CRITICALLY FIXED ANTI-RATIONAL MAPS,

TISCHLER GRAPHS, AND THEIR APPLICATIONS

by

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DEDICATION

To my parents, Kevin and Maria McKay, whose relentless support of my education has made this possible.

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NOMENCLATURE

$\mathbb{N}$	$\{1, 2, 3, \ldots\}$
$\mathbb{R}$	The real numbers
$\mathbb{C}$	The complex plane
$\hat{\mathbb{C}}$	The complex Sphere
$\mathbb{D}$	The unit disc in $\mathbb{R}$
$\mathbb{D}_r$	The disc of radius r centered at 0
$\mathbb{D}^n$	The unit disc in $\mathbb{R}^n$
S^{n-1}	The boundary of $\mathbb{D}^n$
$\mathbb{H}$	The open upper hemisphere of $\hat{\mathbb{C}}$
$f^n(z)$	The n^{th} iterate of f
C_f	The critical set of f
P_f	The post-critical set of f
R_f	The set of repelling fixed points of f
$\text{Fix}(f)$	The set of fixed points of f
$\#S$	The number of elements in a set S
λ	The multiplier of a fixed point
$\mathcal{A}(z_0)$	The basin of attraction of fixed point z_0
$\mathcal{A}_0(z_0)$	The immediate basin of attraction of fixed point z_0
J_f	The Julia set of f
$\mathcal{O}_f$	The topological orbifold of f
$V(G)$	The vertex set of graph G
$E(G)$	The edge set of graph G
$F(G)$	The face set of graph G
$d_G(v)$	The degree of vertex v in graph G
T_f	The Tischler graph associated to critically fixed anti-rational map f
T	A topological Tischler graph

NOMENCLATURE – CONTINUED

g_T	A Schottky map associated to topological Tischler graph T
γ	A simple closed curve
α	An arc
$\gamma \cdot T$	The complexity of a simple closed curve γ with respect to a graph T
$\alpha \cdot T$	The complexity of an arc α with respect to a graph T
$\mathcal{C}(f)$	The set of homotopy classes of simple closed curves in $\hat{\mathbb{C}} \setminus P_f$
$\mathcal{A}(f)$	The global curve attractor of f
γ^{-1}	A pullback of γ
α^{-1}	A pullback of α
∂f	$\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$
$\overline{\partial} f$	$\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$
$\mu_f(z)$	The Beltrami coefficient of f, $\frac{\bar{\partial} f(z)}{\partial f(z)}$

ABSTRACT

We are mainly concerned with maps which take the form of the complex conjugate of a rational map and where all critical points are fixed points, which are known as critically fixed anti-rational maps. These maps have a well understood combinatorial model by a planar graph. We make progress in answering three questions. Using this combinatorial model by planar graph, can we generate all critically fixed anti-rational maps from the most basic example, $z \mapsto \overline{z}^2$? Under repeated pullback by a critically fixed anti-rational map, does a simple closed curve off the critical set eventually land and stay in a finite set of homotopy classes of simple closed curves off the critical set? This is known as the global curve attractor problem and has been an area of interest since its introduction by Pilgrim in 2012. Lastly, anti-rational maps can be used to model the physical phenomenon of gravitational lensing, which is where the image of a far away light source is distorted and multiplied by large masses between the light source and the observer. Maximal lensing configurations are where n masses generate $5n - 5$ lensed images of a single light source. There are very few known examples of maximal lensing configurations, all generated by Rhie in 2003. Can we use these combinatorial models to inspire new examples of maximal lensing configurations? In this dissertation we show one can generate all critically fixed anti-rational maps from the most basic example, $z \mapsto \overline{z}^2$ by a repeated "blow-up" procedure. We also show that all critically fixed anti-rational maps with 4 or 5 critical points have a finite global curve attractor. Lastly we establish a connection between maximal lensing maps and Tischler graphs and generate new examples of maximal lensing maps.

INTRODUCTION

The focus of this thesis is anti-rational maps, defined as the complex conjugate of rational maps. We will mostly deal with critically fixed anti-rational maps, that is where every critical point is a fixed point. We will write the n^{th} iterate of a map f as $f^n = f \circ \cdots \circ f$. Define C_f to be the critical set of f, then we define the post-critical set P_f to be the union over all positive n of $f^n(C_f)$. The set of critically fixed anti-rational maps falls in the broader collection of post-critically finite anti-rational maps, where $\#P_f < \infty$. The study of these maps is interesting in the context of complex dynamics, but also due to their ties to the study of Kleinian groups [LLMM19], and gravitational lensing [KN08].

A natural generalization of post-critically finite (anti)-rational maps are (anti)-Thurston maps. Thurston introduced this topological generalization of post-critically finite rational maps and characterized when they are equivalent to rational maps [DH93]. Geyer extended this to the orientation reversing case and used it to characterize critically fixed anti-rational maps in combinatorial terms as Tischler graphs [Gey20]. If a(n) (anti)-Thurston map is equivalent to a(n) (anti)-rational map, the (anti)-Thurston map is called unobstructed. Tischler graphs are invariant planar graphs associated to critically fixed anti-rational maps. These graphs and Geyer's extension of Thurston's characterization will be our main tools in this thesis.

Pilgrim and Tan Lei introduced a way to generate new examples of Thurston maps by cutting open the sphere along an arc and "gluing" in a disc along the incision. On this disc they define new dynamics and off this disc they preserve the old dynamics of the existing function.

As a result, they generate new Thurston-maps whose degree has increased. They dubbed this procedure a “blow-up” [PL98].

Recently, blow-ups have proven to be useful. Pilgrim and Tan Lei used blow-ups to destroy obstructions, and better understand matings of certain polynomials [PL98]. Since this paper many others have used and studied them. Hlushchanka used them to classify critically fixed rational maps by simultaneously blowing up all the edges of a so called “charge graph” [Hlu19]. More recently Bonk, Hlushchanka, and Iseli use blow-ups to eliminate obstructions from Thurston maps with 4 post-critical points [BHI21].

Thurston’s theory tells us that the hopes of a(n) (anti)-Thurston map being equivalent to an (anti)-rational map lie in the dynamics of simple closed curves under pullback. Pilgrim originally asked if simple closed curves under pullback by an unobstructed Thurston map, must land in a so called “finite global curve attractor” [Pil12]. Many partial results have been proven. A positive result for quadratic polynomials with a periodic finite critical point is proven in [Pil12], and all post-critically finite polynomials in [BLMW20]. There are also positive results for specific examples of rational maps in [Pil12, Lod13], all degree 2 rational maps with 4 post-critical points in [KL19], and blown-up 2×2 Lattés maps in [BHI21]. There is also a positive result for all critically fixed rational maps which uses the blow-up classification from the same paper [Hlu19].

Gravitational lensing is the physical phenomenon where the image of a far away light source is distorted and multiplied due to a collection of masses between the light source and the observer. Gravitational lensing can be modeled by anti-rational maps by choosing a specific anti-rational map with the poles of the map representing the masses. Finding the fixed points of such an anti-rational map is equivalent to finding the position of the lensed images. For some time it was of interest to know, given n masses in the plane, what is the maximal possible number of lensed images? Mao, Petters and Witt provided examples with $3n+1$ lensed images [MPW97], and Rhie provided examples with $5n-5$ lensed images [Rhi03]. Later Khavinson and Neumann showed that $5n-5$ is the theoretical bound [KN06].

Section 2 will introduce the basics of anti-holomorphic dynamics. We will need to know some results about local dynamics and basins of attraction. Section 3 will focus on understanding blow-ups in the setting of critically fixed anti-rational maps. We prove that, with the properly chosen arc to blow-up, an unobstructed Tischler graph remains unobstructed. Furthermore, we will show that we can use blow-ups to generate unobstructed maps from obstructed ones, known as "destroying obstructions". Lastly, our main result of the section is showing that all critically fixed anti-rational maps can be generated from the map $z \mapsto \overline{z}^2$ by repeated blow-up. Note that this is fundamentally different from Hlushchanka's blow-up characterization as they blow-up all edges of a graph at once, unlike our iterative blow-ups of arcs. Section 4 attains a positive result for the global curve attractor problem for critically fixed anti-rational maps with 4 or 5 critical points. Section 5 is concerned with gravitational lensing and the anti-rational maps that model them. We prove that some of these maps have an associated Tischler graph. We also provide new examples of maximal lensing configurations.

In summary we make three main contributions to the existing literature. We will show that for every critically fixed anti-rational map f, there is a repeated blow-up procedure to generate f starting from $z \mapsto \overline{z}^2$ (Theorem 16). We prove there is a finite global curve attractor for critically fixed anti-rational maps with 4 or 5 critical points (Corollary 28). Lastly we provide new examples of maximal lensing configurations (Theorem 38).

BACKGROUND

We say that $r(z) = p(z)/q(z)$ is a *rational map* of degree d if p and q are relatively prime polynomials such that $d = \max\{\deg p, \deg q\}$. We call $f(z) = \overline{r(z)}$ an *anti-rational map* of the same degree as $r(z)$. In general the prefix *"anti-"* will describe the orientation reversing case of familiar concepts in complex analysis and dynamics (anti-holomorphic, anti-Thurston maps, etc.).

(Anti-)Holomorphic Maps

Consider a map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, for $\hat{\mathbb{C}}$ the complex sphere, and denote the n^{th} iterate of f as $f^n = f \circ f \circ \cdots \circ f$. A point $z_0 \in \hat{\mathbb{C}}$ such that $f(z_0) = z_0$ is a *fixed point of* f. The set of fixed points of f will be denoted $\text{Fix}(f)$. The points where f has local degree strictly greater than one are the *critical points of* f. The set of critical points will be denoted $\text{Crit}(f) = C_f$. The set of *critical values* is defined as $f(C_f)$, and the *post-critical set* is defined as the forward orbits of the critical points,

$$P_f = \bigcup_{n=1}^{\infty} f^n(C_f).$$

Definition 1. We call an (anti)-analytic f *critically fixed* if $C_f \subset \text{Fix}(f)$, and *post-critically finite* if $\#P_f < \infty$.

Fixed points and critical points will play a crucial role. The *multiplier* λ *of an anti-rational map* f *at the fixed point* z_0 is defined as

$$\lambda = \frac{\partial}{\partial \overline{z}}\Big|_{z=z_0} f(z).$$

The fixed point z_0 then is:

- A *super attracting fixed point* if $|\lambda| = 0$;
- An *attracting fixed point* if $0 < |\lambda| < 1$;
- An *indifferent fixed point* if $|\lambda| = 1$;
- A *repelling fixed point* if $|\lambda| > 1$.

For a given map f we will denote the set of repelling fixed points as R_f. It will be useful to count the number of fixed points of an anti-rational map. The following result was proved separately using Lefschetz fixed point theorem [Gey08] and harmonic argument principle [KN06].

Theorem 1. *Let f be an anti-rational map of degree $d \geq 2$ with N_{attr} the number of attracting fixed points and N_{rep} the number of repelling fixed points, and no indifferent fixed points. Then $N_{rep} - N_{attr} = d - 1$. In particular, the total number of fixed points is $N = N_{rep} + N_{attr} = 2N_{attr} + d - 1$.*

Definition 2. For $0 \leq |\lambda| < 1$ we can define the *basin of attraction of z_0* as

$$\mathcal{A}(z_0) = \{z \in \hat{\mathbb{C}} : \lim_{n \to \infty} f^n(z) = z_0\}.$$

Furthermore we define the *immediate basin of attraction of z_0* as the connected component of $\mathcal{A}$ containing z_0, denoted $\mathcal{A}_0(z_0)$.

There is a nice description of the local dynamics for attracting and repelling fixed points.

Theorem 2. *[NS03] Let f be an anti-analytic map defined in a neighborhood of a fixed point z_0 with multiplier $|\lambda| \notin \{0, 1\}$. Then there exists an analytic map ϕ defined in a neighborhood of z_0, with $\phi(z_0) = 0$ and $|\phi'(z_0)| = 1$, such that $\phi \circ f(z) = \lambda\overline{\phi(z)}$. That is ϕ conjugates f to*

$w \mapsto \lambda \overline{w}$ in some neighborhood of z_0.

We may hope that for anti-rational maps the *Koenig's map* ϕ near an attracting fixed point would extend to the entire basin of attraction. This must fail since the immediate basin of attraction must contain a critical point [NS03].

Theorem 3. *Let f be an anti-rational map of degree $d \geq 2$ with an attracting fixed point z_0. Then the immediate basin of attraction $\mathcal{A}_0(z_0)$ must contain a critical point of f. If it contains only one critical point of f, then it is simply connected.*

We also have a nice description of the local dynamics for a super attracting point.

Theorem 4. *(Böttcher's Theorem) [Nak93] For g anti-analytic with super attracting fixed point at z_0 there exists a local analytic change of coordinates $w = \phi(z)$, with $\phi(z_0) = 0$, such that $\phi \circ g(z) = \overline{\phi(z)}^n$. That is, ϕ conjugates g to the complex conjugate of the n^{th} power map $w \mapsto \bar{w}^n$ throughout some neighborhood of zero. Furthermore ϕ is unique up to multiplication by an $(n-1)^{st}$ root of unity.*

The map ϕ is known as the *Böttcher map*, and n is the *local degree of f at z_0* defined by the power series

$$f(z) = z_0 + a_n \overline{(z - z_0)}^n + a_{n+1} \overline{(z - z_0)}^{n+1} + \dots.$$

We also may hope that for anti-rational maps Böttcher's Theorem extends to the entire basin of attraction. This is true if the fixed point is the only critical point in the basin.

Theorem 5. *[Gey20] Let f be an anti-rational map of degree $d \geq 2$ with a super-attracting fixed point z_0. If the immediate basin $A_0(z_0)$ contains no other critical points besides z_0, then it is simply connected and the Böttcher map ϕ extends to a conformal map from $A_0(z_0)$ to $\mathbb{D}$.*

Corollary 6. *An anti-rational map f of degree $d \geq 2$ can have at most $2d-2$ attracting or super attracting fixed points.*

Definition 3. A sequence of functions f_n is *normal* if there exists a subsequence which converges locally uniformly. The *Fatou set* of f is the set on which the iterates f^n are normal. Connected components of the Fatou set will be called *Fatou components.* The *Julia set* is the complement of the Fatou set, denoted J_f.

Using Theorem 5 and the fact that the second iterate of an anti-rational map is a rational map, Geyer showed the following.

Theorem 7. *Let f be a critically fixed anti-rational map of degree $d \geq 2$. Then all periodic points of f are either super-attracting or repelling, all Fatou components of f are simply connected, and the Julia set J_f is connected.*

Definition 4. For f an anti-rational map, with critical fixed point z_0 of local degree n, let the Böttcher's map be $\phi : U \to \mathbb{D}_r$. We define the *fixed internal rays* near z_0 as the preimage of the fixed internal rays of $w \mapsto \overline{w}^n$ on $\mathbb{D}_r$. The rays on $\mathbb{D}_r$ are at the angles $\theta = \frac{2(k-1)\pi}{n+1}$, for $k = 1, \dots, n$.

Note: For a critically fixed anti-rational map, each critical point is the only critical point in its immediate basin of attraction, therefore by Theorem 5 these external rays extend to the entire immediate basin of attraction. These rays must land at fixed points in the Julia set, that is repelling fixed points [Gey20]. This will be useful when defining the invariant graph for a critically fixed anti-rational map.

(Anti-)Thurston Maps

A generalization of post critically finite rational and anti-rational maps are *Thurston maps* and *anti-Thurston maps.* Here we discuss properties of Thurston maps and Thurston's characterization of rational maps. Geyer and separately [LLMM19] recently extended these to anti-Thurston maps. In this section we distinguish S^2 (from $\hat{\mathbb{C}}$) as the sphere with no complex structure.

Thurston maps are orientation preserving *branched covers* of the sphere S^2, which are post critically finite. That is $f : (S^2, P_f) \to (S^2, P_f)$ is a branched cover if $f : S^2 \setminus f^{-1}(P_f) \to S^2 \setminus P_f$ is a covering map.

Definition 5. Two Thurston maps $f : (S^2, P_f) \to (S^2, P_f)$ and $g : (S^2, P_g) \to (S^2, P_g)$ are *(Thurston* or *combinatorially) equivalent* if there are orientation preserving homeomorphisms $h_0 : (S^2, P_f) \to (S^2, P_g)$ and $h_1 : (S^2, P_f) \to (S^2, P_g)$ such that the following diagram commutes

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{f} & (S^2, P_f) \\ \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h_1} \\ (S^2, P_g) & \xrightarrow{g} & (S^2, P_g) \end{array}$$

Where we require that h_0 is isotopic to h_1 relative P_f.

If we have a post-critically finite orientation reversing branched cover of the sphere we call it an *anti-Thurston map.* We extend the above definition of (combinatorial)-equivalence to anti-Thurston maps f and g with no changes to the maps h_0 and h_1. For a(n) (anti)-Thurston map f there exists a natural number d such that for all $w \notin f(C_f)$, $\#(f^{-1}(w)) = d$. This d is the degree of f (up to sign) and exists more generally for all branched covers of the sphere.

When are (anti)-Thurston maps equivalent to (anti)-rational maps? We will need some definitions to understand Thurston's characterization of rational maps, and then to understand Geyer's extension to anti-rational maps. We follow [Gey20] closely for this introduction to (anti)-Thurston maps.

To a(n) (anti)-Thurston map $f : S^2 \to S^2$ we can associate a *topological orbifold*, $\mathcal{O}_f$, where the underlying topological space is S^2. We define the *cone points of* $\mathcal{O}_f$ to be all $z \in P_f$. The order $\nu(z)$ of a cone point z is defined as the least common multiple of local degrees of $f^n(z)$ at all points $w \in f^{-n}(z)$, over all $n \geq 1$. If z is in a periodic cycle with a critical point, then we say $\nu(z) = \infty$. We define the *orbifold Euler characteristic* as

$$\chi(\mathcal{O}_f) = 2 - \sum_{z \in P_f} \left(1 - \frac{1}{\nu(z)}\right)$$

We say that $\mathcal{O}_f$ is *hyperbolic* if $\chi(\mathcal{O}_f) < 0$. For more on orbifolds and the orbifold Euler characteristic see [Mil06, Appendix E].

