GRAVITATIONAL WAVE ASTRONOMY USING SPACEBORNE DETECTORS

by

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A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Physics

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Louis Joseph Rubbo IV
To my wife Celeste,
Thanks for being there
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Throughout this dissertation geometrical units, where $G = c = 1$, are used unless explicitly noted. However, all frequencies are reported in Hertz. Sign conventions related to general relativity follow those used in *Gravitation* by Misner, Thorne, and Wheeler [1].

Rank-2 tensors are designated by plain bold symbols, $\mathbf{T}$, while four-vectors are denoted by italicized bold symbols, $\mathbf{x}$. Purely spatial vectors are designated with a vector symbol over the variable, $\vec{x}$.

When working with spacetime indices, Greek letters range over the time and spatial components (0,1,2,3), while Latin letters are reserved for the spatial components only. A bar over the index, or any coordinate label, signifies a change in coordinates,

$$\xi^{\bar{\mu}} = \Lambda^{\bar{\mu}}_{\alpha} \xi^{\alpha}.$$  

Partial differentiation follows the standard comma notation,

$$\xi_{,\nu}^{\mu} = \partial_{\nu} \xi^{\mu} = \frac{\partial \xi^{\mu}}{\partial x^{\nu}}.$$  

Differentiation with respect to the metric is signified with a semicolon,

$$\xi_{;\nu}^{\mu} = \frac{\partial \xi^{\mu}}{\partial x^{\nu}} + \xi^{\alpha} \Gamma^{\mu}_{\alpha \nu}.$$
This dissertation explores the use of spaceborne gravitational wave detectors as observatories for studying sources of gravitational radiation. The next decade will see the launch of the first space-based gravitational wave detector. Planning for several follow on missions is already underway. Before these observatories are constructed, extensive studies into their responses, expected output, and data analysis techniques must be completed. In this dissertation these issues are addressed using the proposed Laser Interferometer Space Antenna as an exemplary model.

The first original work presented here is a complete description of the response of a spaceborne detector to arbitrary gravitational wave signals. Previous analyses worked either in the static or low frequency limits. Part of this investigation is a coordinate free derivation of the response of a general detector valid for all frequencies and for arbitrary motion. Following directly from this result is The LISA Simulator, a virtual model of the LISA detector, in addition to an adiabatic approximation that extends the low frequency limit by two decades in the frequency domain.

Unlike most electromagnetic telescopes, gravitational wave observatories do not return an image of a particular source. Instead they return a set of time series. Encoded within these time series are all of the sources whose gravitational radiation passes through the detector during its observational run. The second original work presented here is the extraction of multiple monochromatic, binary sources using data from multiple time series. For binaries isolated in frequency space and with a large signal to noise ratio, it is shown that these sources can be removed to a level that is below the local effective noise.

A concern for the LISA mission is the large number of gravitational wave sources located within the Milky Way galaxy. The superposition of these sources will form a confusion limited background in the output of the detector. The final original work reported here is a Monte Carlo simulation of the galactic gravitational wave background as it will be observed by LISA. Using this simulation a number of characteristics of the background are calculated, including estimates the number and type of sources LISA will be able to identify, and the average distance in frequency space between bright sources. Also given is a demonstration of how a standard Gaussian test can be used to distinguish the galactic background from the intrinsic detector noise.
CHAPTER 1

GRAVITATIONAL WAVE ASTRONOMY

INTRODUCTION

In his 1905 landmark paper [2], *On the Electrodynamics of Moving Bodies*, Albert Einstein set forth the two postulates of special relativity: the laws of physics are the same for all inertial observers, and no information can travel faster than the speed of light. Ten years later Einstein followed up his original work by publishing the field equations for the general theory of relativity [3, 4]. General relativity describes gravity in a framework that is consistent with special relativity. Specifically, to be compatible with the second postulate of special relativity, gravitational information was found to propagate at the speed of light. As a direct consequence the existence of gravitational waves was inferred.

In the ninety years since the theory of general relativity was published, gravitational waves have eluded direct detection. The preeminent reasons have been a lack of understanding in the physical nature of gravitational radiation, and the difficulties in developing the technology required to make the necessary high precision measurements. In the middle of the last century the problem of understanding the nature of gravitational radiation was resolved by the collective work of a select group of physicists. However, the extremely precise measurements required to detect the anticipated weak gravitational wave signals are just now being realized.

As technology slowly made advances toward the predicted levels required for a direct detection, new understanding of exotic astrophysical bodies, specifically pulsar binaries, allowed for the first indirect detection of gravitational radiation. In the early 1970’s Russell Hulse and Joseph Taylor measured the inspiral rate for a pair of neutron stars as they orbited about their common center of mass [5]. The rate of inspiral agreed with the prediction from general relativity and its description of gravitational wave emission. For their work Hulse [6] and Taylor [7] were awarded the 1993 Nobel prize in physics.
In the thirty years since Hulse and Taylor’s discovery, technology has evolved to a point that allows a reasonable attempt at measuring gravitational waves directly. In turn, large scale terrestrial detectors are either under construction, or already built and are taking their initial data. Leading the way is the NSF funded Laser Interferometer Gravitational-Wave Observatory (LIGO) [8]. LIGO consists of a pair of observatories, each based on a Michelson interferometer design. The Livingston, Louisiana site houses a single interferometer with four kilometer arm lengths, while the Hanford, Washington site contains both a two kilometer and a four kilometer interferometer. Although the LIGO detectors are approaching their planned sensitivity, a direct detection of gravitational waves has yet to be achieved. The first unambiguous detection from LIGO should occur after 2006 when a number of upgrades will improve the overall sensitivity levels at both sites.

Near Cascina, Italy a similar three kilometer interferometer, known as VIRGO, has been constructed by a French and Italian collaboration [9]. Completed in June of 2003, VIRGO is now in a commissioning phase where the detector is monitored as a whole to evaluate any problems associated with construction and to determine its sensitivity levels. Once VIRGO begins its scientific runs, it will join LIGO in forming an international network of interferometer detectors devoted to the search for gravitational waves.

At low frequencies, below a few Hertz, terrestrial gravitational wave detectors are inherently noisy due to seismic and gravity gradient noise. To explore the potentially source rich low frequency portion of the gravitational wave spectrum, future detectors will necessarily be placed in space. The first such proposed mission is the joint ESA/NASA funded Laser Interferometer Space Antenna (LISA) [10]. Scheduled for launch sometime in the next decade, LISA will be comprised of three identical spacecraft orbiting about the Sun in an equilateral triangle formation with arm lengths of five million kilometers. This dissertation is an exploration into using spaceborne detectors, such as LISA, as observational instruments for studying astrophysical systems.

The remainder of this chapter introduces gravitational wave astronomy on a qualitative level. It starts with a historical review of the study of gravitational radiation and initial attempts at detection. The next section explores the physical nature of gravitational waves,
emphasizing their effects on test masses. This leads into a discussion of current attempts to measure gravitational waves using interferometric detectors. Once gravitational waves are detected their use as a probe into astrophysical systems is enormous. The information found through gravitational radiation is both different and complementary to that acquired from electromagnetic radiation. Following the section describing gravitational wave detection is a discussion comparing and contrasting electromagnetic and gravitational radiation. The final section of the chapter discusses the multiple stages associated with the transmission of information between a radiating astrophysical body and an astronomer. These multiple stages lend a natural outline to the rest of the dissertation.

**Historical Enlightenment**

A little over two hundred years prior to Einstein’s work on relativity, Isaac Newton developed his own theory of gravity [11]. In this formulation the apparent attraction between two massive objects is explained by an instantaneous gravitational force. In the years to follow, experiments performed on the Earth and observation of orbiting bodies in the Solar System supported Newton’s theory to within the experimental uncertainty of the day. The acceptance of theory was so secure that in 1845 the existence of Neptune and its position were predicted based on a discrepancy between Uranus’ observed orbit and one prescribed by Newtonian mechanics in the absence of Neptune. However, as the nineteenth century came to a conclusion, and a larger body of experimental data had been collected, a number of discrepancies in Newton’s theory became evident. Most notable was the perihelion precession of Mercury’s orbit.

Although Einstein was aware of the observational discrepancies, his interests lied in the conceptual inconsistencies found when Maxwell’s theory of electromagnetism is merged with Newtonian mechanics. Einstein’s 1905 paper on special relativity was his solution to reconciling the inconsistencies. As part of the explanation special relativity demanded a revolutionary relationship between time and the traditional three spatial dimensions. It was only a few years later that Herman Minkowski, a former professor of Einstein’s, described this union in terms of an absolute four dimensional entity known as *spacetime* [12].
Central to the theory of special relativity is the concept of an inertial reference frame. These are special frames in which the motion of the frame as a whole is governed by its own inertia. That is a frame that is not being acted upon by some kind of force. However, when any reference frame is near a massive body it necessarily experiences the force of gravity. By considering only inertial reference frames, special relativity cannot account for physical situations in which gravity is important. Einstein was aware of this limitation and quickly began working on a formulation of relativity that accounted for gravity.

In November of 1915 Einstein completed his work on incorporating the “gravitational force” into the framework of special relativity. According to the theory of general relativity, gravity should no longer be viewed as an instantaneous force, as Newtonian presumed it to be, but rather a curvature of the four dimensional spacetime. The classic analogy given in public lectures is that of an elastic sheet. The sheet represents spacetime. When a massive object is present, like our Sun, it forms a dimple in the sheet, much like a bowling ball on a trampoline. When a secondary object is introduce, such as the Earth, it does not feel a gravitational force from the Sun as Newton would say, but rather notices a curvature in the spacetime. The orbit of the Earth about the Sun is like a golf ball rolling around in the dimple of the trampoline formed by the bowling ball. The gravitational force envisioned by Newton is nothing more than the illusion of a object following the local curvature of spacetime.\textsuperscript{1}

Within a year of publishing his description of general relativity, Einstein began an investigation into the possibility of propagating disturbances (gravitational waves) in the fabric of spacetime. This amounted to a search for wave-like solutions to the general relativistic field equations. His initial attempt [13] showed that gravitational waves should be a natural consequence of general relativity. However, this initial work was flawed by a few algebraic errors. A few years later a second attempt produced a sound solution based on a linearized form of the field equations [14]. In subsequent years Weyl [15] and Eddington [16] elaborated on Einstein’s work, building a firm understanding of gravitational waves based on a linearized theory of general relativity.

As part of linearized general relativity the self-gravity of the emitting system is neglected.

\textsuperscript{1}In a more precise language, the path followed by an object experiencing a “gravitational force” is that of a geodesic, which is mathematically described by the equation $a^\mu = d^2 x^\mu / d\tau^2 = 0$. 
For most sources of gravitational radiation, such as compact binaries, this is not an appropriate assumption. It was not until the early 1940’s that Landau & Lifshitz published a theory of gravitational wave emission that included self-gravity effects [17]. However, a lack of understanding in the radiation reaction problem raised questions as to weather gravitational waves carry off energy. It took two additional decades of research and ingenious thought experiments by individuals such as Bondi [18, 19], Penrose [20, 21], Isaacson [22, 23] and others to restore confidence in the existence of gravitational waves, and to formulate a sound theory of gravitational wave production.

Experimental searches for gravitational waves started in the 1960’s by Joseph Weber [24]. At that time little was known about possible sources of gravitational waves or the expected intensity of radiation impinging on the Earth. The few initial estimates suggested that the amplitudes for gravitational waves passing through the Solar System would be extremely small and, therefore, would require a revolutionary approach to detection. It took Weber’s unique insight to see the possibility of a measuring a gravitational wave directly. His approach was to use large metallic bars, the first of which was an aluminum cylinder 0.61 m in diameter and 1.5 m in length, weighing more than a metric ton. The idea was to detect gravitational wave induced excitations in the vibrational modes of the bar. In 1969 Weber announced the first positive detection of a gravitational wave [25]. Unfortunately, other research groups were unable to independently confirm his findings, leaving the scientific community in doubt of his results.

At about the same time that Weber announced his possible findings, an alternative approach to gravitational wave detection was beginning to take shape. In a 1962 article Gertsenshtein & Pustovoit [26] described an initial design for a laser interferometer gravitational wave detector, or simply an interferometric detector. This work went mostly unnoticed and in 1970 Rainer Weiss [27] reinvented the approach and gave a detailed account of how such a detector would work. By 1972 a former student of Weber’s, Robert Forward, worked in collaboration with others at Hughes Research Laboratories to build the first working prototype of an interferometric detector [28]. Over the next couple of decades a small number of devoted research groups advanced the design of interferometric detectors. Recently (thirty years later)
their hard work has paid off with the construction of multiple large scale detectors, such as LIGO and VIRGO.

In parallel to the interferometric approaches, the concept of using spacecraft Doppler tracking as a means to measure gravitational waves was in development. The idea was first proposed by Braginsky & Gertsenstein in 1967 [29] and implemented for the first time using preexisting data by Anderson in 1971 [30]. In subsequent years details of the theory were further developed by Estabrook & Wahlquist [31] and Hellings [32, 33, 34], and applied to the Viking [35], Voyager [36], Pioneer [37, 38], and Ulysses [39] missions. While a somewhat straightforward approach to gravitational wave detection, environmental noises from fluctuations in the solar wind, atmospheric water vapor, etc., limit the sensitivity levels of Doppler tracking to a few orders of magnitude greater than the expected gravitational wave amplitudes. However, the negative results returned so far do set upper limits on the strength of gravitational waves possible present in the Solar System.

By combining the fundamental concepts behind ground-based interferometric detectors and spacecraft Doppler tracking, the natural next step in gravitational wave detection was to consider an interferometric detector in space. In the mid 1970’s the first seeds for a spaceborne detector began to surface with the work of Weiss and collaborators [40, 41]. By the early 1980’s a team of physicists at the Joint Institute for Laboratory Astrophysics had put together a full description for a proposed spaceborne gravitational wave detector known as the Laser Antenna for Gravitational-radiation Observation in Space (LAGOS) [42, 43]. Over the next decade LAGOS slowly evolved and was modified into its current proposed incarnation, the Laser Interferometer Space Antenna (LISA) [10]. As of today LISA still awaits full financial backing. Nonetheless, an optimistic launch date is set for sometime around 2011.

**Physical Nature of Gravitational Waves**

To understand the inherent difficulty in measuring gravitational waves it is informative to start with a conceptual review of their physical nature. The previous section alluded to the idea that gravitational waves are themselves a form of curvature. Detecting spacetime curvature requires at least two spatially separated particles. To see the effects of a gravitational
Figure 1.1: As a gravitational wave of magnitude $h$ passes through a pair of test masses initially separated by a distance $L$, the wave causes the masses to oscillate out of phase about their equilibrium positions. The amplitude of the oscillations is given by $\delta L/2 = hL/4$.

wave consider what happens to two test masses separated by a small distance. As shown in Figure 1.1 the gravitational wave will cause the two masses to come together and then apart and finally back to their original position.\(^2\)

A compact way to express the effects shown in Figure 1.1 is to borrow terminology from material sciences, namely the notion of a strain. In general, a strain is the ratio of the change in a length to the original length. Applying this terminology to the situation of the test masses, the magnitude of the gravitational wave may be expressed as

$$h = 2\frac{\delta L}{L},$$

(1.1)

where the factor of two comes about through a careful derivation using the full machinery of general relativity. As result of the relationship expressed in Equation (1.1), $h$ is often referred to as the *wave strain*, which, by its very definition, is a dimensionless quantity.

Figure 1.1 only explores the effects of a gravitational wave in one dimension. If we extend the previous test mass argument to two dimensions, more is revealed about the nature of gravitational waves. To do this consider the rings of test masses shown in Figure 1.2. As a gravitational wave passes perpendicular to one of the rings it distorts the ring in one of two

\(^2\)Figure 1.1 also demonstrates that gravitational radiation acts tidally, stretching and squeezing an extended body.
Figure 1.2: As a gravitational wave passes perpendicular to a ring of test mass it will distort the ring in one of two distinct ways. In one case, (top row) the wave causes the ring to distort into the form of a prolate ellipsoid along the \( x \) axis, followed by the same distortion along the \( y \) axis a half period later. In a similar fashion, the wave can also distort the ring in the diagonal directions (bottom row).

ways. Analogous to electromagnetic radiation, gravitational waves come in two distinct states referred to as the plus (+) and cross (\( \times \)) polarization states. For the plus state the ring may be initially distorted into a prolate ellipse along the \( x \) axis. A half period later the ring is distorted along the \( y \) axis. After a full period of oscillation the ring is returned to its initial shape. In a similar fashion the cross polarization distorts the ring in the same way, but with the effect rotated by \( 45^\circ \). Incidentally, the simultaneous squeezing in one direction while the other is contracting reflects the quadrupole nature of gravitational radiation; a feature that will be explored in the next chapter.

Typical wave strain magnitudes expected to be measured on the Earth are on the order of \( h \sim 10^{-21} \). For LIGO the distance between test masses is a few kilometers. Using Equation (1.1) this translates into a shift in the masses of \( 10^{-16} \) cm for every kilometer of separation. This distance is almost a billionth the size of an individual atom and a factor of a trillion times smaller than the wavelength of light used to measure the shift. These types of numbers are arguably the reason why gravitational waves have not yet been observed directly. Additionally, these values indicate some of the technological feats that must be accomplished in order to measure a gravitational wave directly.
Interferometric Gravitational Wave Detectors

Ground-Based Detectors

Ground-based Michelson interferometers, such as LIGO, consist of at least three widely separated test masses. The masses, which are suspended by vibration-isolated mounts, form an ‘L’ shape with arms of roughly equal length; see Figure 1.3. As a gravitational wave passes perpendicular to the plane of the detector, it causes the interferometer arms to expand and contract. For example, suppose that the arms are aligned with the + polarization of Figure 1.2. As the wave interacts with the detector, it causes the arm lying parallel to the $y$ direction to initially contract, while the remaining perpendicular arm expands. A half period later the reverse occurs. The arm in the $y$ direction expands and the other arm contracts. Since one arm is always stretched while the other is squeezed, the presence of gravitational wave is made evident by the difference in arm lengths,

$$\Delta L(t) = L_x(t) - L_y(t) = L h(t).$$

(1.2)

If we divide through by $L$, the initial arm lengths, then we see that it is appropriate to discuss the strain induced in the detector. The arm length differences are monitored by sending photons between the test masses. For an interesting discourse on how light can be used to monitor the arm lengths in the presence of gravitational wave see Saulson [44].

In general, the detector will not be aligned with any one of the polarization states of the gravitational wave, nor would the incident radiation contain only a single state. Consequently, the detector will measure a combination of the two states. In this case, the response of the detector $h(t)$ is expressed as a sum of responses to each polarization,

$$h(t) = \frac{\Delta L(t)}{L} = h_+(t) F^+ + h_\times(t) F^\times,$$

(1.3)

where $F^+$ and $F^\times$ are the antenna response functions for the detector. They describe the sensitivity of the detector to each polarization state as a function of the source location and

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$^3$The oversimplified view of LIGO developed here will be updated in Chapter 3, where a more in depth discussion of interferometric detectors is presented.
Figure 1.3: A schematic diagram demonstrating the principles behind a Michelson interferometric detector. The mirrors and beam splitter, all of which act as test masses, are suspended vertically allowing them to move in and out as a gravitational wave passes through the detector.

In theory a gravitational wave can be detected using a single interferometer arm, as shown in Figure 1.1. However, phase noise from the non-ideal input laser can mimic a gravitational wave signal. To control the effects of laser phase noise on the measurements, two arms are fed by the same light source. After the original light is split, sent down separate arms, and recombined at the beam splitter, any random variations in the laser phase will cancel.

In addition to laser phase noise, ground-based interferometric detectors also have to contend with seismic, thermal, shot, radiation, and gravity gradient noise. At low frequencies ($f \lesssim 10$ Hz) seismic and gravity gradient noise prevent terrestrial detectors from exploring gravitational wave sources below a few Hertz. This will be true regardless of technological advances in isolation techniques. Gravity gradient noises arise when massive objects such as cars, cows, alligators, atmospheric mass fluctuations, etc., pass by one of the test masses. Since it is impossible to shield the masses from gravitational influences, this form of noise cannot be

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4 The actual list of noise contributions is longer, but those listed here are the dominating forms. See Hughes et al. [45] for a more extensive list, along with a discussion on each source of noise.
overcome. The only way to investigate low frequency gravitational wave sources is to place a detector in space, far from massive bodies.

Space-Based Detectors

The leading candidate for a spaceborne mission is the proposed Laser Interferometer Space Antenna (LISA), a joint venture of NASA and ESA. Under the current proposal, LISA will be comprised of three identical spacecraft placed in exclusive, heliocentric orbits. The spacecraft will form an equilateral triangle that is titled by $60^\circ$ with respect to the ecliptic, and has mean arm lengths of five million kilometers. The center of mass for the constellation, referred to as the guiding center, will be in a circular orbit at 1 AU and $20^\circ$ behind the Earth. As the spacecraft constellation orbits about the Sun, the triangular formation will cartwheel with a retrograde motion (as if it were rolling on a ceiling). LISA’s placement with respect to the Earth is a balance between reducing tidal effects, and the financial and technical costs of telemetering down data from a large distance.

LISA will be sensitive to gravitational wave frequencies between roughly $10 \mu$Hz and 1 Hz. In this range of frequencies the main potential sources of gravitational radiation are supermassive black hole binaries, compact binaries, low period main sequence binaries, and extreme mass ratio inspirals of compact objects into supermassive black holes. One advantage a low
frequency spaceborne detector has over its terrestrial counterparts is that there are guaranteed sources of gravitational waves, such as the cataclysmic binary AM CVn, that have been identified through electromagnetic observations. Unless our understanding of gravitational radiation is flawed, these sources will be quickly identified in the output of the detector.

A NEW WINDOW ON THE UNIVERSE

A common misconception among non-gravitational wave physicists is that the principal objective of a gravitational wave detector is to directly measure gravitational radiation. While the first direct detection will be a celebrated event, their primary purpose is as an observational instrument. The information gained through gravitational waves is both unique and complementary to what is acquired from electromagnetic observations. The next few paragraphs describe some of the main differences in gravitational and electromagnetic radiation.\(^5\)

The emission processes for the two forms of radiation are drastically different. Electromagnetic radiation is produced by the superposition of emissions from individual objects on the atomic scale. Accordingly it tells us something about the thermodynamic state of the emitting system or region of space. Conversely, gravitational waves are produced by the coherent bulk motion of dense regions of mass-energy. From observations of gravitational waves we will be able to attain an understanding of the physics governing the motion of bodies in and around regions of high curvature.

Electromagnetic radiation has a wavelength that is typically much smaller than the size of the emitting system. This allows for detailed images of the source to be formed. The wavelength of a gravitational wave is typically greater than the size of the emitting system. As a result, gravitational wave data cannot be used to form a detailed visual image of the emitting object. Instead the gravitational radiation comes in the form of two data streams, one for each polarization.

The propagation of the two forms of radiation through the universe also differs. Electromagnetic radiation is comprised of coupled oscillations in the electric and magnetic fields, which propagate through the spacetime. By contrast, gravitational waves are propagating

\(^5\)For a similar discussion to what follows, see References [45, 46, 47, 48].
oscillations in spacetime itself and are a form of curvature. This makes defining a gravitational wave a tricky endeavor in some instances.\textsuperscript{6} Unlike electromagnetic radiation, which is easily scattered or absorbed by charged particles, gravitational waves propagate freely through the universe. Consequently, gravitational wave sources that are usually shrouded by optically thick gas, such as the interiors of supernovae, may be explored.

In terms of detection, with a few exceptions, electromagnetic observations are done in a narrow field of view. Gravitational wave detectors, on the other hand, cannot be shielded. As a result they are all sky observatories. They receive a superposition of all sources on the sky whose radiation passes through the detector during their observational runs. Since gravitational waves are weakly coupled to matter, they are difficult to detect and require advance technologies to measure their tiny effects on the detectors.

Lastly, in gravitational wave astronomy one measures the field directly, which falls off as $1/r$, while in most cases of electromagnetic astronomy one measures a flux, which falls off as $1/r^2$. For gravitational wave astronomy this means that a factor of two improvement in the detector sensitivity doubles the distance to which a source can be detected, increasing the volume of the observable universe by a factor of eight. In other words, every factor of two improvement in a gravitational wave detector sensitivity roughly coincides to a full order of magnitude in the volume of observable universe.

There are very few sources of gravitational radiation that also emit large amounts of electromagnetic radiation. Typical astrophysical systems observed in the electromagnetic spectrum are stellar atmospheres, accretion disks, and clouds of interstellar gas, none of which emit significant amounts of gravitational radiation. Conversely, strong sources of gravitational waves are binaries containing dense stellar remnants, and possibly the interiors of supernovae. In the case of the compact binaries, there is little accompanying electromagnetic radiation. For supernovae, dense exterior gas masks the inner environment. There are, however, two classic systems that will emit strongly in both spectra. These are neutron star binaries that contain at least one pulsar, and binary star systems in which there is some level of mass transfer present. In the latter case the system could be either a contact binary where an envelope of

\textsuperscript{6}This issue will be further explored in the introduction to Chapter 2.
Figure 1.5: The four distinct stages of involved in the conveyance of information between an emitting system and an astronomer: production, propagation, detection, and analysis.

gas surrounds the two bodies, or a semi-detach binary in which an accretion disk is formed around one of the components. When these specific systems are found, which may be numerous in our galaxy (see Chapter 5), they will allow a great union of general relativistic and electromagnetic information to study intricate details of the system and to test fundamental theories of physics.

**Gravitational Wave Astronomy Using Spaceborne Detectors**

In any field of astronomy, including gravitational waves, there is a systematic structure from which progress is made. There are sources scattered throughout the universe that are observable by a given detector. These sources produce radiation that propagates from the source to a detector. Along the way the radiation may be absorbed, scattered, or altered in some other fashion. Eventually the radiation reaches the detector. However, the detector acts as a filter in that the astronomer does not have direct access to the received radiation, but rather a processed output signal. It is only through the filtered signal that the astronomer is able to analyze the original radiation and study the emitting source. This discussion demonstrates how there are four distinct stages involved in the conveyance of information between an emitting body and an astronomer: production, propagation, detection, and analysis; see Figure 1.5. The coming chapters of this dissertation explore these stages of transmission as it applies to gravitational wave astronomy with spaceborne detectors.
Chapter 2 introduces a formal description of gravitational wave production based on linearized general relativity. A number of the physical attributes of gravitational radiation that were discussed in this chapter are derived here. Additionally, a derivation of the quadrupole formula, which is central to waveform modeling at the low frequencies covered by spaceborne detectors, is given.

Since gravitational waves interact weakly with matter, their form is rarely altered prior to detection. For this reason, the problem of propagation will be neglected. (See Thorne [49] for a thorough discussion of gravitational wave propagation.)

Chapter 3 gives a detailed account of forward modeling for spaceborne detectors, with a specific focus on the LISA mission. Included here is a derivation of the detector response function that includes the full orbital motion and is valid for all frequencies. Previous treatments of the response function have either assumed a stationary detector with respect to the background sky, or have been limited to the low frequency limit. Also examined in Chapter 3 are three approximations to the full response. By sacrificing some fidelity these approximations are able to accelerate numerical simulations of the full response, in addition to allowing simple analytical expressions for the response function to be written down.

As the previous section discussed gravitational wave observatories do not return an image of an individual source or region of space. Instead they return a collection of time series. Encoded in these times series are the signals from all sources whose radiation passes through the detector during its observational run. Chapter 4 introduces two independent approaches to identifying and subtracting individual sources from the time series: linear least squares fitting and template match fitting. Demonstrations of both approaches are shown in addition to a proposal of their use on the real LISA data.

One of the chief concerns for the LISA mission is the superposition of gravitational wave sources located inside the Milky Way. The potentially overwhelming number of sources \(10^{10}\) would show up in the detector output as a background noise that may hinder other scientific goals for LISA. Chapter 5 presents a study of the galactic gravitational wave background as it would be detected by LISA. Included in this examination is a characterization of the background fluctuations present in the power spectrums, along with a statistical analysis of
bright sources that will be identifiable in the detector output.

Chapter 6 is a summary of the original work presented in this dissertation followed by a discussion of future possibilities for spaceborne detectors as astronomical instruments.
CHAPTER 2

GRAVITATIONAL WAVE PRODUCTION

INTRODUCTION

To define a gravitational wave is a difficult endeavor. Since gravitational waves are themselves a form of curvature, there are situations in which it is impossible to disentangle the curvature produced by a massive body and the actual propagating waves. Nonetheless, there is still an inherent sense that one should be able to define a gravitational wave.

Consider water waves on the ocean. For a sailor on a ship, the ocean waves are easily defined as the propagating ripples on the surface of the water. Similarly, to an astronaut in a spaceship, the ocean waves are easily defined. They are the small scale changes seen on the ocean’s surface. Here, however, the astronaut must be careful to include the reference to the scale the waves. The overall curvature of the Earth also introduces a change in the shape of the ocean’s surface. When the astronaut characterizes the ocean waves as small scale changes on the surface of the water, they are comparing these changes to the overall curvature scale of the Earth.

