omega_F - \omega_H)^2 + \sqrt{-\left(\eta^2 L^2 \omega_H^2 g_{\phi\phi} - 2\eta L \eta E \omega_H g_{t\phi} + \eta^2 E^2 g_{\phi\phi}\right) \left(\frac{16\pi^2 \Sigma}{F_{\theta\phi}^2}\right)}
$$

$$
= g_{\phi\phi} (\Omega_F - \omega_H)^2 + 4\pi \sqrt{-\Sigma g_{\phi\phi}}
$$

$$
= -\frac{(r_H^2 + a^2)^2 \sin^2 \theta}{\Sigma} (\Omega_F - \omega_H)^2 + 4\pi \sqrt{-\Sigma g_{\phi\phi}}
$$

$$
= -\frac{2mr_H}{\Sigma} \left[ 2mr_H \sin^2 \theta (\Omega_F - \omega_H)^2 - 4\pi \Sigma \sin \theta \sqrt{-\Sigma g_{\phi\phi}} \right]. \quad (G.52)
$$

In the force-free limit we have $\eta E \rightarrow \sqrt{-g} F^\theta r \Omega_F / 4\pi$, $\eta L \rightarrow \sqrt{-g} F^\theta r / 4\pi$, and $\mathcal{M}^2 \rightarrow 0$, and the above reduces to the condition that:

$$
2mr_H \sin^2 \theta (\Omega_F - \omega_H)^2 = \Sigma \sin \theta \left| \frac{(\Omega_F - \omega_H)}{F_{\theta\phi}} \sqrt{-g} F^\theta r \right|. \quad (G.53)
$$

This can be immediately recognized as the horizon regularity condition, which is intrinsic to the equations of ideal magnetohydrodynamics.

**Critical Point Root Switching**

We now examine the points at which the physical solution might move from one root to another. It is a property of quartic equations that there are identical roots if and only if the discriminant vanishes, which is to say $\Delta = 0$. Therefore the physical
solution can only switch to another root when:

\[
0 = -16C_4 \rho_\omega^2 \left[ e^2 - \alpha (\mu \eta)^2 \right] \left( \rho_\omega^2 e^2 + \alpha \alpha_{EL} \right) \cdot \\
\cdot \left\{ \left[ \rho_\omega^2 (\mu \eta)^2 + \alpha^2 C_4 + \alpha_{EL} \right]^3 + 27C_4 \rho_\omega^2 \left[ e^2 - \alpha (\mu \eta)^2 \right] \left( \rho_\omega^2 e^2 + \alpha \alpha_{EL} \right) \right\}. \quad \text{(G.54)}
\]

This means that there are four places where the physical solution can move to a different root:

Switch 1 \( \rightarrow e^2 = \alpha (\mu \eta)^2 \),

Switch 2 \( \rightarrow \rho_\omega^2 e^2 = -\alpha \alpha_{EL} \),

Switch 3 \( \rightarrow \left[ \rho_\omega^2 (\mu \eta)^2 + \alpha^2 C_4 + \alpha_{EL} \right]^3 = -27C_4 \rho_\omega^2 \left[ e^2 - \alpha (\mu \eta)^2 \right] \left( \rho_\omega^2 e^2 + \alpha \alpha_{EL} \right) \),

Switch 4 \( \rightarrow \rho_\omega^2 = 0 \). \quad \text{(G.55)}

The Switch 4 condition corresponds to the horizon; from the above we already know that two roots (\( M_1^2 \) and \( M_2^2 \)) both vanish there. That leaves the first three switch points as the only potentially viable places for the solution to switch from the solution valid on the horizon (\( M_3^2 \)) to another root. We examine each of those points in turn.

**Alfvén Switch Point**

The Bernoulli equation may be written as:

\[
\rho_\omega^2 (\mu \eta)^2 \left( 1 + u_p^2 \right) = \frac{\rho_\omega^2 e^2 \alpha - 2\rho_\omega^2 e^2 M^2 - \alpha_{EL} M^4}{(M^2 - \alpha)^2}. \quad \text{(G.56)}
\]

The Alfvén point is defined as the point where \( M^2 = \alpha \) and the denominator vanishes. In order to maintain a physical flow, we must also demand that the numerator vanishes such that \( \rho_\omega^2 e^2 = -\alpha \alpha_{EL} \). This can be recognized as Switch Condition 2, meaning it is possible for a physical flow to switch roots at the Alfvén point. We now evaluate
all four roots at the Alfvén point in order to determine which roots are equivalent there. We begin by evaluating $Q$:

$$Q^3 = \frac{1}{2} \left( \Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3} \right)$$

$$= \frac{1}{2} \left\{ 2 \left[ \rho_\omega^2 (\mu\eta)^2 + \alpha^2 C_4 + \alpha_{EL} \right]^3 + \sqrt{4 \left[ \rho_\omega^2 (\mu\eta)^2 + \alpha^2 C_4 + \alpha_{EL} \right]^6 - 4 \left[ \rho_\omega^2 (\mu\eta)^2 + \alpha^2 C_4 + \alpha_{EL} \right]^6} \right\}$$

$$= \left[ \rho_\omega^2 (\mu\eta)^2 + \alpha^2 C_4 + \alpha_{EL} \right]^3.$$  \hspace{1cm} (G.57)

We next evaluate $S$ at the Alfvén point to find:

$$S = \frac{1}{2} \sqrt{-2p - \frac{1}{3C_4} \left( Q + \Delta_0 \right)}$$

$$= \frac{1}{2} \sqrt{-\frac{2\rho_\omega^2 (\mu\eta)^2 + 2\alpha_{EL} - \alpha^2 C_4}{3C_4} + \frac{1}{3C_4} \left[ 2\rho_\omega^2 (\mu\eta)^2 + 2\alpha^2 C_4 + 2\alpha_{EL} \right]}$$

$$= \frac{1}{2} \sqrt{\alpha^2}$$

$$= \frac{\alpha}{2}.$$ \hspace{1cm} (G.58)

In taking the square root we have assumed a physical flow, such that when $\mathcal{M}^2 = \alpha$ we have $\alpha \geq 0$. Next we evaluate $q$ to find:

$$q = \frac{2\rho_\omega^2 c^2 + \alpha \alpha_{EL} - \alpha \rho_\omega^2 (\mu\eta)^2}{C_4} = -\frac{\alpha}{C_4} \left[ \alpha_{EL} + \rho_\omega^2 (\mu\eta)^2 \right] = -2S \frac{\alpha}{C_4} \frac{\rho_\omega^2 (\mu\eta)^2}{C_4}.$$  \hspace{1cm} (G.59)