Most maps we will be concerned with have *hyperbolic* orbifolds, since for a critically fixed anti-Thurston map, each cone point is a critical fixed point of order ∞. Therefore for a critically fixed anti-Thurston map f,

$$\chi(\mathcal{O}_f) = 2 - \#C_f.$$

The only non-hyperbolic examples we will concern ourselves with are those with two critical fixed points, which will be maps of the form $z \mapsto \bar{z}^n$.

Definition 6. A simple closed curve $\gamma \subset S^2 \setminus P_f$ is *non-peripheral* or *essential* if each component of $S^2 \setminus \gamma$ contains at least two points of P_f. All other simple closed curves in $S^2 \setminus P_f$ are called *peripheral*. If two simple closed curves γ_1 and γ_2 are contained in $\hat{\mathbb{C}} \setminus P_f$ and are homotopic, we will denote this as $\gamma_1 \simeq \gamma_2$.

Since the map f is a local homeomorphism on $\hat{\mathbb{C}} \setminus P_f$, the preimage of a simple close curve γ, denoted $f^{-1}(\gamma)$, is a collection of simple closed curves.

A *multicurve* in $S^2 \setminus P_f$ is a tuple $\Gamma = (\gamma_1, \ldots, \gamma_m)$ of simple, closed, disjoint, non-homotopic, non-peripheral curves $\gamma_k \subset S^2 \setminus P_f$. For any multicurve Γ, we define the associated *Thurston's linear map* $L = L_{f,\Gamma} : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$, and its matrix $A = (a_{kj})_{k,j=1}^\infty$ as follows. Let

$$\gamma_{j,k,\alpha} = \{\gamma \text{ a connected component of } f^{-1}(\gamma_k) : \gamma \simeq \gamma_j \in \Gamma\} \quad \text{and} \quad d_{j,k,\alpha} = \deg f|_{\gamma_{j,k,\alpha}}.$$

Then

$$L(\gamma_k) = \sum_{j=1}^{m} a_{kj}\gamma_j \quad \text{with} \quad a_{kj} = \sum_{\alpha} \frac{1}{d_{j,k,\alpha}} \geq 0.$$

Because A is a non-negative matrix, it has a Perron-Frobenius Eigen value $\lambda(\Gamma, f)$.

Definition 7. For f a(n) (anti)-Thurston map with hyperbolic orbifold $\mathcal{O}_f$, an *obstruction* for f is a multicurve Γ such that $\lambda(\Gamma, f) \geq 1$.

An important type of obstruction is a Levy cycle.

Definition 8. A *Levy cycle* is a multicurve $\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ such that there exists $\gamma'_{k-1} \in f^{-1}(\gamma_k)$ which is homotopic to γ_{k-1}, such that f maps γ'_{k-1} homeomorphically onto γ_k.

We now have Thurston's characterization for Thurston and anti-Thurston maps.

Theorem 8. *[DH93] A Thurston map $f : (S^2, P_f) \to (S^2, P_f)$ with hyperbolic orbifold is equivalent to a rational map if and only if, for any multicurve Γ, we have $\lambda(\Gamma, f) < 1$. That is if an only if f does not have an obstruction. In that case, the equivalent rational map is unique up to Möbius conjugacy.*

Remark: Originally the above theorem was proved for proved for so called "f-stable" multicurves. Later Pilgrim and Tan Lei proved this was not a necessary condition for the orientation preserving case [PL98, Section 3.1]. Geyer later did the same in the orientation reversing case along with the following theorem.

Theorem 9. *[Gey20] Let f be an anti-Thurston map with hyperbolic orbifold $\mathcal{O}_f$. Then f is equivalent to an anti-rational map if and only if f does not have an obstruction. In that case, the equivalent anti-rational map is unique up to Möbius conjugacy.*

Tischler Graphs and Schottky Maps

For some rational maps there are nice combinatorial classifications. For critically fixed rational and anti-rational maps we have *Tischler graphs* [Tis89].

We will only concern ourself with embedded planar graphs. Therefore a *graph* Γ consists of a finite set of points in $\mathbb{C}$ called *nodes* or *vertices*, and a finite set of embeddings of $[0, 1]$ in $\mathbb{C}$ with 0 and 1 mapping onto vertices, called *edges*. We allow multiple edges between vertices, in which case the graph is usually referred to as a multigraph but we will not use this terminology. We also allow edges to loop back onto the same vertex. We will denote the set of all edges in Γ as $E(\Gamma)$ and the set of all vertices in Γ as $V(\Gamma)$. A *planar graph* Γ is an embedded graph in $\mathbb{C}$ or $\hat{\mathbb{C}}$, therefore the edge set is pairwise disjoint. It is natural to think of the connected components of $\hat{\mathbb{C}} \setminus \Gamma$ as *faces of* Γ. We will denote the set of faces induced by Γ as $F(\Gamma)$. Note that since we almost exclusively play in $\hat{\mathbb{C}}$, our figures drawn in in the plane count the face at infinity.

An edge is *incident* to a vertex if the vertex is on the boundary of the edge. Two edges are *adjacent* if they are both incident to the same vertex. The *degree of a vertex* v in graph Γ is the number of edges incident to v, denoted $d_\Gamma(v)$. If the boundary of an edge is a single vertex, the edge gets counted twice in the degree of that vertex.

A *(finite) walk* is a graph Γ is a finite alternating sequence of edges e_j and vertices p_j,

$W = (p_0, e_1, p_1, e_2, \ldots, e_n, p_n)$. The number n is the *length* of the walk, due to the walk seeing n (not necessarily distinct) edges. The vertex p_0 is called the *initial vertex* of walk W and p_1 is called the *terminal vertex* of walk W. If the initial and terminal vertices of a walk are the same then we call W a *closed walk* or *circuit*. Given a face A of Γ we define a *boundary circuit* as a walk along ∂A in a mathematically positive direction, so that $\text{int}(A)$ always lies to the left of the circuit. We call a circuit *simple* if every edge in the walk is unique, and the only repeated vertex is the initial and terminal vertex. We define a *cycle* as a simple circuit.

Recall, given a critically fixed (anti)-rational map f, we have that each immediate basin of attraction of a critical point has some number of fixed internal rays defined by the Böttcher map. These rays must land at fixed points in the Julia set, that is repelling fixed points [Gey20].

Definition 9. The *Tischler graph of a critically fixed anti-rational map* f, T_f is the graph consisting of

- $V(T_f) = \text{Fix}(f) = C_f \sqcup R_f$, and
- $E(T_f) = \{\text{fixed internal rays of } f\}$.

Since vertices which correspond to repelling fixed points have degree 2, we will abuse notation and say $V(T_f) = C_f$ and understand that each edge has a unique repelling fixed point. In Chapter 3 we will need to keep track of these degree 2 vertices, we will call this the *full Tischler graph* and denote the vertex set $V_E T_f = C_f \sqcup R_f$.

This graph is well defined by the following theorem of Geyer. In this theorem a *Jordan domain* is a complementary component of a Jordan curve. A *Jordan curve* is the image of a continuous map $\phi : [0,1] \to \hat{\mathbb{C}}$, such that $\phi(0) = \phi(1)$ and $\phi|_{[0,1)}$ is injective. Or equivalently

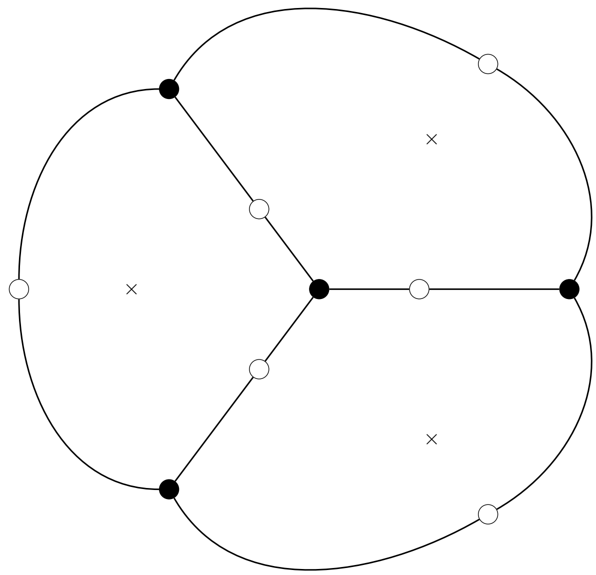


Figure 2.1: The full Tischler graph associated to $f(z) = \frac{3z^2}{\sqrt[3]{2}(z^3+1)}$. Where $\bullet$ represents the critical fixed points, $\circ$ represents the repelling fixed points points, and $\times$ represents the poles of our map f.

a continuous injective image of the unit circle. S^1.

Theorem 10. *[Gey20] Let f be a critically fixed anti-rational map of degree $d \geq 2$. Then its full Tischler graph $T = T_f$ is a connected bipartite graph with $d+1$ faces, whose vertex set $V_E(T) = C_f \sqcup R_f = P_f$. Each vertex that is a critical point of multiplicity m has degree $m+2 \geq 3$, and each vertex which is a repelling fixed point has degree 2. Furthermore, every face A of T is a Jordan domain, and f maps A anti-conformally onto $\hat{\mathbb{C}} \setminus \overline{A}$.*

Geyer also showed for anti-rational maps, the combinatorial dynamics is modeled by Schottky maps as follows. Define first a generalization of the Tischler graph beyond the context of (anti)-rational maps.

Definition 10. A *topological Tischler graph* is a a connected planar graph $T \subset S^2$ such that each vertex has degree ≥ 3 and each face is a Jordan domain.

Later we will again need to take the *full topological Tischler graph* where we take the vertex set to be $V_E T$ to be $V(T)$ along with a unique marked point on the interior of every edge. For a Jordan domain $U \subset S^2$, an associated *topological reflection (in U)* is an orientation reversing homeomorphism $f_U : S^2 \to S^2$ such that $f_U|_{\partial U} = \text{id}$, and $f^2 = \text{id}$ on S^2.

Definition 11. Given a topological Tischler graph T, an associated *Schottky map* f_T is a map such that f_T restricted to every face U of T is a topological reflection associated to U.

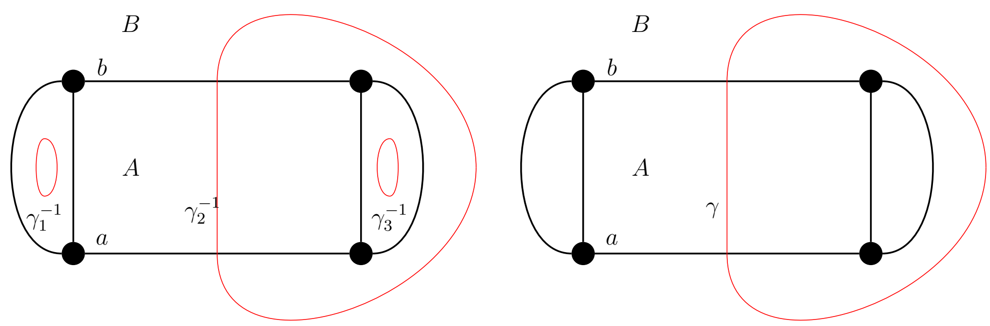
Geyer proved, for a critically fixed anti-rational map f and its Tischler graph $T = T_f$, any associated Schottky map f_T on T is combinatorially-equivalent to f. Using Schottky maps and Thurston theory Geyer was able to characterize obstructions to critically fixed anti-rational maps as follows.

Theorem 11. *[Gey20] Let $T = (P, E)$ be a topological Tischler graph and $f = f_T$ be an associated Schottky map. Then one of the following mutually exclusive cases occurs:*

T **is obstructed** *There is a pair of distinct faces A and B of T sharing two distinct edges $a, b \in \partial A \cap \partial B$. In this case, the map f has a Levy cycle given by a simple closed curve γ which intersects the Tischler graph exactly twice, once in a and once in b. Such a simple closed curve will be called a* Thurston-obstruction.

T **is unobstructed** *For any two distinct faces A and B, there is at most one edge in $\partial A \cap \partial B$. In this case, f is combinatorially-equivalent to an anti-rational map, unique up to Möbius conjugacy.*

Critically fixed anti-rational maps, (topological) Tischler graphs, and Schottky maps will be the main objects of study in this thesis. We need some language around simple closed curves since simple closed curves capture the idea of obstructions. The number of intersections a curve γ has with the Tischler graph T is denoted $\#(\gamma \cap T)$. Since we are only concerned with



(a) (b)

Figure 2.2: An obstructed (Topological) Tischler graph.

γ up to homotopy, we are mostly concerned the minimal number of intersections a homotopy class of curves has with the Tischler graph. If γ_1 and γ_2 are homotopic simple closed curves, we will write $\gamma_1 \simeq \gamma_2$.

Definition 12. The *complexity* or *minimal number of intersections* of γ with T is denoted

$$\gamma \cdot T = \min\{\#(\gamma' \cap T) : \gamma' \simeq \gamma\}.$$

We will say that γ is in *minimal position* if $\gamma \cdot T$ is realized.

Pilgrim and Lei [PL98] applied Thurston theory to define the blow-up of a Thurston map. Applying this process generates new Thurston maps and if done correctly, can destroy obstructions to the original map. Below is an extension of their ideas to anti-Thurston maps. While the core principles are essentially the same, there are some substantial differences in the outcome.

Useful in this section will be the so called "Alexander Trick", listed below.

Lemma 12. *[Hub06, Alexander Trick, Appendix C2] For $\mathbb{D}^n$ the unit disc in $\mathbb{R}^n$, and $S^{n-1} = \partial\mathbb{D}^n$ we have the following:*

1. *Every homeomorphism $f : S^{n-1} \to S^{n-1}$ extends to a homeomorphism $\tilde{f} : \mathbb{D}^n \to \mathbb{D}^n$.*
2. *Any two such extensions $\tilde{f}_1$, $\tilde{f}_2$ are isotopic through a family of homeomorphisms $\tilde{f}_t$ such that $\tilde{f}_t|_{S^{n-1}} = f$ for all $t \in [0, 1]$.*

Note, the statement in Hubbard's book is only for orientation-preserving homeomorphisms, but the proof does not rely on this fact.

Blow-ups of a Tischler graphs

We define the blow-up of a Tischler graph T_f (obstructed or unobstructed) with an associated Schottky map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Here we will take the "full" vertex set $V_E(T_f)$ where we have the vertices corresponding to critical fixed points and one vertex on each edge of degree 2. Recall that in the anti-rational case the degree 2 vertices correspond to the repelling fixed points.

Recall that an arc in this case is defined to be a homeomorphic embedding $\alpha : [0,1] \to \hat{\mathbb{C}}$ such that $\alpha((0,1)) \cap V_E(T) = \emptyset$. We will often refer to α as a set, take this to be an abuse of notation where the set we refer to is $\alpha([0,1])$.

Definition 13. To blow-up along an arc α, the arc must satisfy the following *blow-up conditions*:

1. There exists a face F of the graph T_f such that $\text{int}(\alpha) \subset F$.

2. The two end points of $\alpha = \{p_0, p_1\}$ are in the full vertex set $V_E(T_f)$.

Before outlining the blow-up procedure, let us name some subsets of $\hat{\mathbb{C}}$ which are induced by the arc α, its image $f(\alpha)$, and the Tischler graph, see Figure 3.1. These conditions imply that $\alpha \cup f(\alpha)$ is a simple closed curve, since $f(\text{int}(F)) = \text{int}(F^c)$ and the endpoints of α are fixed. Therefore $\alpha \cup f(\alpha)$ partitions $\hat{\mathbb{C}}$ into two connected components A and B. Let D be a closed topological disk such that $\text{int}(\alpha) \subset \text{int}(D)$ and $T_f \cap D = \partial\alpha = \{p_0, p_1\}$. Therefore $\text{int}(D)$ is partitioned by α into two open connected components $D_A \subset A$ and $D_B \subset B$. It will also be helpful to label the boundary of D in each of these components, let $\partial_A = \partial D \cap \overline{A}$ and $\partial_B = \partial D \cap \overline{B}$. Now we will outline the blowup procedure which takes place on and off the disc D.

Off the disk D, on $\overline{\hat{\mathbb{C}} \setminus D}$, we wish to preserve the dynamics of f. In the face F we pick a continuous homotopy from the identity map to the collapse map $c : \hat{\mathbb{C}} \setminus D \to \hat{\mathbb{C}} \setminus \alpha$ mapping ∂_A and ∂_B homeomorphically onto α. Off of the face F this homotopy is extended by the identity at all times. We then apply the map f to achieve the desired dynamics off F, that is

$$f \circ c : \overline{\hat{\mathbb{C}} \setminus D} \to \hat{\mathbb{C}}.$$

On this disk D we will define the map β as the composition of maps, $\beta : D \xrightarrow{h_1} \overline{\mathbb{D}} \xrightarrow{\tau} \hat{\mathbb{C}} \xrightarrow{h_2} \hat{\mathbb{C}}$ specified below. Since D is a topological disk, there exists an orientation preserving homeomorphism $h_1 : D \to \overline{\mathbb{D}}$ such that $h_1(\alpha) = [-1, 1]$, $h_1(p_0) = -1$, and $h_1(p_1) = 1$. Furthermore $h_1(\partial_A) = \{e^{i\theta} | \theta \in [0, \pi]\}$ and $h_1(\partial_B) = \{e^{i\theta} | \theta \in [\pi, 2\pi]\}$.