In general relativity a similar approach is possible. The Riemann tensor, $R_{\alpha\beta\gamma\delta}$, gives a quantitative measure to the curvature of a given spacetime. Once gravitational waves are produced they propagate through a universe with regions of large scale, but slowly varying curvature produced by large collections of matter. The Riemann tensor that describes these large scale inhomogeneities has an associated length scale $L$ over which changes in the curvature vary. Typically this length scale is much larger than the reduced wavelength $\lambda = \lambda/2\pi$ of the propagating waves.\(^1\) In these cases it is possible to (approximately) split the Riemann tensor into a term that describes the background curvature, $R^B_{\alpha\beta\gamma\delta}$, and a term for the gravitational

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\(^1\)One possible exception are for waves that pass through the central region of a galaxy where supermassive black hole are expected to reside. In these situations the length scale is associated with the Schwarzschild radius of the black hole, and can be on the order of several AU, which is much larger than the reduced wavelength of gravitational radiation emitted from astrophysical sources.
wave, $R_{\alpha\beta\gamma\delta}^{GW}$. The background curvature is found by performing a “Brill-Hartle” average over several wavelengths,

$$R_{\alpha\beta\gamma\delta}^{B} = \langle R_{\alpha\beta\gamma\delta} \rangle,$$

(2.1)

while the gravitational wave term is the remaining portion,

$$R_{\alpha\beta\gamma\delta}^{GW} = R_{\alpha\beta\gamma\delta} - R_{\alpha\beta\gamma\delta}^{B}.$$  

(2.2)

Analogous to the ocean waves, gravitational waves are the small scale variations moving across the surface of a background spacetime.

The above approach is often referred to as the “shortwave approximation” and is due largely to the work of Brill & Hartle [50], and Isaacson [22, 23]. Using such a formalism it is possible to show that gravitational waves interact with matter and curvature in a manner very similar to that of light. They can be absorbed, scattered, and dispersed by matter as well as scattered, focused, and diffracted by the background curvature [49, 51]. The results of such an analysis is that the propagation of gravitational radiation is adequately described by the use of geometric optics.

The shortwave approximation does not solve the problem of defining a gravitational wave near a source of strong curvature. However, it does quantitatively describe our intuitive sense of what gravitational waves are for an observer removed from a region of strong curvature, much like the sailor out on the open sea. It also demonstrates how negligible the effects of matter and large scale curvature are on the evolution of a propagating wave whenever the conditions for the shortwave approximation are applicable [49].

With some sense of the defining properties of a gravitational wave, it only remains to relate the gravitational waveforms $h_{ij}$ measured by a detector to the stress-energy tensor $T^{\mu\nu}$ of the source. To this end, the remainder of the chapter is devoted to the derivation of the waveforms that are expecting to impinge on an interferometric detector. The first section introduces linearized general relativity, from which we will arrive at a differential equation (the wave equation) that directly relates small amplitude waves to the stress-energy tensor. The first approach to solving this equation is to investigate solutions in vacuum where the stress-
energy tensor is zero. It is here that many of the physical characteristics of gravitational waves discussed in Chapter 1 are found. Following this treatment is a discussion on an approximate solution to the wave equation (the quadrupole formula) that is applicable to gravitational wave astronomy.

**Linearized General Relativity**

Possible the most deceptive equation in physics is the Einstein field equation,

\[ G = 8\pi T, \]  

(2.3)

the foundational equation of general relativity. The deception arises from a combination of a geometrical interpretation and a shorthand representation used to convey the theory. Contained in this simple looking expression are ten non-linear, partial differential equations. There is no generalized algorithm to solve such a system of equations. In a few cases, symmetries allow drastic simplifications to a point where the problem is tractable. However, the universe is not always symmetric. In these instances alternative methods must be found.

One approach is to consider perturbations to the flat Minkowski space. Mathematically this procedure begins by expressing the total metric as a linear sum of Minkowski spacetime \( \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \) and a metric perturbation \( h_{\mu\nu} \),

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \]  

(2.4)

The validity of separating the small curvature corrections to the flat spacetime rests on the assumption that \( |h_{\mu\nu}| \ll 1 \). One useful way to interpret the linear decomposition is to think of the perturbation as a new tensor field on a flat background spacetime. While this is a helpful way to conceptually approach the problem, it is not absolutely accurate. The real spacetime contains curvature, just on a very small scale.

Using Equation (2.4) as the metric, it is possible to re-express the field equation given in Equation (2.3). The first step is to evaluate the Christoffel symbols to linear order in the
metric perturbation,

\[ \Gamma_{\mu\nu}^\alpha = \frac{1}{2} g^{\alpha\beta} (g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}) \]

\[ = \frac{1}{2} \eta^{\alpha\beta} (h_{\mu\beta,\nu} + h_{\nu\beta,\mu} - h_{\mu\nu,\beta}) + \mathcal{O}(h^2) \]

\[ = \frac{1}{2} \left( h^{\alpha,\nu} + h^{\alpha,\mu} - h_{\mu,\nu} \right) + \mathcal{O}(h^2). \]  

(2.5)

The last equality used the fact that when working to linear order in the perturbation, indices are raised and lowered using the background metric. Using the full metric would introduce terms on the order of \( h^2 \), which are being neglected.

The linearized form of the Ricci tensor immediately follows from the Christoffel symbols,

\[ R_{\mu\nu} = \Gamma_{\mu\nu,\alpha}^\alpha - \Gamma_{\mu\alpha,\nu}^\alpha + \Gamma_{\beta\alpha}^\beta \Gamma_{\mu\nu}^\beta - \Gamma_{\beta\nu}^\beta \Gamma_{\mu\alpha}^\alpha \]

\[ = \frac{1}{2} \left( h^{\alpha,\nu} + h^{\alpha,\mu} - \Box h_{\mu\nu} \right) + \mathcal{O}(h^2) , \]

(2.6)

where \( h \equiv h^{\alpha}_\alpha = \eta^{\alpha\beta} h_{\alpha\beta} \) is the trace of the metric perturbation and \( \Box = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2 \) is the flat space d’Alembertian operator. Note that the \( \Gamma\Gamma \) terms are second order in \( h \) and are, therefore, neglected. Taking the trace of the Ricci tensor yields the Ricci scalar to linear order,

\[ R = \eta^{\mu\nu} R_{\mu\nu} \]

\[ = h^{\mu\nu}_{\mu\nu} - \Box h + \mathcal{O}(h^2) . \]

(2.7)

Using the above results for the linearized forms of the Ricci tensor and scalar, the Einstein tensor is found to be

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R \]

\[ = \frac{1}{2} \left( h^{\alpha,\nu} + h^{\alpha,\mu} - \Box h_{\mu\nu} - h_{\mu,\nu} - \eta_{\mu\nu} h^{\alpha\beta}_{\alpha,\beta\alpha} + \eta_{\mu\nu} \Box h \right) + \mathcal{O}(h^2) . \]

(2.8)

The components of the Einstein tensor are simplified by defining the trace-reversed perturba-
tion,
\[ \tilde{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \tilde{h}. \] (2.9)

Substitution of this definition into the Einstein tensor reduces it to the form
\[ G_{\mu\nu} = \frac{1}{2} \left( \tilde{h}_{\mu\alpha, \nu} + \tilde{h}_{\nu\alpha, \mu} - \Box \tilde{h}_{\mu\nu} - \eta_{\mu\nu} \tilde{h}_{\alpha\beta} \tilde{h}^{\alpha\beta} \right) + O(h^2). \] (2.10)

Analogous to electromagnetism where the Lorentz gauge choice \( A^\alpha,_{\alpha} = 0 \) simplifies the equations, a similar gauge choice
\[ \tilde{h}_{\mu, \nu} = 0, \] (2.11)

which is also referred to as the Lorentz gauge, simplifies the Einstein tensor. Using this gauge choice the components of the Einstein tensor become
\[ G_{\mu\nu} = -\frac{1}{2} \Box \tilde{h}_{\mu\nu} + O(h^2). \] (2.12)

Hence, to linear order in the metric perturbation, the Einstein field equations are
\[ \Box \tilde{h}_{\mu\nu} = -16\pi T_{\mu\nu}. \] (2.13)

Since all indices are raised and lowered using the background Minkowski metric, the field equations are equivalently expressed by
\[ \Box \tilde{h}^{\mu\nu} = -16\pi T^{\mu\nu}. \] (2.14)

The combination of the metric decomposition (2.4), Lorentz gauge (2.11), and field equations (2.13) are collectively referred to as the linearized theory of general relativity. The linearized theory of gravity is directly applicable to gravitational waves where the metric perturbation, \( h_{\mu\nu} \) corresponds to the wave. Linearized theory is also used to describe the local curvature a spaceborne detector experiences. This issue will be further explored in the next chapter.
Gravitational Wave Description

In vacuum ($T_{\mu\nu} = 0$) the linearized field equations reduce to

$$\Box \bar{h}_{\mu\nu} = 0,$$

(2.15)

which is simply the source free wave equation. The physical solutions to this differential equation are of the form

$$\bar{h}_{\mu\nu} = \Re[A_{\mu\nu} \exp(ik_\alpha x^\alpha)],$$

(2.16)

where $A_{\mu\nu}$ are the components of a complex, constant amplitude tensor, $k^\alpha$ are the components of the wave vector, and $\Re[\cdot]$ is the real-part operator. The meaning of $k$ and its individual components are found from substituting Equation (2.16) back into the vacuum field equations. Doing so yields

$$k_\alpha k^\alpha \bar{h}_{\mu\nu} = 0,$$

(2.17)

which implies that $k$ is a null vector. It is customary to denote the zeroth component of the wave vector as $k^0 = \omega$, from which it follows that the components in a Lorentzian frame are expressed by

$$k \rightarrow (\omega, \vec{k}).$$

(2.18)

Using this wave vector decomposition in Equation (2.16), and applying the real-part operator, produces physical solutions to the wave equation of the form

$$\bar{h}_{\mu\nu} = |A_{\mu\nu}| \cos\left( -\omega t + \vec{k} \cdot \vec{x} + \delta \right),$$

(2.19)

where $\delta$ is a constant related to the real and imaginary parts of the amplitude tensor. In this form it is immediately evident that gravitational waves propagate in the $\vec{k} = \|\vec{k}\|$ direction with an angular frequency of $\omega$ (justifying the earlier notation) and a speed of $\omega/\|\vec{k}\| = 1$. The last statement follows directly from the null-like nature of the wave vector.

The components of the amplitude tensor are restricted by their requirement to obey the Lorentz gauge. Recall that the linear field equations only take their simplified form by imposing
the gauge condition
\[ \bar{h}_{\mu\nu,\nu} = 0. \] (2.20)

Substituting of Equation (2.16) into the above gives
\[ A_{\mu\nu} k^\nu = 0, \] (2.21)
which states that \( A \) must be orthogonal to the vector \( k \).

The amplitude tensor \( A_{\mu\nu} \) is symmetric in its components. This fact can be traced back to the original metric decomposition in linearized general relativity. In general, a symmetric rank-2 tensor in four dimensions has ten independent components. The orthogonality of the amplitude tensor and the wave vector imposes four conditions on the values of \( A_{\mu\nu} \). Further restrictions on the amplitude tensor are found by fixing remaining gauge freedoms.

The further gauge freedoms arise from our ability to transform coordinate systems. For example, it can be shown that the infinitesimal coordinate transformation
\[ x^{\bar{\mu}} = x^\mu + \xi^\mu, \] (2.22)
where \( \xi^\mu \) is small in that \( |\xi_{\mu,\nu}| \ll 1 \), will leave the form of the metric decomposition (2.4) unchanged. When applied to the trace-reversed metric perturbation, this coordinate transformation gives
\[ \bar{h}_{\bar{\mu}\bar{\nu}} = \bar{h}_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} + \eta_{\mu\nu} \xi_{\alpha,\alpha}. \] (2.23)

To arrive at the finalized form for the linear field equations, the Lorentz gauge was imposed on the values of \( \bar{h}_{\mu\nu} \). Doing so here implies that the transformation vector must obey the source free wave equation,
\[ \Box \xi_\mu = 0. \] (2.24)
As with before, the general solution of the wave equation is given by
\[ \xi_\mu = \Re \left[ B_\mu \exp(ik_\alpha x^\alpha) \right], \] (2.25)
where $k$ is the same wave vector from before. In deriving the previous results there was no need to restrict the values of the $B_\mu$ coefficients. The freedom in their choice allows a gauge decision that will further restrict the gravitational wave amplitude tensor.

Substitution of Equation (2.25) into (2.23) and removing the common exponentials gives

$$A\bar{\mu}\bar{\nu} = A_{\mu\nu} - iB_\mu k_\nu - iB_\nu k_\mu + i\eta_{\mu\nu}B^\alpha k_\alpha,$$

which maintains the orthogonality of the amplitude tensor to the wave vector. By a careful selection of the $B_\mu$ coefficients the values of $A\bar{\mu}\bar{\nu}$ are made to follow the restrictions

$$A^\alpha_\alpha = 0 \quad (2.27)$$

and

$$A_{\mu\nu}U^\nu = 0, \quad (2.28)$$

where $U$ is an arbitrary four-velocity with fixed components. Together, Equations (2.21), (2.27), and (2.28) form what is referred to as the transverse-traceless (TT) gauge.

The “traceless” terminology follows directly from Equation (2.27). To understand the “transverse” description consider a background Lorentz transformation to a frame in which $U^\mu = \delta^\mu_0$. From Equation (2.28) it follows that $A_{\mu0} = 0$ for all $\mu$. Also, as part of the background transformation, align the $z$ axis such that the wave vector becomes $k \rightarrow (\omega, 0, 0, \omega)$. In this particular case, Equations (2.21) and (2.28) imply that $A_{\mu z} = 0$ for all $\mu$. The only non-zero components to the amplitude tensor are those that are spatially orthogonal (transverse) to the direction of propagation. In this special coordinate system the matrix representation of the amplitude tensor becomes

$$[A^{TT}_{\mu\nu}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (2.29)$$
The “TT” label indicates that the transverse-traceless gauge conditions have been applied.

The original amplitude tensor started with ten independent components. The orthogonality condition reduces the number down to six. The trace condition further reduces the number to five. The transverse condition contains three more restrictions, bringing the total number of independent components down to two. The remaining two unspecified components correspond to the amplitudes for two polarization states.

Inspection of the previous matrix representation of the amplitude tensor shows that it is the sum two independent matrices,

\[
\begin{bmatrix}
A_{TT}^{\mu\nu} = A_{xx}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}

+ A_{xy}
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Using the direct product, the above is also expressed in tensor form as

\[
A_{TT} = A_+ \epsilon^+ + A_x \epsilon^x,
\]

where the polarization tensors are given by

\[
\epsilon^+ = \hat{x} \otimes \hat{x} - \hat{y} \otimes \hat{y}
\]

\[
\epsilon^x = \hat{x} \otimes \hat{y} + \hat{y} \otimes \hat{x}
\]

and \(A_+ = A_{xx}\) and \(A_x = A_{xy}\). In this representation the gravitational wave is given by

\[
\bar{h}_{TT} = \Re \left[ (A_+ \epsilon^+ + A_x \epsilon^x) \exp(ik \cdot x) \right].
\]

The above discussion assumed a particular coordinate system in which the gravitational wave traveled in the \(\hat{z}\) direction, making the \(x\) and \(y\) axes perpendicular to the propagation direction. This approach is generalized by letting the wave travel in an arbitrary \(\hat{k}\) direction.

\footnote{The transverse condition only introduces three new restrictions, not the expected four. The equation \(A_{\mu\nu} k^\mu U^\nu = 0\) is satisfied for arbitrary values of \(B_\mu\).}
and defining axes $\hat{p}$ and $\hat{q}$ that are perpendicular to the propagation direction. By also requiring that $\hat{q} = \hat{p} \times \hat{k}$, so that the set \{\hat{p}, \hat{q}, \hat{k}\} forms an orthonormal triad, the basis tensors are generalized to the form

$$
\epsilon^+ = \hat{p} \otimes \hat{p} - \hat{q} \otimes \hat{q}
$$

(2.34a)

$$
\epsilon^\times = \hat{p} \otimes \hat{q} + \hat{q} \otimes \hat{p}.
$$

(2.34b)

For a specific orientation of the $\hat{p}$ and $\hat{q}$ axes, known as the principal axes, there is a $\pi/2$ phase delay between the two polarization states. In this case, the gravitational wave of Equation (2.33) is expressed by

$$
\bar{h}_{TT}^{TT} = A_+ \cos \left( 2\pi f (t - \hat{k} \cdot \vec{x}) + \varphi_0 \right) \epsilon^+ + A_{\times} \sin \left( 2\pi f (t - \hat{k} \cdot \vec{x}) + \varphi_0 \right) \epsilon^\times,
$$

(2.35)

where $f$ is the gravitational wave frequency, and $\varphi_0$ is an arbitrary initial phase included here for completeness. In the above $A_+$ and $A_{\times}$ are taken to be real quantities.

The principal axes define a unique coordinate system in which the gravitational wave is the sum of a cosine and sine term. The next section will show how these axes are set by the projection of the gravitational wave source onto the celestial sphere. In other words, for a given observer the principal axes are unique to each source on the sky. A more appropriate basis is one that depends on the observer (or detector in the context of gravitational wave astronomy).

To derive the detector basis consider a fixed rectangular coordinate system associated with the detector. In this coordinate system the propagation direction of the waves becomes a function of the angular position ($\theta, \phi$) of the source on the sky,

$$
\hat{k} = -\sin(\theta) \cos(\phi) \hat{x} - \sin(\theta) \sin(\phi) \hat{y} - \cos(\theta) \hat{z}.
$$

(2.36a)

With this direction set it is now possible to define two unit vectors that are both perpendicular
Figure 2.1: The gravitational wave vector basis is related to the detector basis by a rotation of $\psi$ about the $\hat{k}$ direction. In this schematic the $\hat{k}$ direction is coming out of the paper.

to $\hat{k}$ and to each other,

$$\hat{u} = \cos(\theta) \cos(\phi) \hat{x} + \cos(\theta) \sin(\phi) \hat{y} - \sin(\theta) \hat{z}$$ \hspace{1cm} (2.36b)
$$\hat{v} = \sin(\phi) \hat{x} - \cos(\phi) \hat{y}.$$ \hspace{1cm} (2.36c)

Using this set of basis vectors the detector gravitational wave basis tensors are formed in the following way,

$$\mathbf{e}^+ = \hat{u} \otimes \hat{u} - \hat{v} \otimes \hat{v}$$ \hspace{1cm} (2.37a)
$$\mathbf{e}^x = \hat{u} \otimes \hat{v} + \hat{v} \otimes \hat{u}.$$ \hspace{1cm} (2.37b)

The vector basis sets for the gravitational wave $\{\hat{p}, \hat{q}, \hat{k}\}$ and the detector $\{\hat{u}, \hat{v}, \hat{k}\}$ are related by a rotation about the shared $\hat{k}$ direction by an angle $\psi$; see Figure 2.1. The angle $\psi$ is known as the principal polarization angle. The two basis tensors are related by

$$\mathbf{e}^+ = \cos(2\psi) \mathbf{e}^+ - \sin(2\psi) \mathbf{e}^x$$ \hspace{1cm} (2.38a)
$$\mathbf{e}^x = \sin(2\psi) \mathbf{e}^+ + \cos(2\psi) \mathbf{e}^x.$$ \hspace{1cm} (2.38b)
If we substitute the relationship between the basis tensors into Equation (2.35) we find that,

$$\bar{h}^{TT}(\xi) = h_+(\xi) e^+ + h_\times(\xi) e^\times,$$

(2.39)

where $\xi = t - \hat{k} \cdot \vec{x}$ is called the wave variable, and

$$h_+(\xi) = A_+ \cos \left(2\pi f\xi + \varphi_0\right) \cos 2\psi + A_\times \sin \left(2\pi f\xi + \varphi_0\right) \sin 2\psi$$

(2.40a)

$$h_\times(\xi) = -A_+ \cos \left(2\pi f\xi + \varphi_0\right) \sin 2\psi + A_\times \sin \left(2\pi f\xi + \varphi_0\right) \cos 2\psi.$$  

(2.40b)

Equations (2.39) and (2.40) are central to gravitational wave astronomy. They describe a plane-fronted gravitational wave with frequency $f$ in a detector derived basis. Additionally, by theorems of Fourier analysis the most general solution to the wave equation is the superposition of plane-fronted waves,

$$\bar{h}_{\mu\nu}(x) = \Re \left[ \int_{-\infty}^{\infty} A_{\mu\nu} \exp(ik_\alpha x^\alpha) \, dk \right].$$

(2.41)

Applied to the context of gravitational waves, the above implies that all that is needed is to add up multiple $\bar{h}^{TT}(\xi)$, each with a definite frequency $f_k$ to form a complete, general solution to the linearized field equations. It only remains to derive expression for the polarization amplitudes $A_+$ and $A_\times$, and the functional form for the gravitational wave frequency. To do this requires solving the linearized field equations with the stress-energy tensor included.

**Quadrupole Formula**

The standard way to solve the linear field equations (2.13) is through the use of a Green function. Green functions are derived by solving the original differential equation with the source term replaced by a delta function source. For the d’Alembertian this translate to

$$\square G(x - y) = \delta^{(4)}(x - y),$$

(2.42)

where $\delta^{(4)}$ is the 4-dimensional delta function and $G$ is the Green function. The solution to the
The solution comes in two independent forms, one for advanced waves and one for retarded waves. In the context of gravitational wave astronomy the interest is in the retarded solution,

\[
G(x - y) = -\frac{1}{4\pi \|\vec{x} - \vec{y}\|}\delta^{(3)}(\|\vec{x} - \vec{y}\| - (x^0 - y^0)) \theta(x^0 - y^0).
\] (2.43)

The \(\theta\) function is defined to be unity for \(x^0 > y^0\) and zero otherwise. The full solution to the original linear field equations is the sum of responses to delta function sources,

\[
\bar{h}_{\mu\nu}(x) = -16\pi \int G(x - y)T_{\mu\nu}(y)\,dy
= 4 \int \frac{1}{\|\vec{x} - \vec{y}\|}T_{\mu\nu}(t - \|\vec{x} - \vec{y}\|, \vec{y})\,d\vec{y}.
\] (2.44)

In the above, the substitution \(t = x^0\) has been made and the integration over \(y^0\) was done via the \(\theta\) function. The physical interpretation of Equation (2.44) is that the gravitational wave at \(x \rightarrow (t, \vec{x})\) is the sum of the energy and momentum of sources present at \(\|\vec{x} - \vec{y}\|\) at the retarded time \(t_r = t - \|\vec{x} - \vec{y}\|\). In other words, the gravitational wave amplitude is the summation of all energy-momentum disturbances on the past light cone for the event located at \(x\).

In aiding the evaluation of the integral in Equation (2.44), it is customary to introduce a number of limitations on the source and the region of emission. Specifically, we are interested in solutions far from an isolated source, \(r \gg R_{\text{source}}\) and whose size is much smaller than the emitted gravitational wave wavelength, \(\lambda \gg R_{\text{source}}\). In the context of gravitational wave astronomy the limitation of being far from the source is not much of a hindrance. However, the limitation that the source is small does imply that it must not contain relativistic motions. This follows directly from \(v_{\text{source}} \approx 2\pi R_{\text{source}}/\lambda\).

Applying these limitations to the integration yields,

\[
\bar{h}_{\mu\nu}(x) = \frac{4}{r} \int T_{\mu\nu}(t - r, \vec{y})\,d\vec{y}.
\] (2.45)

\(^3\)Derivations for the d’Alembertian Green function are found in most advanced textbooks on electromagnetism; for example Jackson [52]. In the context of linearized general relativity a derivation is found in Wald [53].
The assumption of being far from the source implied that in the denominator \( \| \vec{x} - \vec{y} \| \to r \),
which then allowed it to be removed from the integral. The long wavelength assumption
allowed the replacement of the \( \| \vec{x} - \vec{y} \| \) argument in the stress-energy tensor by \( r \). This
follows from a Fourier decomposition of \( \mathbf{T} \) from which there will be arguments of the form
\( (2\pi/\lambda)\| \vec{x} - \vec{y} \| \). In the strictest sense the equality in Equation (2.45) only holds as \( r \to \infty \).
For this reason, this solution is referred to as the asymptotic gravitational wave amplitude.

The remaining integrand is further massaged by using the conservation of the energy-momentum tensor in Minkowski space,

\[
T^{\mu\nu} = 0 ,
\]
from which it follows that

\[
\int T^{ij} d^3x = \frac{1}{2} \frac{d^2}{dt^2} \int T^{00} x^i x^j d^3x .
\]

For slowing moving objects, which has already been assumed, the \( T^{00} \) component of the stress-
energy tensor is approximately the mass density of the source, \( \rho \). Therefore,

\[
\int T^{ij} d^3x = \frac{1}{2} \frac{d^2}{dt^2} \int \rho(\mathbf{x}) x^i x^j d^3x \\
= \frac{1}{2} \frac{d^2}{dt^2} I^{ij}(t) ,
\]

where

\[
I^{ij}(t) = \int \rho(\mathbf{x}) x^i x^j d^3x ,
\]
is the quadrupole moment of the mass distribution. Returning to the asymptotic gravitational wave amplitude, we now have

\[
\bar{h}_{ij}(\mathbf{x}) = \frac{2}{r} \frac{d^2}{dt^2} I^{ij}(t_r) ,
\]
where \( t_r = t - r \) is the retarded time. This is the famous quadrupole formula for the gravitational wave amplitude.

As an example of the quadrupole formula in action, consider the unequal mass binary
Figure 2.2: An unequal mass, circular binary system. The coordinate system is defined such that the binary components orbit in the $xy$ plane with the orbital angular momentum aligned with the $z$ axis.

system in a circular orbit shown in Figure 2.2. The positions of the binary components are given by

$$
x_1(t) = R_1 \cos(\Omega t) \quad x_2(t) = -R_2 \cos(\Omega t)
$$

$$
y_1(t) = R_1 \sin(\Omega t) \quad y_2(t) = -R_2 \sin(\Omega t),
$$

where $R_i$ is the orbital radius of the $i^{\text{th}}$ component, and $\Omega = 2\pi f_{\text{orb}}$ is the orbital angular frequency.

If the binary components are treated as point masses, then the quadrupole integral (2.49) goes over to a summation,

$$
I_{ij} = \sum_{\alpha} m_{\alpha} x_{\alpha i} x_{\alpha j}.
$$

Using this expression the individual quadrupole moments are

$$
I_{xx} = \frac{1}{2} \left( M_1 R_1^2 + M_2 R_2^2 \right) \left( 1 + \cos(2\Omega t) \right)
$$

$$
I_{xy} = I_{yx} = \frac{1}{2} \left( M_1 R_1^2 + M_2 R_2^2 \right) \sin(2\Omega t)
$$

$$
I_{yy} = \frac{1}{2} \left( M_1 R_1^2 + M_2 R_2^2 \right) \left( 1 - \cos(2\Omega t) \right)
$$

$$
I_{zz} = I_{zi} = 0.
$$

Substituting these expressions into the quadrupole formula the gravitational radiation emitted
from the binary is found to be

\[
\bar{h}_{ij} = -\frac{4\Omega^2 (M_1 R_1^2 + M_2 R_2^2)}{r} \begin{bmatrix}
\cos(2\Omega t_r) & \sin(2\Omega t_r) & 0 \\
\sin(2\Omega t_r) & -\cos(2\Omega t_r) & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]  

(2.54)

In deriving the quadrupole formula the assumptions was made that the source must be slowly moving. In a consistent manner this assumption allows the application of Newtonian mechanics to derive the binary motion. From Kepler’s third law in barycentric coordinates, the orbital radii are related to the masses and angular frequency according to

\[
R_i = \left(\frac{M_j^3}{(M_i + M_j)^2 \Omega^2}\right)^{1/3},
\]

(2.55)

where \(i \neq j\). Furthermore, the orbital angular frequency is related to the binary separation, \(R = R_1 + R_2\), by

\[
(M_1 + M_2) = \Omega^2 R^3.
\]

(2.56)

Together, these equations allow the amplitude of the gravitational wave tensor to be rewritten in the form

\[
\bar{h}_{ij} = -\frac{4M_1 M_2}{rR} \begin{bmatrix}
\cos(2\Omega t_r) & \sin(2\Omega t_r) & 0 \\
\sin(2\Omega t_r) & -\cos(2\Omega t_r) & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

(2.57)

Physically the above equation describes the trace-reversed metric perturbation emitted from a binary source far from an observer and whose barycenter lies on the observer’s \(z\) axis. It is interesting to note that the gravitational radiation is emitted at twice the orbital frequency. By inspection of Equation (2.49) this is to be expected. Since the positions of the binary components are introduced quadratically, the quadrupole moment is invariant under a \(\pi\) rotation. This then implies that the values of the quadrupole moments are periodic with a frequency of \(2f_{\text{orb}}\).