We now calculate $\mathcal{M}_1^2$ to find:

$$\mathcal{M}_1^2 = \frac{\alpha}{2} - S + \frac{1}{2} \sqrt{-4S^2 - 2p + \frac{q}{S}}$$
We next calculate $M^2_2$ in similar fashion:

\[
M^2_2 = \frac{\alpha}{2} - S - \frac{1}{2} \sqrt{-4S^2 - 2p + \frac{q}{S}}
\]

\[
= -\sqrt{-\frac{\alpha_{EL} + \rho^2_\omega (\mu \eta)^2}{C^4}}.
\] (G.61)

We next calculate $M^2_3$:

\[
M^2_3 = \frac{\alpha}{2} + S + \frac{1}{2} \sqrt{-4S^2 - 2p - \frac{q}{S}}
\]

\[
= \alpha + \frac{1}{2} \sqrt{-4 \left( \frac{\alpha}{2} \right)^2 - 2 \left[ \frac{2\rho^2_\omega (\mu \eta)^2}{2C^4} + \frac{2\alpha_{EL} - \alpha^2C^4}{C^4} \right] + \frac{2\alpha_{EL} + \rho^2_\omega (\mu \eta)^2}{C^4}}
\]

\[
= \alpha.
\] (G.62)

We then finally calculate $M^2_4$ in similar fashion:

\[
M^2_4 = \frac{\alpha}{2} + S - \frac{1}{2} \sqrt{-4S^2 - 2p - \frac{q}{S}}
\]

\[
= \alpha.
\] (G.63)

Therefore there are two possibilities for root switching at the Alfvén point; either $M^2_3 \leftrightarrow M^2_4$ without restriction, or $M^2_3 \leftrightarrow M^2_1 \leftrightarrow M^2_4$ ($M^2_2$ is disallowed because we
must have $\mathcal{M}^2 \geq 0$). For the last switching condition to occur, we demand that:

$$\rho_\omega^2 (\mu \eta)^2 + \alpha^2 \frac{F_{\theta \phi}^2 + \Delta F_{r \phi}^2}{16 \pi \Sigma} = - \left( \eta^2 L^2 g_{tt} + 2 \eta L \eta E g_{t \phi} + \eta^2 E^2 g_{\phi \phi} \right).$$  \hfill (G.64)

If this restriction is met then the Alfén point will allow switching between three roots; otherwise the Alfén point will allow switching between two roots.

**Injection Switch Point**

At the injection point $\mathcal{M}^2 = u_p^2 = 0$; after examining the Bernoulli equation shown above when considering the Alfvén point, this immediately leads to the condition that $\alpha (\mu \eta)^2 = e^2$. This may be recognized as Switch Condition 1. We now consider which roots might be valid there, beginning by evaluating $Q$ to find:

$$Q^3 = \frac{1}{2} \left( \Delta_1 + \sqrt{\Delta_1^2 - 4 \Delta_0^3} \right) = \frac{1}{2} \Delta_1 = \left[ \rho_\omega^2 (\mu \eta)^2 + \alpha^2 C_4 + \alpha_{EL} \right]^3.$$  \hfill (G.65)

We next evaluate $S$ to find:

$$S = \frac{1}{2} \sqrt{- \frac{2p}{3} + \frac{1}{3 C_4} \left( Q + \frac{\Delta_0}{Q} \right)}$$

$$= \frac{1}{2} \sqrt{- \frac{2 \rho_\omega^2 (\mu \eta)^2 + 2 \alpha_{EL} - \alpha^2 C_4}{3 C_4} + \frac{1}{3 C_4} \left[ 2 \rho_\omega^2 (\mu \eta)^2 + 2 \alpha^2 C_4 + 2 \alpha_{EL} \right]}$$

$$= \frac{1}{2} \sqrt{\alpha^2}$$

$$= \frac{\alpha}{2}.$$ \hfill (G.66)

We have applied physical considerations in taking the square root by demanding that the plasma go to rest between the inner and outer light surfaces where $\alpha \geq 0$. We
may now find the roots, beginning with $M^2_1$:

$$M^2_1 = \frac{\alpha}{2} - S + \frac{1}{2} \sqrt{-4S^2 - 2p + \frac{q}{S}}$$

$$= \frac{\alpha}{2} - \frac{\alpha}{2} + \frac{1}{2} \sqrt{-4\left(\frac{\alpha}{2}\right)^2 - 2\rho^2_\omega (\mu \eta)^2 + 2\alpha_{EL} - \alpha^2 C_4} + \frac{2}{\alpha} \frac{\alpha\rho^2_\omega (\mu \eta)^2 + \alpha\alpha_{EL}}{C_4}$$

$$= \frac{1}{2} \sqrt{-2\rho^2_\omega (\mu \eta)^2 + 2\alpha_{EL} C_4} + \frac{2\rho^2_\omega (\mu \eta)^2 + 2\alpha_{EL}}{C_4}$$

$$= 0. \quad (G.67)$$

We next calculate $M^2_2$ in similar fashion:

$$M^2_2 = \frac{\alpha}{2} - S - \frac{1}{2} \sqrt{-4S^2 - 2p + \frac{q}{S}}$$

$$= 0. \quad (G.68)$$

We next calculate $M^2_3$

$$M^2_3 = \frac{\alpha}{2} + S + \frac{1}{2} \sqrt{-4S^2 - 2p - \frac{q}{S}}$$

$$= \alpha + \frac{1}{2} \sqrt{-2\rho^2_\omega (\mu \eta)^2 + 2\alpha_{EL} C_4 - \frac{2\rho^2_\omega (\mu \eta)^2 + 2\alpha_{EL}}{C_4}}$$

$$= \alpha + \sqrt{-\rho^2_\omega (\mu \eta)^2 + \alpha_{EL} C_4}. \quad (G.69)$$

We then finally calculate $M^2_4$ in similar fashion:

$$M^2_4 = \frac{\alpha}{2} + S - \frac{1}{2} \sqrt{-4S^2 - 2p - \frac{q}{S}}$$

$$= \alpha - \sqrt{-\rho^2_\omega (\mu \eta)^2 + \alpha_{EL} C_4}. \quad (G.70)$$
Therefore at the injection point we can switch $\mathcal{M}_1^2 \leftrightarrow \mathcal{M}_2^2$. The other potential switches (conditional on the relationship between $\alpha$ and the radical) are not of significant interest, as outside of the injection point we never want $\mathcal{M}^2 = 0$. 