To introduce new dynamics, τ will be defined almost identically to Pilgrim and Lei, the only difference being the complex conjugation:

$$\tau(z) = S_2(\overline{S_1(z)})^2$$

where

$$S_1(z) = -i\frac{z+1}{z-1} \quad \text{and} \quad S_2 = \frac{z-1}{z+1}.$$

First, S_1 maps the closed unit disk $\overline{\mathbb{D}} \subset \hat{\mathbb{C}}$ onto the closed upper half-plane $\mathbb{H}$ such that $[-1, 1]$ maps onto the closed imaginary axis in the upper half-plane, $\overline{\mathrm{Im}^+}$. More importantly S_1 maps $\{e^{i\theta} | \theta \in [0, \pi]\}$ onto the closed negative real line, $\overline{\mathbb{R}^-}$ and $\{e^{i\theta} | \theta \in [\pi, 2\pi]\}$ onto the closed positive real line, $\overline{\mathbb{R}^+}$. The new dynamics are introduced by $z \mapsto \overline{z}^2$, mapping $\mathbb{H}$ to $\mathbb{C}$, moreover $\overline{\mathbb{R}}$ onto $\overline{\mathbb{R}^+}$ and $\overline{\mathrm{Im}^+}$ onto $\overline{\mathbb{R}^-}$. We then use S_2 to realign, mapping $\overline{\mathbb{R}^+}$ to $[-1, 1]$ and $\overline{\mathbb{R}^-}$ to $\mathbb{R} \setminus (-1, 1)$.

In summary

$$\tau(z) = \overline{\left(\frac{2iz}{1+z^2}\right)}$$

is a critically fixed anti-rational map which maps $\overline{\mathbb{D}}$ to $\hat{\mathbb{C}}$. This map has the property that the upper half disk maps onto the upper half-plane, the lower half disk onto the lower half-plane, and τ fixes -1 and 1.

Lastly, to align our new dynamics with f on $f(\alpha)$, we pick a map $h_2 : \{-1, 1\} \to \{p_0, p_1\}$

such that $h_2(-1) = p_0$ and $h_2(1) = p_1$. By Lemma 12 Part (1), we can then extend h_2 to the interval $[-1,1]$ such that $h_2([-1,1]) = \alpha$ with

1. $h_2 \circ \tau \circ h_1|_{\partial_A} = f \circ c|_{\partial_A}$

2. $h_2 \circ \tau \circ h_1|_{\partial_B} = f \circ c|_{\partial_B}$

Therefore we have a homeomorphism $h_2 : \overline{\mathbb{R}} \to \alpha \cup f(\alpha)$, a homeomorphism on the boundary of Jordan domains, which we can now extend to a homeomorphism $h_2 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

Definition 14. Thus g, the *blow-up of f along α once*, is defined as

$$g(z) = \begin{cases} f \circ c(z), & \text{for } z \in \overline{\hat{\mathbb{C}} \setminus D} \\ \beta(z), & \text{for } z \in D. \end{cases}$$

Theorem 13 (Blowing-up a map once)**.** *Consider a critically fixed anti-rational map f of degree d with Tischler graph T_f, and an arc α satisfying the blow-up conditions above for f. Blowing-up f along α once generates a map g which is Thurston-equivalent to a Schottky map, g_Γ, induced by the new graph $\Gamma = T_f \cup \alpha$. That is $V(\Gamma) = V(T_f) \cup \partial\alpha$ and $E(\Gamma) = E(T_f) \cup \alpha$. The Schottky map g_Γ is unobstructed, therefore g is Thurston-equivalent to a critically fixed anti-rational map of degree $d+1$.*

Proof. By definition α is fixed by the blow-up of f along α. Further, the face of T containing α is partitioned by α into $F_A \subset A$ and $F_B \subset B$. By construction g maps F_A onto $\hat{\mathbb{C}} \setminus F_A$ homeomorphically, as well as F_B onto $\hat{\mathbb{C}} \setminus F_B$ (see Figure 3.1). Thus g is a local homeomorphism off the branch points of f and the endpoints of α. It is then clear that g is an anti-Thurston map with branch points $C_f \cup \partial\alpha$. Since g fixes the endpoints of α by construction, g is a critically fixed anti-Thurston map.

Let $\Gamma = T_f \cup \alpha$. Then g fixes the vertex set $V(\Gamma)$ and maps each edge in the edge set $E(\Gamma)$ homeomorphically onto itself. Thus $g|_\Gamma$ is isotopic to the identity, relative $V(\Gamma)$. Further g maps each face of Γ homeomorphically onto its complement. Thus by an Alexander trick g is isotopic relative $V(\Gamma)$ to g_Γ, an associated Schottky map of Γ. We know the degree of g_Γ is $d+1$ by observing Γ has $d+2$ faces.

Assume g_Γ is an obstructed map and invoking Theorem 11, there exists an essential simple closed curve $\gamma \subset \hat{\mathbb{C}} \setminus V(\Gamma)$ that only intersects two distinct faces of Γ, U and V, and two distinct edges, e_U and e_V. If $\alpha \neq e_U$ and $\alpha \neq e_V$, then e_U and e_V are edges of T_f, implying f is obstructed by γ, a contradiction. Therefore without loss of generality assume $e_U = \alpha$. This implies $U = F_A$ and $V = F_B$, and e_V is adjacent to both F_A and F_B. By construction e_V must also be an edge of T_f, but F_A and F_B are contained in a single face in T_f. Therefore e_V is only adjacent to one face in T_f, therefore that face is not a Jordan domain. This is a contradiction since every face of the Tischler graph T_f is a Jordan domain.

Thus g is Thurston-equivalent to an unobstructed Schottky map g_Γ. Therefore g itself is Thurston-equivalent to a critically fixed anti-rational map of degree $d+1$ with Tischler graph isotopic to Γ. □

Example: Consider blowing-up $s(z) = \overline{z^2}$ along an arc α between two repelling fixed points p_0 and p_1, which satisfies the above blow-up conditions. Figure 3.1 shows the resulting map g on half of the face containing α. By Theorem 13, blowing-up $s(z) = \overline{z^2}$ along such an arc generates a Schottky map on the complete graph with 4 vertices. Each of these vertices has degree 3 and there is only one edge between any two vertices, therefore this is the complete graph on 4 vertices. Thus this map must be combinatorially-equivalent to the tetrahedral map,

$$r(z) = \overline{\left(\frac{3z^2}{\sqrt[3]{2}(z^3+1)}\right)}.$$

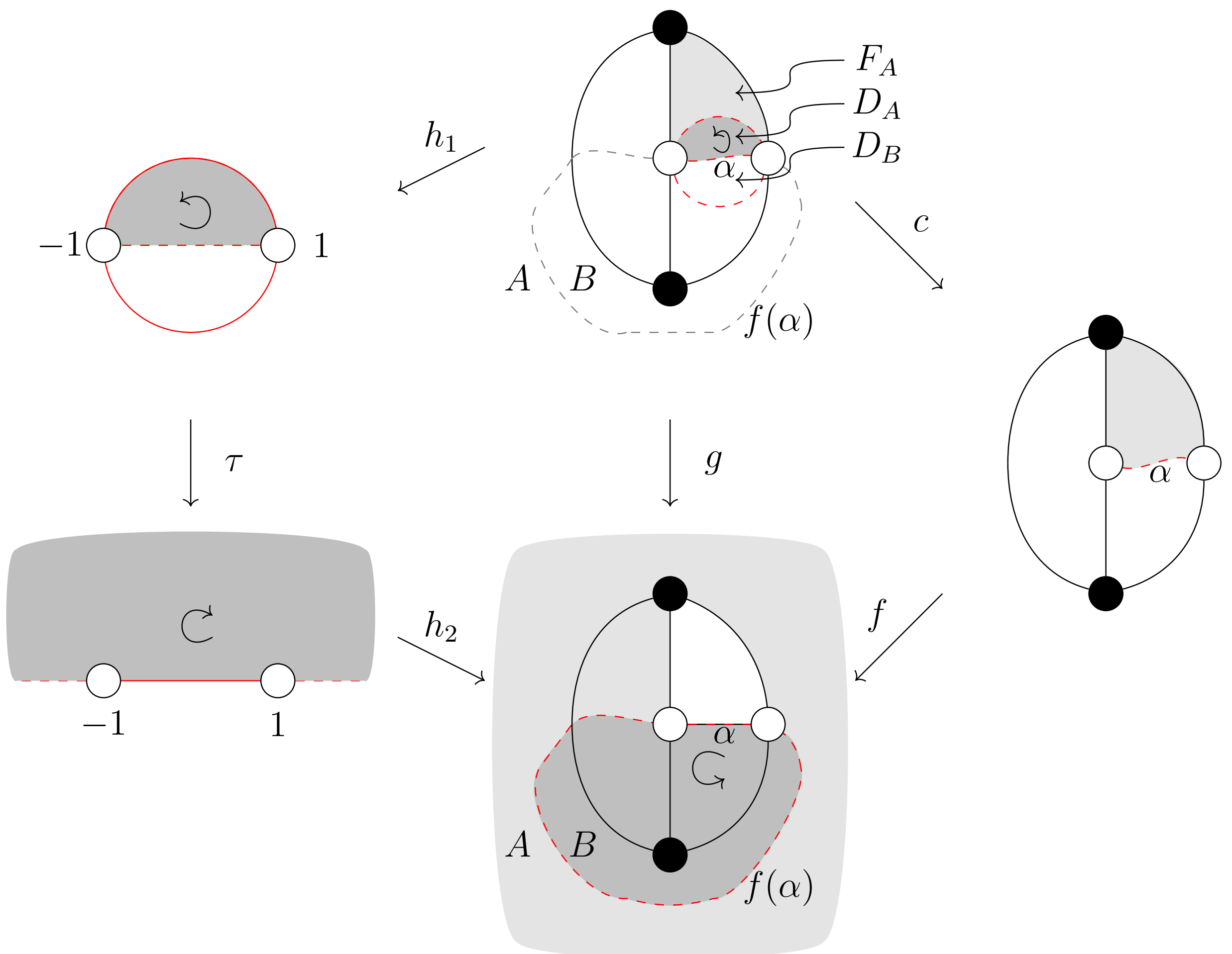


Figure 3.1: The top center gives the Tischler graph associated to $s(z) = \overline{z}^2$ in solid black, as well as associated regions defined for blowing-up $s(z)$ once along α. Specifically F_A is shown as the total shaded region (both dark and light) and D_A is shaded in dark gray. The left hand side of the figure outlines the map g on D, and the right hand side outlines the map g off D. We can see visually how g maps F_A onto B.

Theorem 14. *Let T a topological Tischler graph, f_T be a Schottky map on T, and α an arc meeting the blow-up conditions above. Blow-up f_T along α to generate a new Schottky map g on $\Gamma = T \cup \alpha$. If g is obstructed, then f_T is obstructed. Moreover the obstruction of g cannot intersect α. This implies the blow-up procedure cannot produce "new" obstructions.*

Proof. The start of this proof will be almost identical to that if Theorem 13. By definition α is fixed by the blow-up of f_T along α. Further, the face of T containing α is partitioned by α into $F_A \subset A$ and $F_B \subset B$. By construction g is a critically fixed anti-Thurston map with branch points $C_f \cup \partial\alpha$.

If g is an obstructed map, then there exists an essential simple closed curve $\gamma \subset \hat{\mathbb{C}} \setminus V(\Gamma)$ that only intersects two distinct faces of Γ, U and V, and two distinct edges, e_U and e_V.

First suppose the obstruction intersects α, therefore without loss of generality assume $e_U = \alpha$. This implies $U = F_A$ and $V = F_B$, and e_V is adjacent to both F_A and F_B. By construction e_V must also be an edge of T, but F_A and F_B are contained in a single face in T. Therefore e_V is only adjacent to one face in T_f, therefore that face is not a Jordan domain. This is a contradiction since every face of the topological Tischler graph T_f is a Jordan domain. Therefore if a blow-up of a topological Tischler graph along α is obstructed, the obstruction cannot intersect α.

If g is obstructed and if $\alpha \neq e_U$ and $\alpha \neq e_V$, then e_U and e_V are edges of T, implying f_T is obstructed by γ. □

Applications of Blow-ups

An initial application of blow-ups is to generate unobstructed maps from obstructed ones.

Theorem 15 (Destroying Obstructions)**.** *Consider a topological Tischler graph Γ and an* *associated Schottky map h such that h is obstructed. There exists a finite sequence of arcs $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ which can be blown up to generate an unobstructed map g, a Schottky map associated to the graph $\Gamma' = \Gamma \cup \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$.*

Proof. Since Γ is a finite graph by definition, there are only finitely many pairs of adjacent faces. Therefore, up to homotopy on $\hat{\mathbb{C}} \setminus V(\Gamma)$, there are only finitely many obstructions of h by a simple closed curve γ such that γ intersects exactly two distinct edges of Γ. Let $\{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ be the representatives of all homotopy classes of obstructions of h in minimal position. Let e_n be one of the two edges γ_n intersects for $1 \leq n \leq m$. Let $\{e_1, e_2, \ldots, e_k\}$, for $k \leq m$, be the set of edges such that, for all γ_n there exists an l such that e_l and γ_n intersect. Now we choose a set of arcs $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ such that α_l is isotopic to e_l relative $V(\Gamma)$ and $\text{int}(\alpha_l) \cap \text{int}(e_l) = \emptyset$.

Define g_1 as the blow up of h along α_1. By construction we have effectively "doubled the edge" e_1. By Theorem 14 the blow-up procedure cannot product "new" obstructions. Thus g_1 is not obstructed by any simple closed curve γ_n for $1 \leq n \leq m$ that intersects the edge e_1 as γ_n now intersects three distinct edges. In sequence define g_{l+1} as the blow-up of g_l along α_{l+1}. Again by Theorem 14 the blow-up procedure cannot product "new" obstructions. The map g_{l+1} is not obstructed by any simple closed curve γ_n that intersected $\{\alpha_1, \alpha_2, \ldots, \alpha_{l+1}\}$ by the same argument. By repeated application of Theorem 14, $g = g_k$ is an unobstructed Schottky map on the topological Tischler graph $\Gamma' = \Gamma \cup \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$. □

A Classification of Anti-Rational Maps by Blow-up

Now that we have established we can generate unobstructed maps via obstructed Schottky maps on Tischler graphs, it is natural to ask if all unobstructed maps can be generated via blow-up?

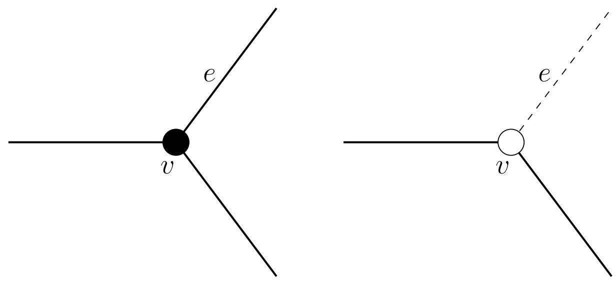


Figure 3.2: On the left, vertex v is incident to edge e and $d_T(v) = 3$. On the right, edge e has been deleted and v now has degree 2, because of this we forget v as a vertex of $T \setminus e$.

Theorem 16 (Classification of Tischler Graphs)**.** *Any critically fixed anti-rational map f is generated by some sequence of blow-ups of $s(z) = \overline{z}^2$.*

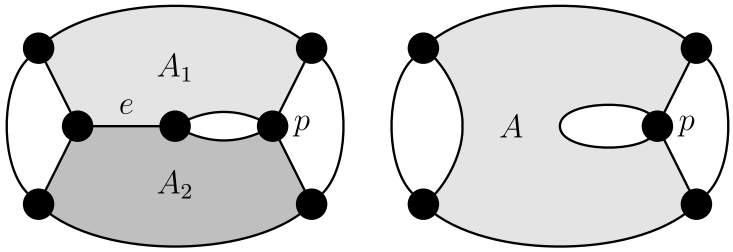
To prove this theorem we show that every unobstructed Tischler graph has an edge such that removing that edge generates a new unobstructed map. Theorem 13 tells us that it is possible to "blow-up" a critically fixed anti-rational map by adding an edge e to the Tischler graph, if e satisfies our blow-up conditions for an arc. We are going to understand what it means to undo this process. Therefore we will need to know how to delete an edge, what possible obstructions this can produce, and how those possible obstructions can interact with each other.

Definition 15. Given a (topological) Tischler graph T and an edge e, the graph produced by *deleting the edge* e is $T \setminus e$, where

- the vertex set, $V(T \setminus e) = V(T) \setminus \{p \in \partial e | d_T(p) = 3\}$ (See Figure 3.2),
- and the edge set, $E(T \setminus e) = E(T) \setminus e$.

We call e *deletable* if $T \setminus e$ is an unobstructed Tischler graph. An edge e is *not deletable* if $T \setminus e$ is an obstructed Tischler graph or is not a Tischler graph.

By Theorem 11, if $T \setminus e$ is an obstructed Tischler graph, there exists a Thurston-obstruction;



(a) T (b) $T \setminus e$

Figure 3.3: Notice T is an unobstructed topological Tischler graph. The graph $T \setminus e$ is not a Tischler graph since A is not a Jordan domain. Therefore $T \setminus e$ is Schottky-obstructed. Furthermore $T \setminus e$ is obstructed since face A and the unbounded face sharing two distinct edges on their boundaries. Therefore $T \setminus e$ is also Thurston-obstructed.

a simple closed curve $\gamma_e \subset \hat{\mathbb{C}} \setminus V(T)$ such that γ_e intersects exactly two distinct edges.

Definition 16. Let T be an unobstructed (topological) Tischler graph and $e \in E(T)$. If γ_e is a Thurston-obstruction for $T \setminus e$, we will call γ_e a *Thurston-unobstruction* for T associated to edge e. More specifically $\gamma \cdot T = 3$ and each connected component of $\hat{\mathbb{C}} \setminus \gamma_e$ contains at least 2 vertices of $T \setminus e$ so that γ_e is non-peripheral.

Now we want to associate terminology and a simple close curve for when $T \setminus e$ is not a Tischler graph. Recall that a topological Tischler graph is defined as a connected planar graph where each vertex has degree ≥ 3 and each face is a Jordan Domain.

Definition 17. A graph Γ is *Schottky-obstructed* if a face of Γ is not a Jordan domain. See Figure 3.3.

A graph Γ has a *cut point* p if as a set $\Gamma \setminus \{p\}$ is disconnected. For $\Gamma \subset \hat{\mathbb{C}}$ this means that there exists two disjoint open sets $U, V \subset \hat{\mathbb{C}}$ such that U contains at least one component of $\Gamma \setminus \{p\}$ and V contains at least one other.