In the previous section the solution to the source free linear field equations were derived. The solutions found were plane-fronted waves with two polarization states, simply described
in a transverse-traceless gauge. For an observer far from the binary, which is already a requirement of the quadrupole formula, the emitted gravitational waves should also be well approximated as being plane-fronted. However, Equation (2.57) is not in the TT gauge for an arbitrary observer. (For the special set of observers who are located on the \( z \) axis, the gravitational wave represented in (2.57) is already in the TT gauge.)

To transform to an arbitrary TT observer, consider gravitational waves emitted in the radial direction \( \hat{k} \), which makes an angle \( \theta \) with respect to the \( z \) axis. Now rotate the original coordinate system about the \( z \) axis by an angle \( \varphi_0 \) so that the propagation direction is confined to the new \( \bar{x}\bar{z} \) plane; see Figure 2.3. In this new coordinate system the components of gravitational wave tensor are

\[
\bar{h}_{\bar{x}\bar{x}} = -\bar{h}_{\bar{y}\bar{y}} = -\frac{4M_1 M_2}{rR} \cos(2\Omega t_r + \varphi_0) \tag{2.58a}
\]
\[
\bar{h}_{\bar{x}\bar{y}} = \bar{h}_{\bar{y}\bar{x}} = -\frac{4M_1 M_2}{rR} \sin(2\Omega t_r + \varphi_0) \tag{2.58b}
\]
\[
\bar{h}_{\bar{z}\bar{z}} = \bar{h}_{\bar{z}\bar{z}} = 0. \tag{2.58c}
\]

As it stands, the new barred coordinate system is not yet in the TT gauge. To arrive at
such a form we must first introduce the projection tensor,

\[ P_{ij} = \delta_{ij} - \hat{k}_i \hat{k}_j, \]  

(2.59)

where \( \hat{k}_i \) is the \( i^{th} \) component of a unit normalized vector that points in the propagation direction. Using the projection tensor, the gravitational wave tensor is transformed into TT coordinates by the combination

\[ \tilde{h}^{TT}_{ij} = (P^k_i P^l_j - \frac{1}{2} P_{ij} P^{kl}) \tilde{h}_{kl}. \]  

(2.60)

Using this relationship the components of the gravitational wave tensor become

\[
\begin{align*}
\tilde{h}^{TT}_{xx} &= \cos^2(\theta) \left( 1 - \frac{1}{2} \sin^2(\theta) \right) \tilde{h}_{xx} \tag{2.61a} \\
\tilde{h}^{TT}_{xz} &= \tilde{h}^{TT}_{zx} = -\frac{1}{2} \sin(2\theta) \left( 1 - \frac{1}{2} \sin^2(\theta) \right) \tilde{h}_{xx} \tag{2.61b} \\
\tilde{h}^{TT}_{xy} &= \tilde{h}^{TT}_{yx} = \cos^2(\theta) \tilde{h}_{xy} \tag{2.61c} \\
\tilde{h}^{TT}_{yy} &= -\left( 1 - \frac{1}{2} \sin^2(\theta) \right) \tilde{h}_{xx} \tag{2.61d} \\
\tilde{h}^{TT}_{yz} &= \tilde{h}^{TT}_{zy} = -\frac{1}{2} \sin(2\theta) \tilde{h}_{xy} \tag{2.61e} \\
\tilde{h}^{TT}_{zz} &= \sin^2(\theta) \left( 1 - \frac{1}{2} \sin^2(\theta) \right) \tilde{h}_{xx} \tag{2.61f}
\end{align*}
\]

The above component values are greatly simplified by introducing the two unit vectors,

\[
\begin{align*}
\hat{p} &= \hat{y} \tag{2.62a} \\
\hat{q} &= \cos(\theta) \hat{x} - \sin(\theta) \hat{z}, \tag{2.62b}
\end{align*}
\]

which are the principal axes discussed in the previous section. Additionally, the angle \( \theta \) is defined with respect to the propagation direction. For astronomers the more appropriate angle is the inclination angle of the binary orbit, \( \iota \). The inclination is defined as the angle between the line of sight from the observer to the source and the angular momentum vector of the binary. As a function of \( \theta \), the inclination angle is given by \( \iota = \pi - \theta \). In this new vector
basis the components of the gravitational wave tensor become

\[
\bar{h}_{pp}^{TT} = -\bar{h}_{qq}^{TT} = \frac{2M_1 M_2}{r R} \left( 1 + \cos^2(\iota) \right) \cos(2\Omega t_r + \varphi_0) \tag{2.63a}
\]
\[
\bar{h}_{pq}^{TT} = \bar{h}_{qp}^{TT} = \frac{4M_1 M_2}{r R} \cos(\iota) \sin(2\Omega t_r + \varphi_0). \tag{2.63b}
\]

To summarize the last few pages, we started with a unequal mass binary in a circular orbit. The orbital motion naturally introduces a rectangular coordinate system in which the masses are confined to the \(xy\) plane and the angular momentum vector of the system points along the \(z\) axis. In this coordinate system the components of the trace-reversed tensor are calculated using the quadrupole formula (2.50). Next, the values for \(\bar{h}_{ij}\) are found for radiation propagating in an arbitrary direction by rotating to a new coordinate system in which \(\hat{k}\) is in the new \(\bar{x}\bar{z}\) plane. To connect to the previous section, the new values of \(\bar{h}_{ij}\) are transformed into the transverse-traceless gauge via the projection tensor. Finally, the wave components are simplified by introducing the principal polarization axes \(\hat{p}\) and \(\hat{q}\).

The important point here is that the binary system is naturally described in one set of coordinates, and the propagating gravitational waves are appropriately described in another. This section stepped through the connection of these coordinate systems by a natural progression of coordinate transformations. The final step left the description of the gravitational wave in a form that is directly comparable to the source free solution of the Einstein field equations discussed in the previous section.
CHAPTER 3

SPACEBORNE GRAVITATIONAL WAVE DETECTORS

INTRODUCTION

The Chapter 1 presentation of a Michelson laser interferometer was an oversimplified view of what is actually implemented. Real interferometric gravitational wave detectors are plagued with various technical issues. Chief among these is the problem of overcoming environmental and detector noise. In an attempt to minimize the noise, the detector’s design must be modified from a simple Michelson structure to a more complex instrument.

Consider the implementation used by the Laser Interferometer Gravitational-Wave Observatory (LIGO) [8]. LIGO is not a true Michelson interferometer, but rather operates as a modified Michelson with Fabry-Perot cavities. The main difference arises from the use of three extra semi-transparent mirrors as shown in Figure 3.1. The first mirror is placed between the input laser and the beam splitter. Its purpose is to recycle the recombined light that comes out of the beam splitter and heads back toward the input laser. The remaining two mirrors are placed inside the detector arms. These mirrors produce what is known as a Fabry-Perot cavity, which is a way to increase the power in the output laser beam and to increase the effective arm length of the detector.

In theory a standard Michelson interferometer can be used to measure the relative difference in its arm lengths. However, the effects of a gravitational wave with an amplitude of $h \sim 10^{-21}$ on a kilometer scale detector will cause a change in the arm length difference on the order of $10^{-16}$ cm. This presents a technological hurdle since such a change would only be detectable through a small variation in the number of photons received at the output photo-detector.

To account for the extremely small variations in the arm lengths, LIGO uses a set of servo systems to insure that the output photo-detector is kept dark. Consequently, the output channel is often referred to as the dark port. The presence of a passing gravitational wave
is encoded in the servo system record, which is constantly trying to counteract the effects of the gravitational wave. By using a feedback loop, LIGO’s response is highly non-linear and, therefore, complicated to simulate.

The implementation of an interferometric detector in space is arguably as difficult as its terrestrial counterparts, but for different reasons. In space a gravitational wave detector is subject to the stochastic environment produced by random solar outbursts and impinging high energy particles that originate from outside the Solar System. Moreover, basic orbital dynamics will cause the distance between the spacecraft to oscillate in time. Each of these effects, and others not discussed, cause variations in the arm lengths that are much greater than those from gravitational waves.

To counteract the effects of the stochastic environment, the spacecraft are designed shelter smaller test masses from external forces. For the most part this will be an effective approach, but it will leave behind residual noise in the detector output. The orbital effects are dealt with
by modeling the spacecraft position using a sophisticated ephemeris of the Solar System.

During the early developmental stages of a planned detector, not all complications make themselves immediately obvious. It is only after building prototypes and end-to-end models that most issues begin to surface. For example, LIGO is based on a scaled down 40 m prototype housed at the California Institute of Technology. The prototype LIGO allows for tests of future technologies and possible implementations that may be incorporated into the fully functioning LIGO detectors.

For the LISA observatory, work is now underway to produce an end-to-end model of the detector [54]. Key ingredients include accurate modeling of the spacecraft orbits and photon trajectories (including the effects of gravitational waves); realistic simulations of the time delay interferometry used to cancel laser phase noise; and experimental characterizations of the various noise contributions. A good end-to-end model helps to make design trade-offs and to avoid costly mistakes.

This chapter introduces one aspect of a LISA end-to-end model: the forward modeling of the detector response to an arbitrary gravitational wave. The fundamental concept behind forward modeling is to take an input gravitational wave signal and calculate the expected output of the detector. The inverse problem is that of data analysis, where the original source parameters are estimated using the detector output. The issue of data analysis as it applies to LISA will be taken up in the next chapter.

The outline of the chapter is as follows. The first section gives a brief overview of the LISA mission and presents a derivation for the spacecraft positions that neglects all Solar System bodies except for the Sun. From here a derivation of the optical path length variation caused by the passage of gravitational wave is given. In the same section is a discussion on how LISA implements the interferometric approach. This naturally leads into a description of The LISA Simulator, a virtual model of LISA which takes an input gravitational wave and returns the simulated response of the detector.

The LISA Simulator consumes a great deal of computer resources and delivers a fidelity that exceeds requirements for many mock data analysis efforts. Indeed, when searching the large parameter spaces that describe various gravitational wave sources, fidelity must be sac-
rificed in favor of speed. To this end, the last three sections introduce three different approximations to the full LISA response. The first approximation, the static limit, is nothing more than a pedagogical exercise of the detector response functions. However, the low frequency limit that follows, accurately describes the response of LISA for most sources below 3 mHz. By relaxing one of the assumptions made in the low frequency limit, the rigid adiabatic limit is able to approximate LISA’s response for most of its frequency band. The chapter concludes with a summary of results that ties all of the approximations together.

**Spaceborne Detectors**

The current proposal for the LISA mission calls for three identical spacecraft to fly in an equilateral triangular formation about the Sun. The center of mass for the constellation, known as the guiding center, will be in a circular orbit at 1 AU and 20° behind the Earth. The triangular formation is itself tilted by 60° with respect to the ecliptic. In addition to the guiding center motion, the formation will cartwheel with a retrograde motion over a one year period (see Figure 1.4).

As will be demonstrated in a later section, the LISA’s orbital motion introduces amplitude, frequency, and phase modulations in the gravitational wave signal. The amplitude modulation is caused by the antenna pattern being swept across the sky. The phase modulation occurs when the differing responses to the two gravitational wave polarizations are combined together. The frequency (Doppler) modulation is due to the motion of the detector relative to the source. Since both the orbital and cartwheel motions have a period of one year, the Doppler modulations will show up in the detector output as sidebands in the power spectrum. Each sideband is separated from the instantaneous carrier frequency by integer values of the modulation frequency, \( f_m = 1/\text{year} \) [55, 56, 57].

The frequencies at which the guiding center and rolling motion impart measurable effects on the signals is easily estimated. Equating the modulation frequency to the standard Doppler shift, \( \delta f \approx (v/c)f \), for motion with velocity \( v \) yields the characteristic frequency \( f_v = cf_m/v \) at which Doppler modulation becomes measurable. Assuming the LISA 5 \( \times 10^9 \) m arm lengths the cartwheel turns with velocity \( v/c = 0.192 \times 10^{-5} \), while the guiding center moves with
velocity \( v/c = 0.994 \times 10^{-4} \). Thus, the Doppler modulation due to the guiding center’s motion becomes measurable at frequencies above \( f_{gc} = 0.3 \) mHz, while the rolling cartwheel motion becomes important above \( f_r = 16 \) mHz. As a result, at low frequencies only the bulk motion of the detector needs to be considered, while at high frequencies the cartwheel motion also needs to be included.

Owing to LISA’s orbital motion about the Sun, a natural set of coordinates for describing the positions of the LISA spacecraft is the heliocentric-ecliptic coordinate system. In these coordinates the Sun is placed at the origin, the \( x \) axis points in the direction of the vernal equinox, the \( z \) axis is parallel to the orbital angular momentum vector of the Earth, and the \( y \) axis is placed in the ecliptic to complete the right handed coordinate system. Ignoring the influence from other Solar System bodies, the individual LISA spacecraft will follow independent Keplerian orbits. The triangular formation comes about through the judicious selection of initial conditions.

To arrive at the positions of each spacecraft as a function of time we start with the parameterized Keplerian orbits,

\[
\begin{align*}
    x_n &= r \left( \cos(i) \cos(\beta_n) \cos(\gamma) - \sin(\beta_n) \sin(\gamma) \right) \\
    y_n &= r \left( \cos(i) \sin(\beta_n) \cos(\gamma) + \cos(\beta_n) \sin(\gamma) \right) \\
    z_n &= -r \sin(i) \cos(\gamma)
\end{align*}
\]

where \( i = \sqrt{3}e \) is the inclination of the orbit, \( \beta_n = 2\pi(n - 1)/3 + \lambda \) \( (n = 1, 2, 3) \) is the relative orbital phase of each spacecraft in the constellation, \( \gamma \) is the ecliptic longitude, and \( r \) is the standard Keplerian radius,

\[
r = \frac{R(1 - e^2)}{1 + e \cos \gamma}.
\]

Here \( R = 1 \) AU is the semi-major axis of the guiding center. The constant \( \lambda \) describes the initial orientation of the constellation as viewed by an observer located at the Sun; see Figure 3.2.

To get the above coordinates as a function of time we first note that the ecliptic longitude
Figure 3.2: As viewed by an observer at the origin, $\lambda$ describes the initial orientation of the spacecraft constellation. The angle $\kappa$ is the initial ecliptic longitude of the guiding center.

is related to the eccentric anomaly $\psi$ by

$$\tan \left( \frac{\gamma}{2} \right) = \sqrt{\frac{1 + e}{1 - e}} \tan \left( \frac{\psi}{2} \right),$$

(3.3)

and the eccentric anomaly is related to the orbital phase $\alpha(t) = 2\pi f_m t + \kappa$ via Kepler’s equation,

$$\alpha - \beta = \psi - e \sin(\psi).$$

(3.4)

Assuming a small eccentricity we may solve Equation (3.4) through an iterative process where we treat the $e \sin(\psi)$ term as being lower order than $\psi$,

$$\psi_n = \alpha - \beta + e \sin(\psi_{n-1}).$$

(3.5)

Through such a procedure we arrive at

$$\psi = \alpha - \beta + e \sin(\alpha - \beta) + e^2 \cos(\alpha - \beta) \sin(\alpha - \beta) + \cdots.$$  

(3.6)

Substituting this result into Equation (3.3) and expanding to second order in the eccentricity gives an ecliptic longitude of

$$\gamma = (\alpha - \beta) + 2e \sin(\alpha - \beta) + \frac{5}{2} e^2 \cos(\alpha - \beta) \sin(\alpha - \beta) + \cdots.$$  

(3.7)

Substituting the ecliptic longitude series into Equation (3.1) and keeping terms up to order $e^2$
gives the Cartesian positions of the spacecraft as functions of time,

\[ x_n(t) = R \cos(\alpha) + \frac{1}{2} Re \left( \cos(2\alpha - \beta_n) - 3 \cos(\beta_n) \right) \]
\[ + \frac{1}{8} Re^2 \left( 3 \cos(3\alpha - 2\beta_n) - 10 \cos(\alpha) - 5 \cos(\alpha - 2\beta_n) \right) \] (3.8a)
\[ y_n(t) = R \sin(\alpha) + \frac{1}{2} Re \left( \sin(2\alpha - \beta_n) - 3 \sin(\beta_n) \right) \]
\[ + \frac{1}{8} Re^2 \left( 3 \sin(3\alpha - 2\beta_n) - 10 \sin(\alpha) + 5 \sin(\alpha - 2\beta_n) \right) \] (3.8b)
\[ z_n(t) = -\sqrt{3} Re \cos(\alpha - \beta_n) + \sqrt{3} Re^2 \left( \cos^2(\alpha - \beta_n) + 2 \sin^2(\alpha - \beta_n) \right) . \] (3.8c)

Using the above coordinates the instantaneous separation between spacecraft is found to equal

\[ L_{12}(t) = L \left[ 1 + \frac{e}{32} \left( 15 \sin \left( \alpha - \lambda + \frac{\pi}{6} \right) - \cos \left( 3(\alpha - \lambda) \right) \right) \right] \] (3.9a)
\[ L_{13}(t) = L \left[ 1 - \frac{e}{32} \left( 15 \sin \left( \alpha - \lambda - \frac{\pi}{6} \right) + \cos \left( 3(\alpha - \lambda) \right) \right) \right] \] (3.9b)
\[ L_{23}(t) = L \left[ 1 - \frac{e}{32} \left( 15 \cos(\alpha - \lambda) + \cos \left( 3(\alpha - \lambda) \right) \right) \right] , \] (3.9c)

where \( L = 2\sqrt{3} e R \). From this it is seen that to linear order in the eccentricity the detector arms are rigid. By selecting the mean arm length equal to those of the LISA baseline, \( L = 5 \times 10^9 \) m, the spacecraft orbits are found to have an eccentricity of 0.00965, which indicates that the second order effects are down by a factor of one hundred relative to the leading order.

**Detector Response: Analytical**

The fundamental idea behind detecting gravitational radiation using interferometric techniques is to monitor the proper distance between two test masses in free fall. One possible way to monitor the separation is to send photons between the masses and monitor changes in the optical path length. Mathematically, this problem reduces to evaluating the integral

\[ \ell_{ij}(t) = \int_i^j \sqrt{g_{\mu\nu} dx^\mu dx^\nu} , \] (3.10)

where here it is assumed that the photons travel from mass \( i \) to mass \( j \). The metric is of the
linearized form discussed in Chapter 2. For a spaceborne detector inside the Solar System, the line element representation of the metric is given by

\[
ds^2 = -(1 + 2\phi)dt^2 + (1 - 2\phi)(dx^2 + dy^2 + dz^2) + h_{ij}dx^i dx^j,
\]

(3.11)

where \(\phi\) denotes the net Newtonian potential from all Solar System bodies and \(h_{ij}\) is the time varying metric perturbation due to gravitational waves. Since both \(\phi\) and \(h\) are small compared to the background Minkowski spacetime, the effects associated with each variable can be evaluated separately.

The Newtonian potential leads to a variety of effects, such as a Shapiro time delay \(\Delta L/L \sim M_\odot/R\), gravitational redshift \(\Delta \nu/\nu \sim M_\odot L/R^2\), deflection of light \(\Delta \theta \sim M_\odot L/R^2\), and tidal flexing \(\Delta L/L \sim M_\odot L^2/R^3\). Each of these effects are considerably larger than any of the effects caused by the passage of a gravitational wave, and they must be subtracted before the data analysis begins. The first step in the subtraction relies on the ability to accurately model the orbital phase shifts using the Solar System ephemeris. The second step in the subtraction employs a high pass filter to remove the residuals from the orbital fit, which occur at harmonics of the detector modulation frequency \(f_m = \text{year}^{-1} \simeq 3.2 \times 10^{-8}\) Hz. The orbital effects and the procedure for their removal should be included in the full end-to-end model, even though they do not directly affect the response of the detector to gravitational waves.

To see the effects of a passing gravitational wave on the optical path length, we set \(\phi\) equal to zero in Equation (3.11), from which it follows that\(^1\)

\[
ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + h_{ij}dx^i dx^j,
\]

(3.12)

For definitiveness we will assume that the passing gravitational wave is traveling along the \(z\) axis of a coordinate system yet to be fully defined. From Equations (2.39) and (2.40) the

\(^1\)The following derivation for the optical path length follows closely the derivations given in Hellings [34] and also Cornish & Larson [58].
Figure 3.3: The physical setup for the derivation of the optical path length variations due to the passage of a gravitational wave. The gravitational wave is assumed to travel along the \( z \) axis. Test mass \( i \) is located at the origin while test mass \( j \) is a distance \( L \) away and confined to the \( xz \) plane. The angle \( \psi \) is the principal polarization angle discussed in Chapter 2.

The metric perturbation is given by

\[
\begin{align*}
\mathbf{h}^{TT}(t, \vec{x}) &= A_+ \cos \left( 2\pi f (t - \hat{z} \cdot \vec{x}) + \varphi_0 \right) \cos(2\psi) (dx \otimes dx - dy \otimes dy) \\
&\quad - A_+ \cos \left( 2\pi f (t - \hat{z} \cdot \vec{x}) + \varphi_0 \right) \sin(2\psi) (dx \otimes dy + dy \otimes dx),
\end{align*}
\]  

(3.13)

where, to simplify the mathematics to come, we have set \( A_\times = 0 \). To further simplify the notation we will let \( h(t, \vec{x}) = A_+ \cos(2\pi f (t - \hat{z} \cdot \vec{x}) + \varphi_0) \). Using the above form for the gravitational wave, the line element given Equation (3.12) now becomes

\[
\begin{align*}
\,ds^2 &= -dt^2 + \left( 1 + h \cos(2\psi) \right) dx^2 + \left( 1 - h \cos(2\psi) \right) dy^2 - 2h \sin(2\psi) dx dy.
\end{align*}
\]  

(3.14)

To proceed we will place test mass \( i \) at the origin and mass \( j \) a distance \( L \) away in the \( xz \) plane and at an angle \( \theta \) from the \( z \) axis; see Figure 3.3. By choosing such a coordinate system
the photon trajectories may be parameterized in the following way

\[ x = \rho \sin(\theta) \quad z = \rho \cos(\theta). \]  

(3.15)

With this parameterization the null-like trajectories of the photons \((ds^2 = 0)\) are now expressed by

\[ dt = \left(1 + h \cos(2\psi) \sin^2(\theta)\right)^{1/2} d\rho \]
\[ \approx \left(1 + \frac{1}{2} h \cos(2\psi) \sin^2(\theta)\right) d\rho, \]  

(3.16)

where in the second line we used the fact that \(h\) is a small perturbation to justify the use of the binomial expansion. Integrating both sides of the above equation we find that

\[ \ell(t_j - t_i) = \int_{t_i}^{t_j} dt = L + \frac{1}{2} \cos(2\psi) \sin^2(\theta) \int_0^L h(t, \rho) d\rho. \]  

(3.17)

Here \(t_i\) is the time of photon emission from test mass \(i\) and \(t_j\) is the time of reception at mass \(j\).

Equation (3.17) represents the optical path length traveled by a photon emitted from mass \(i\) and received at mass \(j\) for the specific orientation shown in Figure 3.3. The coordinate free generalization of this expression is given by

\[ \ell_{ij}(t) = \|\vec{x}_j(t_j) - \vec{x}_i(t_i)\| + \frac{1}{2} \left(\hat{r}_{ij}(t) \otimes \hat{r}_{ij}(t)\right) : \int_{\rho}^{\rho} h(\xi(\rho)) d\rho, \]  

(3.18)

where \(\hat{r}_{ij}(t)\) is a unit vector along the photon trajectory and the colon denotes a double contraction, \(a : b = a^{ij}b_{ij}\). The argument of the metric perturbation \(h_{ij}\) is the parameterized wave variable,

\[ \xi(\rho) = t(\rho) - \hat{k} \cdot \vec{x}(\rho). \]  

(3.19)

Explicitly, the time and position depend on the parameterization in the following way,

\[ t(\rho) = t + \rho \quad \vec{x}(\rho) = \vec{x}_i(t) + \rho \hat{r}_{ij}(t). \]  

(3.20)

The first term in Equation (3.18) represents the Cartesian distance between spacecraft.
Therefore, the variation to the optical path length caused by the passage of the gravitational wave is given by

\[ \delta \ell_{ij}(t) = \frac{1}{2} \frac{\hat{r}_{ij}(t) \otimes \hat{r}_{ij}(t)}{1 - \hat{k} \cdot \hat{r}_{ij}(t)} : \int_{\xi_i}^{\xi_j} h(\xi) d\xi , \] (3.21)

where we used

\[ \frac{\partial \xi(\rho)}{\partial \rho} = 1 - \hat{k} \cdot \hat{r}_{ij}(t) \] (3.22)

to make a change of variables in the integration.

Applying Equation (3.21) to a pair of orbiting spacecraft requires the careful evaluation of the \( \hat{r}_{ij}(t) \) unit vectors. The calculation is complicated by the motion of the spacecraft and the finite speed of light. For a photon emitted from spacecraft \( i \) at time \( t_i \) and received at spacecraft \( j \) at time \( t_j \) the proper evaluation of the unit vector is

\[ \hat{r}_{ij}(t_i) = \frac{\vec{x}_j(t_j) - \vec{x}_i(t_i)}{\ell_{ij}(t_i)} . \] (3.23)

The distance the photon travels between spacecraft is given implicitly through the transcendental relationship

\[ \ell_{ij}(t_i) = \| \vec{x}_j(t_i + \ell_{ij}(t_i)) - \vec{x}_i(t_i) \| . \] (3.24)

Here we have used the fact that the reception time is the emission time plus the time of flight for the photon. We can quantitatively estimate the magnitude of the “point ahead” effect by expanding the photon propagation distance in a \( v/c \) series:

\[ \ell_{ij}(t_i) = L_{ij}(t_i)(1 + \hat{r}_{ij}(t_i) \cdot \vec{v}_j(t_i) + O(v^2)) , \] (3.25)

where \( \vec{v}_j(t_i) \) is the velocity of spacecraft \( j \) and

\[ L_{ij}(t_i) = \| \vec{x}_j(t_i) - \vec{x}_i(t_i) \| \] (3.26)

is the instantaneous spacecraft separation. For the LISA mission, with mean arm lengths of \( 5 \times 10^9 \) m and spacecraft velocities \( v \approx 2\pi f_m R \approx 10^{-4} \), pointing ahead gives a first order effect of approximately \( 10^5 \) m. For comparison the orbital effects given in Equation (3.9) impart a
variation in the photon propagation distance of $10^7$ m.

To demonstrate the equivalence of Equation (3.21) to what is found in the literature we first note that an arbitrary gravitational wave can be decomposed into its frequency components,

$$h(\xi) = \int_{-\infty}^{\infty} \tilde{h}(f) e^{2\pi i f \xi} df .$$

(3.27)

Using the decomposition the anti-derivative of the gravitational wave is now written as

$$\int_{\xi_i}^{\xi_j} h(\xi) d\xi = \int_{-\infty}^{\infty} \tilde{h}(f) \left( e^{2\pi i f \xi_j} - e^{2\pi i f \xi_i} \right) df .$$

(3.28)

From here it directly follows that

$$\delta \ell_{ij}(t) = \ell_{ij}(t) \int_{-\infty}^{\infty} D(f, t, \hat{k}) : \tilde{h} e^{2\pi i \xi} df ,$$

(3.29)

where $D(f, t, \hat{k})$ is the one arm detector tensor,

$$D(f, t, \hat{k}) = \frac{1}{2} (\hat{r}_{ij}(t) \otimes \hat{r}_{ij}(t)) T(f, t, \hat{k}) ,$$

(3.30)

and the transfer function is given by

$$T(f, t, \hat{k}) = \text{sinc} \left( \frac{f}{2 f_{ij}^*} \left( 1 - \hat{k} \cdot \hat{r}_{ij}(t) \right) \right) \exp \left( i \frac{f}{2 f_{ij}^*} \left( 1 - \hat{k} \cdot \hat{r}_{ij}(t) \right) \right) .$$

(3.31)

Here $f_{ij}^* = 1/(2\pi \ell_{ij})$ is the transfer frequency for the $ij$ arm. For a monochromatic gravitational wave Equation (3.29) agrees identically with Cornish [59].

The transfer functions arise from the interaction of the gravitational wave with the detector.\(^2\) For gravitational radiation whose frequency is greater than the transfer frequency the wave period is less than the light propagation time between spacecraft. Consequently, the wave has enough time to at least partially expand and contract (see Figure 1.1) in the time it takes a photon to travel between spacecraft. In other words, above the transfer frequency there is a self-cancellation effect which is accounted for by the transfer functions. Below the

\(^2\)In general, the definition of a transfer function is a filter applied in the frequency domain. It is not always association with a physical property of a system.
transfer frequency the transfer functions are approximately constant in value. Therefore, the
transfer frequency naturally divides the LISA band into high and low frequency regions. This
division will be exploited in later sections where approximations to the detector response are
derived.

The connection between the optical path length variations (3.21) and the detector out-
put depends on the interferometer design. An early LISA proposal suggested the use laser
transponders at the end stations to send back a phased locked signal. Under such a proposal
the detector acted like a modified Michelson interferometer with the phase locking mimicking
a mirror reflection. The current proposal is to have each spacecraft measure two phases dif-
f erences, one for each arm. The phase difference $\Phi_{ij}(t_j)$ as measured on spacecraft $j$, is done
by comparing the phase of the received signal from spacecraft $i$ against the outgoing photon’s
phase that is traveling back to spacecraft $i$. This system is much more akin to the Doppler
tracking methods developed in the late 1970’s.