Fast Magnetosonic Switch Point

Switch Condition 3 occurs when:

$$\left( \frac{\rho_\omega^2 (\mu \eta)^2 + \alpha^2 C_4 + \alpha_{EL}}{3C_4} \right)^3 = -\frac{\rho_\omega^2 e^2 - \alpha (\mu \eta)^2}{C_4} \left( \frac{\rho_\omega^2 e^2 + \alpha \alpha_{EL}}{C_4} \right). \quad (G.71)$$

We will make no attempt to prove that this corresponds to $u_p^2 = u_{FM}^2$, fully justifying its correspondence to the fast magnetosonic point, as the length of such a proof would not be worth its current utility. Rather we simply point out that every other physical critical point of the flow has already been claimed, leaving only the fast magnetosonic point to correspond to this switch condition. As before, we first solve for $Q$ to find:

$$Q^3 = \frac{1}{2} \left( \Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3} \right)$$

$$= \frac{1}{2} \Delta_1$$

$$= \frac{1}{2} \left\{ 2 \left[ \frac{\rho_\omega^2 (\mu \eta)^2 + \alpha^2 C_4 + \alpha_{EL}}{3C_4} \right]^3 + 108C_4 \rho_\omega^2 \left[ e^2 - \alpha (\mu \eta)^2 \right] \left( \frac{\rho_\omega^2 e^2 + \alpha \alpha_{EL}}{C_4} \right) \right\}$$

$$= -\left[ (\mu \eta)^2 + \alpha^2 C_4 + \alpha_{EL} \right]^3. \quad (G.72)$$

We next evaluate $S$ to find:

$$S = \frac{1}{2} \sqrt{-\frac{2p}{3} + \frac{1}{3C_4} \left( Q + \Delta_0 \right)}$$

$$= \frac{1}{2} \sqrt{-\frac{2}{3} \left[ \frac{2\rho_\omega^2 (\mu \eta)^2 + 2\alpha_{EL} - \alpha^2 C_4}{2C_4} \right] + \frac{1}{3C_4} \left[ -2\rho_\omega^2 (\mu \eta)^2 - 2\alpha^2 C_4 + \alpha_{EL} \right]}$$

$$= \frac{1}{2} \sqrt{-\frac{1}{3C_4} \left[ 4\rho_\omega^2 (\mu \eta)^2 + 4\alpha_{EL} + \alpha^2 C_4 \right]}$$
\[
= \frac{1}{2} \sqrt{\alpha^2 - \frac{4}{3C_4} \left[ \rho_\omega^2 (\mu \eta)^2 + \alpha^2 C_4 + \alpha_{EL} \right]}. 
\]

(G.73)

In order to simplify evaluating the roots, we next evaluate the last square root in their expressions to find:

\[
\sqrt{\cdots} = \sqrt{-4S^2 - 2p \pm \frac{q}{S}}
\]

\[
= \sqrt{-\alpha^2 + \frac{4}{3C_4} \left[ \rho_\omega^2 (\mu \eta)^2 + \alpha^2 C_4 + \alpha_{EL} \right] - 2 \left[ \frac{2\rho_\omega^2 (\mu \eta)^2 + 2\alpha_{EL} - \alpha^2 C_4}{2C_4} \right] \pm \frac{q}{S}}
\]

(G.74)

where:

\[
\frac{q}{S} = 2 \frac{2\rho_\omega^2 e^2 + \alpha \alpha_{EL} - \alpha \rho_\omega^2 (\mu \eta)^2}{C_4 \sqrt{\alpha^2 - \frac{4}{3C_4} \left[ \rho_\omega^2 (\mu \eta)^2 + \alpha^2 C_4 + \alpha_{EL} \right]}}. 
\]

(G.75)

To simplify this further we find an expression for \( \rho_\omega^2 e^2 \), which we obtain from the root switch condition:

\[
(\rho_\omega^2 e^2)^2 + \alpha \left[ \alpha_{EL} - \rho_\omega^2 (\mu \eta)^2 \right] \rho_\omega^2 e^2 - \alpha^2 \rho_\omega^2 (\mu \eta)^2 \alpha_{EL} + C_4^2 \left[ \frac{\rho_\omega^2 (\mu \eta)^2 + \alpha^2 C_4 + \alpha_{EL}}{3C_4} \right]^3 = 0. 
\]

(G.76)

Therefore \( \rho_\omega^2 e^2 \) is given by (the substitutions \( \alpha_{EL} + \rho_\omega^2 (\mu \eta)^2)/(3C_4) \to x \) and \( \alpha^2 \to y \) and can be used to reduce the expression under the radical in the quadratic equation to \( (-4x - y/3)(2y/3 - x)^2 \); we will not display the somewhat tedious algebra but merely state the result:

\[
\rho_\omega^2 e^2 = \frac{1}{2} \alpha \left[ \rho_\omega^2 (\mu \eta)^2 - \alpha_{EL} \right]
\]
\[ \pm \frac{C_4}{2} \left| \alpha^2 - \frac{\rho^2 \omega^2 (\mu \eta)^2}{3C_4} + \alpha^2 C_4 + \alpha_{EL} \right| \sqrt{\alpha^2 - \frac{4}{3C_4} \left[ \rho^2 \omega^2 (\mu \eta)^2 + \alpha^2 C_4 + \alpha_{EL} \right]}. \]

(G.77)

Therefore we can simplify the numerator of the expression for \( q/S \) to find:

\[ 4\rho^2 \omega^2 + 2\alpha \alpha_{EL} - 2\alpha \rho^2 \omega (\mu \eta)^2 \]

\[ = \pm 2C_4 \left| \alpha^2 - \frac{\rho^2 \omega^2 (\mu \eta)^2}{3C_4} + \alpha^2 C_4 + \alpha_{EL} \right| \sqrt{\alpha^2 - \frac{4}{3C_4} \left[ \rho^2 \omega^2 (\mu \eta)^2 + \alpha^2 C_4 + \alpha_{EL} \right]}. \]

(G.78)

This means that the quantity under the square root (\( \sqrt{\cdots} \)) in the expression for the roots becomes:

\[ \ldots = 2\alpha^2 - \frac{2}{3C_4} \left[ \rho^2 \omega^2 (\mu \eta)^2 + \alpha^2 C_4 + \alpha_{EL} \right] \pm \left( \pm 2 \left| \alpha^2 - \frac{1}{3C_4} \left[ \rho^2 \omega^2 (\mu \eta)^2 + \alpha^2 C_4 + \alpha_{EL} \right] \right| \right). \]

(G.79)