Lemma 17. *(Cut-Point Lemma) A Tischler graph T does not have any cut-points. If a*

Tischler graph T has a cut-point p, then there exists a simple closed curve γ such that $\gamma \setminus \{p\}$ is contained in a single face F of T, F is not a Jordan domain and γ separates one connected component of $T \setminus e$ from the rest.

Proof. Consider the Tischler graph Γ embedded in $\hat{\mathbb{C}}$. Suppose Γ has a cut-point p. A priori p may or may not be a vertex of the graph. There exists a small open disc Δ containing p such that the closure $\overline{\Delta} \cap (V(\Gamma) \setminus \{p\}) = \emptyset$ and the only edges which intersect Δ are incident to p. Normalize these edges incident to p to be straight lines in Δ. Since p is a cut-point and Δ is open the set $\Gamma \setminus (\Gamma \cap \Delta)$ in $\hat{\mathbb{C}}$ is a disjoint union of compact sets. Therefore there exists a simple closed curve γ separating one connected component of $\Gamma \setminus (\Gamma \cap \Delta)$ from the others. In Δ, adjust the curve γ so it is a union of straight lines terminating at the cut-point p. Therefore the two complementary components of γ form disjoint open sets U and V such that both contain connected components of $\Gamma \setminus \{p\}$. By construction $\gamma \setminus \{p\}$ is contained in a single face F of Γ. The curve γ partitions the walk around the boundary of F into pieces contained in each complementary component of γ. Therefore the face containing $\gamma \setminus \{p\}$ is not a Jordan domain since any continuous map from S^1 to ∂F must map at least two points of S^1 onto p. □

Definition 18. For a graph Γ with a cut-point p we call any such simple closed curve in Lemma 17 a *Schottky-obstruction.*

Proposition 18. *Given an edge e of a (topological) Tischler graph T with at least 4 edges, $T \setminus e$ is not a (topological) Tischler graph if and only if $T \setminus e$ has a Schottky-obstruction, that is, $T \setminus e$ has a cut-point.*

Proof. Let T be a Tischler graph with at least 4 edges, one of those edges being e.

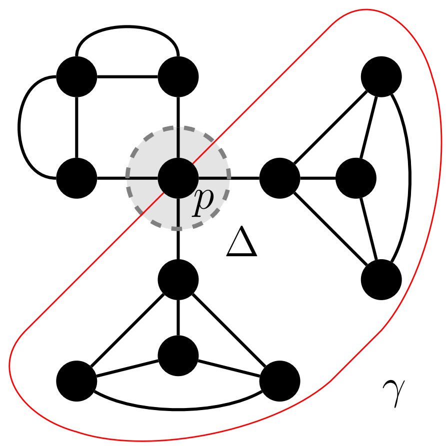


Figure 3.4: A connected planar graph T which is not a topological Tischler graph. We see the small disc Δ (shaded region) about the cut point p, and the simple closed curve γ in red separating one connected component of $T \setminus \{p\}$ from the rest.

First assume $T \setminus e$ has a Schottky-obstruction γ_e. Then there exists a vertex $p \in V(T \setminus e)$ which is a cut-point. By Lemma 17 there exists a face F which is not a Jordan domain, therefore $T \setminus e$ is not a topological Tischler graph.

Now assume $T \setminus e$ is not a topological Tischler graph. Therefore $T \setminus e$ is not connected, not planar, there exists a vertex v such that $d_{T \setminus e}(v) < 3$, or there exists a face F which is not a Jordan domain. If $T \setminus e$ is not connected, then any point in edge e is a cut-point of T, a contradiction by Lemma 17. If $T \setminus e$ is not planar, then T is not planar, a contradiction to T being a topological Tischler graph. By definition of deleting edge e, each vertex in $T \setminus e$ has degree greater than or equal to 3. Therefore we have reduced the problem to showing if $T \setminus e$ has a face F which is not a Jordan domain then $T \setminus e$ has a Schottky-obstruction. If face the F is not a Jordan domain then there is no injective continuous map from the unit circle S^1 to ∂F. Therefore the boundary circuit of F is not simple. Therefore there exists a point $p \in \partial F$ such that any boundary circuit not starting at p around the boundary sees p at least twice. Any walk which starts at p, sees p again before walking the entire boundary of F. Therefore is we start a walk at p, the first time we return to p we have seen a cycle C_1 in ∂F. Continuing this walk we see another cycle C_2 in ∂F such that $C_1 \cap C_2 = \{p\}$. Removing the point p disconnects these two cycles, therefore p is a cut-point and $T \setminus e$ is Schottky obstructed. □

Definition 19. Given a (topological) Tischler graph T and an edge e, if γ_e is a Schottky-obstruction for a graph $T \setminus e$, we call γ_e a *Schottky-unobstruction* for a graph T. More specifically $\gamma_e \cap T \setminus e = \{p\}$ (the cut-point) and γ_e separates the connected components of $(T \setminus e) \setminus p$.

Note if the cut-point p of $T \setminus e$ is contained in an edge then by definition the original graph T is Thurston-obstructed. Therefore the cut-point p is a vertex of T. We call this vertex the *vertex associated to* γ_e.

Proposition 19. *An edge e of unobstructed Tischler graph T is not deletable if and only if there exists one of the following unobstructions associated to e, by which we mean obstructions for $T \setminus e$ that were not obstructions for T. Either there exists a Thurston-unobstruction associated to e or there exists a Schottky-unobstruction associated to e.*

Proof. If $T \setminus e$ is an obstructed Tischler graph, then there exists a Thurston-obstruction $\gamma_e \subset \hat{\mathbb{C}} \setminus V(T \setminus e)$. This curve γ_e must intersect e since T is an unobstructed topological Tischler graph. Therefore γ_e is a Thurston-unobstruction associated to e.

If $T \setminus e$ is not a Tischler graph, then by Proposition 18 there exists a Schottky-obstruction γ_e and an associated cut-point at vertex p. Again, γ_e must intersect e since T is a Tischler graph, and Tischler graphs do not have cut-points by Lemma 17. Therefore γ_e is a Schottky-unobstruction associated to e. □

An unobstruction of either type will be assumed to be a representative of the homotopy class having minimal intersection number with the Tischler graph given, that is, in minimal position. Note if γ is a Schottky-unobstruction then we do not count the vertex intersection in $\gamma \cdot T$.

Recall two unobstructions, γ_1 and γ_2, are homotopic if there exists an isotopy relative the marked points, denoted $\gamma_1 \simeq \gamma_2$. Note that a Thurston-unobstruction cannot be homotopic to a Schottky-unobstruction since a Thurston-unobstruction must be disjoint from the vertex set and a Schottky-unobstruction must intersect a vertex. Let $\gamma_1 \cdot \gamma_2$ be the minimum number of intersections between the two unobstructions up to homotopy. If γ_1 and γ_2 are Schottky-unobstructions and share a vertex, do not count this toward the intersection number. If a pair of unobstructions realize this minimal intersection number we say they are in *minimal position with respect to each other.* We say two unobstructions *intersect trivially* if there exists a point p such that $\gamma_1 \cap \gamma_2 = \{p\}$. If p is a vertex then γ_1 and γ_2 must both be Schottky-unobstructions which share the associated vertex, thus $\gamma_1 \cdot \gamma_2 = 0$. If p is not a vertex then we can homotopy γ_1 such that $\gamma_1 \cap \gamma_2 = \emptyset$, thus $\gamma_1 \cdot \gamma_2 = 0$. Therefore in order for two unobstructions in minimal position to intersect trivially, they both must be Schottky-unobstructions and the intersection point must be a vertex.

Lemma 20. *Let T be an unobstructed topological Tischler graph, and γ_1 and γ_2 be two non-homotopic unobstructions. The two unobstructions can simultaneously be written in minimal position with respect to the Tischler graph and each other.*

Proof. Let γ_1 and γ_2 be two non-homotopic unobstructions such that they are in minimal position with respect to T, and all intersections (not at a vertex) are transversal.

If γ_1 and γ_2 intersect on an edge of the Tischler graph, we can homotopy that intersection into a face without increasing the number of edge intersections of γ_1 and γ_2. So we will assume that all $\gamma_1 \cap \gamma_2$ occur in a face or at a vertex.

A *bi-gon* is connected component $B \subset \hat{\mathbb{C}} \setminus (\gamma_1 \cup \gamma_2)$ such that $\#(\gamma_1 \cap \gamma_2 \cap \overline{B}) = 2$ and $\partial B \cap V(T) = \emptyset$. If a bi-gon B exists such that $B \cap C_f = \emptyset$, then the bi-gon can be removed. That is if B is homotopic to a point, then the two intersections of γ_1 and γ_2 on ∂B can be removed by homotopy. If some edge segment of T in B only has boundary on γ_k, we would contradict γ_k being in minimal position with respect to T. Thus any edge of the Tischler graph which intersects B must connect γ_1 to γ_2 by a segment in B. So there exists a homotopy preserving these edge crossings and not adding any new ones.

Thus by repeated intersection removal we can minimize Tischler graph intersections of both γ_1 and γ_2 while also minimizing intersections of γ_1 with γ_2 □

Lemma 21. *Let T be an unobstructed topological Tischler graph, and γ_1 and γ_2 be two non-homotopic unobstructions, then $\gamma_1 \cdot \gamma_2 = 0$.*

The rough outline of the proof is to first normalize the graph to standardize some language. Now suppose $\gamma_1 \cdot \gamma_2 \neq 0$ then $\gamma_1 \cdot \gamma_2 \geq 2$ and the two curve generate some number of "bi-gons". We then have to argue that γ_1 and γ_2 have too many intersections with the Tischler graph to be unobstructions.

Proof. Normalized γ_1, γ_2, and T such that

- the graph T has a vertex at infinity not associated to γ_1 or γ_2 (only needed γ_1 or γ_2 is a Schottky-unobstruction),
- the unobstructions γ_1 and γ_2 intersect T minimally,
- the unobstructions γ_1 and γ_2 intersect each other minimally,

- any intersections points in $\gamma_1 \cap \gamma_2$ are in a face or are a vertex (again only needed if γ_1 or γ_2 is a Schottky-unobstruction).

Suppose that two unobstructions γ_1 and γ_2 intersect non-trivially. Therefore $\gamma_1 \cdot \gamma_2 \geq 2$. Let $\{B_1, \ldots B_m\}$ be the set of bi-gons generated by γ_1 and γ_2. That is B_j is a connected component of $\hat{\mathbb{C}} \setminus (\gamma_1 \cup \gamma_2)$ such that

- ∂B_j contains no critical points,
- $\#(\partial B_j \cap \gamma_1 \cap \gamma_2) = 2$.

There exists at least one bi-gon since γ_1 and γ_2 intersect non-trivially. Since γ_1 and γ_2 are in minimal position with respect to each other, each B_j contains at least one vertex in $V(T)$.

Consider how the Tischler graph can interact with these bi-gons. Let $j \in \{1, 2, \ldots, m\}$,

- if $\partial B_j \cdot T = 1$, by Lemma 17, the graph T has a face which is not a Jordan domain.
- if $\partial B_j \cdot T = 2$, by Theorem 11 the boundary ∂B_j is an obstruction for T since we know B_j and its complement must contain more than one vertex to have all vertices of T have degree at least 3.

This implies $\partial B_j \cdot T \geq 3$. We also know, as unobstructions, $\gamma_k \cdot T \leq 3$ for $k = 1, 2$, and we have equality if and only if γ_k is a Thurston-unobstruction by Proposition 19. Therefore,

$$(\bigcup_{j=1}^{m} \partial B_j) \cdot T \leq \gamma_1 \cdot T + \gamma_2 \cdot T \leq 3 + 3 = 6. \tag{3.1}$$

Now we consider the the values of $\gamma_1 \cdot \gamma_2$. It is often helpful to reference the complementary components of the unobstructions. Let V_j be bounded complementary component of γ_j and U_j be the unbounded complementary component of γ_j.

Case 1: $\gamma_1 \cdot \gamma_2 = 2$.

The pictures must look like those in Figure 3.5. If both γ_1 and γ_2 are Schottky unobstructions then there is only one bi-gon, see Figure 3.5a. The boundary of this bi-gon forms an obstruction as it intersects two distinct edges and two distinct faces.

If γ_2 is a Thurston-unobstruction and γ_1 is a Schottky-unobstruction then γ_2 can be decomposed as the union of two arcs, see Figure 3.5b. Let $\alpha_{2,1}$ be the portion of γ_2 in V_1 and let $\alpha_{2,2}$ be the portion of γ_2 in U_1. The Schottky-unobstruction γ_1 is also partitioned by γ_2, but only one component is an arc, let that be $\alpha_{1,1}$. It is clear that $\alpha_{2,1} \cdot T + \alpha_{2,2} \cdot T = \gamma_2 \cdot T$. If $\alpha_{2,k} \cdot T = 0$ for $k = 1$ or $k = 2$, then $\alpha_{1,1} \cup \alpha_{2,k} \cdot T = 1$ and defines a cut point. Therefore without loss of generality $\alpha_{2,1} \cdot T = 1$ and therefore $\alpha_{2,1} \cup \alpha_{1,1}$ form an obstruction.

If both γ_1 and γ_2 are Thurston-unobstructions then the picture looks like Figure 3.5c. That is there are 4 bi-gons, and similar to the argument above we can decompose the two simple closed curves into arcs, two of which must have a single intersection. The union of these two arcs forms an obstruction.

Thus if γ_1 and γ_2 are to intersect only twice, then the original graph was obstructed, a contradiction.

Case 2: $\gamma_1 \cdot \gamma_2 > 2$.

We claim there are too many bi-gons for the limited number of edge intersections with the unobstructions. If both γ_1 and γ_2 are Schottky unobstructions then $\gamma_1 \cdot T + \gamma_2 \cdot T = 2$. But there must be at least one bi-gon B with $\partial B \cdot T \geq 3$, a contradiction to Equation 3.1. Thus two Schottky unobstructions must only intersect trivially.

Without loss of generality, assume γ_2 is a Thurston-unobstruction, we claim there are at

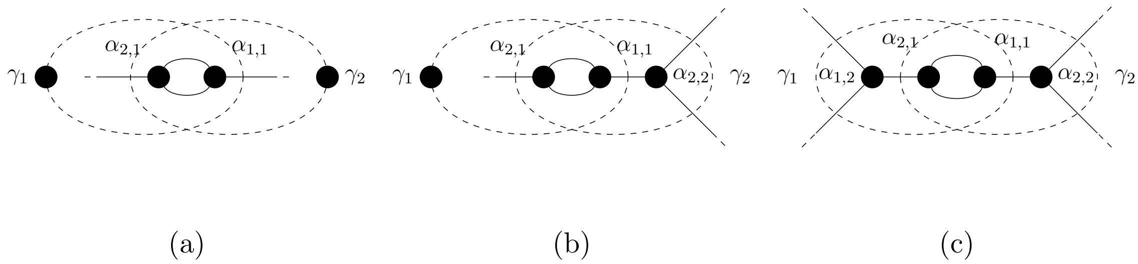


Figure 3.5: Above are sections of Tischler graphs in solid black and various unobstructions as the dashed lines. Figure 3.5a shows two Schottky-unobstructions. Figure 3.5b shows one Schottky-unobstruction, one Thurston-unobstruction. Figure 3.5c shows two Thurston-unobstructions. In all three examples we find the original graph must be obstructed.

least 3 bi-gons, such that no pair of bi-gons share more than 2 points on their boundary. To see this claim to be true, focus on V_2 and consider how γ_1 can intersect this region. First assume γ_1 is a Thurston unobstruction. Since $\gamma_1 \cdot \gamma_2 > 2$, there exists a portion of γ_1 which partitions V_2 into two components β_1 and β_2. If there are no other portions of γ_1 in them, each of these components is a bi-gon. Because $\gamma_1 \cdot \gamma_2 > 3$, there exists at least one more component of γ_1 in V_2. Each portion of γ_1 that we see, partitions β_j into a new potential bi-gon $\beta_{j,1}$ and a non-bi-gon $\beta_{k,2}$. Since there are only finitely many segments of γ_1 in V_2, there are always at least two bi-gons in V_2. The same argument can be made in U_2, therefore if both γ_1 and γ_2 are Thurston-unobstruction $m \geq 4$. If γ_1 is a Schottky-unobstruction then one of these bi-gons can be destroyed by the vertex associated to γ_1 and the number of bi-gons is $m \geq 3$.

We now see that $m \geq 3$ and $(\bigcup_{j=1}^{m} \partial B_j) \cdot T \geq 9$. This is a contradiction as $(\bigcup_{j=1}^{m} \partial B_j) \cdot T \leq \gamma_1 \cdot T + \gamma_2 \cdot T \leq 6$. Thus a Thurston-unobstruction cannot intersect any other unobstruction. See Figure 3.6.

Finally we have our conclusion, $\gamma_1 \cdot \gamma_2 = 0$. □

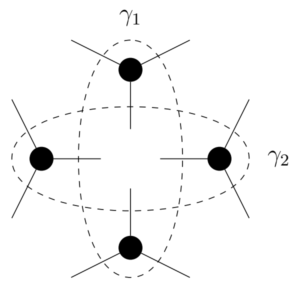


Figure 3.6: Multiple bi-gons all of which have boundary intersection with the Tischler graph greater than 3.

Lemma 22. *Let g be a critically fixed anti-rational map of degree $d \geq 3$ with Tischler graph T_g. Then there exists a deletable edge e of T_g.*

Proof. Select an edge e_1 of T_g. There are two cases.

Case 1: Edge e_1 is deletable. Let $e = e_1$ and we are done. We get a new unobstructed map f as a Schottky map defined on $T_f = T_g \setminus e$.