Inherent in the phase difference measurements are both the gravitational wave signal and
noise contributions from laser phase noise $C(t)$, shot noise $n_s(t)$, and acceleration noise $n_a(t)$:

$$\Phi_{ij}(t_j) = C_{ji}(t_i) - C_{ij}(t_j) + 2\pi\nu_0 \left( n^s_{ij}(t_j) - n^a_{ij}(t_j) + n^a_{ji}(t_i) + \delta \ell_{ij}(t_i) \right).$$

Here the time $t_i$ is implicitly found through the transcendental equation $t_i = t_j - \ell_{ij}(t_i)$. The
subscripts on the noise components indicate the directional dependence of the component:
$C_{ij}$ is the laser phase noise introduced by the laser on spacecraft $j$ that is pointed toward
spacecraft $i$, $n^s_{ij}$ is the shot noise in the photo-detector on spacecraft $j$ that is receiving a
signal from spacecraft $i$, and $n^a_{ij}$ is the projected acceleration noise from the accelerometer on
spacecraft $j$ in the direction of spacecraft $i$. The position noise and gravitational path length
variations are converted into a phase difference by multiplying by the angular frequency of the
laser, $2\pi\nu_0$.

Once the six phase differences are measured, the different interferometer signals are syn-
thesized in software. For example, the Michelson signal formed by using spacecraft 1 as the
vertex craft is

$$S_1(t) = \Phi_{12}(t_{21}) + \Phi_{21}(t) - \Phi_{13}(t_{31}) - \Phi_{31}(t),$$

(3.33)
where \( t_{21} \) and \( t_{31} \) are found from

\[
\begin{align*}
t_{21} &= t - \ell_{21}(t_{21}) \quad (3.34a) \\
t_{31} &= t - \ell_{31}(t_{31}). \quad (3.34b)
\end{align*}
\]

However, due to the relatively large laser phase noise, the Michelson signal is not a viable option for LISA. Instead a number of so called time delay interferometer (TDI) signals will be used [60, 61, 62]. These signals are built by combining time delayed Michelson signals in such a way as to reduce the overall laser phase noise down to a tolerable level. A particular example of a TDI variable is the \( X \) signal [63]:

\[
X(t) = \Phi_{12}(t_{21}) + \Phi_{21}(t) - \Phi_{13}(t_{31}) - \Phi_{31}(t) \\
-\Phi_{12}(t'_{21}) - \Phi_{21}(t_{13}) + \Phi_{13}(t'_{31}) + \Phi_{31}(t_{12}), \quad (3.35)
\]

where the new times \( t_{12}, t_{13}, t'_{21}, \) and \( t'_{31} \) are defined through the implicit relationships

\[
\begin{align*}
t_{12} &= t_{21} - \ell_{12}(t_{12}) \quad (3.36a) \\
t_{13} &= t_{31} - \ell_{13}(t_{13}) \quad (3.36b) \\
t'_{21} &= t_{13} - \ell_{21}(t'_{21}) \quad (3.36c) \\
t'_{31} &= t_{12} - \ell_{31}(t'_{31}). \quad (3.36d)
\end{align*}
\]

By permutations of the indices in Equation (3.35) similar forms for the so called \( Y \) and \( Z \) signals are also found.

By writing the response of the detector in a coordinate free manner we are able to apply this formalism to an arbitrary space mission. All that has to be changed are the spacecraft orbits and the scalings of the noise elements. It should also be emphasized that the response is calculated entirely in the time domain. In later sections approximations to the full response are developed by working in a hybrid time/frequency domain. This hybrid approach assumes extra information about the sources, which allows explicit expressions for the detector response to be given.
Formalism

As an application of the equations presented in the previous section, we have simulated the response of the proposed LISA mission. *The LISA Simulator* [64, 65] is designed to take an arbitrary input gravitational wave and output the full response of the detector. To apply the equations we have elected to work entirely in the before mentioned heliocentric-ecliptic coordinate system. As a result all times are evaluated in terms of Solar System barycentric time. The conversion to the detector time is through the standard relationship

\[ d\tau = \sqrt{1 - v^2(t)} \, dt, \]

but since we only work to leading order in \( v \) the distinction is not made. (In practice there will be difficulties in synchronizing the clocks on the spacecraft [66], but they do not trouble the simulations.)

The positions of the spacecraft are calculated to second order in the eccentricity via Equation (3.8), which includes the leading order flexing motion of the array. Solar System tidal effects, and third order terms in the eccentricity are neglected for now.

Applications

In the LISA band there are two main sources of gravitational radiation. The first are binaries located either within the Milky Way galaxy or nearby globular clusters. These sources have component masses on the order of a solar mass and distances measured in kiloparsecs. The other major source for LISA are the coalescence of supermassive black holes at cosmological distances.\(^3\)

One of the guaranteed sources for the LISA mission is the cataclysmic variable AM Canum Venaticorum. This galactic binary star system is comprised of a low mass helium white dwarf that is transferring material to a more massive white dwarf by way of Roche lobe overflow. AM CVn’s orbital frequency of 0.972 mHz, and close proximity to the Earth (\( \sim 100 \) pc) make it a good calibration binary for LISA. Shown in Figure 3.4 is the simulated response of LISA to AM CVn, expressed as a spectral amplitude \( h_f(f) = \sqrt{S_h}. \) (For the simula-

\(^3\)Recent research has focused on a possible third class of gravitational wave sources for LISA: extreme mass ratio inspirals. Here a compact object falls into a supermassive black hole that is located at the center of a galaxy.
Figure 3.4: The simulated X signal response of AM CVn, demonstrating the induced modulations caused by LISA’s orbital motion. For reference, the red line is the average intrinsic detector noise in this region of the spectrum.

Another LISA source is the merger of two supermassive black holes. There is mounting evidence that at a supermassive black hole resides in the nuclei of every galaxy. Additionally, observations out to large red shifts suggests that galaxy collisions may be a common scenario in the evolution of the Universe. If dynamical friction mechanism are able to bring the central black holes close enough together, then radiation reaction effects will drive them to coalescence. Shown in Figure 3.5 is the simulated response of LISA to two $10^6 M_\odot$ black holes coalescing at a redshift of $z = 1$ (luminosity distance $D_L = 6.63$ Gpc) with a random location and orientation on the sky. For this example the time to coalescence was chose such that the observation tracks the final year to merger.
Figure 3.5: The Michelson response of LISA to two $10^6 \, M_\odot$ black holes coalescing at $z = 1$ ($D_L = 6.63 \, \text{Gpc}$). The red line is the average Michelson noise for LISA.

Intrinsic Noise

Laser phase noise, photon shot noise, and acceleration noise are the expected dominant forms of noise in spaceborne detectors. As previously discussed, time delay interferometry is used to reduce the effects of the laser phase noise to a tolerable level. *The LISA Simulator* assumes that the TDI signal processing is properly implemented and, therefore, neglects laser phase noise in its simulations.

The noise simulation is done in the time domain by drawing random deviates at each time step from a Gaussian distribution with unit variance and zero mean. For the white photon shot noise we scale the random number by the spectral density tolerance limit defined in the LISA Pre-Phase A Report [10] ($S_{ps} = 1.0 \times 10^{-22} \, \text{m}^2/\text{Hz}$). For the colored acceleration noise we begin by generating a white noise time series scaled by the accelerometer’s tolerance limit ($S_{acc} = 9.0 \times 10^{-30} \, \text{m}^2/\text{s}^4/\text{Hz}$), and integrate twice to arrive a colored time series for the acceleration noise. The integration introduces the $f^{-4}$ falloff in the power spectrum that is characteristic of acceleration noise. The result of this procedure for the Michelson signal (3.33) is shown in Figure 3.6.
Figure 3.6: A realization of the Michelson noise for the LISA mission. The $\sqrt{f^{-4}}$ falloff observed at low frequencies is from the acceleration noise. At higher frequencies the white colored photon shot noise dictates the noise levels. The red line is a 128 bin running average of the noise. The green, V-shaped line is the standard sensitivity curve.

Comparing the noise realization to a standard LISA sensitivity curve [67], a number of differences are apparent. The most obvious is the lack of rise in the noise for high frequencies. This is because the standard sensitivity curve folds the average detector response into the noise curve. The plot from the Sensitivity Curve Generator includes polarization and all sky averaged transfer functions, which equals $3/5$ at low frequencies and grows as $f^2$ (on a power spectrum) above the transfer frequency. A second difference is in the overall normalization. The Sensitivity Curve Generator scales the path length variations by the interferometer mean arm length $L$, while The LISA Simulator scales the path length variations by the optical path length $2L$. This difference in scaling accounts for a portion of the separation in the plots.

To arrive at a simulation of the $X$ noise, the noise elements are combined as dictated by Equation (3.35). Doing so gives the results displayed in Figure 3.7, which agrees with the predicted results. To see this, we start with the analytical expression of the average Michelson
noise curve shown in Figure 3.6,

\[ h_f^M(f) = \frac{1}{2L} \sqrt{4S_{ps} + 8 \left( 1 + \cos^2 \left( \frac{f}{f_*} \right) \right) \frac{S_{acc}}{(2\pi f)^4}}, \]  

(3.37)

which is derived in the appendix of Cornish [59]. In the above \( f_* = 1/(2\pi L) \) is the mean transfer frequency for an arm. Next, we note that the \( X \) signal is formed by differencing two Michelson signals, one time delayed by roughly twice the light travel time between spacecraft. Accordingly, the noise will enter in the \( X \) signal as

\[ n_X(t) = n_M(t) - n_M(t - 2L), \]  

(3.38)

where the \( X \) and \( M \) subscripts refer to the \( X \) and Michelson signals respectively. Performing a Fourier transform of \( n_X(t) \) gives

\[ \tilde{n}_X(f) = \tilde{n}_M(f) \left( 1 - e^{-2if/f_*} \right), \]  

(3.39)
The spectral amplitude of the $X$ noise is then given by

$$h_{X}^{X}(f) = \sqrt{S_{X}(f)} = 2 \left| \sin \left( \frac{f}{f_{*}} \right) \right| h_{M}^{M}(f).$$  

(3.41)

Shown in Figure 3.8 is a plot of $h_{X}^{X}(f)$ along with the running average X noise from Figure 3.7. Although the above derivation of the $X$ noise spectral amplitude assumed constant arm lengths, we see that there is excellent agreement between the predicted results of Equation (3.41) and the simulation, which included the variations in the arms.

Although Equations (3.21), (3.32), and (3.35) describe the full response of a spaceborne detector, they are analytically difficult to handle and time consuming to evaluate. For this rea-
son we will now explore three approximations to the full response that uses extra information about the input waveforms and a simplified description of the detector. These approximations not only aid in the development of data analysis techniques, but give greater insight into the workings of the detector.

**Static Limit**

**Formalism**

As a point of reference, we can apply our general, coordinate-free methodology to a static, equal arm detector interacting with a monochromatic, plane-fronted gravitational wave. From Chapter 2, Equation (2.35), the gravitational wave can be expressed as

\[ h(f, \xi) = A_+ \cos(2\pi f(t - \hat{k} \cdot \vec{x}) + \varphi_0) \epsilon^+ + A_\times \sin(2\pi f(t - \hat{k} \cdot \vec{x}) + \varphi_0) \epsilon^\times, \quad (3.42) \]

where \( \hat{k} \) is the propagation direction, \( f \) is the gravitational wave frequency, and \( \epsilon^+ \) and \( \epsilon^\times \) are the basis tensors in the gravitational wave frame. To get to the detector frame we must rotate the basis tensors by an amount \( \psi \) (the principle polarization angle) about the \( \hat{k} \) direction. The rotation gives

\[
\begin{align*}
\epsilon^+ &= \cos(2\psi)\epsilon^+ - \sin(2\psi)\epsilon^\times, \\
\epsilon^\times &= \sin(2\psi)\epsilon^+ + \cos(2\psi)\epsilon^\times. 
\end{align*}
\]

(3.43a) (3.43b)

The gravitational wave in the detector basis is now expressed by

\[ h(\xi) = h_+(\xi) \epsilon^+ + h_\times(\xi) \epsilon^\times, \quad (3.44) \]

where \( \xi = t - \hat{k} \cdot \vec{x} \) is the usual wave variable and

\[
\begin{align*}
h_+(\xi) &= A_+ \cos(2\pi f\xi + \varphi_0) \cos(2\psi) + A_\times \sin(2\pi f\xi + \varphi_0) \sin(2\psi), \\
h_\times(\xi) &= -A_+ \cos(2\pi f\xi + \varphi_0) \sin(2\psi) + A_\times \sin(2\pi f\xi + \varphi_0) \cos(2\psi). 
\end{align*}
\]

(3.45a) (3.45b)
In the static limit the arms are rigid with a set length $L$. For a noiseless detector, the $X$ signal given in Equation (3.35) becomes

$$X(t) = 2\pi \nu_0 \left( \delta \ell_{12}(t - 2L) - \delta \ell_{12}(t - 4L) + \delta \ell_{21}(t - L) - \delta \ell_{21}(t - 3L) - \delta \ell_{13}(t - 2L) + \delta \ell_{13}(t - 4L) - \delta \ell_{31}(t - L) + \delta \ell_{31}(t - 3L) \right). \quad (3.46)$$

In terms of a $X$ signal strain, $x(t) = X(t)/(2\pi \nu_0 L)$, the above is equivalent to (after some lengthy algebra)

$$x(t) = \sin^2 \left( \frac{f}{f_\ast} \right) D(f, \hat{k}) : h(f, \xi) \quad (3.47)$$

where $f_\ast = 1/(2\pi L)$ is the static transfer frequency and

$$D(f, \hat{k}) = \frac{1}{2} \left( (\hat{r}_{12} \otimes \hat{r}_{12}) T(f, \hat{k} \cdot \hat{r}_{12}) - (\hat{r}_{13} \otimes \hat{r}_{13}) T(f, \hat{k} \cdot \hat{r}_{13}) \right). \quad (3.48)$$

Here $T(f, \hat{k} \cdot \hat{r}_{ij})$ is the round trip transfer function given by

$$T(f, \hat{k} \cdot \hat{r}_{ij}) = \frac{1}{2} \left( \text{sinc} \left( \frac{f}{2f_\ast} \left( 1 - \hat{k} \cdot \hat{r}_{ij} \right) \right) \exp \left( -i \frac{f}{2f_\ast} \left( 3 + \hat{k} \cdot \hat{r}_{ij} \right) \right) \right. $$

$$+ \left. \text{sinc} \left( \frac{f}{2f_\ast} \left( 1 + \hat{k} \cdot \hat{r}_{ij} \right) \right) \exp \left( -i \frac{f}{2f_\ast} \left( 1 + \hat{k} \cdot \hat{r}_{ij} \right) \right) \right). \quad (3.49)$$

The above expression for the round trip transfer function agrees identically to the coordinate free expression given in Equation (5) of Cornish & Larson [58]. Additionally, for the assumed orientation taken in Schilling [68] the above transfer function reduces to his Equation (4) (with the proper normalization taken into account).

In the static limit the $X$ signal is related to the Michelson signal from vertex 1 by the relationship

$$s_1(t) = \sin^{-2} \left( \frac{f}{f_\ast} \right) x(t) = D(f, \hat{k}) : h(f, \xi), \quad (3.50)$$

which agrees with Equation (6) of Reference [58].

**Applications**

In the limit where the wavelength of the gravitational wave is much larger than the arm
lengths of the detector ($\lambda \gg L$), the round trip transfer functions are approximately unity. Consequently, the Michelson response given in Equation (3.50) is further simplified to the form

$$s_1(t) = A_+ F^+ (\theta, \phi, \psi) \cos \left( 2\pi f (t - \hat{k} \cdot \hat{x}) + \varphi_0 \right) + A_\times F^\times (\theta, \phi, \psi) \sin \left( 2\pi f (t - \hat{k} \cdot \hat{x}) + \varphi_0 \right), \quad (3.51)$$

where the antenna beam pattern factors are given by

$$F^+ (\theta, \phi, \psi) = \frac{1}{2} (\hat{r}_{12} \otimes \hat{r}_{12} - \hat{r}_{13} \otimes \hat{r}_{13}) : \epsilon^+ \quad (3.52a)$$

$$F^\times (\theta, \phi, \psi) = \frac{1}{2} (\hat{r}_{12} \otimes \hat{r}_{12} - \hat{r}_{13} \otimes \hat{r}_{13}) : \epsilon^\times. \quad (3.52b)$$

This is the case that applies to most detections by LIGO (the exception being periodic sources, where the modulation effects due to the Earth’s motion must be included). In a similar way we can investigate the long wavelength, static limit for the LISA mission.

For definitiveness we will orient the detector as shown in Figure 3.9. The unit vectors are then given by

$$\hat{r}_{12} = \cos \left( \frac{\pi}{12} \right) \hat{x} + \sin \left( \frac{\pi}{12} \right) \hat{y} \quad (3.53a)$$

$$\hat{r}_{13} = \cos \left( \frac{5\pi}{12} \right) \hat{x} + \sin \left( \frac{5\pi}{12} \right) \hat{y}. \quad (3.53b)$$
Substitution of these basis vectors into Equation (3.52) yields

\[
\langle \hat{r}_{12} \otimes \hat{r}_{12} \rangle : e^+ = \frac{1}{2}\left( (1 + \cos^2(\theta)) \sin(2\phi + \pi/3) - \sin^2(\theta) \right)
\]  \hspace{1cm} (3.54a)
\[
\langle \hat{r}_{12} \otimes \hat{r}_{12} \rangle : e^x = \cos(\theta) \sin(2\phi - \pi/6)
\]  \hspace{1cm} (3.54b)
\[
\langle \hat{r}_{13} \otimes \hat{r}_{13} \rangle : e^+ = \frac{1}{2}\left( (1 + \cos^2(\theta)) \sin(2\phi - \pi/3) - \sin^2(\theta) \right)
\]  \hspace{1cm} (3.54c)
\[
\langle \hat{r}_{13} \otimes \hat{r}_{13} \rangle : e^x = -\cos(\theta) \sin(2\phi + \pi/6)
\]  \hspace{1cm} (3.54d)

Using these results the antenna beam pattern factors are found to have their familiar form for a detector with an opening angle of 60° [69],

\[
F^+ = \frac{\sqrt{3}}{2} \left( \frac{1}{2} \left( 1 + \cos^2(\theta) \right) \cos(2\phi) \cos(2\psi) - \cos(\theta) \sin(2\phi) \sin(2\psi) \right)
\]  \hspace{1cm} (3.55a)
\[
F^x = \frac{\sqrt{3}}{2} \left( \frac{1}{2} \left( 1 + \cos^2(\theta) \right) \cos(2\phi) \sin(2\psi) + \cos(\theta) \sin(2\phi) \cos(2\psi) \right)
\]  \hspace{1cm} (3.55b)

In our calculations the overall factor of \( \sqrt{3}/2 \), which is related to the opening angle, is a natural result of the mathematics.

**Low Frequency Limit**

**Formalism**

The application of the long wavelength, static limit to the LISA mission is little more than pedagogical exercise. The motion of the detector is evident even at low frequencies and only gets more prevalent toward higher frequencies. However, as seen in previous sections, the full response of a spaceborne detector is complicated by the intrinsic arm length fluctuations, pointing ahead, and the signal cancellation accounted for in the transfer functions. As a second approximation to the response of LISA we will neglect all of these effects. That is, we will work to linear order in the spacecraft positions, evaluate all spacecraft locations at a common time, and set the transfer functions to unity. It should be noted that this approximation was originally worked out by Cutler [69] and can be viewed as an extension of the LIGO response to spaceborne detectors with motion. The transfer function \( T(f, \hat{k} \cdot \hat{r}_{ij}) \) can only be set to
unity for frequencies $f \ll f_*$. For the LISA mission, whose bandwidth is $10^{-5}$ to 1 Hz, the
transfer frequency has a mean value of $f_* = 0.00954$ Hz $\approx 10^{-2}$ Hz. Consequently, we only
expect the low frequency limit to hold up to 10 mHz.

In the limits of $f \ll f_*$ and $f/f_* \ll L$ the variation in the optical path length due to a
passing gravitational wave (3.21) reduces to\footnote{The condition that $f/f_* \ll L$ simply requires that the gravitational wave frequency remain roughly constant
over the light travel time between spacecraft. For all but the most relativistic sources, this condition is trivially
satisfied.}

$$
\delta \ell_{ij}(t) \approx \frac{1}{2} \frac{\hat{r}_{ij}(t) \otimes \hat{r}_{ij}(t)}{1 - \hat{k} \cdot \hat{r}_{ij}(t)} : \mathbf{h}(\xi(t))(\xi_j - \xi_i),$$

$$
= L \left( \hat{r}_{ij}(t) \otimes \hat{r}_{ij}(t) : \mathbf{h}(\xi(t)) \right). \tag{3.56}
$$

Working in terms of strains and neglecting noise, the Michelson signal from spacecraft 1 is
given by

$$
s_1(t) = \frac{1}{2L} \left( \delta \ell_{12}(t - 2L) + \delta \ell_{21}(t - L) - \delta \ell_{13}(t - 2L) + \delta \ell_{31}(t - L) \right)

\approx \frac{1}{2L} \left( \delta \ell_{12}(t) + \delta \ell_{21}(t) - \delta \ell_{13}(t) + \delta \ell_{31}(t) \right). \tag{3.57}
$$

The last line follows from the low frequency condition $f \ll f_*$. In an analogous fashion to the static limit, the strain can be re-expressed as

$$
s_1(t) = h_+(\xi(t)) F^+ + h_\times(\xi(t)) F^\times \tag{3.58}
$$

where

$$
\xi(t) = t - \hat{k} \cdot \vec{x}_1(t)

= t + R \sin \theta \cos (\alpha(t) - \phi), \tag{3.59}
$$

is the wave variable at the guiding center. The antenna beam pattern factors, $F^+(t)$ and
\( F^\times(t) \), are now given by

\[
F^+(t) = \frac{1}{2} \left( \cos(2\psi)D^+(t) - \sin(2\psi)D^\times(t) \right),
\]

\[
F^\times(t) = \frac{1}{2} \left( \sin(2\psi)D^+(t) + \cos(2\psi)D^\times(t) \right),
\]

where the projection functions are given by

\[
D^+(t) = \left( \hat{r}_{12}(t) \otimes \hat{r}_{12}(t) - \hat{r}_{13}(t) \otimes \hat{r}_{13}(t) \right) : e^+
\]

\[
D^\times(t) = \left( \hat{r}_{12}(t) \otimes \hat{r}_{12}(t) - \hat{r}_{13}(t) \otimes \hat{r}_{13}(t) \right) : e^\times.
\]

Working to linear order in the eccentricity, the Keplerian orbits given in (3.8) yield

\[
D^+(t) = \sqrt{3} \cdot \frac{1}{64} \left[ -36 \sin^2(\theta) \sin(2\alpha(t) - 2\lambda) + (3 + \cos(2\theta)) \left( \cos(2\phi) \left( 9 \sin(2\lambda) \\
- \sin(4\alpha(t) - 2\lambda) \right) + \sin(2\phi) \left( \cos(4\alpha(t) - 2\lambda) - 9 \cos(2\lambda) \right) \right) \\
-4\sqrt{3} \sin(2\theta) \left( \sin(3\alpha(t) - 2\lambda - \phi) - 3 \sin(\alpha(t) - 2\lambda + \phi) \right) \right];
\]

\[
D^\times(t) = \frac{1}{16} \left[ \sqrt{3} \cos(\theta) \left( 9 \cos(2\lambda - 2\phi) - \cos(4\alpha(t) - 2\lambda - 2\phi) \right) \\
-6 \sin(\theta) \left( \cos(3\alpha(t) - 2\lambda - \phi) + 3 \cos(\alpha(t) - 2\lambda + \phi) \right) \right].
\]

Equations (3.58) to (3.62b) constitute the analytical formalism for the Low Frequency Approximation. These equations are numerically quick to evaluate and can be handled analytically.

To test the range of validity for this approximation we used The LISA Simulator (TLS) as a template to calculate the correlation between the full response and the Low Frequency Approximation (LFA),

\[
r(f) = \frac{\langle s_{TLS}|s_{LFA} \rangle}{\sqrt{\langle s^2_{TLS}\rangle \langle s^2_{LFA}\rangle}}.
\]
The angle brackets denote an inner product defined as

$$\langle a(f)|b(f)\rangle = \int_{-\infty}^{\infty} (a(f)b^*(f) + a^*(f)b(f)) \, df ,$$

(3.64)

where the asterisk denotes complex conjugation. Using fixed random choices for the source location and orientation, we systematically varied the gravitational wave frequency and calculated the correlation at each frequency. The results of such a calculation are shown in Figure 3.10.

We found that the *Low Frequency Approximation* has a strong correlation to the true response for frequencies below 3 mHz, at which point the correlation drops to 95%. The steep turn down in the correlation prior to the transfer frequency is to be expected since the *Low Frequency Approximation* neglects the self-cancellation effects encoded in the transfer functions. The oscillations at higher frequencies are due to the transfer functions present in full response template $s_{\text{TLS}}$. The precise structure of these oscillations depends on the source
location through the $\hat{k} \cdot \hat{r}_{ij}(t)$ dependence in the transfer functions. However, the turn down at 3 mHz is location independent. The location dependence is not evident until the correlation value has already dropped to nearly zero.

The significance of a particular correlation value is dependent upon the signal-to-noise ratio of the source. For high signal-to-noise the effects neglected in the approximation will be detectable. Conversely, for a low signal-to-noise one may continue to use the approximation at higher frequencies as the difference would not be detectable.

Applications

As previously discussed the motion of the detector introduces amplitude, frequency, phase modulations into the gravitational wave signal. Using the formalism of the Low Frequency Approximation the physical origins of these modulations are easily understood.

To see where the modulations originate, we begin by rewriting the strain given in Equation (3.58) as

$$s_1(t) = A_+ F^+(t) \cos \left( 2\pi f(t - \hat{k} \cdot \vec{x}_1) + \varphi_0 \right)$$

$$+ A_\times F^\times(t) \sin \left( 2\pi f(t - \hat{k} \cdot \vec{x}_1) + \varphi_0 \right),$$

(3.65)

Using the trigonometric relationship

$$A \cos(\xi) + B \sin(\xi) = \sqrt{A^2 + B^2} \cos \left( \xi + \tan^{-1}(B/A) \right)$$

(3.66)

the strain is re-expressed as

$$s_1(t) = A(t) \cos \left( 2\pi ft + \phi_D(t) + \phi_P(t) + \varphi_0 \right),$$

(3.67)
where

\[ A(t) = \sqrt{(A_+ F^+(t))^2 + (A_\times F^\times(t))^2} \] (3.68a)

\[ \phi_D(t) = -2\pi f \hat{k} \cdot \bar{x}_1(t) = 2\pi f R \sin(\theta) \cos(\alpha(t) - \phi) \] (3.68b)

\[ \phi_P(t) = -\tan^{-1}\left( \frac{A_\times F^\times(t)}{A_+ F^+(t)} \right). \] (3.68c)

Using these expressions it is possible to immediately read off the origin of each modulation. The amplitude modulation, \( A(t) \), is caused by the antenna pattern being swept across the sky during the course of an orbit. The phase modulation, \( \phi_P(t) \), arises when the differing responses to the two gravitational wave polarizations are combined together to form the signal. The frequency (Doppler) modulation, \( \phi_D(t) \), is due to the motion of the detector relative to the source. It is also viewed as the difference in the gravitational wave phase at the Solar System barycenter and the detector. Since both the orbital and cartwheel motion have a period of one year, these modulations will show up as sidebands in the power spectrum separated from the instantaneous carrier frequency by integer values of the modulation frequency, \( f_m = 1/\text{yr} \).

**Rigid Adiabatic Limit**

**Formalism**

The failure of the *Low Frequency Approximation* at high frequencies is due to neglecting the transfer functions. As a third approximation to the LISA response we will now include transfer functions, but continue to hold the detector rigid by working to leading order in the spacecraft positions and evaluating all spacecraft locations at the same instant of time. Such an approximation has been worked out before for the case of a stationary detector by Cornish & Larson [58] and Cornish [59], but here we extend it to include the motion of the detector.

Physically this approximation can be viewed in the following way. At an instant of time we hold the detector fixed and send photons up and back along the interferometer arms and calculate the phase differences. We then increment the time by a small amount, moving the rigid detector to its new position in space, and repeat the process. The sequence of stationary
states is the origin of the term “adiabatic” for describing the approximation.

In the limit of a slowly changing gravitational wave frequency \((f / \dot{f} \ll L)\) the optical path length variations (3.21) reduces to

\[
\delta \ell_{ij}(\xi) = L \sum_{n} D(f_n, t, \hat{k}) : h_n(\xi),
\] (3.69)

where the one arm detector tensor is given by

\[
D(f, t, \hat{k}) = \frac{1}{2} \left( \hat{r}_{ij}(t) \otimes \hat{r}_{ij}(t) \right) T(f, t, \hat{k}),
\] (3.70)

and the associated transfer function is

\[
T(f, t, \hat{k}) = \text{sinc} \left( \frac{f}{2f_*} \left( 1 - \hat{k} \cdot \hat{r}_{ij}(t) \right) \right) \exp \left( i \frac{f}{2f_*} \left( 1 - \hat{k} \cdot \hat{r}_{ij}(t) \right) \right).
\] (3.71)

The summation in Equation (3.69) allows for sources in which the emitted gravitational waves are produced in harmonics of some fundamental frequency. For example, both eccentric [70] and post-Newtonian binaries [71] produce radiation in harmonics of the orbital frequency.