Depending on the various signs, this can either vanish or not. Therefore we have two potential sets of roots; we will either have:

\[ M_1^2 = \frac{\alpha}{2} - \frac{1}{2} \sqrt{-\frac{1}{3} \alpha^2 - \frac{4}{3C_4} \left[ \rho^2 \omega^2 (\mu \eta)^2 + \alpha_{EL} \right]} + \sqrt{\frac{2}{3} \alpha^2 - \frac{1}{3C_4} \left[ \rho^2 \omega^2 (\mu \eta)^2 + \alpha_{EL} \right]}, \]

\[ M_2^2 = \frac{\alpha}{2} - \frac{1}{2} \sqrt{-\frac{1}{3} \alpha^2 - \frac{4}{3C_4} \left[ \rho^2 \omega^2 (\mu \eta)^2 + \alpha_{EL} \right]} - \sqrt{\frac{2}{3} \alpha^2 - \frac{1}{3C_4} \left[ \rho^2 \omega^2 (\mu \eta)^2 + \alpha_{EL} \right]}, \]

\[ M_3^2 = \frac{\alpha}{2} + \frac{1}{2} \sqrt{-\frac{1}{3} \alpha^2 - \frac{4}{3C_4} \left[ \rho^2 \omega^2 (\mu \eta)^2 + \alpha_{EL} \right]}, \]

\[ M_4^2 = \frac{\alpha}{2} + \frac{1}{2} \sqrt{-\frac{1}{3} \alpha^2 - \frac{4}{3C_4} \left[ \rho^2 \omega^2 (\mu \eta)^2 + \alpha_{EL} \right]}, \]

(G.80)
or we will have:

\[
\begin{align*}
\mathcal{M}_1^2 &= \frac{\alpha}{2} - \frac{1}{2} \sqrt{-\frac{1}{3} \alpha^2 - \frac{4}{3 C_4} [\rho_\omega^2 (\mu \eta)^2 + \alpha_{EL}]}, \\
\mathcal{M}_2^2 &= \frac{\alpha}{2} - \frac{1}{2} \sqrt{-\frac{1}{3} \alpha^2 - \frac{4}{3 C_4} [\rho_\omega^2 (\mu \eta)^2 + \alpha_{EL}]}, \\
\mathcal{M}_3^2 &= \frac{\alpha}{2} + \frac{1}{2} \sqrt{-\frac{1}{3} \alpha^2 - \frac{4}{3 C_4} [\rho_\omega^2 (\mu \eta)^2 + \alpha_{EL}]} + \sqrt{\frac{2}{3} \alpha^2 - \frac{1}{3 C_4} [\rho_\omega^2 (\mu \eta)^2 + \alpha_{EL}]}, \\
\mathcal{M}_4^2 &= \frac{\alpha}{2} + \frac{1}{2} \sqrt{-\frac{1}{3} \alpha^2 - \frac{4}{3 C_4} [\rho_\omega^2 (\mu \eta)^2 + \alpha_{EL}]} - \sqrt{\frac{2}{3} \alpha^2 - \frac{1}{3 C_4} [\rho_\omega^2 (\mu \eta)^2 + \alpha_{EL}]}.
\end{align*}
\]

(G.81)

In other words (absent the additional restriction of one of the radicals vanishing) we will either have \(\mathcal{M}_1^2 \leftrightarrow \mathcal{M}_2^2\) or \(\mathcal{M}_3^2 \leftrightarrow \mathcal{M}_4^2\). From physical considerations, however, we can conclude that this is a \(\mathcal{M}_3^2 \leftrightarrow \mathcal{M}_4^2\) switch point. This switch point corresponds to the fast magnetosonic point, at which point the flow should be super-Alfvénic \((M^2 > \alpha)\); that would not be true in general if this were an \(\mathcal{M}_1^2 \leftrightarrow \mathcal{M}_2^2\) switch point.

**Remarks on Root Switching**

From the above, we concluded that \(\mathcal{M}_3^2\) is the valid root on the horizon, that at the fast magnetosonic point we can generally have an \(\mathcal{M}_3^2 \leftrightarrow \mathcal{M}_4^2\) switch point, that at the Alfvén point we can generally have an \(\mathcal{M}_3^2 \leftrightarrow \mathcal{M}_4^2\) switch without restriction or an \(\mathcal{M}_3^2 \leftrightarrow \mathcal{M}_1^2 \leftrightarrow \mathcal{M}_4^2\) switch with restriction, and that at the injection point we have an \(\mathcal{M}_1^2 \leftrightarrow \mathcal{M}_2^2\) switch point.

It might seem that the additional restriction on the Alfvén point might be required in order to result in an \(\mathcal{M}_1^2 \rightarrow \mathcal{M}_4^2 \rightarrow \mathcal{M}_3^2\) solution from the injection point to the horizon. In general, however, that is not the case. The switch conditions we have found merely guarantee that at a switch point two roots will be equal; they do not guarantee anything else, such as the continuity of any given root across a switch point.
Practical application of the above formulas suggests that for reasonable parameter spaces $\mathcal{M}_2$ is valid between the injection point and the Alfvén point, where it will generally always smoothly transition to $\mathcal{M}_3^2$. If the parameters are very precisely tuned then $\mathcal{M}_2^3$ will remain real until it transitions to $\mathcal{M}_3^2$ at the fast magnetosonic point, which will then remain the valid solution until the horizon. After crossing the horizon the solution can transition to $\mathcal{M}_1^2$ or other root (which is to say $\mathcal{M}_3^2$ can be discontinuous across the horizon).

If the parameters are not precisely tuned then one of two things will generally happen. First, $\mathcal{M}_2^3$ will match $\mathcal{M}_3^2$ at an “inner fast magnetosonic point”, they will become complex conjugates for some extent, then they will meet again at an “outer fast magnetosonic point”, after which $\mathcal{M}_2^3$ will remain real until the horizon. Second, there will never be a “fast magnetosonic point”; $\mathcal{M}_3^2$ will increase for awhile after the Alfvén point, but never rise to meet $\mathcal{M}_3^2$. In that case $\mathcal{M}_2^3$ will generally remain real everywhere interior to the Alfvén point, it will simply rise for a bit before going to zero at the horizon.

Precisely tuning the parameters such that a physical $\mathcal{M}_2^2 \rightarrow \mathcal{M}_3^2$ transition occurs is one of the core problems of finding ideal magnetohydrodynamic magnetospheres. The difference between a very badly matching parameter set and a roughly matching parameter set can often be thousandths of a percent (and sometimes much less). Additionally, the matching of $\mathcal{M}_2^3$ to $\mathcal{M}_3^2$ must be made such that derivatives of $\mathcal{M}^2$ are smooth across the fast magnetosonic surface. That adds even more sensitivity to the problem; although exactly matching $\mathcal{M}_2^3$ to $\mathcal{M}_3^2$ would result in a smooth derivative, extreme parameter sensitivity can be required in order to numerically match them to a good enough degree such that derivatives $\mathcal{M}^2$ are smooth.
APPENDIX H

MAGNETOFRICTIONAL METHOD
In this appendix we discuss the convergence of the magnetofrictional method and explicitly list the computational terms used in its application and in the determination of its error.