Case 2: Edge e_1 is not deletable. Then there exists an associated unobstruction γ_{e_1}. Here and throughout the proof we can safely assume all unobstructions are in minimal position with respect to T_g. Consider the two connected components A_1 and B_1 of $\hat{\mathbb{C}} \setminus \gamma_{e_1}$. Without loss of generality pick an edge e_2 completely contained in A_1. Such an edge exists if γ_{e_1} is a Schottky- or Thurston-unobstruction since:

If γ_{e_1} is a Schottky-unobstruction then $\gamma_{e_1} \cdot T_g = 1$. By definition γ_{e_1} is based at a vertex v_1 and is essential. Therefore there exists a vertex $v_2 \neq v_1$ in A_1, such that e_1 is incident to v_2. By definition $\deg(v_2) \geq 3$, therefore there are at least two other edges incident to v_2. If neither of those edges are contained in A_1, they intersect γ_{e_1}, a contradiction.

If γ_{e_1} is a Thurston-unobstruction then $\gamma_{e_1} \cdot T_g = 3$. By definition there are at least two distinct vertices v_1 and v_2 contained in A_1 incident to edges e_1 and e_1' intersecting γ_{e_1}. Assume e_1 is incident to v_1 and e_1' is incident to v_2. We also have $d_T(v_1) \geq 3$ and $d_t(v_2) \geq 3$. If none of the edges incident to v_1 and none of the edges incident to v_2 are contained in A_1, then $\gamma_{e_1} \cdot T_g \geq 6$, a contradiction.

Again we have the same two cases within A_1.

Case 2.1: Edge e_2 is deletable. In which case we are done.

Case 2.2: Edge e_2 is not deletable, then by Lemma 21, its associated unobstruction γ_{e_2} cannot intersect γ_{e_1} non-trivially. We select A_2 and B_2, the connected components of $\hat{\mathbb{C}} \setminus \gamma_{e_2}$, such that $A_2 \subsetneq A_1$. This ensures A_2 contains at least one less edge than A_1, namely e_2. As well as one less vertex, namely the endpoint of e_2 contained in B_2. We then select edge e_3 completely contained in A_2 and continue the process. Again e_3 exists by the above argument.

In general we continue the search for a deletable edge by selecting edge e_{n+1} contained in $A_n \subset A_{n-1}$. If this edge is not deletable select A_{n+1} to be the connected component of $\hat{\mathbb{C}} \setminus \gamma_{e_{n+1}}$ such that $A_{n+1} \subsetneq A_n$. See Figure 3.7 for an example.

The graph T_g is finite so this process must terminate at some step $n+1$ where we find a deletable edge, or some final step K. By the above argument A_K must contain an edge and as the process has terminated, A_K does not contain an unobstruction. Thus any edge in A_K is deletable. □

Proof of Theorem 16. Suppose we have a map f of degree $d+1 \geq 2$. By Lemma 22 the Tischler graph associated to f, T_f has a deletable edge e_1. The new graph $T_1 = T_f \setminus e_1$ generates an unobstructed Schottky map f_1 of degree d.

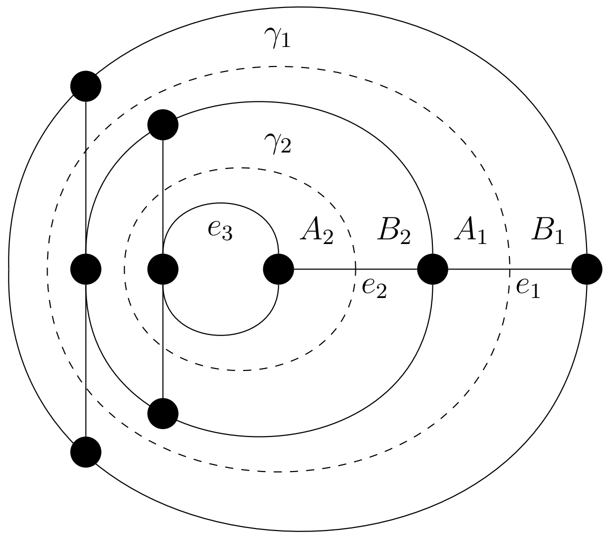


Figure 3.7: Although there may be a more efficient selection of edges, this example illustrates the proof of Lemma 22. Notice γ_1 and γ_2 could have just as easily been chosen to be Schottky-unobstructions associated to e_1 and e_2 respectively and the process does not change. Edge e_3 is deletable.

Theorem 11 tells us if f_n is an unobstructed Schottky map on topological Tischler graph T_n, f_n is Thurston-equivalent to a critically fixed anti-rational map.

By Lemma 22, if $\deg(f_n) \geq 3$ then there exists an edge e_{n+1} of T_n which is deletable. The new graph $T_{n+1} = T_n \setminus e_{n+1}$ generates an unobstructed Schottky map f_{n+1} such that $\deg(f_{n+1}) = \deg(f_n) - 1$.

If $\deg(f_n) = 2$ then the number of critical points is bounded by $2\deg(f_n) - 2 = 2(2) - 2 = 2$. As we have a map on the sphere f_n has exactly 2 critical points. As each vertex must have degree at least 3, thus T_n must be the graph with 2 vertices and 3 edges, that is f_n is Thurston equivalent to $s(z) = \overline{z}^2$.

This process implies that starting with $s(z)$ we can successively blow-up the arcs

$$\{e_{n-1}, e_{n-2}, \ldots, e_1\}$$

to generate a map Thurston-equivalent to f. □

THE GLOBAL CURVE ATTRACTOR PROBLEM

Thinking about the pullbacks of multicurves is natural in Thurston Theory. Pilgrim was first interested in studying the dynamics on these curves in [Pil12] and inspired many others to do the same, as mentioned in the introduction. We need to introduce a bit of terminology to state the problem. Consider a post-critically finite (anti)-rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with post-critical set P_f. We define the set $\mathcal{C}(f)$ as the set of all simple closed curves contained in $\hat{\mathbb{C}} \setminus P_f$.

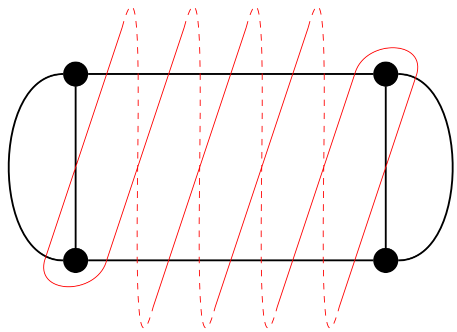
Definition 20. A *pullback* of a curve $\gamma \in \mathcal{C}(f)$ is defined as a connected component of $f^{-1}(\gamma)$.

The global curve attractor problem asks: Does there exists a finite subset $\mathcal{A}(f) \subset \mathcal{C}(f)$ such that, for every $\gamma \in \mathcal{C}(f)$ there exists $n \in \mathbb{N}$ such that for all $m \geq n$, every pullback of γ by f^m is isotopic relative P_f to an element $\mathcal{A}(f)$? If such a set exists we call $\mathcal{A}(f)$ the finite global curve attractor for f. The general conjecture is as follows.

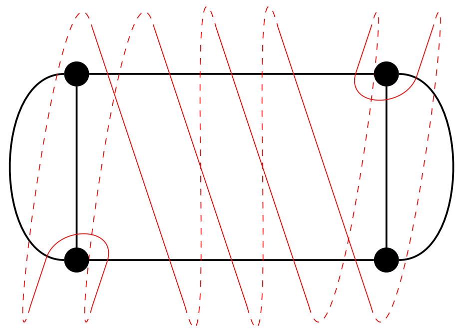
Conjecture 23 (Global Curve Attractor Problem)**.** *Every post-critically finite rational map or anti-rational map (which is not a Lattés map) has a finite global curve attractor.*

Obstructed maps and Lattés maps do not have finite global curve attractors. For obstructed maps see Figure 4.1, an example of a topological Tischler graph with 2 distinct faces which share 2 distinct edges. We can construct an essential simple closed curve with arbitrarily many intersections on these two edges such that under iterated pullback the curve is 2-periodic. Thus we have generated infinitely many curves which are non-homotopic and do not "simplify" under pullback.

Lattés maps are defined as rational maps f such that there exists an affine map $L : \mathbb{T} \to \mathbb{T}$ of



(a) A simple closed curve γ.



(b) The pullback of γ.

Figure 4.1: Schottky maps associated to obstructed Tischler graphs do not have a finite global curve attractor. Note we could add more edge intersections and still have an invariant 2-cycle of curves under pullback, thus the global curve attractor is not finite.

the 2-torus and a holomorphic map $\Theta : \mathbb{T} \to \hat{\mathbb{C}}$ such that $f = \Theta \circ L \circ \Theta^{-1}$ [Mil06, Definition 7.4, page 72]. As an example consider $L(z) = 2z \mod (1)$ and let $\Theta : \mathbb{T} \to \mathbb{T}/(z \sim -z)$ be the quotient map. By the Riemann-Hurwitz formula $\mathbb{T}/(z \sim -z)$ is isomorphic to the sphere $\hat{\mathbb{C}}$. This induces f, a degree 4 rational map on the sphere. There is a nice visual of this map as a "pillow map", see Figure 4.2. Pilgrim showed these maps do not have a finite global curve attractor [Pil12, page 21]. For more on Lattés maps see [BM17].

The main result of this section is Corollary 28, which says that critically fixed anti-rational maps with 4 or 5 critical points have a finite global curve attractor.

Recall: A curve γ is called *essential* or *non-peripheral* if each complementary component of γ contains at least two points in P_f, otherwise γ is called *peripheral*. We will understand that two curves γ_1 and γ_2 are homotopic, denoted $\gamma_1 \simeq \gamma_2$, if they are isotopic relative P_f.

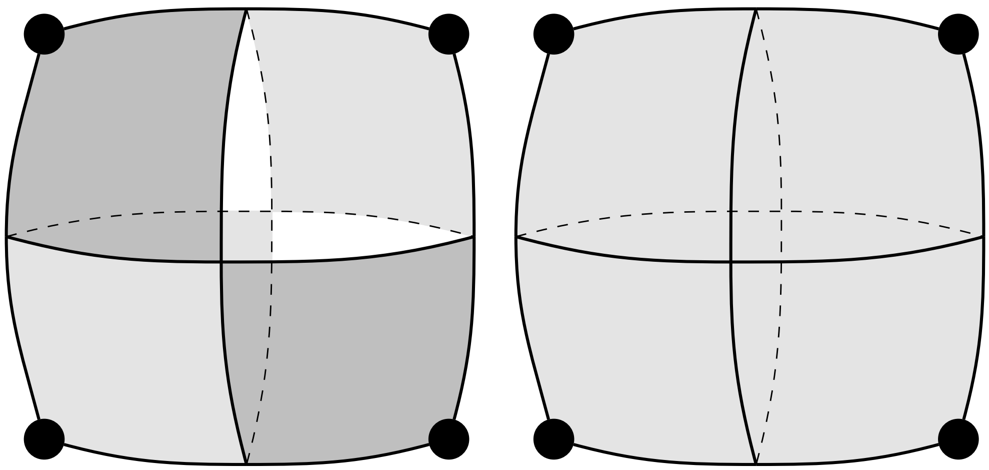


Figure 4.2: The left pillow maps onto right pillow where the white panels map to the front of the pillow on the right, and the grey panels map to the back of the pillow on the right. The marked points are the post-critical set of this map.

Our goal is show that complexity decreases under pullback. Since we will restrict to critically fixed anti-rational maps f, recall "complexity" will be the minimal number of intersections a curve has with the Tischler graph of f. Specifically defined and denoted as

$$\gamma \cdot T = \min\{\#(\gamma' \cap T) : \gamma' \simeq \gamma\}.$$

We say that γ is in *minimal position* if $\gamma \cdot T$ is realized.

Arcs and Curves

We will restrict to the setting where f is a critically fixed anti-rational map with 4 or 5 critical points. Therefore we have that $P_f = C_f$ and thus $\mathcal{C}(f)$ is the set of simple closed curves which avoid the critical points. In this setting an essential simple closed curve γ separates 2 critical points from the rest. That is, there are connected components $\Delta_\gamma \sqcup \Delta'_\gamma = \hat{\mathbb{C}} \setminus \gamma$, without loss of generality we will assume $\Delta_\gamma \cap C_f = \{p_1, p_2\}$. It is natural to then think of a *critical arc* α connecting these two marked points in Δ_γ.

Definition 21. Specifically $\alpha : [0,1] \to \Delta_\gamma$ such that, $\alpha|_{(0,1)}$ is a homeomorphism mapping into $\Delta_\gamma \setminus C_f$, $\alpha(0) = p_1$, and $\alpha(1) = p_2$.

We will also measure the "complexity" of a critical arc α by counting the number of edge crossings with the Tischler graph.

Definition 22. That is, the *complexity of a critical arc* α is the number of intersections the interior of α has with the Tischler graph,

$$\alpha \cdot T = \min\{\#(\mathrm{int}(\alpha') \cap T) : \alpha' \simeq \alpha\},$$

where α_1 is homotopic to α_2, denoted $\alpha_1 \simeq \alpha_2$, if they are isotopic relative C_f. Again we say that α is in *minimal position* if it realizes $\alpha \cdot T$.

Since Δ_γ is a topological disk and the only two marked points are p_1 and p_2, any two critical paths α_1 and α_2 from p_1 to p_2 contained in Δ_γ are isotopic relative C_f.

Definition 23. An *arc* α *associated to a simple closed curve* γ is a critical arc in Δ_γ. Similarly given α, a *curve* γ *associated to* α is the boundary of some small neighborhood U of α, such that $U \cap C_f \setminus \{p_1, p_2\} = \emptyset$. The connection between essential γ and arcs α will be our main tool in this section.

Since the simple closed curve γ is homotopic to the boundary of some arbitrarily small neighborhood of α it is useful to pullback α and recover the pullback(s) of γ. By "arbitrarily small neighborhood of the arc", we mean a subset of an ϵ-neighborhood of α which is simply connected and whose closure contains no critical points not on the boundary of α. Furthermore we mandate that the boundary of such a neighborhood intersects the Tischler graph finitely many times and transversely at every intersection.

Definition 24. The *pullback of an arc* α is again defined as a connected component of $f^{-1}(\alpha)$.

If α^{-1} is a pullback of α there is some small neighborhood V of α^{-1} such that $f(V) = U$, by continuity of f. Therefore there is at least one pullback of γ which is homotopic to the boundary of a small neighborhood of α^{-1}, we call this pullback γ^{-1} *the pullback of* γ *associated to* α^{-1}. It is possible that α^{-1} is some non-trivial arc-system (read collection of arcs). In which case ∂V would be some collection of pullbacks of γ. We will more clearly state the structure of a pullback of a critical arc in Lemma 25. We will say a pullback of an arc is *essential* if the arc system is not homotopic to a point. The first goal is to associate the complexity of critical arcs with the complexity of simple closed curves. The following lemma holds in general, not just in the setting of 4 or 5 critical points.

Lemma 24. *Consider f a critically fixed anti-rational map with Tischler graph T and a critical arc α connecting critical points p_1 and p_2 in minimal position. Let γ be the boundary of some arbitrarily small neighborhood of α.*

1. *If $\alpha \cdot T = 0$, then*

$$\gamma \cdot T \leq d_T(p_1) + d_T(p_2).$$

2. *If $\alpha \cdot T \neq 0$, then γ is in minimal position. That is if $\alpha \cdot T > 0$ then*

$$\gamma \cdot T = d_T(p_1) + d_T(p_2) + 2(\alpha \cdot T).$$

Where $d_T(p_j)$ the degree of the vertex $p_j \in V(T)$.

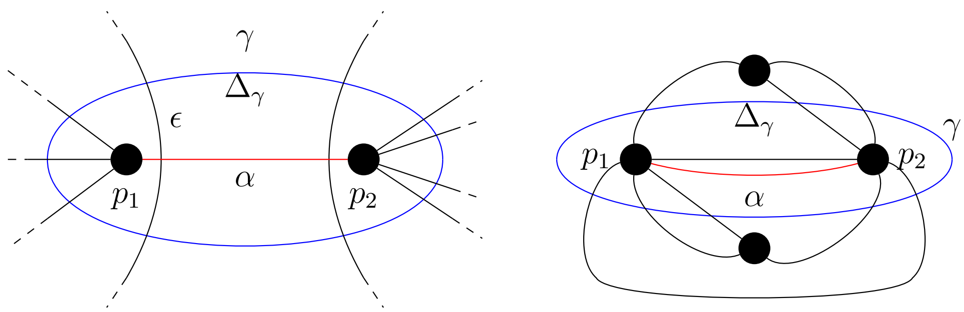
Proof. Let γ be the boundary of an arbitrarily small neighborhood of α, where α is a critical arc between p_1 and p_2 in minimal position. Thus γ has finitely many intersections with the Tischler graph.

Assume, without loss of generality, Δ_γ is the complementary component of γ such that $\Delta_\gamma \cap C_f = \{p_1, p_2\}$ and consider how the Tischler graph T can intersect this topological disk. Any edge $e \in E(T)$ which intersects γ is partitioned into a collection of components outside of Δ_γ and a collection of components inside of Δ_γ. By the Lemma 17 (Cut-Point Lemma), any component contained in Δ_γ must connect to a critical point, or exit Δ_γ through its boundary. The number of components terminating at a critical point in Δ_γ is bounded above by $d_T(p_1) + d_T(p_2)$. The number of components which have both boundary points in γ is bounded above by $2(\alpha \cdot T)$. For if one of these components did not intersect α, just homotopy γ to be the boundary of some smaller neighborhood. Thus we establish the following upper bound

$$\gamma \cdot T \leq d_T(p_1) + d_T(p_2) + 2(\alpha \cdot T).$$

If $\alpha \cdot T = 0$, we have proved (1) above. See Figure 4.3b.