The Michelson signal is given by the appropriate combinations of the optical path length variation (3.69),

\[
s_1(t) = \frac{1}{2L} \left( \delta \ell_{12}(t - 2L) + \delta \ell_{21}(t - L) - \delta \ell_{13}(t - 2L) + \delta \ell_{31}(t - L) \right),
\] (3.72)

which may now be expressed in the compact form

\[
s_1(t) = \sum_{n} D(f_n, t, \hat{k}) : h_n(\xi),
\] (3.73)

where the round trip detector tensor takes the form

\[
D(f, t, \hat{k}) = \frac{1}{2} \left( (\hat{a} \otimes \hat{a}) T(f, t, \hat{k} \cdot \hat{a}) - (\hat{b} \otimes \hat{b}) T(f, t, \hat{k} \cdot \hat{b}) \right),
\] (3.74)
and the round trip transfer function is

\[ T(f, t, \hat{k} \cdot \hat{a}) = \frac{1}{2} \left[ \text{sinc} \left( \frac{f}{2f_s} (1 - \hat{k} \cdot \hat{a}) \right) \exp \left( -i \frac{f}{2f_s} (3 + \hat{k} \cdot \hat{a}) \right) 
+ \text{sinc} \left( \frac{f}{2f_s} (1 + \hat{k} \cdot \hat{a}) \right) \exp \left( -i \frac{f}{2f_s} (1 + \hat{k} \cdot \hat{a}) \right) \right]. \] (3.75)

This is the same result as the static limit, but now the unit vectors are a function of time,

\[ \hat{a}(t) = \frac{\vec{x}_2(t) - \vec{x}_1(t)}{L}, \]
\[ \hat{b}(t) = \frac{\vec{x}_3(t) - \vec{x}_1(t)}{L}. \] (3.76)

Collectively Equations (3.73) - (3.76) are the analytical formalism for the Rigid Adiabatic Approximation. As with the Low Frequency Approximation, the expressions are computationally quick to evaluate and can be easily manipulated analytically.

Figure 3.11 shows the correlation between the full response and the Rigid Adiabatic Approximation for a monochromatic gravitational wave. Note that by including the transfer functions we are able to extend agreement with the full response two decades in frequency beyond where the Low Frequency Approximation broke down. The turn down at \( \sim 500 \) mHz comes about through neglecting the second order terms in the spacecraft positions. As described earlier, the second order orbital effects are down by two orders of magnitude in comparison to the linear order. This is evident in the Rigid Adiabatic Approximation through the transfer frequencies, which are evaluated for a rigid detector. Normally the transfer frequencies are given by

\[ f_{ij}^s(t) = \frac{1}{2\pi \ell_{ij}(t)}, \] (3.77)

but for a rigid detector this reduces to the static form \( f_s = 1/(2\pi L) \). The extension to higher orders in the orbital eccentricity can be done. The trade off is that the expressions become more complicated since the transfer frequencies would then become functions of time. In turn, this would require that each transfer frequency be evaluated along each arm during each time step rather than using one constant value throughout the entire calculation. Additionally, the normalization of the unit vectors in Equation (3.76) would need to be evaluated at each
Figure 3.11: The correlation between the *Rigid Adiabatic Approximation* and the full response of the LISA detector for monochromatic source. The turn down at $\sim$500 mHz is due to neglecting the higher order effects in the spacecraft positions.

...step since the arm lengths would vary as a function of time via Equation (3.9). Such an approach would be appropriately called the *Flexing Adiabatic Approximation* since the arm lengths would now oscillate in time about a mean value of $L$. Although the expressions would become analytically complicated, the numerical evaluation would not be significantly slower since the additional steps are straightforward to evaluate.

Applications

Utilizing the computational speed of the *Rigid Adiabatic Approximation* we may investigate various data analysis questions. Here we provide one concrete example by determining when phase evolution of a binary system due to radiation reaction needs to be included in the source modeling.

For our calculations we use the post-Newtonian approximation from Blanchet *et al.* [71], whereby the gravitational wave amplitude is calculated to zeroth order in $(M/R)$, while the phase evolution is calculated to second order. The justification for this is that LISA will be far more sensitive to the gravitational wave phase than to the amplitude [69]. It is also ad-
vantageous not to include the higher order amplitude terms since they introduce harmonics of the orbital frequency, and each harmonic would require a separate transfer function according to Equation (3.73).

Even though we include the evolution of the gravitational wave frequency we can still approximate the wave as being monochromatic over the period of one light travel time between spacecraft. This allows us to trivially evaluate the integral in Equation (3.29) and arrive at Equation (3.73) as we did for the monochromatic source. This is only permissible for sources that are accurately described by the post-Newtonian expansion. For highly relativistic sources, whose frequency evolution occurs at a much higher rate, we would be forced to carefully evaluate the integral at each step. Quantitatively, this requirement is the same $f \ll \dot{f}$ assumed for the Low Frequency Approximation.

To quantify the importance of including the evolution of the gravitational wave phase, we calculated the correlation between a monochromatic Rigid Adiabatic Approximation to one in which the phase evolution is included. Figure 3.12 shows the correlation for three types of binaries expected to reside inside our own galaxy: a close white dwarf binary with mass components $0.5 \, M_\odot$, a neutron star - neutron star binary with individual masses of $1.4 \, M_\odot$, and a $10 \, M_\odot$ black hole with a neutron star companion.

We found that the frequency at which the monochromatic signal diverges from an adiabatic approximation that includes phase evolution, depends on the masses of the binary components. The reason for this comes from the expression for $\dot{f}$, which contains a mass dependent coefficient. For the stellar mass binaries we studied, the drop in the correlation happens to coincide with the breakdown of the Low Frequency Approximation. The majority of Milky Way sources have a frequency below $3 \, \text{mHz}$, and thus can be modeled as a monochromatic binary using the Low Frequency Approximation. However, a considerable number of the resolvable sources have frequencies above $3 \, \text{mHz}$. As a result, must be modeled using either the full detector response or the Rigid Adiabatic Approximation (see Figure 5.9 and accompanying text).

Another way to represent the same data is to set the independent variable equal to the change in the frequency, scaled by a bin width, $\delta f = (f_f - f_i)/\Delta f$, where for one year of observation the bin width is $\Delta f = 1/\text{yr} \approx 3.2 \times 10^{-8} \, \text{Hz}$. Figure 3.13 shows the correlation
Figure 3.12: The correlation between a monochromatic Rigid Adiabatic Approximation and one that includes a 2PN phase evolution. The red curve is for a WD-WD binary with mass components of 0.5 $M_\odot$, the green curve is for a NS-NS binary with masses 1.4 $M_\odot$, and the blue curve is for a 10 $M_\odot$ black hole and a 1.4 $M_\odot$ neutron star binary.

as a function of the scaled change in frequency $\delta f$. Unlike with the standard correlation representation of a monochromatic and coalescing signal, the results of this calculation are independent of the system’s masses. It is also interesting to note that these results imply that it will be possible to detect a change in frequency within a bin width. This result is not in conflict with the Nyquist theorem, which states that the frequency resolution is limited by the inverse of the observation time. However, Nyquist’s theorem does not make any assumptions on the functional form of the signal, which is known in our case. As a result, it is possible to measure the gravitational wave frequency to better than a bin width in resolution.

**Concluding Remarks**

This chapter examined forward modeling of spaceborne gravitational wave detectors with a special emphasis on LISA. Forward modeling will play two distinct roles in the development of spaceborne observatories. The first is as part of a complete end-to-end model that takes
Figure 3.13: The correlation as a function of the fractional bin width change in the frequency, \( \delta f = (f_f - f_i)/\Delta f \), between a monochromatic Rigid Adiabatic Approximation one that includes a 2PN phase evolution. Note that all three types of sources previously considered are included in this plot.

into account every conceivable physical effect. The second role is as an intermediary between source simulation and the development of data analysis techniques. Here we focused on the latter role, and to that end we have studied two approximations to the full response, the Low Frequency Approximation and the Rigid Adiabatic Approximation.

We found that the Rigid Adiabatic Approximation could be used in place of the full response for a wide range of data analysis projects. For example, the relatively simple analytical form of the Rigid Adiabatic Approximation is well suited to the calculation of Fisher information matrices in studies of astrophysical parameter extraction. On the other hand, The LISA Simulator is available if we need to simulate the response for a highly relativistic gravitational wave source, such as the final stages in the coalescence of two black holes.
CHAPTER 4

SIGNAL IDENTIFICATION AND SUBTRACTION

INTRODUCTION

Unlike most electromagnetic telescopes, gravitational wave observatories do not return an image of an individual source or a region of space. Instead they return a collection of time series. Encoded in these time series are the signals from all sources whose radiation passes through the detector. While elementary data analysis techniques, such as power spectrums and correlation calculations, aid in the identification of possible signals, they are not able to estimate the individual parameter values that uniquely describe the emitting bodies. To turn a gravitational wave detector into an astronomical observatory requires the development of advance data analysis techniques that are able to identify and subtract out individual signals from the data streams.

Inside the Milky Way galaxy there are $\sim 10^{10}$ binaries whose emitted gravitational waves have a frequency inside the LISA band [72]. For most of these sources the measured signals will be buried in the intrinsic noise of the detector. However, there will still be thousands of sources whose signals are strong enough to swamp the response of the detector at low frequencies ($f \lesssim 3$ mHz). The superposition of the stronger sources will form a confusion limited background which will act as a form of noise in the detector output.

Lying above the confusion background will be a number of sources whose signals are larger than the local (in the frequency domain) rms value of the background. Due to the relative brightness of these sources, they will be identifiable in the LISA data streams. Additionally, at higher frequencies, where the confusion background drops below the detector noise, individual sources will also be identified. For each of the identified galactic binaries the values of the orbital frequencies and chirp masses indicate that radiation reaction effects will not drive the binaries to coalescence during the lifetime of the LISA mission. In turn, their signals will be
ever present in the output of the detector. It is these galactic binaries that are prime targets for data analysis techniques used for identification and subtraction.

The ability to remove galactic binaries in the region of the LISA noise floor (between 3 and 10 mHz) is crucial. Galactic binary signals will interfere with the ability of data analysis techniques to identify other gravitational wave sources, such as those from the inspiral of compact objects into supermassive black holes [73, 74].

This chapter describes two approaches to signal identification and subtraction. The first method described is linear least squares fitting. In this approach an initial guess for the source parameters is used to identify the true values by minimizing the difference in the detector response and a preliminary model for the source. The second method discussed is that of template match fitting. Here the approach is to evaluate the cross-correlation between a template and the original response. If the template accurately models the true signal, then the cross-correlation will be maximized. As part of the discussion for each method is a demonstration in its ability to remove a low frequency, monochromatic signal. Following the descriptions of each method is a discussion on fitting multiple sources whose bandwidths overlap. The chapter ends with a summary of the two approaches, discussing their advantages and disadvantages and suggesting a combination of the two that may be required in the problem of source identification and subtraction for the real LISA data.

**Linear Least Squares Fitting**

**Theory**

LISA will return a collection of time series, each with a uniform sampling cadence. Within these time series are the signals from individual sources \( h_a(t, \vec{\lambda}_a) \) and random detector noise \( n(t) \). Each source is itself described by a set of \( M \) parameters, \( \vec{\lambda} \rightarrow (\lambda_1, \lambda_2, \cdots, \lambda_M) \). For a linear detector, such as LISA, the net response is the sum of the individual signals and the intrinsic noise of the detector,

\[
s_i^{(j)} = \sum_a h_a^{(j)}(t_i, \vec{\lambda}_a) + n_i^{(j)},
\]  

(4.1)
Here the subscript \( i \) denotes a particular instant of time, while the superscript \((j)\) refers to a particular channel (times series) of information. The responses \( s^{(j)}(t) \) are the only accessible quantities from the detector. This is one sense in which the detector acts as a filter between the signals and the astronomer.\(^1\)

The first step in any removal scheme is to identify the parameter values \( \lambda_{\alpha} \) that uniquely describe a particular source. Since the only accessible quantities are the time series, which include not only the desired signal, but also the sum of the remaining signals and the detector noise, the estimation of the source parameters will necessarily be contaminated by the other contributions. As a result, it impossible to solve for the true parameter values exactly. Instead what is done is to derive an estimation of the true values along with a stated uncertainty. One such approach that performs the estimation along with returning a quantitative uncertainty is the method of *linear least squares fitting*.

In the method of least squares fitting we start with an initial guess to the true parameter values \( \vec{\lambda} \). Using the initial guess a model for the system \( h'(t, \vec{\lambda}') \) is formed and subtracted from the original time series. The result is a quantity known as the residue, which is expressed as

\[
x_i \equiv s_i - h'_i(\vec{\lambda}') = h_i(\vec{\lambda}) + n_i - h'_i(\vec{\lambda}) .
\]  

(4.2)

If the initial guess to the parameter values are close to their true values, then the residue may be expanded to linear order (thus the origin of the term “linear” in the name *linear least squares fitting*),

\[
x_i = -\Delta \lambda_{\alpha} \frac{\partial h_i}{\partial \lambda_{\alpha}} + n_i .
\]  

(4.3)

Here the repetition in the \( \alpha \) index implies a summation over the \( M \) variables that describe the source. The \( \Delta \lambda_{\alpha} \) represent the corrections that must be applied to the initial guesses to arrive at the best fit parameters for the signal. Therefore, it is these values that we wish to derive.

---

\(^1\)The signals \( h_{\alpha}(t, \vec{\lambda}_{\alpha}) \) from the individual sources also include the location and orientation dependent detector response functions as described in Chapter 3, which also acts as a form of detector filtering.
The values of $\Delta \lambda$ that lead to the best fits are those that minimize the power in the residue. That is, we want to minimize the quantity

$$P = x_i x_i$$

$$= \left( n_i - \Delta \lambda \frac{\partial h_i}{\partial \lambda} \right) \left( n_i - \Delta \lambda \frac{\partial h_i}{\partial \lambda} \right)$$

$$= n_i n_i - 2 n_i \Delta \lambda \frac{\partial h_i}{\partial \lambda} + \Delta \lambda \Delta \lambda \frac{\partial h_i}{\partial \lambda} \frac{\partial h_i}{\partial \lambda}$$

(4.4)

In the above, and for what is to follow, the repetition in the $i$ index implies a summation over all time samples, while the repetition in the Greek indices imply a sum over the source dependent variables.

The noise is an inherently random quantity that is not known separately from the detector output. However, if the initial guesses to the parameter values are close to their true values, which has already been assumed, then the noise will be strongly approximated by the calculated residue using the initial parameters; that is $n_i \approx x_i' (\lambda')$.

The power in the residue may now be minimized with respect to the corrections,

$$\frac{\partial P}{\partial \Delta \lambda} = -2 x_i' \frac{\partial h_i}{\partial \lambda} + 2 \Delta \lambda \frac{\partial h_i}{\partial \lambda} \frac{\partial h_i}{\partial \lambda} = 0.$$  

(4.5)

The above expression is simplified by defining the $M \times M$ square matrix

$$\Gamma_{\alpha \beta} = \frac{\partial h_i}{\partial \lambda} \frac{\partial h_i}{\partial \lambda} ,$$

(4.6)

which is known as the Fisher information matrix. The nomenclature comes from the idea that as more data points are included in the sum over $i$, more information is included about the correct parameter values.\(^2\) Using the definition of the Fisher matrix the minimization expression is now rewritten as

$$-2 x_i' \frac{\partial h_i}{\partial \lambda} + 2 \Delta \lambda \Gamma_{\alpha \beta} = 0 ,

(4.7)$$

\(^2\) The actual definition of the Fisher matrix varies slightly in the literature. In some instances the definition includes a scale factor equal to the inverse of the rms noise squared. For pre-whitened data, the end results of using either definition are the same.
which has the solution

\[ \Delta \lambda_\alpha = \Gamma^{-1}_{\alpha\beta} x'_i \frac{\partial h_i}{\partial \lambda_\beta}, \]  

(4.8)

where \( \Gamma^{-1}_{\alpha\beta} \) is the inverse of the Fisher matrix. These are the desired corrections that must be applied to the initial guesses to arrive at the best fit parameters for the source.

Once the best fit parameters are found the next question to address is how well are they determined. In other words, what is the uncertainty in the fit values? As part of the answer to this question is the important point that the calculated best fit parameters depend on the effective noise in the detector.\(^3\) For a different noise realization the derived fits will be different. The reason for this is that the residue is the sum of a deterministic signal and random noise. However, if the noise has zero mean, then the expectation value of the best fit parameters would equal the true values, and the variance would equal the uncertainty in the parameter estimates.

Mathematically what is required to arrive at an estimate of the fit uncertainties is to calculate the squared deviation in the parameter values from their means, and then perform an ensemble average,

\[ \sigma_\alpha = \left\langle (\Delta \lambda_\alpha - \langle \Delta \lambda_\alpha \rangle) (\Delta \lambda_\alpha - \langle \Delta \lambda_\alpha \rangle) \right\rangle, \]  

(4.9)

where \( \langle \cdots \rangle \) returns the expectation value of its argument. It turns out that the above equation is a single element of a more general statistical quantity known as the covariance matrix,

\[ C_{\alpha\beta} = \left\langle (\Delta \lambda_\alpha - \langle \Delta \lambda_\alpha \rangle) (\Delta \lambda_\beta - \langle \Delta \lambda_\beta \rangle) \right\rangle. \]  

(4.10)

The elements of the covariance matrix describe how random fluctuations in one parameter are correlated to each of the other parameters.

To calculate the covariance elements the corrections from Equation (4.8) are substituted into the above general expression for \( C_{\alpha\beta} \). Before doing so note that the only random quantities are the corrections \( \Delta \lambda_\alpha \) and the initial residue \( x'_i \). Moreover, since the corrections are a function of the residue, the only random quantity in which the expectation value must be taken is that

\(^3\) The adjective “effective” is used to emphasize that there are other signals contained in the detector response that are not being solved for at this time. The sum of these signals and the intrinsic detector noise form an effective noise for the signal that is being solved for.
of the residue. Hence,

\[ C_{\alpha\beta} = \Gamma^{-1} \frac{\partial h_i}{\partial \Delta \lambda_\gamma} \Gamma^{-1} \frac{\partial h_j}{\partial \Delta \lambda_\delta} \left( (x'_i - \langle x'_i \rangle)(x'_j - \langle x'_j \rangle) \right). \tag{4.11} \]

If the effective noise is Gaussian distributed and uncorrelated, then the expectations simplify to

\[ \left( (x'_i - \langle x'_i \rangle)(x'_j - \langle x'_j \rangle) \right) = \sigma_{\text{rms}}^2 \delta_{ij}, \tag{4.12} \]

where \( \sigma_{\text{rms}} \) is the rms value of the noise and \( \delta_{ij} \) is the Kronecker delta. With this result the covariance matrix becomes

\[ C_{\alpha\beta} = \sigma_{\text{rms}}^2 \Gamma^{-1} \frac{\partial h_i}{\partial \Delta \lambda_\gamma} \Gamma^{-1} \frac{\partial h_j}{\partial \Delta \lambda_\delta}, \tag{4.13} \]

which simplifies to

\[ C_{\alpha\beta} = \sigma_{\text{rms}}^2 \Gamma^{-1}_{\alpha\beta}. \tag{4.14} \]

It turns out that the covariance matrix is the inverse of the Fisher information matrix up to an overall scaling associated with the rms of the noise.

Returning to Equation (4.9) the uncertainties in the individual parameters are seen to equal the square root of the diagonal elements of the covariance matrix. Since the covariance matrix is directly related to the Fisher matrix the uncertainties are also given by

\[ \sigma_\alpha = \sigma_{\text{rms}} \sqrt{\Gamma^{-1}_{\alpha\alpha}}, \tag{4.15} \]

where the repetition of the \( \alpha \) index on the right hand side does not imply a summation. Notice that the uncertainties depend linearly on the effective noise in the detector. Additionally, the elements of the Fisher matrix scale as the signal squared. Therefore, the parameter uncertainties scale as the inverse of the signal-to-noise ratio, as one would expect.

To summarize the steps involved in the linear least squares fitting, we start with an estimate of the true parameter values. The estimated values must be close enough to the true values to allow the residue to be accurately expanded to linear order. Next, the power in the residue is minimized with respect to the desired parameter corrections. The solution of the minimization,
Equation (4.8), gives how much the original guesses must be corrected to arrive at the best fit parameters for the signal. The uncertainties in the fits are then given by Equation (4.15).

So far the discussion has used only a single time series in calculating the best fit parameters. To include multiple, orthogonal channels we begin by rewriting the net detector response as a combination of the individual time series,

\[ s_i = s_i^{(1)} \cup s_i^{(2)} = \left( h_i^{(1)}(\vec{\lambda}) + n_i^{(1)} \right) \cup \left( h_i^{(2)}(\vec{\lambda}) + n_i^{(2)} \right), \]  

(4.16)

where the superscripts denote the channel. In this example we work with only two time series since LISA will only return two independent channels of information on the gravitational wave sources. (A third possible channel, referred to as the symmetrized Sagnac signal, will monitor the detector noise [75].) The ‘∪’ symbol used to combine the two channels indicates that the responses are not directly added together point by point. Instead the channels are treated as if they originated from separate, independent vector spaces and are concatenated together. However, since each time series is recorded in parallel over the same observational period the combination is not a true concatenation.

The net residue is formed using the same combination as the net time series,

\[ x_i = x_i^{(1)} + x_i^{(2)} \]
\[ = \left( h_i^{(1)}(\vec{\lambda}) - h_i^{(1)}(\vec{\lambda}') + n_i^{(1)} \right) \cup \left( h_i^{(2)}(\vec{\lambda}) - h_i^{(2)}(\vec{\lambda}') + n_i^{(2)} \right) \]
\[ \approx \left( -\Delta \lambda_\alpha \frac{\partial h_i^{(1)}}{\partial \lambda_\alpha} + n_i^{(1)} \right) \cup \left( -\Delta \lambda_\alpha \frac{\partial h_i^{(2)}}{\partial \lambda_\alpha} + n_i^{(2)} \right). \]  

(4.17)

Since the two channels are orthogonal and independent, the cross terms in calculating the residue power are zero. By virtue of not having cross terms, the rest of the two channel derivation follows identically as before with each channel acting independently. The result is

---

4The sense in which the channels are orthogonal is that the noise realizations are uncorrelated across multiple time series. See Cutler [69] for a discussion on channel orthogonalization in the context of LISA.
that the Fisher information matrix becomes the sum of individual matrices,

$$\Gamma_{\alpha\beta} = \Gamma^{(1)}_{\alpha\beta} + \Gamma^{(2)}_{\alpha\beta}.$$ \hfill (4.18)

The final expression for the corrections is also adjusted accordingly,

$$\Delta \lambda_\alpha = \Gamma^{-1}_{\alpha\beta} \left( x_i^{(1)} \frac{\partial h_i^{(1)}}{\partial \lambda_\beta} + x_i^{(2)} \frac{\partial h_i^{(2)}}{\partial \lambda_\beta} \right).$$ \hfill (4.19)

**Application**

As an application of linear least squares fitting, consider the problem of removing a monochromatic gravitational wave signal from the low frequency region ($f \ll f_*$) of the LISA band after one year of observation. Below the transfer frequency the detector response is accurately described by the *Low Frequency Approximation* as given in Equation (3.58). Since the gravitational wave is assumed to be monochromatic, the response is fully determined by the seven source parameters $\vec{\lambda} \rightarrow (\mathcal{A}, f, \theta, \phi, \psi, \iota, \varphi_0)$, where $\mathcal{A}$ is the intrinsic amplitude of the gravitational wave, see Equation (2.63),

$$\mathcal{A} = \frac{2M_1M_2}{rR}.$$ \hfill (4.20)

For the purpose of this example the effective noise is taken to be Gaussian. Also, the initial guesses for the parameter values are set equal to the true values. This will ensure that we are within the linear regime. The results of the identification and removal are shown in Table 4.1 and in Figure 4.1.

As seen in Table 4.1 even though the initial guesses were the true values, the calculated best fit values differ. The contamination from the effective noise caused the solution to wander away from the true values. Linear least squares fits to a pseudo-source that is the combination of the true signal and the effective noise. As a result, when the subtraction occurs it also removes a portion of the local effective noise.

The previous example is for single realization of the noise. If multiple realizations are processed sequentially for the same source, then the derived corrections will fluctuate due to
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Original Value ((\lambda_a))</th>
<th>Uncertainty ((\sigma_\alpha))</th>
<th>Correction ((\Delta\lambda_a))</th>
<th>Fit Value ((\lambda'_a + \Delta\lambda_a))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>(2.448 \times 10^{-20})</td>
<td>(3.183 \times 10^{-22})</td>
<td>(4.279 \times 10^{-22})</td>
<td>(2.49079 \times 10^{-20})</td>
</tr>
<tr>
<td>(f)</td>
<td>(5.000 \times 10^{-4})</td>
<td>(2.129 \times 10^{-10})</td>
<td>(-5.756 \times 10^{-10})</td>
<td>(4.99999424 \times 10^{-4})</td>
</tr>
<tr>
<td>(\theta)</td>
<td>(1.736)</td>
<td>(9.784 \times 10^{-3})</td>
<td>(-10.06 \times 10^{-3})</td>
<td>(1.72594)</td>
</tr>
<tr>
<td>(\phi)</td>
<td>(4.652)</td>
<td>(9.105 \times 10^{-3})</td>
<td>(18.28 \times 10^{-3})</td>
<td>(4.67026)</td>
</tr>
<tr>
<td>(\psi)</td>
<td>(0.2354)</td>
<td>(7.560 \times 10^{-3})</td>
<td>(6.402 \times 10^{-3})</td>
<td>(0.241802)</td>
</tr>
<tr>
<td>(\iota)</td>
<td>(1.325)</td>
<td>(7.361 \times 10^{-3})</td>
<td>(5.615 \times 10^{-3})</td>
<td>(1.33061)</td>
</tr>
<tr>
<td>(\varphi_0)</td>
<td>(2.865)</td>
<td>(26.84 \times 10^{-3})</td>
<td>(2.367 \times 10^{-3})</td>
<td>(2.86737)</td>
</tr>
</tbody>
</table>

Table 4.1: The results of the linear least squares fitting. The signal-to-noise ratio for this source is one hundred. The gravitational wave frequency is in units of Hertz, while all angles are in radians.

the random noise in the residues. If the noise is normally distributed, then the distributions of the corrections will also be Gaussian with mean zero and a standard deviation equal to the predicted uncertainty from the covariance matrix. Figure 4.2 shows a histogram of correction values, normalized by their predicted uncertainty from the covariance matrix, calculated for ten thousand realizations of the noise. As expected, the distributions reflect the Gaussian noise.

Occasionally the calculated uncertainty for a variable will be larger than the range of possible values (for the angular variables) or an appreciable size of the original value \((A, f)\). To understand why this occurs, recall that least squares fitting is based on a search for a local extremum in parameter space. When an unreasonable uncertainty is returned, this is an

Figure 4.1: A demonstration of a signal identification and subtraction using the linear least squares fitting procedure. The red line is the spectral amplitude of the detector response prior to the removal of the signal, while the blue is what is left after the signal is removed.
Figure 4.2: A histogram of the parameter corrections calculated for $10^4$ realizations of the noise using the linear least squares fitting routine. Each correction value is normalized by its corresponding uncertainty from the covariance matrix. The normalization allows all seven parameters to placed on a single histogram plot and compared directly.

indication that in the region of parameter space where the sources resides, one of the parameter directions is functionally “flat.” In turn, there are a range of values that equally return a best fit solution; see Figure 4.3. Consequently, the fitting routine is unable to make a meaningful estimate of the original parameter values.

When a “bad” parameter is encountered, we remove it from the Fisher information matrix and calculate the remaining uncertainties based on the remaining sub-matrix. For the signal subtraction that follows, we use the either the true parameter value (when it is known) or the original guess. We are justified for doing so since any value is as good as any another.

**Template Match Fitting**

**Theory**

While linear least squares fitting allows for the best fit parameters to be solved for along with placing a quantitative uncertainty on the solution, it requires an initial guess for the
Figure 4.3: The correlation between the original signal and a sequence of templates with identical parameters, save for $\psi$ (blue) and $\iota$ (red). When a fit for a parameter is possible, such as with $\psi$, the corresponding correlations have a single extremum value (in this case $\psi$ has a pair of degenerate peaks). For a “bad” parameter, such as $\iota$, there are range of values that equally return a best fit solution.

parameter values that is sufficiently close to be in the linear regime. This means that least squares is not a stand-alone procedure for signal identification and removal. It depends on external information to be applicable. An alternative approach that does not require an initial guess for the parameter values is template match fitting.