Proof of Convergence

In this subsection we show that the magnetofrictional method inevitably drives a given vector potential $A_\phi$ towards a force-free state. We do this in two steps; first, we show that a force-free state is a minimum (extremum) in the electromagnetic energy of a given volume. Second, we show that the magnetofrictional method reduces the electromagnetic energy contained in a given volume, thereby inevitably driving that volume towards a minimum energy and force-free state. For simplicity we assume Boyer-Lindquist coordinates, that a force-free state exists, and that no pathologies develop (e.g. magnetic reconnection, if necessary, is assumed to occur via numerical diffusion).

We begin by defining the energy $W$ contained within the fields at a given coordinate time to be:

$$ W \equiv \int T^{t}_{t} \sqrt{\gamma} d^3x. \quad (H.1) $$

Here $\sqrt{\gamma} d^3x$ is the appropriate proper volume element and $T_{\alpha\beta}$ is the electromagnetic stress energy tensor. Differentiating with respect to coordinate time, we find:

$$ W_{,t} = \int T^{t}_{t;\cdot t} \sqrt{\gamma} d^3x $$

$$ = \int (T^{\alpha}_{t;\cdot \alpha} - T^{\alpha}_{t;\cdot \cdot \alpha}) \sqrt{\gamma} d^3x $$

$$ = -\int F_{\beta\beta} J^{\beta} \sqrt{\gamma} d^3x - \oint T^{A}_{t} d\Sigma_{A}. \quad (H.2) $$

Here $d\Sigma_{A}$ is the appropriate directed surface element bounding the region such that the second term is a measure of the net Poynting flux through the region’s boundary.
We have assumed no field pathologies, so any field line entering the volume also exits the volume. Energy flux per unit field line is conserved, so there can be no net Poynting flux into the region along any magnetic field lines and the second term necessarily vanishes. In a force-free configuration $F_{\alpha\beta}J^\beta = 0$ and the first term also vanishes; therefore a force-free configuration extremises the field energy $W$ in coordinate time.

The core conceit of the magnetofrictional method is that the excess momentum flux of a non force-free configuration may be converted into the coordinate velocity of a fictitious plasma; mathematically, this may be stated as:

$$-F_{\alpha\beta}J^\beta = \frac{1}{\nu} v^\alpha. \quad (H.3)$$

Here the coordinate velocity $v^\alpha$ of the plasma is defined in terms of its four velocity $u^\alpha$ as $v^\alpha \equiv u^\alpha / u^t$. Both the field line angular velocity $\Omega_F$ and the toroidal field $\sqrt{-g}F^{\theta r}$ are assumed to be functions of $A_\phi$ (i.e. conserved along magnetic field lines), so $v^\phi = 0$.

We now make the simplifying assumption that $\Omega_F$ vanishes (i.e. $F_{tr} = F_{t\theta} = 0$ and there is no electric field from the perspective of a distant observer); we will address the $\Omega_F \neq 0$ case later. We then add in a fictitious electric field that allows $v^A$ to fulfill the condition of a perfectly conducting plasma, $F_{\alpha\beta}u^\beta = 0$:

$$F_{tr} = -F_{\theta r}v^\theta, \quad F_{t\theta} = F_{\theta r}v^r, \quad F_{t\phi} = -F_{r\phi}v^r - F_{\theta\phi}v^\theta. \quad (H.4)$$

This is analogous to defining $\vec{E} = -\vec{v} \times \vec{B}$. From Maxwell’s equations we note that $\nabla \times \vec{E} = -\partial_t \vec{B}$, so defining the electric field in this way should be interpreted as
defining the time rate of change of the magnetic field. Now that the condition of a perfectly conducting plasma has been met, we simplify the first integrand for the rate of change of the field energy \( W, t \) (Equation [H.2]) using \( F_{\alpha \beta} u^\beta = 0 \) to find:

\[
F_{t \beta} J^\beta = (F_{bc} v^c) J^b = v^c (F_{\beta c} J^\beta - F_{tc} J^t) = \frac{1}{\nu} v^c v_c.
\]  

Therefore the rate of change of the field energy for a non force-free configuration under application of the magnetofrictional method is given by:

\[
W, t = - \int \frac{1}{\nu} v^A v_A \sqrt{\gamma} d^3x.
\]

For \( \nu < 0 \) the right hand side is always negative (recall the signature of our metric) and using a fictitious plasma and electric field to evolve \( A_\phi \) via the condition of a perfectly conducting plasma inevitably drives the field energy towards an extremum and a force-free state.

We now address the case of \( \Omega_F \neq 0 \), where the electric field according to a distant observer does not originally vanish. Treating this case separately is not necessary, but makes the above somewhat more transparent and allows us to explore the meaning of the field line angular velocity. First, consider a subvolume over which \( \Omega_F \) may be considered to be a constant. In that subvolume the vector potential \( A_\alpha = (A_t, A_r, A_\theta, A_\phi) \) is given by:

\[
A_\alpha = (-\Omega_F A_\phi, A_r, A_\theta, A_\phi).
\]

Suppose we now make a boost to a new frame via \( \phi' = \phi - \Omega_F t \). Then \( A_{\alpha'} \) is given
by:

\[ A_{\alpha'} = (0, A_r, A_\theta, A_\phi) . \]  

(H.8)

Noting that \( F_{tr} = -A_{t,r} \) and \( F_{t\theta} = -A_{t,\theta} \), we see that in the new frame \( F_{tr}, F_{t\theta}, \) and \( \Omega_F \) all vanish (suggestive of the fact that \( \Omega_F \) may be interpreted as a measure of the rotation of magnetic field lines referenced to zero angular momentum frames).

In this frame we are free to define an electric field that will evolve the magnetic field towards a force-free configuration using the procedure outlined above. Noting that \( t = t', \ r = r', \ \theta = \theta', \) and \( v' = v \) we find that \( A_{\phi'}(r', \theta') = A_\phi(r, \theta) \), so \( A_{\phi',t'} = -v' A_{\phi',A'} = -v A_{\phi,A} = A_{\phi,t} \). Therefore application of the magnetofrictional method in the original frame inevitably moves the vector potential towards a force-free configuration for all field line angular velocities.