If $\alpha \cdot T > 0$, there exists a component of an edge ϵ contained in Δ_γ which does not connect to a p_k for $k = 1, 2$. This edge component must partition Δ_γ into two connected components. See Figure 4.3a If one of these components does not contain a critical point, then it is null-homotopic, therefore assign $C_1 \sqcup C_2 = \Delta_\gamma \setminus \epsilon$, such that C_j contains p_j. The arc α must intersect ϵ, since α connects p_1 to p_2 and cannot intersect γ. Therefore every edge intersection of α accounts for two edge intersections of γ. Furthermore each edge incident to p_j must exit C_j through γ. Thus our upper bound is in fact a least upper bound. This establishes item (2) and finishes our proof. □



(a) $\alpha \cdot T = 2 \implies \gamma \cdot T = 3 + 4 + 2(2)$ (b) $\alpha \cdot T = 0 \implies \gamma \cdot T \leq 5 + 5$

Figure 4.3: Above we have a simple closed curve γ (in blue) and one of its complementary components Δ_γ which contains two critical points p_1 and p_2. To γ we have the associated arc α (in red) in minimal position to the edge components which intersect Δ_γ in black.

Note it is tempting to say that if $\alpha \cdot T = 0$, then

$$\gamma \cdot T = d_T(p_1) + d_T(p_2) - 2|\overrightarrow{p_1 p_2}|,$$

where $|\overrightarrow{p_k p_j}|$ is the number of edges incident to both p_k and p_j. But this is just not true, see Figure 4.3b.

Pullback of Arcs

With the connection between critical arcs and curves established for critically fixed anti-rational maps with 4 or 5 critical points, we need a better understanding of what happens when we pullback an arc. It is useful to consider a *lift of an arc* α, denoted $\tilde{\alpha}$. The lift $\tilde{\alpha}$ is defined to be the closure of a connected component of $f^{-1}(\text{int}(\alpha))$. Notice that $f|_{\tilde{\alpha}}$ is a homeomorphism onto α.

Lemma 25. *Let f be a critically fixed anti-rational map on $\hat{\mathbb{C}}$. Let α be a critical arc, then there is at most one essential pullback of α. Furthermore any essential pullback is homotopic to a union of critical arcs, all having the same two boundary points.*

Proof. Let α be a critical arc between p_1 and p_2. A pullback α^{-1}, is a connect set of lifts. Let $\tilde{\alpha}$ and $\tilde{\alpha}'$ be two distinct lifts contained in α^{-1}. It is enough to show that if $\partial\tilde{\alpha} \neq \{p_1, p_2\}$ then $\tilde{\alpha}$ can be homotopied away from α^{-1}.

Suppose $\partial\tilde{\alpha} \neq \{p_1, p_2\}$. Since f is critically fixed and continuous, there exists $p^{-1} \in \partial\tilde{\alpha}$, a preimage of a critical point such that $p^{-1} \notin C_f$. If $\tilde{\alpha}'$ is another lift of α such that $p^{-1} \in \partial\tilde{\alpha}'$, then for all neighborhoods U of p^{-1}, $\deg(f|_U) \neq 1$. Therefore p^{-1} is critical, a contradiction. We have then that any lift $\tilde{\alpha}$ that is not a critical arc is isotopic to a point relative C_f, even when considered in a pullbackα^{-1}. We have thus established there is at most one essential component of α^{-1} and up to homotopy α^{-1} is a union of critical arcs between p_1 and p_2. See Figure 4.4b. □

The following lemma tells us when an arc α is "complex enough" so that under pullback the complexity reduces.

Lemma 26. *Given a critically fixed anti-rational map f with Tischler graph T, consider α a critical arc in minimal position, and a lift of α, $\tilde{\alpha}$. If $\alpha \cdot T > 1$, then $\tilde{\alpha} \cdot T < \alpha \cdot T$.*

Proof. Since $\alpha \cdot T \geq 2$, there exists two distinct edges $e_1, e_2 \in E(T)$ on the boundary of some face $F \in F(T)$ such that there is a connected component of $\alpha \cap F$ which is a path from e_1 to e_2, denoted α_F. Since α_F is contained in a face it has no edge intersections. Furthermore any of its lifts, $\tilde{\alpha_F}$, are outside of F and only has edge intersections on $\partial\tilde{\alpha_F} \subset e_1 \cup e_2$.

Suppose there exists a lift $\tilde{\alpha_F}$ such that $\partial\tilde{\alpha_F} = \partial\alpha_F$. Then $\overline{\alpha_F \cup \tilde{\alpha_F}}$ defines a simple closed curve γ. Since α_F is contained in the face F, $\mathrm{int}(\tilde{\alpha_F})$ is contained in some other face G, therefore $\gamma \cdot T = 2$ and γ defines an obstruction as in Theorem 11. This is a contradiction to f being anti-rational.

Thus every lift $\tilde{\alpha_F}$ must have at most one shared boundary point with α_F, reducing the number of edge intersections. Specifically $\alpha_F \subset \alpha$, every lift of α contains exactly one lift of α_F therefore

$$\tilde{\alpha} \cdot T \leq (\alpha \cdot T) - 1.$$

□

As advertised, to prove the following theorem we use our understanding of pullbacks of arcs to prove that the complexity of the associated simple closed curve will reduce under pullback.

Theorem 27. *Let f be a critically fixed anti-rational map. Let $\gamma \in \mathcal{C}(f)$ such that γ is associated to a critical arc α. If $\alpha \cdot T \geq 2$, then $\gamma^{-1} \cdot T < \gamma \cdot T$.*

Proof. Assume $\partial\alpha = \{p_1, p_2\} \subset C_f$. By Lemma 24,

$$\gamma \cdot T = d_T(p_1) + d_T(p_2) + 2(\alpha \cdot T).$$

By Lemma 25 the only essential pullback of α is a union of critical arcs between p_1 and p_2. Define for $1 \leq j \leq n$, α_j to be a critical arc such that

$$\alpha^{-1} \simeq \bigcup_{j=1}^{n} \alpha_j.$$

If $n = 0$, then all preimages of γ are peripheral.

If $n = 1$, then γ^{-1} is associated to $\alpha^{-1} \simeq \tilde{\alpha}$, a lift of α which is a critical arc. By Lemma 24

1. If $\alpha^{-1} \cdot T = 0$, then

$$\gamma^{-1} \cdot T \leq d_T(p_1) + d_T(p_2) < \gamma \cdot T.$$

2. If $\alpha^{-1} \cdot T \neq 0$, then by Lemma 26

$$\gamma^{-1} \cdot T = d_T(p_1) + d_T(p_2) + 2(\tilde{\alpha} \cdot T) < \gamma \cdot T.$$

If $n \geq 2$ then γ has multiple pullbacks associated to the critical arc system $\alpha^{-1} = \alpha_1 \cup \cdots \cup \alpha_n$. Assume these α_j are ordered such that for $j \in \{1, \ldots, n\}$, $\alpha_j \cup \alpha_{j+1}$ is the boundary of some open topological disc D_j such that $\alpha_k \cap D_j = \emptyset$ for all $k \notin \{j, j+1\}$. For ease of notation assume $D_j = D_{j \mod (n+1)}$ and $\alpha_j = \alpha_{j \mod (n+1)}$. Since γ is the boundary of an arbitrarily small neighborhood of α and f is continuous, we know that for each $j \in \{1, \ldots, n\}$, there exists $\gamma_j \subset D_j$, a pullback of γ. We also know

$$\bigcup_{j=1}^{n} \overline{D_j} = \hat{\mathbb{C}}.$$

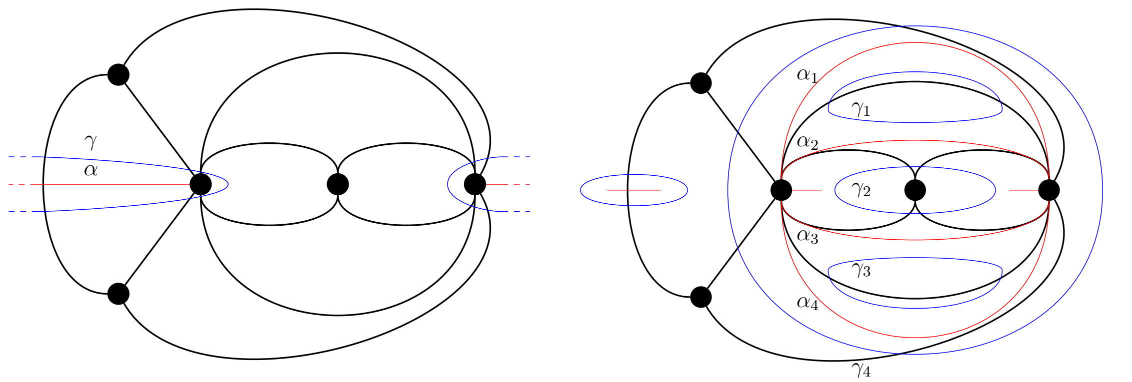
See Figure 4.4.

If there exists a k such that $D_k \cap C_f = C_f \setminus \{p_1, p_2\}$, then γ_k is the only essential pullback of γ. Therefore the other complementary component of $\alpha_k \cup \alpha_{k+1}$ is

$$(\bigcup_{j \neq k} \overline{D_k}) \setminus (\alpha_k \cup \alpha_{k+1}),$$

which contains no critical points. Therefore $\alpha_j \simeq \alpha_l$ for all $1 \leq j, l \leq n$ and $\gamma^{-1} = \gamma_k$. Furthermore, there exists D_m for $m \neq k$ and associated γ_m. If γ_m does not intersect an edge of T, then α_m and α_{m+1}, by definition, live in the same face F. Therefore F must map onto its complement 2-to-1, a contradiction to Theorem 10. Thus $\gamma_m \cdot T > 1$ and since each edge is forward invariant the number of edge crossings cannot increase under pullback,

$$\gamma^{-1} \cdot T \leq d_T(p_1) + d_T(p_2) - \gamma_m \cdot T < \gamma \cdot T.$$



(a) The arc α is in red, and the curve γ is in blue. Both suggestively go through ∞.

(b) The preimage of α is in red, and the preimage of γ is in blue.

Figure 4.4: Notice for the pullback of α, the only essential components are critical arcs. Furthermore each pair of adjacent critical arcs form a disk, which contains associated pullback of γ.

If there exists an m and k such that $D_m \cap C_f \neq \emptyset$ and $D_k \cap C_f \neq \emptyset$ first assume $\gamma_m \cdot T = 0$. The critical points in D_m are then disconnected from the critical points not in D_m. This is a contradiction since the Tischler graph must be connected. This same argument applies to γ_k and we establish that $\gamma_m \cdot T \neq 0$ and $\gamma_k \cdot T \neq 0$. Without loss of generality if we assume $\gamma^{-1} = \gamma_k$, then

$$\gamma^{-1} \cdot T \leq d_T(p_1) + d_T(p_2) - \gamma_m \cdot T < \gamma \cdot T.$$

This shows if $\alpha \cdot T \geq 2$, then $\gamma^{-1} \cdot T < \gamma \cdot T$. □

Corollary 28. *Critically fixed antirational maps with 4 or 5 critical points have a finite global curve attractor.*

Proof. First we should note that the set of homotopy classes of simple closed curves $\gamma \in \hat{\mathbb{C}} \backslash C_f$

with associated arc α such that $\alpha \cdot T \leq 1$ forms a finite set, call this set $\mathcal{I}$. The set of homotopy classes of simple closed curves $\gamma \in \hat{\mathbb{C}} \setminus C_f$ with no associated arc, that is peripheral γ, also form a finite set, call this set $\mathcal{P}$. We claim that $\mathcal{I} \cup \mathcal{P}$ contains the global curve attractor of f, $\mathcal{A}(f)$.

Let f be a critically-fixed anti-rational map with Tischler graph T. Consider an essential $\gamma \in \mathcal{C}(f)$, with associated arc α.

By Theorem 27 if $\alpha \cdot T < 2$, then $\gamma^{-1} \cdot T < \gamma \cdot T$, that is $\gamma^{-1} \in \mathcal{I} \cup \mathcal{P}$.

By Theorem 27 if $\alpha \cdot T \geq 2$, then

$$\gamma^{-1} \cdot T < \gamma \cdot T.$$

Then $\gamma_2 = \gamma^{-1}$ is a simple closed curve which is either peripheral, or separates 2 critical points from the rest. If γ_2 is peripheral then $\gamma_2 \in \mathcal{I}$. If γ_2 is non-peripheral then it separates two critical points from the rest. Therefore there exists a critical arc α_2 such that γ_2 is homotopic to the boundary of some arbitrarily small neighborhood of α_2. If $\alpha_2 \cdot T < 2$, then $\gamma_2 \in \alpha_2 \cdot T \geq 2$. If $\alpha_2 \cdot T \geq 2$, pullback to reduce the complexity again. Repeat this process until $\alpha_n \cdot T < 2$ therefore $\gamma_n \in \mathcal{I} \cup \mathcal{P}$.

Therefore $\mathcal{A}(f) \subset \mathcal{I} \cup \mathcal{P}$, a finite set. Thus f has a finite global curve attractor. $\square$

LENSING GRAPHS

The goal of this section is to shed some light on which topological Tischler graphs are Tischler graphs of maximal lensing maps (defined below). Recall that gravitational lensing is the phenomenon where the path of light from some far away light source is effect by the gravity of a single mass or collection of point masses between the light source and the observer. We will assume that the distance between the masses is much smaller than the distance between the observer to the point masses and the light source to the point masses, we project each point mass onto a single plane, the *lens plane*. There is another type of lensing called multi-plane lensing which leads to more complicated equations. We assume that the straight line path from the light source to the observer intersects this plane at the origin.

Definition 25. We therefore define a *lensing map* of degree n as a function which takes the form

$$L(z) = \overline{\sum_{j=1}^{n} \frac{\sigma_j}{z - z_j}},$$

where z_j represents the position of a mass in the lens plane and the mass of z_j is given by

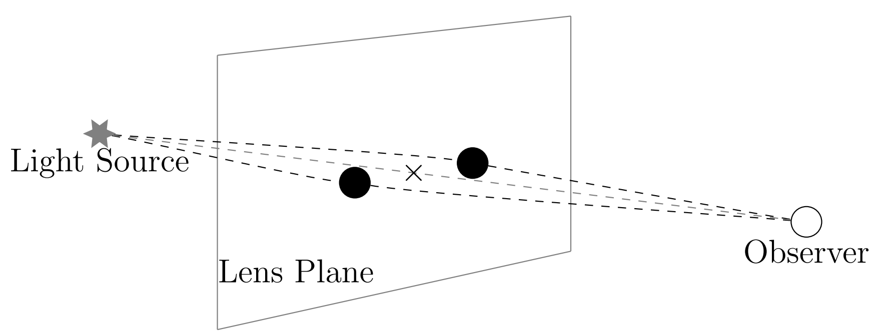


Figure 5.1: A cartoon of a lens plane where × represents a mass at the origin, and the filled in circles are the lensed images.

σ_j [KN08]. Thus σ_j must be real and positive for all $j \in \{1, 2, \ldots, n\}$. The lensed images in this system are represented by the fixed points of L.

Recall Theorem 1 which tells us for f an anti-rational map of degree $d \geq 2$, with no indifferent fixed points we have a relationship between the number of attracting fixed points, the number of repelling fixed points, and the degree of f. Let N_{attr} be the number of attracting fixed points, N_{rep} be the number of repelling fixed points, and $d = \deg(f)$. Then we have the total number of fixed points is $N = N_{rep} + N_{attr} = 2N_{attr} + d - 1$. By Theorem 3 each basin of attraction contains at least one critical point, therefore $N_{attr} \leq 2d - 2$, and we have

$$N \leq 2(2d-2) + d - 1 = 5d - 5.$$

Definition 26. A lensing map L of degree d is a *maximal lensing map* if L has $5d-5$ fixed points.

By [LSL15, Theorem 3.1] a maximal lensing map does not have indifferent fixed points, therefore by Theorem 1 we know $2d-2$ must be attracting, and $3d-3$ must be repelling.

The data of a maximal lensing map $L(z)$ can be phrased as follows:

1. The map $L(z)$ is anti-rational of degree d with d simple poles,
2. $2d-2$ attracting fixed points,
3. $3d-3$ repelling fixed points,
4. each pole has real positive residue, and
5. $L(\infty) = 0$.

Existing Examples

The existing examples of maximal lensing maps in every degree are attributed to Rhie [Rhi03]. Constructing these examples involves a small perturbation of work done by Mao, Petters, and Witt [MPW97].

Definition 27. *Mao-Petters-Witt maps (MPW maps)* are critically fixed anti-rational maps of degree $n \geq 3$ which, for positive real number p, take one of two forms,

$$L_{n,+p}(z) = \frac{na_p\overline{z}^{n-1}}{\overline{z}^n + p^n} \quad \text{and} \quad L_{n,-p}(z) = \frac{na_p\overline{z}^{n-1}}{\overline{z}^n - p^n}.$$

Immediately we see a critical fixed point at 0 of multiplicity $n-2$. We also have simple critical points equally spaced, separated by an angle of $2\pi/n$. For $L_{n,+p}$ the non-zero critical points are $\{z = p(n-1)^{1/n}e^{(2k)\pi i/n} | k = 1, \ldots, n\}$. We can pick $a_p = (n-1)^{(2-n)/n}p^2$ to ensure all these critical points are fixed. For $L_{n,-p}$ the non-zero critical points are $\{z = p(n-1)^{1/n}e^{(2k-1)\pi i/n} | k = 1, \ldots, n\}$. We can also pick $a_p = (n-1)^{(2-n)/n}p^2$ to ensure all these critical points are fixed. From here on we will drop the subscript and understand $a = a_p$ depends on p and is chosen such that the MPW map is critically fixed; Therefore, it is clear that MPW maps have associated Tischler graphs. All of this information is summarized in Figure 5.2.

The last useful fact we need is that MPW maps have some symmetries. First they are are self conjugate under an associated rotation, meaning

$$L_{n,\pm}(e^{i2\pi/n}z) = e^{i2\pi/n}L_{n,\pm}(z).$$

Second there is also a rotational symmetry between the two different forms, that is

$$L_{n,+p}(e^{\pi i/n}z) = e^{\pi i/n}L_{n,-p}(z).$$

Lastly we can rescale the map by conjugation by some non-zero scalar s, that is

$$\frac{1}{s}L_{n,+p}(sz) = L_{n,+p/s}(z).$$

When $n = 3$ we can choose $a = 1/\sqrt[3]{2}$ such that $L_{3,+p}(z)$ has real critical fixed points $c_0 = 0$ and $c_1 = \sqrt[3]{2}$. By evaluating $L_{3,+p}|_{\mathbb{R}}$, we can show the interval $[c_0, c_1]$ is invariant under $L_{3,+p}$. We can also show there is a repelling fixed point on the negative real axis. Using rotational symmetry by $2\pi/3$ we find 10 fixed points. A similar argument also works for $L_{3,-p}(z)$ and for more masses. Tischler graphs for 3 and 4 masses are shown in Figure 5.2.