The fundamental concept behind template matching is that the product of a deterministic signal and a random process is itself a random quantity. Furthermore, the cross-correlation of a deterministic signal with random noise will tend to zero as the number of sample points increases. That is, if we start with a linear response,

$$s_i = h_i(\lambda) + n_i,$$

and calculate the zero-lag, cross-correlation of the time series with the original signal, we find
that

\[ r = \frac{1}{N} \sum_{i=1}^{N} s_i h_i \]

\[ = \frac{1}{N} \sum_{i=1}^{N} h_i^2(\bar{x}) + \frac{1}{N} \sum_{i=1}^{N} n_i h_i(\bar{x}). \]  \hspace{1cm} (4.22)

The first summation, representing the average power in the signal, is a deterministic quantity. Conversely, the second summation is over the product of deterministic signal and a random variable. The net result is a random quantity. In general, the sum over a set of random variable will grow as the square root of the number of data points. Since the cross-correlation is itself scaled by the inverse of the number of points, the second term in \( r \) goes as \( 1/\sqrt{N} \). Therefore, as \( N \) approaches infinity the last term approaches zero, leaving the square of the signal as the remaining quantity in the correlation.

The above description of template matching used the original signal in the calculation of the cross-correlation with the detector response. However, in the problem of signal identification and subtraction there is no \textit{a priori} knowledge of the source parameters. Their values must be a derivable quantity.

Template matching arrives at an estimate of the best fit parameter values by first constructing a family of templates. A template is formed by combining a generalized model for the source \( \tau(t, \cdot) \), which also includes the detector response functions, with a unique set of parameter values \( \vec{\lambda} \rightarrow (\lambda_1', \lambda_2', \cdots, \lambda_M') \). In a typical template matching scheme the model is set by the type of system that is being searched for. What is left are the choices for the parameter values. Without prior knowledge, the only way to derive an estimate of the true parameter values is to construct a set, referred to as a family, of templates that contain all possible combinations of the values. Of course, if the variables are continuous this approach would require an infinite number of templates. A realistic compromise is to space the parameter values in successive templates by a chosen incremental value. The result is a finite set of templates with a set resolution in each parameter.

The issue of how to space the templates in order to keep the total number down to a reasonable size without compromising resolution is an important question. Owen [76] and
Owen & Sathyaprakash [77] addressed this issue in the context of LIGO source detection. Recently a paper by Cornish & Larson [57] addressed the same problem for low frequency sources in the LISA band. The suggested approach used in these papers is to construct a metric on the parameter space. A metric allows one to measure a “distance” (referred to as overlap in the context of template spacing) between templates. By setting a cutoff overlap, a count of the number of templates required to search for a particle system is found. A balance is reached by maximizing the overlap while minimizing the number of templates.

Once a family of templates is constructed, the search for the true parameter values proceeds by calculating the cross-correlation between each template and the original time series. The template with the greatest correlation will be based on the best fit parameters. Although straightforward, for a large template family this procedure can be time consuming to implement. Additionally, the true parameter values are only found to within the resolution set by the template overlap.\(^5\)

One possible solution in balancing a large family of templates versus the loss of resolution is to build a hierarchal search scheme. The idea is to do a sequence of searches with each new search focusing in on a region of parameter space that is suspected to contain the true source parameter values. At the highest level, where the family of templates must cover the entire space, the templates are widely spaced. After the cross-correlation has been calculated for all of the first level templates, a new family of templates are produced with a reduced parameter spacing and with the range of tested values limited to the area of parameter space about the point that produced the highest correlation. The cross-correlations for the second level template family are then calculated with finer parameter resolution and with less templates than would have been necessary if the same resolution was used at the first level. The hierarchal scheme may be repeated as many times as necessary to reach a predetermined resolution, while at the same time reducing the total number of required templates.

While the above description of template match fitting accurately describes the general methodology, it disguises some of the issues of implementation. These issues include the combination of multiple channels of information and the calculation of the intrinsic amplitude

\(^5\)The effective noise will also disrupt the fit. The issue of noise in the template fitting process will be addressed in the next section.
in the gravitational wave signal.

Most of the LISA signals that are of interest for identification and subtraction are monochromatic and located at low frequencies. For these types of systems it is possible to generate templates directly in the frequency domain using the prescription given in Cornish & Larson [57]. The advantage of working in the frequency domain is that the number of relevant Fourier coefficients (those that contain a high percentage of the power) are much smaller than the number of sample points in the full time series. Consequently the computational costs are greatly reduced. In the frequency domain a normalized cross-correlation is given by

$$ r = \frac{\langle \tilde{s}(f) | \tilde{\tau}(f) \rangle}{\sqrt{\langle \tilde{s}(f) | \tilde{s}(f) \rangle} \sqrt{\langle \tilde{\tau}(f) | \tilde{\tau}(f) \rangle}}, \quad (4.23) $$

where $\tilde{s}(f)$ is the Fourier transform of the original time series and $\tilde{\tau}(f)$ is the template in the frequency domain. The $\langle \cdot | \cdot \rangle$ represents an inner product defined as

$$ \langle \tilde{a}(f) | \tilde{b}(f) \rangle = \int_{-\infty}^{\infty} \left( \tilde{a}(f) \tilde{b}^*(f) + \tilde{a}^*(f) \tilde{b}(f) \right) df. \quad (4.24) $$

The asterisk denotes complex conjugation.

The cross-correlation given in Equation (4.23) is for a single channel. For two orthogonal channels this equation must be modified accordingly. The appropriate way is to add the single channel correlations, each weighted by the fractional power in the associated channel,

$$ r = \frac{\langle \tilde{s}^{(1)} | \tilde{s}^{(1)} \rangle}{\langle \tilde{s}^{(1)} | \tilde{s}^{(1)} \rangle + \langle \tilde{s}^{(2)} | \tilde{s}^{(2)} \rangle} \left( \frac{\langle \tilde{s}^{(1)} | \tilde{\tau}^{(1)} \rangle}{\sqrt{\langle \tilde{s}^{(1)} | \tilde{s}^{(1)} \rangle} \sqrt{\langle \tilde{\tau}^{(1)} | \tilde{\tau}^{(1)} \rangle}} \right) $$

$$ + \frac{\langle \tilde{s}^{(2)} | \tilde{s}^{(2)} \rangle}{\langle \tilde{s}^{(1)} | \tilde{s}^{(1)} \rangle + \langle \tilde{s}^{(2)} | \tilde{s}^{(2)} \rangle} \left( \frac{\langle \tilde{s}^{(2)} | \tilde{\tau}^{(2)} \rangle}{\sqrt{\langle \tilde{s}^{(2)} | \tilde{s}^{(2)} \rangle} \sqrt{\langle \tilde{\tau}^{(2)} | \tilde{\tau}^{(2)} \rangle}} \right). \quad (4.25) $$

Note that in the limit of one of the channels going to zero, the two channel correlation reduces to the single channel case.

Inspection of the two channel cross-correlation expression shows that it is independent of the scale in the templates and the detector responses. This implies that in searching for the greatest correlation among the family of templates, the intrinsic amplitude does not effect the correlation values and, therefore, must be derived separately.
As discussed in Chapter 3 the detector’s orbital motion introduces a location and orientation dependent modulation signature on the gravitational wave signal. The act of modulating the signal causes the spectral power to spread across multiple frequency bins. Coupled with the detector’s antenna beam pattern sweeping across the sky, this implies that the effective strain amplitude measured in the detector is different than the intrinsic amplitude of the gravitational wave signal. To see this, consider the power in the signal calculated over a complete orbit,

\[ A^2 = \langle \tilde{h}(\vec{x})|\tilde{h}(\vec{x}) \rangle = \langle \tilde{h}^2(\vec{x}) \rangle. \]  

(4.26)

Using the Low Frequency Approximation as given in Equation (3.58) the above is evaluated to

\[ A^2 = \frac{1}{2} A^2 \left( (1 + \cos^2(\iota))^2 \langle F^2_+ (\theta, \phi, \psi) \rangle + 4 \cos^2(\iota) \langle F^2_\times (\theta, \phi, \psi) \rangle \right), \]  

(4.27)

where \( \langle F^2_+ \rangle \) and \( \langle F^2_\times \rangle \) are the spectral powers associated with the antenna beam pattern factors. By rearranging the above equation we arrive at a relationship that expresses the intrinsic amplitude as a function of the rest of the source parameters and the amplitude measured in the detector,

\[ \mathcal{A} = A \left( \frac{1}{2} \left( (1 + \cos^2(\iota))^2 \langle F^2_+ (\theta, \phi, \psi) \rangle + 4 \cos^2(\iota) \langle F^2_\times (\theta, \phi, \psi) \rangle \right) \right)^{-1/2}. \]  

(4.28)

At this point in the template matching procedure fits for the other parameters have already been made. This allows us to calculate the detector response terms using the best fit template with the intrinsic amplitude set to unity,

\[ \langle \dot{\chi}^2(A = 1) \rangle \approx \frac{1}{2} \left( (1 + \cos^2(\iota))^2 \langle F^2_+ \rangle + 4 \cos^2(\iota) \langle F^2_\times \rangle \right), \]  

(4.29)

where the approximately equals sign reflects the fact that the left hand side uses the best fit, rather than the true parameter values.

Next, we must estimate the value of the detector measured amplitude. This is found by minimizing the difference in the net detector response with a unit template scaled by the
desired amplitude,
\[
\chi^2 = \langle (\tilde{s} - A\hat{\tau})^2 \rangle ,
\] (4.30)

where \(\hat{\tau}\) is a normalized template in the sense that \(\langle \hat{\tau}^2(f) \rangle = 1\). The solution for the minimization of \(\chi^2\) is given by
\[
A = \langle \tilde{s} | \hat{\tau} \rangle .
\] (4.31)

Combining Equations (4.28), (4.29), and (4.31) we find that an estimate of the intrinsic amplitude is
\[
A' = \frac{\langle \tilde{s} | \hat{\tau} \rangle}{\sqrt{\langle \hat{\tau}^2(A = 1) \rangle}} .
\] (4.32)

In analogy with the two channel correlation, when working with multiple channels of information the best fit intrinsic amplitude is the weighted sum from the individual channels,
\[
A' = \frac{\langle \tilde{s}^{(1)} | \hat{\tau}^{(1)} \rangle}{\langle \tilde{s}^{(1)} | \hat{\tau}^{(1)} \rangle + \langle \tilde{s}^{(2)} | \hat{\tau}^{(2)} \rangle} \left( \frac{\langle \tilde{s}^{(1)} | \hat{\tau}^{(1)} \rangle}{\sqrt{\langle \hat{\tau}^{(1)}(A = 1) | \hat{\tau}^{(1)}(A = 1) \rangle}} \right) \\
+ \frac{\langle \tilde{s}^{(2)} | \hat{\tau}^{(2)} \rangle}{\langle \tilde{s}^{(1)} | \hat{\tau}^{(1)} \rangle + \langle \tilde{s}^{(2)} | \hat{\tau}^{(2)} \rangle} \left( \frac{\langle \tilde{s}^{(2)} | \hat{\tau}^{(2)} \rangle}{\sqrt{\langle \hat{\tau}^{(2)}(A = 1) | \hat{\tau}^{(2)}(A = 1) \rangle}} \right) .
\] (4.33)

**Application**

To directly compare the source removal capabilities of template match fitting to that of linear least squares fitting, reconsider the situation defined previously. That is, the removal of a monochromatic source from the low frequency region of the LISA spectrum with assumed Gaussian noise.

To make the situations comparable, and to reduce the computation time, a hierarchal family of templates was constructed about the true source parameters. At the first level each parameter was separated by 1.5 times the predicted uncertainty from the covariance matrix and had a total range defined by \(\pm 3\sigma_\alpha\). With this choice in parameter spacing the first level search required \(6^5 = 7,776\) templates. (Recall that the intrinsic amplitude is calculated separately.) For each sequential level of refinement the parameter separation was halved from the previous level. For example, at the second level the parameter separation was 0.75 times the uncertainty. For all refinement levels the template parameters range from minus the parameter separation
Figure 4.4: A schematic diagram demonstrating the hierarchal search used in the template match fitting. The initial template family was confined to the region around the true source parameters. This diagram also demonstrates how a hierarchal scheme saves on computational costs. With a two level hierarchal search each parameter has 9 unique values. To achieve the same resolution with a non-hierarchal search would require 23 unique values per parameter.

to plus the parameter separation. This meant that the template count for each refinement level was $6^3 = 216$. Figure 4.4 pictorial reiterates the hierarchal scheme used in the search. Note that at least one generated template is identical to the original input signal. Table 4.2 and Figure 4.5 summarize the results of the template matching removal using the same source and noise realization from the linear least squares fitting example.

Although the search is centered on the true values, the returned best fit parameters are different than the original values. As with before, the contamination from the effective noise has caused the best fit values to shift away from the true parameter values. In terms of

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Original Value $(\lambda_\alpha)$</th>
<th>Uncertainty $(\sigma_\alpha)$</th>
<th>Correction $(\Delta \lambda_\alpha)$</th>
<th>Fit Value $(\lambda'<em>\alpha + \Delta \lambda</em>\alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$2.448 \times 10^{-20}$</td>
<td>$3.183 \times 10^{-22}$</td>
<td>$4.265 \times 10^{-22}$</td>
<td>$2.49065 \times 10^{-20}$</td>
</tr>
<tr>
<td>$f$</td>
<td>$5.000 \times 10^{-4}$</td>
<td>$2.129 \times 10^{-10}$</td>
<td>$-5.988 \times 10^{-10}$</td>
<td>$4.9999999401 \times 10^{-4}$</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$1.736$</td>
<td>$9.784 \times 10^{-3}$</td>
<td>$-10.09 \times 10^{-3}$</td>
<td>$1.72591$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>$4.652$</td>
<td>$9.105 \times 10^{-3}$</td>
<td>$17.93 \times 10^{-3}$</td>
<td>$4.66993$</td>
</tr>
<tr>
<td>$\psi$</td>
<td>$0.2354$</td>
<td>$7.560 \times 10^{-3}$</td>
<td>$6.378 \times 10^{-3}$</td>
<td>$0.241778$</td>
</tr>
<tr>
<td>$\iota$</td>
<td>$1.325$</td>
<td>$7.361 \times 10^{-3}$</td>
<td>$5.521 \times 10^{-3}$</td>
<td>$1.33052$</td>
</tr>
<tr>
<td>$\varphi_0$</td>
<td>$2.865$</td>
<td>$26.84 \times 10^{-3}$</td>
<td>$2.516 \times 10^{-3}$</td>
<td>$2.86752$</td>
</tr>
</tbody>
</table>

Table 4.2: The results of the template match fitting. The signal to noise ratio for this source is one hundred. The gravitational wave frequency is in units of Hertz, while all angles are in radians.
Figure 4.5: A demonstration of a signal identification and subtraction using the match template fitting procedure. The red line is the spectral amplitude of the detector response prior to the removal of the signal, while the blue is what is left after the signal is removed.

the theory of template matching the noise contamination arises from the second term in Equation (4.22). For finite $N$ the correlation between the noise and the template contributes a non-zero factor. It is this factor that accounts for the drift away from the true parameter values. (In a general template matching search it is also possible to miss the true values due to the discrete spacing of the templates. However, in this example it is known that at least one template was the original input signal.)

As with least squares fitting, by processing a sequence of noise realization for the same source, a histogram for the correction values can be made. Figure 4.6 shows the results of such a calculation using the template match fitting. As with before the corrections are distributed in a Gaussian manner, reflecting the type of noise included.

**Multiple Source Fitting**

In the examples of signal identification and subtraction presented so far, the source was implicitly assumed to be isolated in frequency space. LISA’s actual output will contain multiple sources with overlapping bandwidths (see Figure 5.9 and accompanying text). When multiple sources overlap, there are three possible approaches to identifying and removing signals from the data streams: (1) sequentially identify each source and then remove all of the sources at once, (2) identify and remove each source one by one, and (3) simultaneously fit and remove
Figure 4.6: A histogram of the parameter corrections calculated for $10^4$ realizations of the noise using the template match fitting routine. Each correction value is normalized by its corresponding uncertainty from the covariance matrix. The normalization allows all seven parameters to placed on a single histogram plot and compared directly.

all of the sources at once.

Figure 4.7 shows a sequential identification and final removal procedure based on template match fitting. In this example twenty sources with random parameter values were spread randomly over one hundred frequency bins. To distinguish between effective noise and sources that may be identified, the signal-to-noise ratio for each source was calculated using the standard formula [76],

$$(SNR)^2 = \int_0^{\infty} \frac{2|h(f, \vec{\lambda})|^2}{S_n(f)} df,$$

(4.34)

where $S_n(f)$ is the one-sided noise power spectral density. If the signal-to-noise was larger than five, then the routine tried to fit for the source. In this particular example, seventeen of the original twenty sources had a signal-to-noise greater than five.

As evident from Figure 4.7, the sequential identification and final removal is not effective approach to multi source fitting. The reason for this is that as each source is being fitted, the other sources act as noise. Recall that in both template matching and least squares fitting
Figure 4.7: A sequential identification and final removal procedure using template match fitting. Inside the frequency window are twenty randomly generated sources. If the signal had a signal-to-noise ratio greater than five, then the routine attempted to identify the signal in the original data streams. The red curve is the original spectral amplitude, the green curve is the intrinsic detector noise, and the blue curve is the residue after removal.

what is actually solved for is a pseudo-source that is the sum of the target signal and the effective noise. For an isolated source with reasonable signal-to-noise, the pseudo-source is the target source plus a small perturbation due to the noise. Conversely, when multiple signals overlap, the pseudo-source is the sum of the target plus other comparable random sources. Consequently, the fitting procedure fits to a new random source that is not actually in the data. It is this false source that gets subtracted and causes the sequential fit to fail.

Figure 4.8 shows the results of fitting and removing the sources one by one using the same set of data. The order in which the sources were identified and removed was based on their signal-to-noise ratios. The brightest source was identified first, followed by the second, and so forth. Although the final residues appear reasonable, only twelve of the attempted seventeen fits had a correlation with the original signals greater than 0.5. For the other five sources, the routine produced best fits to sources that did not exist in the data streams.

Both linear least squares and match template fitting are able to simultaneously fit for multiple signal. In the case of template matching the optimal template is one formed from the sum of the individual best fit templates. Since both approaches to sequential fitting fails, each of the best fit templates must be found simultaneously. As a result, the parameter space grows geometrically with the number of sources. For this reason template match fitting quickly becomes computationally impractical when large number of source have overlapping
Figure 4.8: A sequential identification and removal procedure using template match fitting. The order in which the sources were identified and removed was based on the brightest to the dimmest. Although the final residues appear reasonable, only twelve of the attempted seventeen fits were acceptable.

bandwidths.

For linear least squares, the extension to multiple source fitting is accomplished by first rewriting the residue (4.2) to include all $N$ sources that are being fit for,

$$x_i = \sum_{a=1}^{N} \left( h_a(t_i, \vec{\lambda}_a) - h'_a(t_i, \vec{\lambda}'_a) \right) + n_i, \quad (4.35)$$

If we have initial guesses for the source parameters that are sufficiently close to their true values, then we may write

$$x_i = -\Delta \lambda_{\alpha} \frac{\partial H_i}{\partial \lambda_{\alpha}} + n_i, \quad (4.36)$$

where the sum on the $\alpha$ index ranges over all parameters for all sources, $\alpha = (\lambda_1, \lambda_2, \cdots, \lambda_{N\cdot M})$, and $H_i$ is the sum of all sources,

$$H_i = \sum_{a=1}^{N} h_a(t_i, \vec{\lambda}_a). \quad (4.37)$$

From here the derivation for multiple source least squares fitting follows the single source prescription given earlier. The one notably difference is that the Fisher information matrix is now $(M \cdot N) \times (M \cdot N)$ in size. The far off diagonal elements describe the correlations between parameters across different sources, which are not necessarily zero.
Figure 4.9: A simultaneous identification and removal procedure based on linear least squares fitting. In this case, each of the seventeen fits had a high correlation to the original signals.

Figure 4.9 shows a simultaneous fitting procedure using a linear least squares fitting approach. The input time series were the same as those used in the sequential approaches. As seen in the plots, when the sources are fit simultaneously the resulting residues are near or below the original noise levels. Additionally, for all seventeen of the attempted fits the correlations between the original and the fit sources were 87% or greater.

For the multiple source fitting an iterative method was also implemented. As with the previous examples, the initial parameter guesses were set equal to the true parameter values. After an initial use of the least squares routine, a second calculation was performed using the best fit parameters from the first run as the initial guesses for the second. In the above example, five iterations were performed on the data, at which point the solutions appear to converge on the best fit parameters.

**Concluding Remarks**

This chapter introduced two independent approaches to source identification and subtraction. Each methodology has its own advantages and disadvantages. The method of linear least squares fitting is quick and returns solutions with a resolution limited only by numerical roundoff errors and the precision of the input signals. Additionally, least squares is able to quantitatively assign an uncertainty to the fit parameters. The main weakness of least squares is that requires an initial guess to the parameters that are close to the true values. Therefore,
this method cannot stand on its own, but rather depends on prior knowledge to be applicable.

Conversely, template matching is a straightforward and robust method that does not require external information. The shortcoming of template matching is that it is a time consuming process only able to determine the parameters down to a preset resolution determined by the template spacing. Even with a hierarchal scheme, the number of templates required to sufficiently search the parameter space can still be large enough to make the procedure impractical. However, in the absence of any prior knowledge, it is the appropriate way to proceed.

When it comes to the actual LISA data, a combination of the two methods will probably be the best way to attack the problem of signal identification and subtraction. By using a coarse template match search to arrive in the region of parameter space that contains the source, linear least squares can take over and calculate the best fit parameters. However, the initial coarse search must have a high enough resolution to arrive in a region of parameter space where the residue is accurately approximated to linear order.
CHAPTER 5

GALACTIC GRAVITATIONAL WAVE BACKGROUND

INTRODUCTION

To lowest order, gravitational waves are produced by a time varying quadrupole moment. The simplest such example is that of a binary star system in which two stellar objects orbit about a common center of mass. Surveys performed on the Milky Way galaxy suggests that roughly one hundred billion stars reside inside the galaxy. Of these, approximately two-thirds are a member of a binary system. The superposition of gravitational wave signals from such a large collection of galactic binaries could potentially dominate the response of LISA over a portion of its frequency band.

The first investigation into a potential galactic gravitational wave background was carried out in 1966 by Mironovskii [78]. His original work focused on a background produced by a population of W UMa binaries; a special class of binary systems in which two main sequence stars orbit close enough for their surfaces to come in contact. In the low period range where W UMa binaries are confined ($P = 0.25$ to $1.0$ days), their space density is greater than that of another type of binary. Moreover, since gravitational wave luminosity scales as $P^{-10/3}$ it was believed that the net gravitational wave luminosity of the galaxy would be determined solely by W UMa binaries. For these reasons Mironovskii’s work became the lore about galactic backgrounds for the next two decades.

With the exception of a survey article by Douglas & Braginsky [79], very little work was done on galactic backgrounds until the mid 1980’s. It was at this time that the first proposal for a spaceborne gravitational wave detector was set forth [42, 43]. Motivated by the possibility of a low frequency ($f < 1$ Hz) detector, a number of research groups began to explore what sources may lie in this region of the gravitational wave spectrum. Two particular investigations focused on sources within the Milky Way galaxy. Lipunov & Postnov [80] studied the expected
signal strengths from low and intermediate mass binaries through a Monte Carlo model of binary evolution, while Evans, Iben, & Smarr [81] estimated signal levels from a population of double white dwarf binaries. The main conclusion from these and other studies was that in the low frequency region of the gravitational wave spectrum, where spaceborne detectors would reside, the superposition of gravitational wave signals could conspire to form a background in a detector’s output.

In 1990 a detailed and comprehensive analysis of the galactic background was conducted by Hils, Bender, & Webbink [72]. Their paper calculated the expected spectral flux of gravitational radiation incident on a future low frequency spaceborne detector.\textsuperscript{1} Their study included unevolved, cataclysmic, neutron star - neutron star, black hole - neutron star, and close white dwarf binaries. Additionally, they revisited the work of Mironovskii, where they made a number of improvements to the estimated contribution from W UMa binaries. Their work produced an estimate of the spectral amplitude for the galactic background as a function of gravitational wave frequency. Specifically, they concluded that the superposition of galactic binaries would produce a signal greater than the detector noise for frequencies between 0.1 and 10 mHz.

The analysis by Hils, Bender, & Webbink was based on calculating the average luminosity from a population of sources. From the luminosity they derived a representative strain amplitude based on the standard formula [82],

\[
h = \left( \frac{8}{\pi f_{gw}^2} \frac{L_{gw}}{4\pi r^2} \right)^{1/2}.
\]  

While this approach has the advantage of being easy to implement, it disguises possible frequency dependent fluctuations in the background. A recent paper by Nelemans, Yungelson, & Portegies Zwart [83] addressed the averaging issue by calculating the strain amplitude for each source. They found that the background does in fact produce large fluctuations. The origin of the fluctuations is due to the spatial distribution of binaries in the galaxy. For the rare nearby sources, the measured signals are above the galactic background and, therefore,

\textsuperscript{1}At this time LISA had not yet come into its modern form. The LAGOS detector discussed in Hils et al. is an early incarnation of LISA.
are ideal systems for identification and subtraction. Conversely, the overwhelming number of
distant sources combine to form a true background for LISA.

Lacking from all previous studies of the galactic background are the effects imparted by
the detector on the individual signals. LISA’s orbital motion will modulate the signals across
multiple frequency bins. It will also rescale the intrinsic strain levels through an efficiency
factor associated with the time dependent antenna pattern being swept across the sky. The
net result is that the galactic background will be reduced from the predicted Solar System
barycenter levels as viewed in the actual output of the detector.

The aim of the work presented here is to improve our understanding of how the galactic
gravitational wave background will be encoded in LISA’s output data streams. Our investi-
gation of the background is done in two phases. The first phase is to build a Monte Carlo
simulation of the background by modeling each source and processing it through a model of
LISA. In modeling the individual types of binaries we follow the descriptions given in Hils,
Bender, & Webbink, which from here will be referred to as HBW. The second phase is to sta-
tistically characterize the features of the background. Among the quantities we investigate are
tests of Gaussianity in the distribution of Fourier coefficients, the number and type of bright
sources, and the density (in frequency space) of bright sources. Our interest in bright source
statistics stems from the idea that they will be identifiable in the data, and thus removable.

Since the study of the galactic gravitational wave background naturally divides itself into
two sections, modeling and characterization, the outline of the chapter will follow suite. The
first three sections are devoted to a description of the Monte Carlo simulation. It is here that
we describe how the individual sources are modeled and processed through LISA using the
response approximations from Chapter 3. Also included is a direct comparison of our Monte
Carlo results to the averaging approach used by HBW. The two sections that follow calculate a
number of statistical properties associated with the galactic background. Next, is a discussion
on the definition of what a true confusion limited background is. The chapter concludes with
a discussion on the various assumptions used in the simulation and how changes in these
assumptions would alter our results.
Modeling the Milky Way Galaxy

The first step in modeling the galaxy is to model a single source and then build up the galaxy by linearly adding each source together. In general, a binary system is composed of two stellar objects, each of which is on an elliptical orbit. Peters & Mathews [70] showed that the gravitational waves emitted from an elliptical binary are produced in harmonics of the orbital frequency, $\Omega = (M/R^3)^{1/2}$. For highly eccentric orbits the power radiated in gravitational waves is spread across a large range of harmonics. For example, an eccentricity of 0.7 produces relatively intense output for harmonics $n = 4$ through 18, with a peak emission at $n = 10$ (see Figure 3 of Reference [70]). For lower eccentricities the power radiated at higher harmonics approaches zero with more than 60% of the power radiated in the second harmonic. In the limiting case of a circular binary, all of the radiative power is found in the second harmonic. This is consistent with the derivation for the waveforms from a circular binary given in Chapter 2. There the gravitational wave frequency was found to equal $2\Omega$.

For eccentric binaries the peak emission occurs as the binary passes through periastron. Since gravitational waves carry away energy and angular momentum, the emission of gravitational radiation causes an eccentric orbit to circularize during the course of its lifetime [84]. Furthermore, for semi-detached and contact binaries, the transfer of material also causes the orbit to circularize [85]. The rate in which a binary becomes circular is greater than the inspiral rate, which also arises from the loss of energy through the emission of gravitational waves. As a result, most galactic binaries observed today should be approximately circular. For this reason, we make the assumption that all binaries are circular.