**Expanded Magnetofrictional Terms**

In this subsection we expand the advection equation of the magnetofrictional method, \( A_{\phi,t} = -v A_{\phi,A} \), in order to explicitly show the form of the equations being used. We begin by noting that the advection equation may be rewritten as:

\[ A_{\phi,t} = -\nu f \left( g^{rr} A_{\phi,r}^2 + g^{\theta\theta} A_{\phi,\theta}^2 \right) . \]  

(H.9)

Here we have exploited the fact that the coordinate velocity may be rewritten as \( v_A = -\nu F_{A\beta} J^\beta = \nu f A_{\phi,A} \) for a function \( f(r, \theta, A_{\phi}, \Omega_F, \sqrt{-g F^{\theta r}}) \). We then weight the coefficient of friction \( \nu \) by a measure of the poloidal magnetic field strength:

\[ \nu = \nu_0 \frac{1}{|B_{p}|^2} . \]  

(H.10)
Here \( \nu_0 \) is a constant; it is negative so that the magnetofrictional method converges and its magnitude is selected for numerical stability. We define the magnitude of the poloidal field \( |B_p|^2 \) as:

\[
|B_p|^2 = \frac{1}{\sum \Delta \sin^2 \theta} \left( A_{\phi,\theta}^2 + \Delta A_{\phi,r}^2 \right). \tag{H.11}
\]

This weighting of \( \nu \) is chosen primarily for convenience, in that it prevents regions with relatively sharp gradients in \( A_\phi \) (i.e. large poloidal field) from evolving too rapidly and its factor of \( \Delta \) removes the coordinate singularity on the horizon. Using this weighting, the poloidal velocities \( v^r \) and \( v^\theta \) are given by:

\[
v^r = -\nu_0 \frac{f}{\sum |B_p|^2} \Delta A_{\phi,r},
\]

\[
v^\theta = -\nu_0 \frac{f}{\sum |B_p|^2} A_{\phi,\theta}. \tag{H.12}
\]

The common prefactor may be expanded as:

\[
\frac{f}{\sum |B_p|^2} = \frac{1}{A_{\phi,\theta}^2 + \Delta A_{\phi,r}^2} \frac{1}{4\pi \Sigma^2} \left[ C_{B_\phi} \frac{d}{dA_\phi} \left( \sqrt{-\Omega g F^{\theta r}} \right)^2 + C_r A_{\phi,r} + C_{rr} A_{\phi,rr} + C_{\Omega r} \Omega F_{,r} 
+ C_\theta A_{\phi,\theta} + C_{\theta\theta} A_{\phi,\theta\theta} + C_{\Omega \theta} \Omega F_{,\theta} \right]. \tag{H.13}
\]

The factor \( C_{B_\phi} \) is given by:

\[
C_{B_\phi} = -\frac{1}{2} \Sigma^2. \tag{H.14}
\]

\( C_r \) is given by:

\[
C_r = -\Sigma \Delta \alpha_{,r}
= -2m \frac{\Delta}{\Sigma} \left( r^2 - a^2 \cos^2 \theta \right) + \frac{4am \Delta}{\Sigma} \left( r^2 - a^2 \cos^2 \theta \right) \sin^2 \theta \cdot \Omega_F
\]
\[ + \frac{2 \Delta}{\Sigma} \left( r^5 + 2a^2r^3 \cos^2 \theta - a^2mr^2 \sin^2 \theta + a^4r \cos^4 \theta + a^4m \cos^2 \theta \sin^2 \theta \right) \sin^2 \theta \cdot \Omega_F^2 \\
- 4amr \Delta \sin^2 \theta \cdot \Omega_{F,r} \\
+ \Delta \left[ r^4 + a^2r^2 (2 - \sin^2 \theta) + 2a^2mr \sin^2 \theta + a^4 \cos^2 \theta \right] \sin^2 \theta \cdot 2\Omega_F \Omega_{F,r}. \quad \text{(H.15)} \]

\( C_{rr} \) is given by:

\[
C_{rr} = -\Sigma \Delta \alpha \\
= -\Delta \left( r^2 - 2mr + a^2 \cos^2 \theta \right) - 4amr \Delta \sin^2 \theta \cdot \Omega_F \\
+ \Delta \left[ r^4 + a^2r^2 (2 - \sin^2 \theta) + 2a^2mr \sin^2 \theta + a^4 \cos^2 \theta \right] \sin^2 \theta \cdot \Omega_F^2. \quad \text{(H.16)}
\]

\( C_{\theta} \) is given by:

\[
C_{\theta} = -\Sigma \sin \theta \left[ \frac{\alpha}{\sin \theta} \right]_\theta \\
= \frac{1}{\Sigma} \left[ r^4 - 2mr^3 + 2a^2r^2 \cos^2 \theta - 2a^2mr \left( 1 - 3 \sin^2 \theta \right) + a^4 \cos^4 \theta \right] \cos \theta \frac{\sin \theta}{\sin \theta} \\
- \frac{4amr}{\Sigma} \left[ r^2 + a^2 \left( 1 + \sin^2 \theta \right) \right] \sin \theta \cos \theta \cdot \Omega_F \\
+ \frac{1}{\Sigma} \left[ r^6 + a^2r^4 \left( 3 - 2 \sin^2 \theta \right) + 6a^2mr^3 \sin^2 \theta + a^4r^2 \left( 3 - \sin^2 \theta \right) \cos^2 \theta \right. \\
+ a^4mr \left( 6 - 2 \sin^2 \theta \right) \sin^2 \theta + a^6 \cos^4 \theta \left] \sin \theta \cos \theta \cdot \Omega_F^2 \right. \\
- 4amr \sin^2 \theta \cdot \Omega_{F,\theta} \\
+ \left[ r^4 + a^2r^2 \left( 2 - \sin^2 \theta \right) + 2a^2mr \sin^2 \theta + a^4 \cos^2 \theta \right] \sin^2 \theta \cdot 2\Omega_F \Omega_{F,\theta}. \quad \text{(H.17)}
\]

\( C_{\theta\theta} \) is given by:

\[
C_{\theta\theta} = -\Sigma \alpha \\
= - \left( r^2 - 2mr + a^2 \cos^2 \theta \right) - 4amr \sin^2 \theta \cdot \Omega_F \\
+ \left[ r^4 + a^2r^2 \left( 2 - \sin^2 \theta \right) + 2a^2mr \sin^2 \theta + a^4 \cos^2 \theta \right] \sin^2 \theta \cdot \Omega_F^2. \quad \text{(H.18)}
\]
\( C_{\Omega r} \) is given by:

\[
C_{\Omega r} = \Sigma \Delta G_{\phi} A_{\phi,r} = \Sigma \Delta \left( g_{t\phi} + g_{\phi\phi} \Omega_F \right) A_{\phi,r} \\
= 2amr \Delta A_{\phi,r} \sin^2 \theta \\
- \Delta \left[ r^4 + a^2 r^2 \left( 2 - \sin^2 \theta \right) + 2amr \sin^2 \theta + a^4 \cos^2 \theta \right] \sin^2 \theta \cdot \Omega_F A_{\phi,r}. \tag{H.19}
\]