For MPW-maps with $n > 3$ the critical point at zero is no longer simple, therefore MPW maps are not maximal lensing maps as we only generate a total of $3n + 1$ lensed images. Intuitively we have "wasted" repelling fixed points on the one critical point at 0. Rhie's idea was to add a small mass at zero which has the effect of "splitting" the critical point at 0.

Definition 28. *Rhie maps* are a small perturbation of a MPW-map, for p and b real positive numbers and $|b|$ small relative to p, they take the following form:

$$R_{n+1,\pm p}(z) = L_{n,\pm p}(z) + \frac{b^2}{\overline{z}} = \frac{(na + b)\overline{z}^n \pm b^2 p^n}{\overline{z}(\overline{z}^n \pm p^n)}.$$

Note that we have preserved the rotational symmetries

$$R_{m,\pm p}(e^{i2\pi/n}z) = e^{i2\pi/n}R_{m,\pm p}(z) \quad \text{and} \quad R_{m,\pm p}(e^{i\pi/n}z) = e^{i\pi/n}R_{m,\mp p}(z).$$

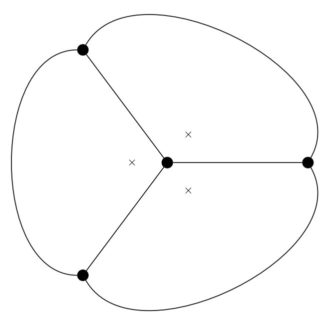
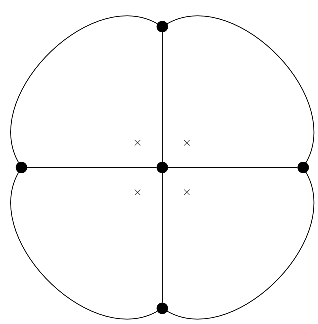
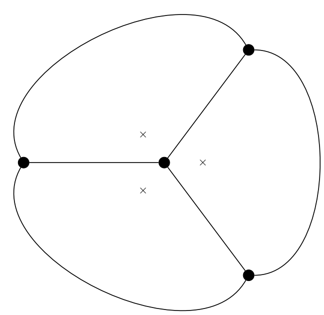
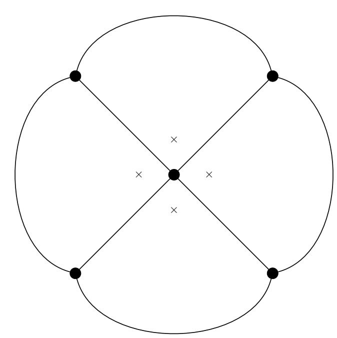
Function	$L_{n,+p}(z) = \frac{na\overline{z}^{n-1}}{\overline{z}^n + p^n}$	$L_{n,-p}(z) = \frac{na\overline{z}^{n-1}}{\overline{z}^n - p^n}$
Poles	$z = pe^{(2k-1)\pi i/n}$	$z = pe^{2k\pi i/n}$
Critical Points	$z = 0$, and $z = p\sqrt[n]{n-1}e^{2k\pi i/n}$	$z = 0$, and $z = p\sqrt[n]{n-1}e^{(2k-1)\pi i/n}$
a-Value	$a = (n-1)^{(2-n)/n}p^2$	$a = (n-1)^{(2-n)/n}p^2$
Tischler Graph	 	 

Figure 5.2: A summary of the two forms of MPW maps. Assume $k = 1, \cdots, n$.

Rhie shows for $|b|$ small, non-zero fixed points at worst become attracting or repelling fixed points nearby. Analyzing the behavior on the real line, Rhie finds added fixed points near zero. For n odd, there is an added repelling fixed point on the negative real line, and an added attracting fixed point on the positive real axis. Away from zero our behavior stays the same. Thus the real line accounts for 5 fixed points. By our rotational symmetry we find $5n$ lensed images for the $n+1$ masses in $\mathbb{C}$ and there we have a maximal lensing map. For n even, there is an added attracting fixed point on $\mathbb{R}$. After rotating by $e^{\pi i/n}$ Rhie finds one added repelling fixed point on $\mathbb{R}$. Therefore again we found $2n$ new fixed points near 0, and maintain $3n$ fixed points away from zero achieving the maximal lensing for $n+1$ masses.

Quasiconformal Surgery

This is a brief introduction to quasiconformal maps, for more Branner and Fagella's book is a great resource [BF14]. We will follow their definitions and notation.

For $z \in \mathbb{C}$, let x and y be real numbers such that $z = x + iy$. Consider the partial derivatives of a function f with respect to x and y. At all points where $\partial f/\partial x$ and $\partial f/\partial y$ exist, we define

$$\partial f = \frac{1}{2}\Big(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\Big) \quad \text{and} \quad \overline{\partial} f = \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big).$$

Definition 29. Let $f : U \to V$ be an orientation-preserving homeomorphism between domains $U, V \subseteq \mathbb{C}$, and let $K \geq 1$ be a constant. Then f is K-*quasiconformal* if f is ACL (absolutely continuous on almost all horizontal and vertical lines), and at almost all points $z \in U$ we have that

$$\left|\bar{\partial} f(z)\right| \leq \frac{K-1}{K+1} \left|\partial f(z)\right|.$$

The function f is *quasiconformal* if such a K exists.

Note this ACL property guarantees the partial derivatives almost everywhere.

For $f : U \to V$ a quasiconformal map, we have the following properties.

- Since f is a continuous open mapping, with partial derivatives almost everywhere, f is $\mathbb{R}$-differentiable almost everywhere. [BF14, Theorem 1.9, page 24]

- If f is K-quasiconformal, then f^{-1} is K-quasiconformal. [BF14, Property 1, page 31]

- If f is quasiconformal, then $\partial f \neq 0$ almost everywhere. More over the Jacobian determinant $Jf(z) = |\partial f(z)|^2 - |\overline{\partial} f(z)|^2 > 0$ almost everywhere. [BF14, Corollary 1.16, page 32]

- If f is 1-quasiconformal, then f is conformal. [BF14, Theorem 1.14 (Weyl's Lemma), page 32]

Definition 30. The *complex dilatation* (or *Beltrami coefficient*) of a quasiconformal map is defined as

$$\mu_f(z) = \frac{\bar{\partial} f(z)}{\partial f(z)}.$$

By the results quoted above, for f quasiconformal, μ_f is defined a.e., is in $L^\infty(U, \mathbb{C})$, and that $\|\mu_f\|_\infty \leq \kappa := \frac{K-1}{K+1} < 1$. We will use κ and K interchangeably, always related by the equations $\kappa = \frac{K-1}{K+1}$ and $K = \frac{1+\kappa}{1-\kappa}$.

If we fix a point z where f is differentiable and $|\mu_f(z)| \leq \kappa$, we can see that $\mu_f(z)$ determines an equivalence class of ellipses up to scaling in the tangent space at z which the derivative map $Df(z)$ maps to circles in the tangent space of $f(z)$. Here $|\mu_f(z)|$ encodes the eccentricity of the ellipse at z (0 is a circle and close to 1 is almost a line segment), and $\arg \mu_f(z)$ encode orientation. Because of this, the complex dilatation of f is often called an *ellipse field*. Notice this is only well-defined almost everywhere, and varies measurably, but generally not continuously with respect to z.

We now define a class of functions which "look like" complex dilatations of quasiconformal maps and state the fundamental result which says you can find a quasiconformal map with a given complex dilatation.

Definition 31. A *Beltrami coefficient* in U is a function $\mu \in L^\infty(U, \mathbb{C})$ such that $\|\mu\|_\infty = \kappa < 1$.

Theorem 29 (Measurable Riemann Mapping Theorem)**.** *[BF14, Theorem 1.27, page 40] Let $\mu \in L^\infty(\mathbb{C}, \mathbb{C})$ be a Beltrami coefficient in $\mathbb{C}$. Then there exists a quasiconformal map $f : \mathbb{C} \to \mathbb{C}$ with $\mu_f = \mu$ almost everywhere. The set of all such maps is given by $g = h \circ f$, where $h(z) = az + b$ is an affine map, with $a, b \in \mathbb{C}$, $a \neq 0$.*

We can always normalize our solution to fixed two points (usually 0 and 1) to get a unique solution. Commonly, these homeomorphisms f are viewed as (non-standard) charts on $\mathbb{C}$ as a Riemann surface, and Beltrami coefficients are viewed as *(measurable) almost complex structures.* The Measurable Riemann Mapping Theorem says that any almost complex structure (ellipse field we "declare" to be circles) can be integrated and induces a complex structure. The standard complex structure is represented by $\mu_0 = 0$. This whole theory makes sense on Riemann surfaces, and can be used to study deformations of complex structures in *Teichmüller theory.*

Definition 32. We say f is *quasirational* if and only if there exists a quasiconformal map ϕ and a rational map g such that $f = \phi \circ g \circ \phi^{-1}$.

A map $g : U \to \mathbb{C}$ is *quasiregular* if and only if there exists $\phi : U \to \phi(U)$ which is K-quasiconformal and $f : \phi(U) \to g(U)$ which is holomorphic such that $g = f \circ \phi$. There is also the following equivalent definition which characterizes quasiregular maps locally [BF14, Definition 1.35, page 57].

Definition 33. Let $U \subset \mathbb{C}$ be open and $K < \infty$. A mapping $g : U \to \mathbb{C}$ is K-*quasiregular* if and only if for every $z \in U$, there exists neighborhoods N_z and $N_{g(z)}$ of z and $f(z)$ respectively, a K-quasiconformal mapping $\phi : N_z \to \mathbb{D}$ and a conformal mapping $f : N_{f(z)} \to \mathbb{D}$, such that $(f \circ g \circ \phi^{-1})(z) = z^d$, for some $d \geq 1$.

Theorem 30 (Second Shishikura Principle)**.** *[BF14, Prop 5.5, page 183] Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be quasiregular. Let $X \subset \hat{\mathbb{C}}$ be open and satisfy:*

1. *$\partial_{\bar{z}} f = 0$ a.e. on $\hat{\mathbb{C}} \setminus X$ and f is quasiconformal on X; and*
2. *there exists $N \geq 1$ such that $max_{z \in X} |\mathcal{O}(z) \cap X| \leq N$ (i.e. the orbits of f pass through X at most N times).*

Then f is quasirational.

Lensing Graphs

The goal of this section is to prove that we can associate to a maximal lensing map a Tischler graph, and give us some tools to generate new maximal lensing maps.

First let us show that all attracting fixed points can be quasiconformally deformed into super-attracting fixed points and therefore, given a maximal lensing map we have an associated critically fixed anti-rational map. The converse is conjectured to be true, but remains unproven.

Theorem 31. *For an anti-rational map f with all immediate basins of attraction simply connected, and with all critical points contained in immediate basins of attraction. Then f is quasiconformally conjugate to a critically fixed anti-rational map g on a neighborhood of the Julia set $\mathcal{J}_f$.*

Proof. The idea of the proof is to first construct a critically fixed map $\tilde{f}$, which is equal to f off an open set containing the critical points and attracting fixed points. Then we show $\tilde{f}$ is quasirational.

First for every attracting fixed point z_0, we construct $\tilde{f}$ in the immediate basin of attraction $\mathcal{A}_0 = \mathcal{A}_0(z_0)$. Let $h : \mathcal{A}_0 \to \mathbb{D}$ be the conformal map from the simply connected basin of attraction to the unit disk such that $h(z_0) = 0$. Therefore $B \coloneqq h \circ f \circ h^{-1}(z)$ is a Blaschke product of $\overline{z}$ of degree $m \geq 2$ with attracting fixed point at 0. There exists an $r \in (0, 1)$ such that the disk $\mathbb{D}_r$ contains all of the critical points of f in $\mathcal{A}_0$. By the Schwarz Lemma the closure $\overline{\mathbb{D}_r}$, maps strictly inside $\mathbb{D}_r$ by $B(z)$.

Define the set $D' \coloneqq B^{-1}(\overline{\mathbb{D}}_r)$, therefore $\overline{\mathbb{D}}_r \subset D'$. We now show that D' must be both connected and simply connected. Assume first that D' is not connected. As $\mathbb{D}_r$ is forward invariant there exists a connected component D'_1 such that $\mathbb{D}_r \subset D'_1$. Therefore the map $B : D'_1 \to \overline{\mathbb{D}_r}$ is m-to-1 since $\mathbb{D}_r \subset D'_1$ containing all m critical points of $B(z)$. Thus the map is of maximal degree on D'_1, and $D' = D'_1$ must be connected. Let γ be a simple closed curve in D', and G be the bounded complementary component of γ. By definition $B(\gamma) \subset \mathbb{D}_r$ and by the maximum principle $G \subset \mathbb{D}_r$, therefore $G \subset D'$. As this is true for any simple closed curve γ, D' must be simply connected. See Figure 5.3.

We define $L(z)$ to be a quasiregular, linear-interpolating map on the annulus $A_1 = D' \setminus \mathbb{D}_r$ as follows (see Figure 5.4). We will call $\partial D'$ and $\partial \mathbb{D}_r$ the outer and inner boundary of annulus A_1, denoted γ_1^{out} and γ_1^{in} respectively. Define $A_2 = \mathbb{D}_r \setminus \mathbb{D}_{r^m}$, as $B(D') = \mathbb{D}_r$ and $\mathbb{D}_r$ maps onto $\mathbb{D}_{r^m}$ by $\overline{z}^m$. We will call $\partial \mathbb{D}_r$ and $\partial \mathbb{D}_{r^m}$ the outer and inner boundary boundary of annulus A_2, denoted γ_2^{out} and γ_2^{in} respectively. Therefore we define $L_\partial : \partial A_1 \to \partial A_2$ as $L_\partial|_{\gamma_1^{in}} = \overline{z}^m$ and $L_\partial|_{\gamma_1^{out}} = B(z)$.

Each of these annuli are covered by the infinite strip, that is there is a projection map

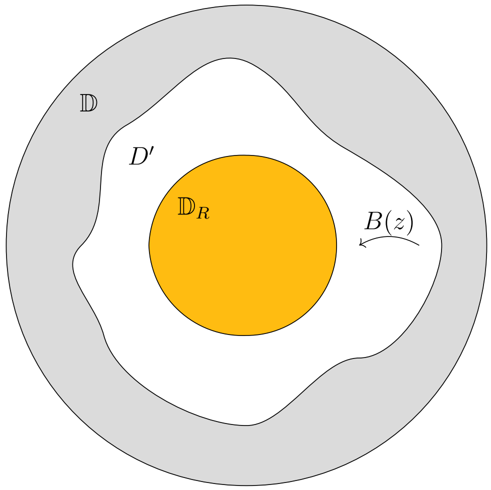


Figure 5.3: Changing the dynamics of a(n) (anti)-rational map near an attracting fixed point to make a critical fixed point.

$\Pi_j : S_j \to A_j$, for $j = 1, 2$. Pre-composition by scaling we can assume the upper boundary of the strip l_j^{up} maps to the outer boundary of the annulus γ_j^{out} and similarly the lower boundary of the strip l_j^{low} to the inner boundary of the annulus γ_j^{in}. Furthermore we can choose projections such that $\Pi_j(z) = \Pi_j(w)$ if and only if $z - w \in \mathbb{Z}$. The map L_∂ lifts to a map $F_\partial : \partial S_1 \to \partial S_2$. Extend to a map $F : S_1 \to S_2$ by linear-interpolation. That is for $z \in S_1$, we have a pair of points on the boundary, $z^{low} \in l_1^{low}$ and $z^{up} \in l_1^{up}$ such that $\text{Re } z = \text{Re } z^{low} = \text{Re } z^{up}$. Define $F(z) = w$, the point on the straight line connecting $F_\partial(z^{low})$ and $F_\partial(z^{up})$ such that $\text{Im } w = \text{Im } z$. This extension of ∂F to F defines an extension of ∂L to $L : A_1 \to A_2$ since $F(z+1) = F(z) - m$.

We define $\tilde{B}$ as

$$\tilde{B}(z) = \begin{cases} B(z) & \text{for } z \in \mathbb{D} \setminus D', \\ L(z) & \text{for } z \in D' \setminus \mathbb{D}_r, \\ \overline{z}^m & \text{for } z \in \overline{\mathbb{D}}_r. \end{cases}$$

It is clear that by construction $\tilde{B}(z)$ is quasiregular on $\mathbb{D} \setminus (\partial D' \cup \partial \mathbb{D}_r)$. Note the map $B(z)$

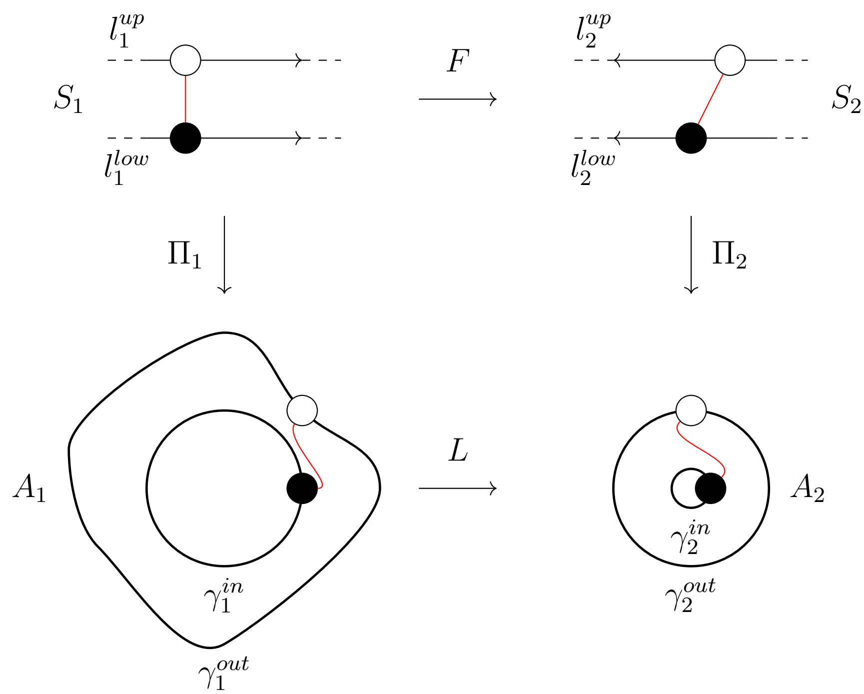


Figure 5.4

has not critical points on ∂A_1, therefore the outer boundary of A_1 is an analytic curve as the locally analytic preimage of the circle under $B(z)$. Therefore the boundaries of the annuli ∂A_1 and ∂A_2 are analytic curves and $\tilde{B}$ is quasiregular on all of $\mathbb{D}$ [BF14]. We therefore have defined $\tilde{B}$ to be quasiregular.