In Chapter 2 the gravitational waveforms for the $+$ and $\times$ polarizations were derived for a circular binary system with unequal masses, Equation (2.63). At the Solar System barycenter these reduce to

$$h_+(t) = A_+ \cos(2\psi) \cos(2\Omega t + \phi_0) + A_\times \sin(2\psi) \sin(2\Omega t + \phi_0)$$

(5.2a)

$$h_{\times}(t) = -A_+ \sin(2\psi) \cos(2\Omega t + \phi_0) + A_\times \cos(2\psi) \sin(2\Omega t + \phi_0),$$

(5.2b)
where the polarization amplitudes are given by

\[ A_+ = \frac{2M_1M_2}{r} \left( \frac{\Omega^2}{M_1 + M_2} \right)^{1/3} \left( 1 + \cos^2(\iota) \right) \]  
\[ A_\times = \frac{4M_1M_2}{r} \left( \frac{\Omega^2}{M_1 + M_2} \right)^{1/3} \cos(\iota). \]  

(5.3a)

(5.3b)

In the above expressions for \( A_+ \) and \( A_\times \) Kepler’s third law was used to replace the orbital separation previously found in the denominator. Inspection of the above expressions shows that a monochromatic binary is uniquely determined by a set of nine parameters: \( M_1, M_2, \Omega, \psi, \iota, \varphi_0, \) and \( r \), plus the angular location variables \( \theta \) and \( \phi \). To model an individual circular binary requires an accurate representation of these nine parameters.

The source parameters are separable into two categories, extrinsic and intrinsic parameters. The extrinsic variables \( \{ r, \theta, \phi, \psi, \iota, \varphi_0 \} \) do not depend on the type of binary system under consideration. Instead they depend on the location of the observer with respect to the source. The remaining variables \( \{ M_1, M_2, \Omega \} \) are a function of the type of binary system being modeled. For instance, the masses of all observed neutron stars are roughly 1.4 \( M_\odot \), while the mass components of an unevolved binary can range from a few tenths to tens of solar masses.

**Extrinsic Parameters**

For the set of extrinsic variables there is a further separation into those that locate the source in space \( \{ r, \theta, \phi \} \) and those that describe the orientation as viewed by a particular observer \( \{ \psi, \iota, \varphi_0 \} \). To derive a unique location for each source we start with a cylindrically symmetric disk model of the galaxy with an exponential falloff in both the radial and vertical directions,

\[ \rho = \rho_0 e^{-r/r_0} e^{-|z|/z_0}. \]  

(5.4)

Here \( \rho_0 \) is the space density of sources at the center of the galaxy, \( r_0 \) is the radial scale length, and \( z_0 \) is the vertical scale length of the galactic disk. The values of \( r_0 \) and \( z_0 \) vary with the different types of systems, but all sources are assumed to obey the above model. From the above galactic model the probability distribution functions for the different coordinates are
Figure 5.1: The angular distribution of 3,000 close white dwarf binaries in the galaxy. The assumed radial scale factor is 3.5 kpc while the vertical scale factor is 0.090 kpc. The galactic center is located near $\theta \approx 90^\circ$ and $\phi \approx 270^\circ$.

immediately read off,

\begin{align}
    p(r) &= \frac{r}{r_0^2} \exp \left( -\frac{r}{r_0} \right) \quad 0 \leq r < \infty \quad (5.5a) \\
    p(\phi) &= \frac{1}{2\pi} \quad 0 \leq \phi < 2\pi \quad (5.5b) \\
    p(z) &= \frac{1}{z_0} \exp \left( -\frac{|z|}{z_0} \right) \quad -\infty < z < \infty \quad (5.5c)
\end{align}

To arrive at a unique source location we draw random coordinate values from the above galactic position distributions. The means by which the random draws are performed use either the transformation method or rejection method as described in Reference [86].

The binary positions are simply described in galactocentric-cylindrical coordinates. The natural coordinate system for the LISA mission is heliocentric-ecliptic coordinates. Therefore, once the positions for the binaries are drawn using the galactic position distributions, they are translated to the LISA coordinate system. The results of such a procedure for a fraction of the close white dwarf binaries are shown in Figure 5.1. The galactic center is located at roughly
\[ \theta \approx 90^\circ \text{ and } \phi \approx 270^\circ. \] Recall from Equation (3.68b) that the Doppler shift scales as \( \sin(\theta) \), meaning that sources located at the galactic center are maximally Doppler shifted. However, as shown in Chapter 4, the uncertainty in the parameters are dictated by the derivative of the signal. Consequently, LISA’s sensitivity to angular position is poorest for sources located at the galactic center.

The observed orientation of a binary system is set by the principal polarization angle \( \psi \), the inclination angle \( \iota \), and the initial phase \( \varphi_0 \). The inclination angle is defined to be the angle between the line of sight to the binary \( \hat{n} \) and the angular momentum vector of the binary \( \vec{L} \). The quantity \( \hat{n} \cdot \vec{L} \) is uniformly distributed between zero and \( \pi \). The principle polarization angle is uniformly distributed between zero and \( \pi \). The distribution of \( \varphi_0 \) is uniform between zero and \( 2\pi \).

**Intrinsic Parameters**

The distributions for each of the intrinsic parameters \( \{M_1, M_2, \Omega\} \) depend on the binary type under consideration. In the literature there are two distinct ways in which the intrinsic parameter distributions are derived. The first approach, which we refer to as the analytical method, combines observational information with a physical model for the system’s evolution to arrive at an analytical estimation for the individual parameter distributions. The studies by Mironovskii on W UMa binaries [78] and Webbink on close white dwarf binaries [87] are two relevant examples that use such an approach. The main failing of the analytical method is that in some instances there is limited observational data to guide a theoretical model. The lack of data leads to speculations and best guess assumptions that may, or may not, accurately describe the real physics associated with the binary system. For binary systems where a large collection of observational data does exist, such as the case with cataclysmic variables, the derived distributions are able to reproduce the data.

An alternative to analytical modeling is a technique known as *population synthesis*.\(^2\) Here an initial mass function is used to describe the mass distribution for the primary (initially more massive) star. Additional distributions for the initial mass ratio, orbital separation, and

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\(^2\)Two predominant examples of population synthesis found in the literature are the SeBa code by Portegies Zwart & Verbunt [88] and the Scenario Machine by Lipunov, Postnov, & Prokhorov [89].
sometimes eccentricity are also assumed. Using the initial mass function, and other distributions, a collection of binaries are constructed with a random set of initial conditions and then evolved in a computer simulation. One of the end products of the evolution are the intrinsic parameter distributions for the various binaries that contribute to the galactic background. As with the analytical models, there are a number of failings associated with population synthesis. The two main problems are the choice of an initial mass function and other initial distributions, and the detailed physics used in the simulation of the binary evolution. One particular difficulty is in modeling the mass transfer rate for semi-detached and contact binaries. As Verbunt [90] and also Portegies Zwart, Yungelson, & Nelemans [91] have shown, even with the same set of initial conditions different population synthesis codes will evolve the system differently, resulting in different distributions.

For our simulation of the galactic background the preeminent difficulty with population synthesis codes is that they can take years of writing and modifying in order to arrive at an end product that is somewhat realistic. For this reason we have elected to follow the first approach of using analytically derived results to arrive at the intrinsic parameter distributions. Moreover, we used the parameter distributions given in the comprehensive HBW paper. For this reason our galactic backgrounds include only those sources that are describe in their paper, which are W UMa (3 × 10^7), unevolved (7 × 10^6), cataclysmic (1.8 × 10^6), neutron star - neutron star (10^6), black hole - neutron star (5 × 10^5), and close white dwarf binaries (3 × 10^6). The quantity in parentheses indicates how many are included in our Monte Carlo simulation. The following subsections contain short reviews for each type of binary system.

W Ursae Majoris Binaries:

W UMa binaries are composed of two main sequence stars with an orbital separation small enough for their surfaces to come into contact. Due to their close proximity, both stars share a common envelope of gas. W UMa binaries are divided into two subclasses, A and W-type. The A-type subclass are massive stars of spectral type A or F. Their large mass, and thus larger radii, mean that their orbital periods are limited to the range of 0.4 to 0.8 days. The second subclass of W UMa’s, W-type, are composed of cooler G or K stars and have corresponding shorter periods of 0.22 to 0.4 days.
Historically, the interest in W UMa binaries arose from the idea that in the narrow range of orbital periods that they are confined to, their space density is greater than that of any other type of binary. Moreover, since gravitational wave luminosity scales as $P^{-10/3}$, W UMa binaries have the potential to dominate the integrated gravitational wave luminosity of the galaxy. It was for these reasons that led Mironovskii to base his study of a possible galactic background on W UMa binaries. However, subsequent studies have shown other types of binaries, most notably unevolved binaries, contribute a substantial part to the net gravitational wave luminosity of the galaxy.

Our Monte Carlo simulation of the W UMa binaries follows closely to the presentation given in HBW. The one place we diverge from HBW is in a modification to Mironovskii’s original period distribution. Figure 1 of Mironovskii’s paper shows a plot of the W UMa period distribution based on published data. From this plot he fits a single curve to the entire range periods. At high periods the curve fails to accurately reproduce the data. We have corrected for this by introducing a piecewise defined that uses Mironovskii’s curve for low and intermediate periods and ours for high periods.

**Un-evolved Binaries:**

Un-evolved binaries consist of two main sequence stars, neither of which fill their Roche lobes. Orbital periods for un-evolved binaries range from as low as 0.28 days up to millions of years. Correspondingly, they are located at very low frequencies in the LISA band where the response of the detector is dominated by acceleration noise.

In HBW’s original analysis they considered $7 \times 10^{10}$ un-evolved systems. Due to computational limitations in modeling and simulating the detector response for such a large number of sources, we have elected to simulate only $7 \times 10^6$ un-evolved binaries. As will be seen in a later section, un-evolved binaries will be buried in LISA’s acceleration noise. Therefore, we can safely model a fraction of the total number without compromising our final overall results.

**Cataclysmic Binaries:**

In a cataclysmic binary a low mass, main sequence companion star is transferring material by way of Roche lobe overflow to a more massive white dwarf. As with W UMa binaries
the orbital periods for cataclysmic are confined to a small range of values \((P = 0.052\) to 0.963 days). Although observational studies of cataclysmic systems are more limited than for W UMa and unevolved systems, the roughly one hundred known cataclysmic binaries do allow for an accurate period and mass distribution to be formulated.

Neutron Star - Neutron Star Binaries:

Neutron star - neutron star binaries are very clean systems to study. For neutron star binaries inside the LISA band, the orbital separations are large enough and the bodies small enough to accurately treat each object as a point mass. Additionally, without the mass transfer seen in W UMa and cataclysmic systems, the phase evolution due is exclusively to radiation reaction effects.

At present, the few observed neutron star binary systems are not enough to construct a testable initial period distribution function. In light of the shortage of data, the assumption made in HBW is that the initial period distribution is uniform with limits set by a combination of production models and observational data from the few known neutron star binaries. A current day period distribution is found by evolving the initial periods using the standard relationship \[84]\,

\[
\frac{dP}{dt} = -\frac{95}{5} (2\pi)^{8/3} \frac{M_1 M_2}{(M_1 + M_2)^{1/3}} P^{-5/3}.
\] \tag{5.6}

Observations of neutron stars indicate that all neutron stars have a mass close to \(1.4 M_\odot\) with very little variation. For this reason all of the simulated neutron stars in our Monte Carlo model have masses of exactly \(1.4 M_\odot\).

Black Hole - Neutron Star Binaries:

Although no black hole - neutron binaries have been observed, there is strong observational evidence that stellar mass black holes do exist as binary components in x-ray systems. In these systems mass transfer from a companion star forms an accretion disk around the unseen black hole. In the accretion disk dynamical friction heats the material causing the emission of x-rays detectable by Earth orbiting satellites. The formation of a black hole - neutron star system could result from the further evolution of an x-ray system.

Since there is no observational data to guide a derivation of the period distributions, much
of the calculations for black hole - neutron star binaries follow closely to the neutron star binary cases. This includes the assumption of a uniform initial period distribution. Since the production scenarios for black holes in binaries differs slightly to those of neutron stars [92], the extreme values of the period distributions are adjusted accordingly. For simplicity all black holes are assumed to have a mass of $10 M_\odot$.

**Close White Dwarf Binaries:**

Theoretical predications for the evolutionary track of unevolved binaries with initially intermediate masses suggest that close white dwarf binaries may be a common occurrence in the galaxy [87, 93, 94]. However, the formation scenarios for close white dwarf binaries are much more complicated than for neutron star - neutron star and black hole - neutron star binaries. As two intermediate mass main sequence stars evolve, at least one stage of mass transfer is expected to occur. During this stage it is not clear how to calculate the loss of angular momentum. In addition, only a handful of white dwarf systems have been observed. Consequently, the initial period distribution for close white dwarf binaries is not well understood.

The detailed analysis of white dwarf binaries given in HBW implies that there should be $3 \times 10^7$ such systems in the galaxy. In light of the uncertainties in the evolutionary models and observational data, they reduced their total to a more conservative $3 \times 10^6$. Our Monte Carlo simulations adopted the 10% reduced value. For each of the extrinsic parameters, we followed the presentation given in HBW. However, a number of distributions published in their paper are functions of distributions found elsewhere. For this reason we had to supplement the information given in HBW with distributions found in Webbink’s original paper on white dwarf binaries [87] and Miller & Scalo [95].

**Barycentric Background**

Previous studies of the galactic background approached the problem by calculating the gravitational wave luminosity as a function of frequency. From the luminosity they then
derive a strain amplitude using\(^3\)

\[
h = \left( \frac{8}{\pi f_{gw}^2} \frac{L_{gw}}{4\pi r^2} \right)^{1/2},
\]  

(5.7)

where \(r\) is the distance to the source, and \(f_{gw}\) and \(L_{gw}\) are the gravitational wave frequency and luminosity respectively.

In order to make comparisons between our results and the previous studies, we must relate the above expression for the strain amplitude to quantities calculated in our Monte Carlo simulation. To this end, we first note the relationship between gravitational wave luminosity and flux,

\[
\frac{L_{gw}}{4\pi r^2} = F_{gw} = \frac{1}{16\pi} \langle \dot{h}_+^2 + \dot{h}_x^2 \rangle,
\]  

(5.8)

where the angle brackets denote an average over several wavelengths. Using this relationship the strain amplitude is rewritten as

\[
h = \left( \frac{1}{2\pi^2 f_{gw}^2} \langle \dot{h}_+^2 + \dot{h}_x^2 \rangle \right)^{1/2}.
\]  

(5.9)

From the waveforms given in Equation (5.2) we find that

\[
\langle \dot{h}_+^2 + \dot{h}_x^2 \rangle = 2\pi^2 f_{gw} (A_+^2 + A_x^2),
\]  

(5.10)

where \(\Omega = \pi f_{gw}\) was used to relate the angular orbital frequency to the gravitational wave frequency. Returning to the strain amplitude we find that it can be equivalently expressed as

\[
h = (A_+^2 + A_x^2)^{1/2}.
\]  

(5.11)

The polarization amplitudes, \(A_+\) and \(A_x\), are functions of the binary masses, distance to the source, orbital period, and inclination angle (see Equation (5.3)), all quantities that our simulation evaluates.

\(^3\)In the literature the formula for \(h\) contains an inconsistent factor of \(\sqrt{2}\). For later comparisons we adopted the apparent definition used in HBW, which is a factor of \(\sqrt{2}\) larger than the description found in Nelemans, Yungelson, & Portegies Zwart [83]
Before applying Equation (5.11) it is important to note its proper use when considering the net strain amplitude from a collection of sources. As gravitational waves converge on the detector, they interfere with each other constructively and destructively. Equation (5.11) correctly accounts for the interference in the same way as in standard data analysis uncorrelated, random errors add quadratically with the square root taken after all errors have been summed together. Similarly for the background the net strain amplitude is the square root of all of the individual polarization amplitudes from the individual sources added quadratically.

Figure 5.2 compares our Monte Carlo results to the original findings of HBW. (Note that what is plotted is the spectral amplitude $h_f$, not the strain amplitude $h$. The two are related by $h_f = \sqrt{T} h$ where $T$ is the observational period, which is set to one year for our realizations.) For each type of binary we find great agreement. The smearing effect seen at high frequencies is due to empty bins in the spectrum. Each of the orbital period distributions have a small, but finite, probability at low periods. Consequently it takes a large number of draws against the period distribution to produce a source with an extremely low period. In the cases of the neutron star - neutron star and close white dwarf binaries the probability in the period distribution tails becomes small enough that for the number of sources included in our simulated background one would not expected an extremely low period (high gravitational wave frequency) source. This is why the HBW data extends beyond any of our simulated sources.

The difference in the unevolved binaries is accounted for by the fact that we only model $7 \times 10^6$ systems. A factor of $10^4$ less than what is included in the analysis by HBW. In general, background levels grow as $\sqrt{N}$. This is evident from Equation (5.11) where the square root is over $N$ summations. Since our realization of the unevolved binaries is short by a factor of $10^4$ the difference between our results and HBW should be, and is, two decades.

Figure 5.3 is a plot of the net galactic background as viewed at the barycenter of the Solar System. The discontinuity in the background at $f = 10^{-4.6}$ Hz is an artifact of not including a complete realization of the unevolved binaries. If the full $7 \times 10^{10}$ unevolved binaries were included, the galactic background would be roughly constant between 1 and 100 $\mu$Hz at a level of $h_f \approx 10^{-17}$ Hz$^{-1/2}$.

As Nelemans, Yungelson, & Portegies Zwart [83] found with their population synthesis
Figure 5.2: A comparison of our Monte Carlo results to the original findings of HBW. The disagreement in the unevolved binaries plot is due the fact that we modeled $10^{-4}$ less systems than HBW. To aid in the comparison, the pictured realization of the close white dwarfs binaries includes $3 \times 10^7$ systems, a factor of ten greater than what our simulations actually include.

models of the galaxy, when the individual sources are modeled the background appears “noisy”.

This feature, although suspected to occur, could not be derived using the averaging techniques in the studies by Evans, Iben, & Smarr [81] or HBW. In later sections we will find that the large fluctuations are a result of the sources being spatially distributed throughout the galaxy.
Figure 5.3: A realization of the galactic background as measured at the Solar System barycenter. The red line is the predicted Michelson noise spectral amplitude for LISA.

**Detector Response**

Previous studies of the galactic background treated LISA as both motivation and a point of reference for their work (for example, see References [72, 81, 83]). They did not include the detector response in their calculations of the background levels. By doing so they neglected modulation spreading and location dependent detector efficiencies, both of which cause the measured amplitudes to decrease.\(^4\)

Chapter 3 presented a numerical model for the full response of LISA valid for all frequencies and arbitrary input waveforms. While fully robust in its application, *The LISA Simulator* calculates the detector response in the time domain. As a consequence, to prevent aliasing in a low frequency signal \((f < 10 \text{ mHz})\) requires \(\sim 10^6\) data points for an observational period of one year. The computation time for a single source at this cadence is measured in minutes, which is not reasonable for the 4.33 \(\times 10^7\) sources that must be processed. Conversely, if we could work directly in the frequency domain the number of relevant Fourier coefficients per

---

\(^4\)Although the previous studies did not include the detector response for each source, by plotting their results on a graph with the LISA sensitivity curve they do include an orbit averaged detector efficiency in their comparisons.
source roughly thirty. Here the concept of “relevant” are those coefficients that contain some high percentage of the power in the signal.

Of the $4.33 \times 10^7$ sources modeled only 0.002% are above 3 mHz, which is approximately the frequency at which both the transfer functions and the radiation reaction effects become appreciable. Below 3 mHz the evolution of the gravitational wave phase is not detectable, which allows the waveforms to be well approximated as monochromatic (see Figure 3.12). Additionally, the Low Frequency Approximation can be appropriately used to simulate LISA’s response (Figure 3.10). However, the form of the Low Frequency Approximation presented in Chapter 3 calculated the response in the time domain. To accelerate the simulated response to each binary we use the frequency domain description of the Low Frequency Approximation as given in Cornish & Larson [57].

For the 0.002% of sources located above 3 mHz the Low Frequency Approximation does not accurately describe the LISA’s response. To include the effects of the transfer functions and the phase evolution requires the use of the Rigid Adiabatic Approximation with waveform modeling to second post-Newtonian (2PN) order in the phase.\(^5\) Unfortunately, at this time we do not have a frequency domain version of the Rigid Adiabatic Approximation, which means that the few high frequency sources must be processed directly in the time domain.

In terms of the phase evolution, the cutoff between when the Low Frequency and the Rigid Adiabatic Approximations are used, is chosen such that the change in frequency is less than a tenth of a frequency bin. The choice of a tenth originates from Figure 3.13, which showed that the correlation between a monochromatic and 2PN Rigid Adiabatic Approximation drops below 95% at roughly a tenth of a bin. Furthermore, calculations using the linear least squares fitting suggest that the frequency resolution of LISA is fine enough to detect changes at this level. The change in frequency is found by expanding Equation (5.1) of Cutler [69],

$$\Delta f = \frac{3}{40} (8\pi)^{8/3} f_i^{11/3} M^{5/3} T^2,$$

where $f_i$ is the initial gravitational wave frequency, $M = (M_1 M_2)^{3/5}/(M_1 + M_2)^{1/5}$ is the chirp

\(^5\)As discussed in Chapter 3, we can consistently model a source to zeroth post-Newtonian order in the amplitude while working to second order in the phase since LISA’s amplitude resolution is much lower than its frequency resolution.
mass, and $T$ is the observational time.

Since the change in frequency scales with the chirp mass, more massive massive systems, such as the black hole - neutron star binaries, may evolve by more than a tenth of a bin, even if their initial frequencies are below 3 mHz. To account for this possibility, the conditions to use the Low Frequency Approximation are that the source’s initial frequency is below 3 mHz, and that its change in frequency is less than a tenth of a bin. If this is not the case, then the Rigid Adiabatic Approximation is used to simulate the LISA’s response.

The Rigid Adiabatic Approximation works in the time domain while the Low Frequency Approximation works directly in the frequency domain. To combine the results from each approximation we first simulate all of the sources that trigger the Rigid Adiabatic Approximation and add them linearly in the time domain. From here we perform an inverse Fast Fourier Transform to arrive at the associated Fourier coefficients. For the sources that are simulated in the frequency domain directly, we coherently add them together by summing the real and imaginary parts of their respective coefficients at each frequency. By adding the coefficients we are able to maintain phase information which dictates the constructive and destructive interference of the signals. The final detector response is the sum of the Fourier coefficients from the low frequency and the adiabatic results. It is also at this time that we add in a detector noise realization. Figure 5.4 shows a particular realization for one of the Michelson signals.

In comparing the background as observed in the output of the detector versus in the barycenter frame, the most striking feature is that it is lower by a full decade across the entire spectrum. The reduction is due to two effects associated with the measurement of a gravitational wave signal by LISA. The first effect is the detector efficiency. In the discussion on template matching in Chapter 4, it was shown that the measured strain in the detector is less than the intrinsic amplitude of the gravitational wave (see Equation (4.28)). The physical origin of the detector efficiency comes from LISA’s orbital motion sweeping the antenna beam pattern across the sky, and thus producing a time varying response to any given location on the sky. Table I and Figure 3 of Cornish & Larson [57] shows the effects of detector efficiency on the six nearest known interacting white dwarf binaries. In each case the reduction is at
The second cause in the background reduction is due to the modulation spreading of the signals across multiple frequency bins. At high frequencies the effects of spreading are evident by power showing up in bins that were previously empty in the barycenter frame. At lower frequencies one would expect that the spreading from adjacent bins would cancel out due to the numerous sources found in each bin. However, as will be shown in the next few sections, the background levels are dominated by a few bright sources. When the bright sources are spread out there is not a compensating bright source in the adjacent bin, and so as a result the galactic background is also reduced at low frequencies.

**Identification of the Background**

Of great interest to the LISA mission is the identification of the galactic background in the detector output. Since LISA is omnidirectional it detects all one hundred billion galactic binaries simultaneously. (Our Monte Carlo simulation only includes $\sim 10^7$ sources.) A vast majority of these sources will be buried in the intrinsic noise of the detector, but a few thousand...
will have signals strong enough to reach above the detector noise level.

Even without identifying individual sources, the overall level of the galactic background is able to set limitations on quantities such as population counts and orbital period distributions. The background shown in Figure 5.4 is based on a particular set of models for each type of binary system. For many of these sources the total numbers, or even their existence, is not completely accepted. For example, the original HBW estimate for the number of close white dwarf binaries located in the galaxy was $3 \times 10^7$. This total was reduced by a factor on ten in the discussion section of their paper due to conflicting values between their predicted space density and one implied by observations. However, by changing just the number of sources the observed power spectrum of the galactic background shifts up or down accordingly.

In addition to the number of sources, the shape of the overall background indicates the orbital period distributions for the different types of systems. Most sources that lie above the detector noise (especially at high frequencies) are close white dwarf binaries. As a result, studies in the shape of the background can give an indication of the underlying period distribution and space densities for white dwarf binaries. For the other types of systems the analysis is complicated by the overlap of the period distributions in the spectrum and the relative strengths of the produced signals. By looking for windows in the spectrum where a particular type of source is predicted to dominate the background, studies into their orbital period distributions are still possible.

Before the above studies can be made, the galactic background must be identified in the detector output. By inspection of the spectrum shown in Figure 5.4, the galaxy is evident by the jaggedness between 0.1 and 10 mHz. Outside this region the galactic binary signals are weaker than the detector noise. This is seen in the plot by the relative smoothness across multiple frequency bins. One way to identify the background is to statistically study the Fourier coefficient distributions in different regions of the spectrum. In particular, we know that the simulated intrinsic noise of the detector is Gaussian distributed. Therefore, if the galactic background is non-Gaussian, its presence would be evident in a test for Gaussianity.

Prior to testing the Fourier coefficients for Gaussianity, the detector output must be whitened in order to make objective comparisons across the entire spectrum. The basic con-
cept behind whitening is to remove the frequency dependent trend observed in the power spectrum. This is accomplished by dividing the original output of the detector by the local (in frequency space) average value of the spectral amplitude,

\[
\hat{s}(f) \rightarrow \frac{\hat{s}(f)}{h_f(f)},
\]

where \(\hat{s}(f)\) is the Fourier decomposition of the detector output, and the bar over \(h_f(f)\) represents a local average. The problem with whitening the data in this way is that the average value is strongly influenced by bright (outlier) sources. A more appropriate calculation is the median, which is much more stable against the presence of an outlier data point. By calculating a median first and then converting to a mean, the general frequency dependent trends in the data are accurately modeled.

To arrive at the relationship between the mean and median values consider two random deviates, \(x\) and \(y\), which are Gaussian distributed. The deviates represent the real and imaginary parts of the Fourier coefficients. We are now interested in deriving the probability density function for the spectral amplitude function, \(f(x, y) = \sqrt{x^2 + y^2}\). This is done by first noting that the deviates \(x\) and \(y\) are independent, allowing the joint probability distribution function to be written as the product of the individual distributions,

\[
p(x, y)\, dx\, dy = p(x)p(y)\, dx\, dy
\]

\[
= \frac{1}{2\pi \sigma_x \sigma_y} \exp \left( -\frac{(x - \bar{x})^2}{2\sigma_x^2} \right) \exp \left( -\frac{(y - \bar{y})^2}{2\sigma_y^2} \right)\, dx\, dy.
\]

For deviates with zero mean (\(\bar{x} = \bar{y} = 0\)) and equal standard deviations (\(\sigma_x = \sigma_y = \sigma\)) the above simplifies to

\[
p(x, y)\, dx\, dy = \frac{1}{2\pi \sigma^2} \exp \left( -\frac{x^2 + y^2}{2\sigma^2} \right)\, dx\, dy.
\]

The goal is to arrive at the distribution function for the spectral amplitude \(f(x, y) = \sqrt{x^2 + y^2}\), which implies that the next step is to make a change of variables from rectilinear to polar
coordinates. Such a procedure yields,

\[ p(r, \theta) \, dr \, d\theta = \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2} r \, dr \, d\theta . \]  

(5.16)

Integrating out the \( \theta \) dependence gives the desired result,

\[ p(r) \, dr = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} dr . \]  

(5.17)

The above represents the distribution of spectral amplitude values for Gaussian distributed Fourier coefficients.

The average of the spectral amplitude distribution is calculated using the standard formula,

\[ \bar{r} = \int_{0}^{\infty} r \, p(r) \, dr , \]  

(5.18)

which gives \( \bar{r} = \sigma \sqrt{\pi/2} \). The median value is found by solving the integral equation

\[ \int_{0}^{\hat{r}} p(r) \, dr = \frac{1}{2} , \]  

(5.19)

which has the solution \( \hat{r} = \sigma \sqrt{2 \ln(2)} \). Combining these two results gives the average spectral amplitude as a function of the median value

\[ \bar{h}_f(f) = \frac{1}{2} \sqrt{\frac{\pi}{\ln(2)}} \hat{h}_f(f) , \]  

(5.20)

where we have made the substitutions \( \bar{r} = \bar{h}_f(f) \) and \( \hat{r} = \hat{h}_f(f) \).

To summarize the whitening procedure, we begin by calculating a running median of the strain spectral amplitude of the detector output over a small frequency window (512 bins). We then convert the median values into means via Equation (5.20). Lastly, we divide the Fourier coefficients of the detector output by the local mean values of the spectral amplitudes. The result of this procedure is to produce a flat power spectrum.