\( C_{\Omega \theta} \) is given by:

\[
C_{\Omega \theta} = \Sigma G_{\phi} A_{\phi,\theta} = \Sigma \left( g_{t\phi} + g_{\phi\phi} \Omega_F \right) A_{\phi,\theta} \\
= 2amr A_{\phi,\theta} \sin^2 \theta \\
- \left[ r^4 + a^2 r^2 \left( 2 - \sin^2 \theta \right) + 2amr \sin^2 \theta + a^4 \cos^2 \theta \right] \sin^2 \theta \cdot \Omega_F A_{\phi,\theta}. \tag{H.20}
\]

In this form it is obvious that there are no divergences on the horizon; as mentioned above this is a consequence of scaling \( \nu \) by the magnitude of the poloidal field. There is however a \( \sin \theta \) divergence in \( C_{\theta} \); this is enforcing \( A_{\phi,\theta} = 0 \) on the axis as a consequence axisymmetry. Note also that inside the horizon \( C_{rr} \) and \( C_{\theta\theta} \) differ in sign; this leads to the magnetofrictional method becoming intrinsically anti-diffusive and numerically unstable there. As mentioned in the main text, outside the light surfaces an overall factor of \(-1\) is added to the above in order to maintain numerical stability, as otherwise the sign of \( \alpha \) in the \( C_{rr} \) and \( C_{\theta\theta} \) terms leads to anti-diffusive numerical instabilities. We finally note that all derivatives of \( A_{\phi} \) used to evaluate \( v^A \) may be taken using central finite differencing. A one-sided derivative appropriate to upwind differencing is only taken to evolve the \( A_{\phi,A} \) derivative in \( A_{\phi,t} = -v^A A_{\phi,A} \).

The derivative of the toroidal field (and field line angular velocity, in cases where it is not uniform) is taken to be an unknown function of \( A_{\phi} \). Practically this is implemented using a lookup table that is modified during runtime to diminish any
kinks that develop on the inner light surface.

Expanded Percent Terms

In this subsection we detail our measure of how force-free a given configuration is. Note that the force-free equations may be written as:

$$-F_{\alpha\beta}J^\beta = X_\alpha.$$  \hspace{1cm} (H.21)

For a configuration to be force-free we must have $X_\alpha = 0$. If the toroidal field $\sqrt{-g}F^{\theta r}$ is conserved along magnetic field lines, as it must be when implemented as a function of $A_\phi$, then to within numerical error $X_t = X_\phi = 0$. The poloidal components of the momentum flux may be written as:

$$X_A = f \cdot A_{\phi,A}.$$  \hspace{1cm} (H.22)

Here $f$ is a function of $r, \theta, \Omega_F, \sqrt{-g}F^{\theta r}$, and $A_\phi$; setting $f = 0$ yields the transfield equation. To measure how force-free a given solution is, we must measure how close to zero $f$ is; explicitly, $f$ is given by:

$$f = -\frac{1}{4\pi \Sigma \Delta \sin^3 \theta} [D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7].$$  \hspace{1cm} (H.23)

The functions $D_i$ are given by:

$$D_1 = \frac{1}{2} \Sigma \sin \theta \frac{d}{dA_\phi} (\sqrt{-g}F^{\theta r})^2 = -\frac{\sin \theta}{\Sigma} \cdot C_{B_\phi} \frac{d}{dA_\phi} (\sqrt{-g}F^{\theta r})^2,$$

$$D_2 = \Delta \alpha \cdot r \sin \theta A_{\phi,r} = -\frac{\sin \theta}{\Sigma} \cdot C_r A_{\phi,r},$$

$$D_3 = \Delta \alpha \cdot \sin \theta A_{\phi,rr} = -\frac{\sin \theta}{\Sigma} \cdot C_{rr} A_{\phi,rr},$$
\[
D_4 = \sin^2 \theta \left( \frac{\alpha}{\sin \theta} \right) A_{\phi,\theta} = -\frac{\sin \theta}{\Sigma} \cdot C_{\theta} A_{\phi,\theta}, \\
D_5 = \alpha \sin \theta A_{\phi,\theta\theta} = -\frac{\sin \theta}{\Sigma} \cdot C_{\theta\theta} A_{\phi,\theta\theta}, \\
D_6 = -\Delta G_{\phi} \sin \theta A_{\phi,\theta} \Omega_{F,r} = -\frac{\sin \theta}{\Sigma} \cdot C_{\Omega r} \Omega_{F,r}, \\
D_7 = -G_{\phi} \sin \theta A_{\phi,\theta} \Omega_{F,\theta} = -\frac{\sin \theta}{\Sigma} \cdot C_{\Omega \theta} \Omega_{F,\theta}. 
\] 

(H.24)

To establish a measure of force-freeness, we take the sum \( \sum_i D_i \) and compare it to the (absolute) maximum \( D_i \) term, and insist that the ratio fall below a given threshold:

\[
\epsilon = \frac{\sum_i D_i}{\text{Max} (|D_i|)}. 
\] 

(H.25)

In this work we insisted that \( \epsilon < 1\% \) over the entire domain. In practice, when a solution is found we typically find \( \epsilon \ll 1\% \) over most of the domain with \( \epsilon \approx 1\% \) on only a few segments of the inner light surface.
APPENDIX I

JET-DISK MAGNETOSPHERE MODEL
In this appendix we provide brief review of the quantities used to discuss the Jet-Disk magnetosphere model developed in Chapter 8.

**Disk Energy, Momentum, and Field Line Angular Velocity**

In this subsection we provide a brief review of the specific energy, angular momentum, and field line angular velocity associated with a disk in the equatorial plane of a Kerr spacetime in Boyer-Lindquist coordinates. Using a $(+, -, -, -)$ metric signature, the conserved specific energy $e$ and angular momentum $l$ of a particle with four velocity $u^\alpha$ are given by:

\[ e = k^\alpha u_\alpha, \]
\[ l = -l^\alpha u_\alpha. \]  
(I.1)

Here $k^\alpha$ and $l^\alpha$ are the Killing vectors associated with the temporal and azimuthal symmetries of the spacetime. The normalization of the four velocity $u_\alpha u^\alpha = 1$ with $u^r = u^\theta = 0$ then yields:

\[ -\frac{1}{\rho^2_\omega} \left(g_{\phi\phi}e^2 + 2g_{t\phi}el + g_{tt}l^2\right) = 1. \]  
(I.2)