Now we define the function $\tilde{f}$ as follows.

$$\tilde{f}(z) = \begin{cases} f(z) & \text{for } z \notin h^{-1}(D'), \\ h^{-1} \circ \tilde{B} \circ h(z) & \text{for } z \in h^{-1}(\mathbb{D}) = \mathcal{A}_0. \end{cases}$$

By definition $\tilde{B}(D' \setminus \mathbb{D}_r) \subset \mathbb{D}_r$, therefore if we let $X \coloneqq h^{-1}(D' \setminus \mathbb{D})$, $\tilde{f}(X) \cap X = \emptyset$. Furthermore $\tilde{f}$ is quasiregular on $\hat{\mathbb{C}} \setminus X$. Therefore by Shishikura's Second Principle there exists an anti-rational map g such that $\tilde{f}$ is quasiconformally conjugate to g. Since $\tilde{f}$ fixes the critical point at z_0, so does g. □

Corollary 32. *For f an anti-rational map with a maximal number of attracting fixed points, there exists g a critically fixed anti-rational map, such that f and g are quasiconformally conjugate in a neighborhood of $\mathcal{J}_f$.*

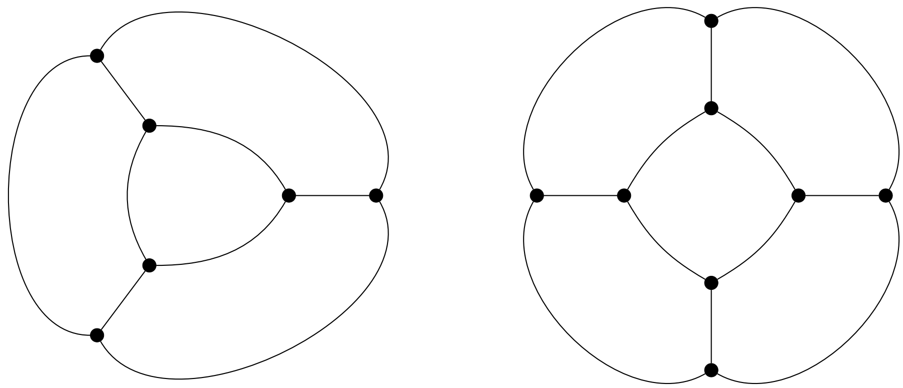
Proof. Since f has the maximal number of attracting fixed points 2deg $f - 2$, each critical point is in an immediate basin of attraction by Theorem 3. Applying Theorem 31, we construct $\tilde{f}$ which is critically fixed and equal to f on some neighborhood of $\mathcal{J}_f$. By construction $\tilde{f}$ is quasirational, therefore quasiconformally equivalent to some critically fixed anti-rational map g. □

Corollary 33. *Every maximal lensing map can be represented by a Tischler graph.*

We can see that the Tischler graphs associated to Rhie maps take the forms shown in Figure 5.5a and Figure 5.5b. They take the form of an annulus with n “spokes” connected the inner boundary to the outer boundary. See Figure 5.5a and Figure 5.5b.

Question: If T is an unobstructed Tischler graph such that each vertex $v \in V(T)$ has $deg_T(v) = 3$, is the Schottky map $L_T(z)$ associated to T combinatorially-equivalent to a maximal lensing map?

It is clear that L_T is combinatorially equivalent critically fixed anti-rational map. The difficulty in answering this question lies in the property of the residue at poles of L_T. A priori Tischler graphs give us no information about the coefficients of the associated anti-rational maps.



(a) Degree 4 Rhie map (b) Degree 5 Rhie map

Figure 5.5: Tischler graphs for existing maximal lensing maps in degrees 4, and 5.

New Examples

The idea for generating new examples was inspired by "gluing" together Tischler graphs of Rhie maps. The mathematical intuition is to make a small perturbation of the MPW-map by adding multiple masses near 0 which preserve rotational symmetry.

Theorem 34. *[Tu11, Implicit Function Theorem, Appendix B2] Let U be an open subset in $\mathbb{R}^n \times \mathbb{R}^m$ and $F : U \to \mathbb{R}^m$ a C^∞ map. Write $(x, y) = (x^1, \ldots, x^n, y^1, \ldots, y^m)$ for a point in U. At a point $(a, b) \in U$ where $f(a, b) = 0$ and the determinant $det[\partial f^i/\partial y^j(a, b)]$ is nonzero, there exists a neighborhood $A \times B$ of (a, b) in U and a unique function $h : A \to B$ such that in $A \times B \subset U \subset \mathbb{R}^n \times \mathbb{R}^m$,*

$$f(x, y) = 0 \iff y = h(x).$$

Moreover h is C^∞.

Lemma 35. *For $I \subset \mathbb{R}$ an open interval, let $f : I \to \mathbb{R}$ be a smooth map. Let $x_0 \in I$ be an* *attracting (or repelling) fixed point of f. Let τ be another open interval in $\mathbb{R}$ containing 0, such that $g_t(x) : \tau \times I \to R$ such that $g_t(x)$ is smooth for $t \in \tau$, smooth for $x \in I$, $g_0(x) = 0$, and $g_0'(x) = 0$. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that $f_t(x) = f(x) + g_t(x)$ has an attracting (repelling) fixed point at x_t such that $|x_0 - x_t| < \epsilon$.*

Proof. Define $F(t, x) = f_t(x) - x$, therefore $F(0, x_0) = 0$ and $\frac{\partial F}{\partial x}|_{(x_0,0)} = f'(x_0) - 1 \neq 0$.

Implicit function theorem implies there exists a smooth map $t \to x_t$ such that for $|t| < \delta_1$, $F(t, x_t) = 0$. Note, the multiplier map $\lambda(t) = f_t'(x_t)$ is continuous.

If x_0 is attracting, $|\lambda(t)| < 1$ for $t < \delta \leq \delta_1$, then $f_t(x) = f(x) + g_t(x)$ has an attracting fixed point at x_t such that $|x_0 - x_t| < \epsilon$.

If x_0 is repelling, $|\lambda(t)| > 1$ for $t < \delta \leq \delta_1$, then $f_t(x) = f(x) + g_t(x)$ has a repelling fixed point at x_t such that $|x_0 - x_t| < \epsilon$. □

Theorem 36. *The map $\mathcal{L}_{2n,+}(z) = L_{n,+1}(z) + L_{n,-t}(z)$ is a lensing map with $8n + 1$ lensed images for proper choice of t. Of these lensed images $3n + 1$ are attracting fixed points and $5n$ are repelling fixed points.*

Proof. For $K = (n - 1)^{(2-n)/n}$ we have

$$\mathcal{L}_{2n,+}(z) = \frac{K\bar{z}^{n-1}}{\bar{z}^n + 1} + \frac{Kt^2\bar{z}^{n-1}}{\bar{z}^n - t^n}.$$

If n is odd, then $L_{n,+1}(z)$ has one non-zero critical fixed point at $x = \sqrt[n]{n-1}$, one repelling fixed point $r \in \mathbb{R}^+$, such that $r < x$ and one repelling fixed point $R \in \mathbb{R}^-$. Let $T = \min\{r/2, 1\}$. Restricting to $z \in \mathbb{R}$, let $g_t(z) = L_{n,-t}(z)$, $I_+ = (r/2, 2x)$, and $\tau = (-T, T)$. By construction τ and I_+ are disjoint, therefore $z \in I$ is away from possible poles, and away from zero. Therefore $g_t(z)$ is smooth in t, smooth in x, $g_0(z) = 0$ and $g_0'(z) = 0$. We apply Lemma 35 and see $\mathcal{L}_{2n,+}(z)$ must also have an attracting fixed point near x and a repelling fixed point near r. Furthermore applying the same lemma for $z \in I_- = (1, 2R)$ we see $\mathcal{L}_{2n,+}(z)$ must also have a repelling fixed point near R. Furthermore for $|t|$ very small $L_{n,-t}(z) \approx \frac{Kt^2}{\bar{z}}$, therefore we have a second positive real attracting fixed point according to Rhie. In summary on $\mathbb{R}$ we see $\mathcal{L}_{2n,+}(z)$ has 2 repelling fixed points and two attracting fixed points. By rotational symmetry by $e^{2\pi i/n}$, we have detected $2n$ repelling fixed points and $2n$ attracting fixed points of $\mathcal{L}_{2n,+}(z)$.

We can rescale and rotate $\mathcal{L}_{2n,+}(z)$ to define

$$\mathcal{L}'_{2n,-}(z) = (te^{\pi i/n})^{-1}\mathcal{L}_{2n,+}(te^{\pi i/n}z) = L_{n,-1/t}(z) + L_{n,+1}(z) = \frac{Kt^{n-2}z^{n-1}}{t^n z^n - 1} + \frac{Kz^{n-1}}{z^n + 1}$$

We see the fixed points discussed above have also been scaled by $(te^{\pi i/n})^{-1}$.

Now investigate the contribution of $L_{n,+1}(z)$ to the lensed images of $\mathcal{L}'_{2n,-}(z)$. Again, $L_{n,+1}(z)$ has one non-zero critical fixed point at $x' = \sqrt[n]{n-1}$, and one repelling fixed point $r' \in \mathbb{R}^+$. Restrict z to $\mathbb{R}$ and apply Lemma 35, now for $g_t'(z) = L_{n,-1/t}(z)$, $I'_+ = (r'/2, 2x')$, and $\tau' = (-r'/2, r'/2)$. We find $\mathcal{L}'_{2n,-}(z)$ has a real attracting fixed point near x' and a real repelling fixed point near r'. Again by the same rotational symmetry we have discovered n more attracting and n more repelling fixed points. Lastly we note the one super attracting fixed point at 0 to give us a total of $3n + 1$ attracting fixed points.

So far we have discovered only $3n$ repelling fixed points. Note rescaling $\mathcal{L}'_{2n,-}(z)$ back to $\mathcal{L}_{2n,+}(z)$ and keeping track of the intervals I and I' used we notice we have accounted for $|z| < 2t\sqrt[n]{n-1}$ and $r/2 < |z|$.

By Corollary 33 we can realize this lensing map as a Tischler graph. We need to attach

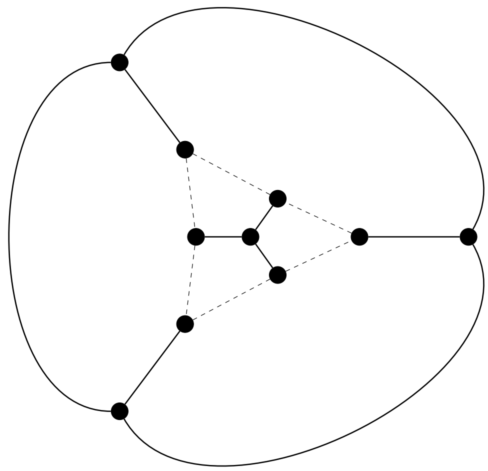


Figure 5.6: Tischler graph associated to $\mathcal{L}_{2(3),+}(z)$.

the critical points on $|z| = t\sqrt[n]{n-1}$ to critical points on $|z| = \sqrt[n]{n-1}$, inner and outer critical points respectively. Due to rotational symmetry these connections must preserve cyclic order. Furthermore due to symmetry with respect to the real line, there is only one way to do this, and that is to have each edge go from an inner critical point c_i to an outer critical point c_o such that $|\arg(c_o) - \arg(c_i)| = \frac{\pi}{n}$. See Figure 5.6, where these connecting edges are dashed lines. Since each of the inner and outer critical point needs 2 more incident edges, we must add a total of $2n$ edges. Recall each edge corresponds to a repelling fixed point, and therefore we have our final count on lensed images of $8n + 1$.

If n is even, repeat the same proof but now the map is $e^{\pi i}$ rotationally symmetric, so only focus on the positive real axis. Realize $2n$ attracting fixed points and n repelling fixed points from the outer map on $\mathbb{R}$. Rotate first to realize the n repelling fixed points away from zero. Now scale that rotation to consider the inner map. Find n attracting fixed points and one more at 0, as well as n repelling fixed points. Finally glue the inner spokes to the outer spokes to see the $8n+1$ lensed images. □

Corollary 37. *The map* $\mathcal{L}_{6,+}(z) = L_{3,+1}(z) + L_{3,-t}(z)$ *is a maximal lensing map for proper choice of* t*.*

Proof. By Theorem 36 $\mathcal{L}_{6,+}(z)$ is a lensing map with 25 lensed images for proper choice of t. See Figure 5.6. □

Theorem 38. *The map* $M_{2n+1,-}(z) = L_{n,-1}(z) + L_{n,+t}(z) + b^2/\overline{z}$ *is a maximal lensing map for proper choice of* $b < t$*.*

Proof. Recall $L_{n,+p}(e^{\pi i/n}z) = e^{\pi i/n}L_{n,-p}(z)$, therefore $\mathcal{L}_{2n,-}(e^{\pi i/n}z) = e^{\pi i/n}\mathcal{L}_{2n,+}(z)$. Therefore by Theorem 36 we can find t such that $\mathcal{L}_{2n,-}(z) = L_{n,-1}(z) + L_{n,+t}(z)$ is a lensing map of $8n+1$ lensed images.

Just like the Rhie perturbation, for small b we split the critical point at 0 into n attracting fixed points and n repelling fixed points. To see this, label the fixed points of $\mathcal{L}_{2n,-}(z)$ on the positive real line as 0, r_1 as the small repelling fixed point, c_1 as the critical fixed point, and r_2 as the large repelling fixed point. Realize that $\mathcal{L}_{2n,-}(z)$ is increasing in the interval $[0, c_1]$, therefore in order for r_1 to be repelling, $\mathcal{L}_{2n,-}(z) < z$ for $z \in [0, r_1]$. By the triangle inequality

$$|M_{2n+1,-}(z)| \leq |\mathcal{L}_{2n,-}| + |\frac{b^2}{\overline{z}}| \quad \text{and} \quad |M'_{2n+1,-}(z)| \leq |\mathcal{L}'_{2n,-}| + |\frac{b^2}{z^2}|.$$

Therefore we can choose b small such that $M_{2n+1,-}(z) \leq z$ for some interval contained in $(0, r_1]$. Choose $s \in (0, r_1)$ such that $M_{2n+1,-}(z) < z$ on (s, r_1). Furthermore we can choose b small such that $|M'_{2n+1,-}(z)|$ is within ϵ of $|\mathcal{L}'_{2n,-}|$ on (s, r_1). Therefore we show for sufficiently small b, there exists an attracting fixed point of $M'_{2n+1,-}(z)$ in the interval $(0, r_1)$. We will call these the "attracting fixed points near zero".

By rotational symmetry of $M'_{2n+1,-}(z)$ we see n attracting fixed points near zero. Therefore we have a degree $d = 2n + 1$ map with $2d - 2 = 4n$ attracting fixed points. By Theorem 1 we must have $3d - 3 = 6n$ repelling fixed points. The map $\mathcal{L}_{2n,-}(z)$ sees $5n$ repelling fixed points away from zero, therefore by the triangle inequality $M_{2n+1,-}(z)$ does too. The remaining fixed points occur near zero. To see this realize

$$M_{2n+1,-}(z) = L_{n,-1}(z) + R_{n+1,+t}(z),$$

therefore we find the repelling fixed points near zero from Rhie's argument.

Therefore we have $10n = 5(2n+1) - 5$ lensed images resulting from $2n+1$ masses, a maximal lensing configuration. See Figure 5.7 for the Tischler graph. $\square$

Corollary 39. *The sum of a finite number of MPW maps of degree n and a Rhie map of degree $n + 1$ of alternating type is a maximal lensing lensing map for a proper choice of real poles. More specifically*

$$\sum_{k=1}^{m} L_{n,(-1)^k p_k}(z) + \frac{b^2}{\overline{z}}$$

is a maximal lensing map of degree $m \cdot (n + 1)$ for the proper choice of $b < p_m < \cdots < p_1$.

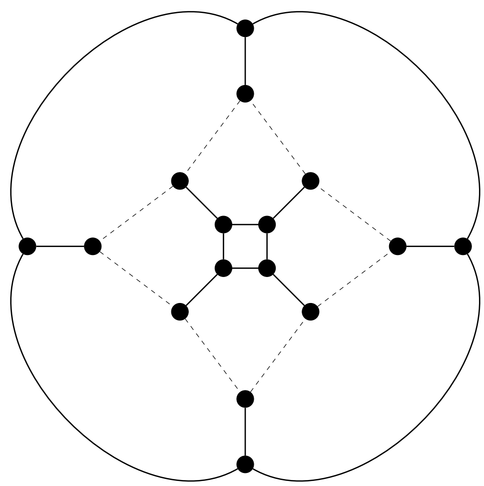


Figure 5.7: The Tischler graph associated to $M_{2(4)+1,+}(z)$. Where the dashed edges correspond to the edges connecting inner and outer critical points from the proof of Theorem 36.

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