With the Fourier coefficients properly whitened we perform a running Gaussian test across the spectrum. Tests for Gaussianity are done using both a \( \chi^2 \) and a Kolmogorov-Smirnov
Figure 5.5: A running KS Gauss test applied to the real Fourier coefficients of the Michelson signal from vertex one. The presence of the galactic background is apparent by the low $p$ values about $f = 10^{-3}$ Hz. Similar results are found for the imaginary coefficients, for the other data channels, and when using a $\chi^2$ test.

(KS) test. Figure 5.5 shows the results of the KS test performed on the real coefficients for the Michelson signal from vertex one.

The $p$ values plotted along the ordinate axis are the probabilities that the set of measured deviates are Gaussian distributed. While it is impossible to say with certainty that a set of measured deviates are necessarily distributed with a specific distribution, they can be disproven. However, $p$ values of few hundredth are usually accepted as agreeing with the tested distribution function. For $p$ values below a hundredth, which are seen in the milliHertz region of the detector output, indicate that the measured set of data is not Gaussian distributed.

Comparing the results of the Gaussian test (Figure 5.5) and the original LISA output (Figure 5.4) indicates that the detector response is non-Gaussian for frequencies in which the background is above the intrinsic noise level. Outside these regions, where the detector’s noise dominates the output, the returned $p$ values are consistent with a Gaussian distribution, as they should since the simulation of the noise is based on Gaussian distributions. Similar results are found for the imaginary coefficients, other channels of data, and when calculated using a
The standard belief in the gravitational wave community is that the galactic background should be Gaussian distributed. This assumption is based on the Central Limit Theorem, which states that for a large sample of random deviates, regardless of their parent distribution, the distribution of average values will be approximately Gaussian. However, the Central Limit Theorem is not directly applicable to the galactic background since the net power in a single frequency bin may be dictated by a single bright source. Moreover, in a small region of the spectrum there are very few bright sources. Far less than the large number of data points required for the Central Limit Theorem to be applicable.

In support of this hypothesis Figure 5.8 shows a portion of the Gaussian test in the region of the spectrum where the background becomes sparse enough that there are isolated sources in the frequency domain and the underlying detector noise is known to be Gaussian. In a region near a source the Gaussian test fails, while in between the sources (for example near $f = 10^{-1.9}$ Hz) the returned $p$ values suggest that the response is Gaussian.
Chapter 4 introduced two techniques for removing an individual signal from LISA’s data streams. As part of the description for linear least squares fitting the assumption was made that the effective noise is Gaussian distributed. However, the previous section demonstrates that this is not the case in regions where the galactic background is above the intrinsic noise of the detector. Large fluctuations originating from bright sources cause the tails of the expected Gaussian distributions to be enlarged. If the bright sources were removed from the data streams, then the remaining background would be Gaussian. To demonstrate this we must identify each bright sources, remove it, and then retest the remaining background for Gaussianity.

Identifying a bright source is not as simple as scanning the power spectrum for excess power in a particular frequency bin. The modulation effects caused by LISA’s orbital motion spread the spectral power from a source across multiple bins. Although the bandwidth over which a source will spread is a known function of the gravitational wave frequency [96], if multiple sources are stacked in a small region of frequency space their bandwidths will overlap and disguise the true number of sources in the region.

For the real LISA data the issue of identify individual sources in regions of bandwidth overlap is a difficult problem not fully solved. For our model of the galaxy, where we generate each binary, we know the parameter values for each source. With this extra fortuitous information we can quickly and accurately identify the bright sources.

Our approach to identifying bright sources is to categorize them according to their signal-to-noise ratio using the standard formula [76],

\[(SNR)^2 = \int_0^\infty \frac{2|h(f, \vec{\lambda})|^2}{S_n(f)} df, \]  

where \(S_n(f)\) is the one-sided power spectral density of the noise. In order to properly use Equation (5.21) the interpretation of what the noise is needs to be clarified. We are interested in removing sources that are bright relative to the local power spectrum level. Therefore, the
\( S_n(f) \) curve must be a superposition of the detector noise and the galactic background. It is the effective noise discussed in Chapter 4.

To derive the power spectral density curve for the effective noise we start with the median curve calculated in the whitening procedure. When performing the whitening the median curve was evaluated for each frequency bin. For the signal-to-noise calculations we do not require the same resolution and, therefore, may use a simple piecewise function that describes the spectral trends. We find that the curve

\[
S_n(f) = \begin{cases} 
10^{-51.0}f^{-4.0} & f \leq 10^{-4.0} \\
10^{-48.8}f^{-3.45} & 10^{-4.0} < f \leq 10^{-3.0} \\
10^{-51.29}f^{-4.28} & 10^{-3.0} < f \leq 10^{-2.4} \\
10^{-44.5028}f^{-1.452} & 10^{-2.4} < f \leq 10^{-2.15} \\
10^{-41.381} & 10^{-2.15} < f 
\end{cases}
\] (5.22)

accurately follows the power spectrum for the detector output. Note that the last section of \( S_n(f) \) is independent of frequency because it is a fit to the white photon shot noise. Similarly, the first section describes the acceleration noise for the detector.

Using the effective noise curve given in Equation (5.22) we calculated the signal-to-noise ratio for each binary in relation to the total output of the detector (noise and background). In the LISA literature the general rule of thumb for a source that is considered identifiable is if its signal-to-noise ratio is greater than five. (Occasionally a signal-to-noise of three is used as the cutoff.) Following suite we considered all binaries with a signal-to-noise greater than five as a bright source. If a source is identified as bright, then we remove it from the data stream using the same detector response approximation that generated it.

Shown in Figure 5.7 is the first Michelson channel after the bright sources have been removed. Visual inspection of the spectrum shows that without the bright sources the bin to bin fluctuations are much smaller; an indication that the Fourier coefficients may be Gaussian distributed. Figure 5.8 confirms this hypothesis. When the bright sources are removed from the data streams the galactic background is Gaussian in nature.

The underlying Gaussian structure of the galactic background is an important result in
Figure 5.7: The Michelson signal from vertex one after the bright sources have been removed from the data streams.

Figure 5.8: A running Gauss test applied to the detector output after the bright sources have been removed. Unlike before, all returned $p$ values are consistent with a Gaussian distribution.
of itself. As discussed in the previous chapter, the method of linear least squares fitting is a powerful and accurate technique for identifying individual sources in the output of LISA. However, a necessary assumption made in the derivation of the results is that the effective noise is Gaussian distributed. The above results imply that this is indeed the case.

Bright Source Statistics

An interesting question to ask is what makes a particular binary bright in the output of the detector? To answer this recall the functional form of the intrinsic amplitude,

$$A = \frac{2M_1 M_2}{r} \left( \frac{4\pi^2 f_{\text{orb}}^2}{M_1 + M_2} \right)^{1/3}.$$  \hspace{5cm} (5.23)

Given a particular frequency bin the value of $f_{\text{orb}}$ cannot vary by more than bin width, leaving only the masses and the distance to the source to determine if a signal is bright in the detector output. For the binaries that makeup the galactic background, the range in component masses are confined to the small window between 0.1 and 10 $M_\odot$. For most binary systems the window is actually smaller. Only black hole - neutron star binaries, which are predicted to be the least abundant of any type of binary system, have a considerable amount of mass. As a result, the variable that dictates if a binary will be bright is $r$, the distance to the source. This is consistent with the range of values that $r$ can take on. Distances can range from $\sim 3$ pc up to $\sim 30$ kpc, over four decades in spread. The bright sources we identify in the background are typically located relatively nearby.

Of the $4.33 \times 10^7$ sources included in the Monte Carlo simulation of the galactic background, only 4,526 were identified as bright (SNR $> 5$). Table 5.1 lists each binary type and how many were flagged as bright. It is interesting to note that the only types of systems that were removed contain compact objects. Only compact binaries are able to reach the higher frequency regions of the LISA band (see Figure 5.2). At these higher frequencies the acceleration noise in the detector reduces to a level that allows the compact object backgrounds to be detectable in LISA’s data streams.

The close white dwarf binaries dominant the list of removed types due to their strong presence above 1 mHz. Not only do the white dwarfs extend roughly a half decade higher in
Table 5.1: A table of removed sources. The “Percentile of Total Include” is the percentage of binaries removed in comparison to the number of that binary type that were included. The “Percentile of Total Removed” is the percentage of the binaries removed in comparison to the total number of all binaries removed.

frequencies than any other binary type, but they are also the most abundant of the compact binaries. As discussed previously, the background levels grow as $\sqrt{N}$. By simply having more close white dwarf binaries than any other type of compact binaries, they dominant the number of bright sources.

For issues concerning data analysis, an interesting quantity to know is the density of bright sources, or equivalently, the number of bright sources per bin. The bright sources represent signals that are identifiable in LISA’s output. By understanding their location and separation (in the frequency domain) proper data analysis tools can be developed and applied in the search for these signals in the output time series.

Figure 5.9 is a plot of the number of bright sources per frequency bin. The peak density corresponds to an average of one source every fourteen bins. The bandwidth over which the modulation effects spread a source is given by [96]

$$B = 2(1 + 2\pi f R \sin(\theta)) f_m ,$$

(5.24)

where $R$ is the orbital radius of LISA (equal to 1 AU), $\theta$ is the colatitude of the source on the sky, and $f_m = 1$/year is the modulation frequency. For the peak density at 3 mHz the bandwidth is approximately twenty-one bins. As a result, in this region of the spectrum there are bright sources whose power at least partially overlap. Of course, Figure 5.9 only represents the average distance between bright sources. In a particular region of the spectrum, all of the bright sources may actually be located in a smaller subregion.
Figure 5.9: The number of bright sources per bin. The peak density at 3 mHz corresponds to, on average, a bright source every fourteen bins.

**Confusion Limited Background**

The superposition of all galactic binaries will form a confusion limited background in the detector output. Previous definitions of the confusion background rely on a particular cutoff frequency below which there are more than one source per frequency bin. However, this definition is inappropriate for two reasons. The first is that there is not a particular frequency in which the background drops below a source per bin. Figure 5.10 shows the fraction of bins, as calculated in the barycenter frame, that contain more than one source per bin (red curve), exactly one source per bin (green curve), and less than one source per bin (blue curve). Empty frequency bins first show up as earlier as 0.6 mHz, while bins with multiple sources are located at frequencies well beyond 4 mHz.

The second way in which prior definitions are inappropriate, is that below the traditional frequency cutoff that defined the confusion background \( f \sim 1 \text{ mHz} \), it was thought that individual sources could not be identified. However, as shown in previous sections the galactic background levels prior to source removals are dictated by the few bright sources in the data.
Figure 5.10: A plot of the fraction of bins with more than one source per bin (red), exactly one source per bin (green), and less than one source per bin (blue). As evident from the plot, there is not a particular frequency at which the galactic background “turns off.”

streams. These sources can be identified due to their relative brightness compared to the rest of the galactic background and, therefore, should not be included in the definition of a confusion background.

A true confusion limited background is what remains after a full data analysis procedure has removed all of the identifiable signals. At present such an algorithm does not exist. The work presented in the previous chapter represents the seeds for such a procedure. In the section on multiple source removal, it was shown that a sequential removal scheme does not effectively subtract sources when their bandwidths overlap. This then led us to consider the simultaneous removal of a collection of sources. At present, our simultaneous removal routines have only been applied to twenty overlapping singles. According to the estimates in the previous section, there will be between four and five thousand galactic binaries that are identifiable in LISA’s data streams. For such a large population, the size of the information matrix that requires an inversion is at the limits of what can be reasonable performed without numerical roundoff errors invalidating the solutions.\footnote{Since the identifiable sources are spread over two frequency decades, the information matrix will be block}
Even without a full removal algorithm, we are still able to make a reasonable estimate of the confusion limited background based on our results from the previous section. Figure 5.7 represents a close approximation to how the confusion background will appear in LISA’s output. It is not an exact representation since our procedure for removing bright sources was to subtracted an exact replica of the sources from the data stream. In a real data analysis subtraction, contamination from the effective noise corrupts the signal fits. Consequently, an exact fit and removal is not possible. Instead a pseudo-source, which contains both the target source and a portion of the effective noise, is removed.

For our estimate of the confusion limited background we calculated the median curve of the power spectral density for the detector response after the bright sources were removed. We found that the curve

\[
S_{CLB}(f) = \begin{cases} 
10^{-51.0} f^{-4.0} & f \leq 10^{-4.0} \\
10^{-49.2} f^{-3.55} & 10^{-4.0} < f \leq 10^{-3.0} \\
10^{-53.831} f^{-5.1} & 10^{-3.0} < f \leq 10^{-2.6} \\
10^{-46.031} f^{-2.1} & 10^{-2.6} < f \leq 10^{-2.25} \\
10^{-41.981} f^{-0.3} & 10^{-2.25} < f \leq 10^{-2.0} \\
10^{-41.381} f^{-2} & 10^{-2.0} < f 
\end{cases}
\]  

(5.25)

accurately recreates the frequency trends found in the galactic background after the removals. Figure 5.11 is a plot of Equation (5.25), represented as a spectral amplitude \( h_f = \sqrt{S_{CLB}} \). For reference, the median fit prior to removal and the average Michelson noise for LISA are also included.

**Concluding Remarks**

An important point to keep in mind about the results presented here is that they assumed a particular set of descriptions for the galaxy and the individual binary types. A different collection of models may return a slightly different set of results. However, the main conclusions diagonal. For such a matrix, there are inversion techniques that will accurately invert the matrix, but at present they have not been incorporated into our analysis.
Figure 5.11: The blue curve represents an estimate of the confusion limited background due to the remaining galactic binaries after a data analysis algorithm has subtracted out the identifiable sources. For reference, the green curve is the median fit to the galactic background prior to removal (Equation (5.22)), and the red curve is the average Michelson noise in the LISA detector.

drawn here are determined by only two parameters, the distance to the sources and the total number of binary systems included.

The distance to the sources dictates the statistics governing the bright sources. If there were a larger number of local sources, then the median value of the background levels higher than what our realizations found. In terms of our assumptions, the distance to the sources are solely dictated by the choice in the galactic model. Our model for the shape of the galaxy, Equation (5.4), is one of two standards found in the literature, the other being

\[
\rho = \rho_0 e^{-r/r_0} \operatorname{sech}^2 \left( \frac{z}{z_0} \right),
\]  

(5.26)

which is the one assumed by Nelemans, Yungelson, & Portegies Zwart. The main difference in the two galactic models is that the above tends to place move sources at the galactic center, while our exponential falloff in the z direction spreads the sources out in the central region. In the Solar neighborhood, which is roughly 9 kpc from the galactic center, the two models are
similar. This implies that the number of bright sources is not strongly effective by the choice in galactic model.

Conversely, the number of sources included in a model of the galaxy can impart a noticeable difference. As the sources are linearly added together, their random differences in phase will cause constructive and destructive interference. Statistically the problem is analogous to a random walk. As a result, the net spectral amplitude per frequency bin will grow as \( \sqrt{N} \). A factor of one hundred difference in the number of sources would raise or lower the galactic background by a full decade. For low frequency sources (W UMa, unevolved, and cataclysmic binaries) electromagnetic surveys of the galaxy have placed strong limits on the total number of such binaries. However, for the high frequency sources (neutron star - neutron star, black hole - neutron star, and white dwarf binaries) equivalent surveys have failed to place strict bounds on the number of compact binaries. One return of the LISA mission will be to place limits on the population sizes for each of the compact binaries by measuring the galactic background median levels and the number of bright sources from each population.
CHAPTER 6

SUMMARY AND FUTURE ANALYSES

INTRODUCTION

The field of observational astronomy separates itself from other scientific disciplines in that most instances human intervention is impossible. To collect data requires the use of telescopes able to receive transmitted information from distant regions of space. It is for this reason that the telescope necessarily acts as a filter between the astronomer and the measured signal. Key to the study of astrophysical objects is the ability to interpret the output of the detector in order to recreate a representative model of the emitting source.

Gravitational wave astronomy is no different. The large scale interferometers currently built, or planned for construction in the coming years, measure gravitational radiation produced by distant objects. The most powerful emitters of gravitational waves are typically electromagnetically dim, but this is what makes gravitational wave astronomy so exciting. For the first time astronomers will be able to directly study the spacetime around black holes; measure the event rate of supermassive black hole mergers suspected to occur at the center of colliding galaxies; and they may even detect the gravitational radiation emitted during the formation of the Universe. These are just a few topics the could be explored with information extracted from the gravitational wave spectrum.

Furthermore, the combination of information attained from the electromagnetic and gravitational wave spectra will aid in supporting or rejecting accepted theories. For example, current electromagnetic observations suggest that at the center of every galaxy lies a supermassive black hole. Moreover, galaxy surveys out to large red shifts imply that galaxy collisions are a common occurrence in the history of the universe. Together these measurements imply that the central black holes should eventually merge into a single, larger black hole. However, such an event has not yet been witnessed. In addition, theoretical models, usually based on some
form of dynamical friction, cannot fully explain how the black holes would reach a close enough separation to allow gravitational radiation reaction effects to drive them to coalescence. With the use of gravitational wave observations, the merger rate of supermassive black holes in the universe can be calculated. If this rate agrees with prediction based on the number of galaxy collision, then we would know that there is a mechanism, which may or not be dynamical friction, that is able to drive the black holes together.

As we await the first unambiguous detection of gravitational waves, almost a full century after their predicted existence, we continue to develop a stronger theoretical understanding of gravitational wave sources and the data analysis techniques for extracting these signals from the data. This dissertation has contributed to this effort by modeling the response of spaceborne gravitational wave detectors, and investigating how signals can be identified and subtracted from their output.

**“The Big Picture” Summary**

In the source rich low frequency ($f < 1$ Hz) region of the gravitational wave spectrum, the proposed LISA observatory will detect the merger of supermassive black holes, the inspiral of compact objects into supermassive black holes, and the combined signals from billions of galactic binaries. The first step in studying these, or any other possible sources, is to model the response of the detector (Chapter 3). With an understanding of the detector response in hand, a detailed data analysis procedure (Chapter 4) can be applied to the detector output. The primary goal of the data analysis is to extract the source parameters for a desired type of system, for example an unequal mass, circular binary (Chapter 2). Specific to the LISA mission is the signal contamination formed by the superposition of an overwhelming number of galactic binaries. In order to confidently understand the analysis of a given source, or to study the general response of the detector, a firm understanding of how the galactic background will be present in LISA’s output is necessary (Chapter 5).

**Gravitational Wave Production**

To lowest order, gravitational waves are produced by a time varying quadrupole moment.
The simplest and most abundant sources of gravitational radiation in the LISA band are binary star systems. For eccentric binaries Peters & Mathews [70] showed that the frequencies of the emitted waves are harmonics of the fundamental orbital frequency, $\Omega = (M/R^3)^{1/2}$. In other words, eccentric binaries appear as a set of circular binaries with the same source parameters save for the frequency.

Peters [84] demonstrated that eccentric binaries will slowly circularize due to stronger emission of radiation at periastron. Furthermore, in semi-detached and contact binaries the transfer of material causes the orbit to circularize [85]. The time scales for these processes are much smaller than the time to coalescence. For this reason most of the sources in the LISA band are expected to be roughly circular, with waveforms described by Equation (2.63).

**Spaceborne Detectors**

For most sources in the LISA band the effects of radiation reaction are minimal, indicating that the signals will be continually present in the detector data streams during the lifetime of the mission. The constant presence of these signals means that the motion of the detector will impart amplitude, frequency (Doppler), and phase modulations into the signal. The main effect of the modulations is to spread the spectral power across multiple frequency bins. In general, the motion of the detector complicates the time dependent response of LISA [97].

A detailed model of the detector response is needed in order to develop data analysis techniques that will be able to identify and subtract individual signals from the data streams. To this end, *The LISA Simulator* [64, 65, 98] has been developed. The Simulator is a numerical model of the LISA detector which returns a set of time series (one for each of the constellation’s vertices) for arbitrary incident gravitational waveforms. Included in the Simulator are the modulation effects discussed above, in addition to the transfer functions that account for self cancellation effects for gravitational waves that “fit inside” the detector arms.

The equations that *The LISA Simulator* are based on describe the full response of a spaceborne detector. However, they are analytically difficult to handle and time consuming to evaluate. For these reasons the *Low Frequency Approximation* and the *Rigid Adiabatic*
Approximation were developed. By neglecting high frequency effects and by requiring slow variations in the gravitational wave frequencies \( \frac{f}{\dot{f}} \ll L \), these approximations are able to accelerate the time required to simulate the detector response to an assumed waveform. Furthermore, their analytical forms are more conducive to theoretical studies.

Data Analysis

Unlike most electromagnetic telescopes, gravitational wave observatories do not return an image of an individual source or a region of space. Instead they return a collection of time series. Encoded in these time series are all the measured signals from each source whose radiation passes through the detector. (These are what are returned by The LISA Simulator and its two approximations.) The goal of gravitational wave data analysis is to extract the signals from the data streams and identify the individual parameters that describe the emitting source.

Chapter 4 introduced two independent approaches to the problem of signal identification and subtraction: linear least squares fitting and template match fitting. The former works by minimizing the difference in the detector output and an initial guess model for the source. The minimization procedure arrives at a set of best fit parameters for the source. In addition to the estimated parameter values, linear least squares returns a quantitative uncertainty for each parameter fit. Template match fitting works by calculating the zero-lag, cross-correlation between the detector output and a model of the source, referred to here as a template. To arrive at an estimate of the original source parameters, a family of templates are generated, each with a different combination of parameter values. By searching for the largest correlation between the detector output and the templates, an estimate for the original source parameters is made.

When fitting to a particular signal with either approach there is necessarily contamination due to the intrinsic noise in the detector and from other sources in the same region of frequency space. The contamination means that it is impossible to solve for the true source parameters exactly. Instead what is done is to fit to a pseudo-source which is a combination of the target source and the local effective noise. In the end, what is removed is most of the original target

\(^2\)The work presented in Chapter 4 is currently being prepared for publication [99].
source and a portion of the noise.

Included in the discussion of each data analysis technique is a demonstration of its ability to identify and remove a low frequency, monochromatic signal from simulated LISA data streams. Given the same source and noise realization, both methods return parameter estimations that agree to within numerical tolerances.

**Galactic Background**

Inside the Milky Way galaxy there are $\sim 10^{10}$ binaries whose emitted gravitational waves fall within the LISA band. The superposition of signals from these galactic sources will form a background signal in the LISA data streams. To properly understand the detector’s intrinsic noise and to formulate robust data analysis techniques, it is vital that an accurate model of the detector’s response to the galactic background is formulated.

Previous analyses of the galactic background were based on the average gravitational wave luminosity. While this approach has the advantage of being easy to implement, it fails to capture the significance of the relatively bright sources. The bright sources weight the average values and lead to predicted levels for the confusion background that are larger than what will actually be observed in LISA. Additionally, prior work used LISA’s all sky and orientation averaged sensitivity curve as a point of reference to demonstrate how prevalent the galactic background will be. By doing so they neglect modulation and detector efficiency effects associated with the response of each source.

The Monte Carlo simulations of the galactic background presented in Chapter 5 modeled each binary source and processed it through the detector using either the *Low Frequency* or the *Rigid Adiabatic Approximations* of Chapter 3. By modeling each source we are able to statistically analyze the fluctuations in the galactic background. We found that in the frequency regions where the galactic background dominates the response of the detector, the Fourier coefficients are not Gaussian distributed. This allows the background to be identified separate from the intrinsic noise of the detector. Additionally, we found that the fluctuations are associated with nearby compact binaries. After these sources are removed from the data streams, the remaining galactic background in Gaussian in nature.

Our work also demonstrates that the detector efficiency and modulation spreading causes
the background levels to decrease by roughly an order of magnitude relative to the barycenter frame. This implies that the galactic background may not be as much of a hindrance in extracting signals from other gravitational wave sources as previously thought.³

**FUTURE WORK AND ANALYSES**

With gravitational wave astronomy still in its infancy, there are many important issues that still need to be addressed. The work presented in this dissertation only scratches the surface of what needs to be done to prepare for future spaceborne detectors. Over the course of the next few years new studies will demonstrate how to use LISA as an observational tool capable of making profound discoveries and contributions to our understanding of astrophysical objects and the Universe. Beyond LISA, next generation spaceborne gravitational wave detectors will improve on LISA’s reach, and open up new areas of discovery. The final two subsections outline two such topics which build on the work presented here.

Combined Electromagnetic and Gravitational Wave Observations

Through electromagnetic surveys a handful of close white dwarf and interacting binaries are already known to exist with orbital frequencies inside the LISA band. These binaries are not only guaranteed sources for the LISA mission, but they also present ideal systems for the study of binary physics. For example, one of the outstanding problems in simulating semi-detached and contact binaries is how to model the mass transfer process accurately. By combining information from electromagnetic and gravitational wave observations it is possible to directly measure the mass transfer rate.

By themselves the gravitational wave and electromagnetic spectrums are not able to completely identify the source parameters due to correlations between individual variables. Consider spectroscopic binaries where a single radial velocity curve is measured. In this case the component masses and orbital inclination are coupled together allowing only a single mass function to be determined. By adding a second spectrum of data some of the correlations are broken, allowing stronger limits to be set on the parameter values.

³Our analysis of the galactic gravitational wave background is currently being prepared for publication [100].
In addition to breaking variable correlations, each spectrum is able to measure certain parameters to exquisite accuracy. Electromagnetic telescopes are able to pinpoint the sky location of a source, while LISA has poor sky location resolution [69, 101]. Conversely, electromagnetic observations are unable to easily measure the orbital frequency of a binary, while LISA is able to measure the frequency to roughly ten significant figures.

The methodology of combining the two spectra follows the linear least squares fitting routine in an analogous way that multiple channels are combined together. For each spectra their is a corresponding Fisher information matrix. The net Fisher matrix is found by linearly adding together the individual matrices,

$$\Gamma_{ij} = \Gamma_{ij}^{GW} + \Gamma_{ij}^{EM}. \quad (6.1)$$

Since the uncertainties in the source position from electromagnetic measurements are small, the corresponding Fisher matrix will contain relatively large values in the diagonal elements for the angular position variables and zeroes in the associated off-diagonal elements. Therefore, when added to the gravitational wave Fisher matrix, the net diagonal elements describing the angular positional variables will be dictated solely by the electromagnetic measurements. One way to think of this is that the information from the gravitational wave observations is no longer being used to try to locate the source. Instead the gravitational wave information is used to estimate the remaining source parameters. Consequently, better fits for each of the parameters can be calculated.

The Big Bang Observer

During the early 1980’s a select group of physicists explained how an early period of exponentially rapid expansion in the universe, known as inflation, could answer a number of unresolved issues left by standard Big Bang theory [102, 103, 104]. As a byproduct, inflationary theory makes three testable predictions: first, the universe should be spatially flat. Second, the amplification of quantum fluctuations during inflation would lead to a nearly scale invariant spectrum of density fluctuations. Third, there will be relic gravitational waves observable today. Through electromagnetic measurements performed in the last few years, most
Figure 6.1: The initial design for BBO calls for four LISA-like spacecraft constellations, two of which are overlaid and the remaining two leading or trailing the overlaid pair by 120°. The mean arm lengths for the BBO observatories are $5 \times 10^4$ km, a factor of 100 less than called for in the LISA mission.

notably those by WMAP, the first two predictions have been tested and verified, leaving the measurement of the relic gravitational waves to future experiments.

To this end, a follow on mission to LISA, known as the Big Bang Observer (BBO), has been proposed [105]. The primary goal of the BBO is to directly measure the relic gravitational radiation predicted by inflationary models. The best window on the gravitational wave spectrum for such a detection places BBO between LISA and LIGO with its peak sensitivity in the range of 0.01 to 10 Hz.

In many ways the BBO mission will be an updated version of LISA, see Figure 6.1. Initial plans for BBO use four LISA-like spacecraft constellations, two of which are overlaid either 20° in front of or behind the Earth in its orbit. The remaining two constellations are located $\pm120°$ away from the overlaid pair. The multi constellation arrangement is required since the primary sources of gravitational waves in BBO’s band (other than the relic waves) will be coalescing binaries with a neutron star and/or black hole. Because the binaries are in the act of coalescing, their presence in the detector data streams will be temporary (just a few months). As a result, there will be limited Doppler modulation information available for locating the sources on the celestial sphere. By using multiple constellations, time of arrival data will allow BBO to triangulate the source location.
As with all missions, forward modeling of the detector prior to launch is essential for testing design ideas and developing data analysis strategies. Since BBO will follow the LISA design closely, much of the forward modeling presented in Chapter 3 can be easily converted to BBO specifications. To construct a BBO Simulator based on The LISA Simulator would only require adjustments to the individual noise tolerance levels, and to the spacecraft orbits. Additionally, a BBO Simulator would require a set of four simulations to account for each constellation of spacecraft.

**Concluding Remarks**

The Big Bang Observer, along with LIGO and LISA, demonstrate how the field of gravitational wave astronomy naturally integrates itself into the larger discipline of observationally astronomy. By opening a new window on the Universe the information gained through gravitational wave detection enhances our current observations with electromagnetic telescopes, while at the same time allowing studies into regions of space that were previously unobservable due to their low electromagnetic emissions. Gravitational wave astronomy is one detection away from becoming the newest observational science.
BIBLIOGRAPHY