Here $\rho^2_\omega$ is the cylindrical radius, defined as $\rho^2_\omega \equiv g_{t\phi}^2 - g_{tt}g_{\phi\phi} = \Delta \sin^2 \theta$. We then demand that the centripetal force on the particle balance the gravitational force in the radial direction to find:

\[ 0 = \Gamma_{\alpha\beta}^\gamma u^\alpha u^\beta \]
\[ = -\frac{g_{rr}^\gamma}{2\rho^4_\omega} \left[ g_{tt,r} (g_{\phi\phi}e + g_{t\phi}l)^2 - 2g_{t\phi,r} (g_{\phi\phi}e + g_{t\phi}l) (g_{t\phi}e + g_{tt}l) + g_{\phi\phi,r} (g_{t\phi}e + g_{tt}l)^2 \right]. \]  
(I.3)
Exploiting the definition of the cylindrical radius $\rho_\omega^2$ this reduces to:

$$0 = -\frac{g_{rr}}{2\rho_\omega^2} \left[ g_{\phi\phi,r}e^2 + 2g_{t\phi,r}el + g_{tt,r}l^2 + \rho_\omega^2 \right]. \quad (I.4)$$

For the gravitational and centripetal forces to balance the quantity in the square brackets must vanish. Inserting the expressions for the metric components in the equatorial plane, the velocity normalization condition and force balance condition then yield two conditions that $e$ and $l$ must satisfy:

$$0 = 2r \left( 1 - e^2 \right) - 2m + \frac{2m(l - ae)^2}{r^2},$$

$$0 = -2mr + (1 - e^2) \left( r^2 + a^2 \right) + l^2 - \frac{2m(l - ae)^2}{r}. \quad (I.5)$$

These equations may be solved for $e$ and $l^2$ to find:

$$e = \pm \sqrt{1 + \frac{l^2 - 4mr}{3r^2 + a^2}},$$

$$l^2 = \frac{C_1 \pm C_2}{C_3}. \quad (I.6)$$

The functions $C_i$ are given by:

$$C_1 = mr^6 - 3m^2r^5 + 2ma^2r^4 + 6m^2a^2r^3 + ma^2(a^2 - 12m^2)r^2 + 5m^2a^4r,$$

$$C_2 = 2ma(a^2 + 3r^2) \Delta \sqrt{mr},$$

$$C_3 = r^2 (r^3 - 6mr^2 + 9m^2r - 4ma^2). \quad (I.7)$$

What signs are chosen in $e$ and $l$ depend on what conditions are desired:

$$e > 0 \rightarrow l = -\sqrt{\ldots + \ldots},$$
In Figure 8.10 we demanded a disk with positive energy co-rotating with the black hole, so the second solution was selected.

In order to determine the field line angular velocity associated with the disk we use the condition that the plasma is a perfect conductor, $F_{\alpha\beta}u^\beta = 0$, to find two expressions for $u^t$ and $u^\phi$ under the assumption that $u^r = u^\theta = 0$:

$$F_{rt}u^t + F_{r\phi}u^\phi = 0,$$
$$F_{\theta t}u^t + F_{\theta\phi}u^\phi = 0.$$

When applying either definition of field line angular velocity, $F_{rt} \equiv -F_{r\phi}\Omega_F$ or $F_{\theta t} \equiv -F_{\theta\phi}\Omega_F$, both of the above expressions yield the same condition; $u^\phi = \Omega_F u^t$.

Replacing $u^t$ and $u^\phi$ with $e$ and $l$ from their definitions in terms of Killing vectors from Equation (I.1) and solving for $\Omega_F$ yields:

$$\Omega_F = \frac{-g_{t\phi} e - g_{t\theta} l}{g_{\phi\phi} e + g_{t\phi} l}.$$  

Inserting the appropriate metric components in the equatorial plane, we finally conclude that:

$$\Omega_F = \frac{2mr_H}{a} \left[ \frac{2mae + (r - 2m) l}{(r^3 + a^2 r + 2ma^2) e - 2mal} \right] \omega_H.$$  

We introduced a factor of $r_H = m + \sqrt{m^2 - a^2}$ in order to scale by the angular velocity
of the horizon \( \omega_H = a/2mr_H \). The distributions of \( e, l \) and \( \Omega_F \) for a co-rotating positive energy disk around a black hole with spin parameter \( a = 0.8m \) are shown in Figure 8.10.

**Terminal Jet Lorentz Factor**

In this subsection we provide a brief derivation of the terminal Lorentz factor of material along a magnetic field line, as it relates to the force-free Jet-Disk model discussed in Chapter 8. When plasma inertia effects are important, the conserved energy and angular momentum fluxes generalize to:

\[
\eta E = \mu \eta u_t + \frac{1}{4\pi} \sqrt{-g} F^{\theta r} \Omega_F,
\]

\[
\eta L = -\mu \eta u_\phi + \frac{1}{4\pi} \sqrt{-g} F^{\theta r}.
\]  

(I.12)

In other words the flow’s total flux of energy or angular momentum is conserved, but the ratio of the energy or momentum carried by the plasma or transported as a Poynting flux can vary. If for simplicity we assume a cold flow, then the relativistic enthalpy \( \mu \) reduces to the mass of the plasma’s constituent particles and is conserved. Conservation of energy then yields a restriction between the initial (I) state of the flow, presumed to be near the black hole, and the final (F) state, presumed to be far away:

\[
\mu \eta u_t(I) + \frac{1}{4\pi} B_\phi(I) \Omega_F = \mu \eta u_t(F) + \frac{1}{4\pi} B_\phi(F) \Omega_F.
\]  

(I.13)

Here we have replaced \( \sqrt{-g} F^{\theta r} \) with \( B_\phi \) to highlight the importance of the toroidal magnetic field; the exact correspondence in flat space is \( \sqrt{-g} F^{\theta r} = r \sin \theta B_\phi \), where \( B_\phi \) corresponds to a standard orthonormal spherical coordinate system. If we assume that some percentage \( \kappa \) of the electromagnetic Poynting flux is converted into plasma
energy, then we have $B_{(F)}^\phi = (1 - \kappa)B_{(I)}^\phi$. Solving for $u_{t(F)}$ we then find:

$$u_{t(F)} = u_{t(I)} + \kappa \frac{B_{(I)}^\phi}{\mu \eta} \frac{\Omega_F}{4\pi}.$$  \hspace{1cm} (I.14)

We took an absolute value for simplicity of interpretation; both $\eta$ and $B^\phi$ are directed but in this instance combine to form a positive quantity. In flat space far from the black hole $u_t$ corresponds to the Lorentz factor of the flow. In other words so long as the toroidal field is large enough and the plasma flow isn’t overly massive almost arbitrarily large Lorentz factors can be achieved. While remaining within the context of stationary and axisymmetric ideal plasma flows the primary limiting factor of such a “bead on a rotating wire” acceleration mechanism will be the geometry of the field lines.