Robert Grosseteste, and the History of the Actual Infinite

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Abstract: The problems with the notion of infinity that plagued pre-modern philosophers and mathematicians ever since the introduction of Zeno's paradoxes are thought to see their first solution in the original and singular accomplishments of the late-19th century German mathematician Georg Cantor. In this paper I argue that a select few Medieval philosophers advanced the concept of the actual infinite from its largely Aristotelian conception to a stage that foreshadowed Cantor's accomplishments. I emphasize, in particular, the contributions of the 13th century scholastic philosopher Robert Grosseteste, whose work in this arena seems especially under-recognized and deserving of tribute.

I. Introduction – infinity in modernity

The history of infinity, as far as modern mathematics is concerned, may consist of little more than the success story of one Georg Cantor. In the late-19th and early-20th century Cantor, a German mathematician and philosopher, founded modern set theory and used it to formalize the notion of infinity for the first time in meaningful and definitive mathematical terms. He proved that there are infinite sets of different sizes (or *transfinite* sets), namely the uncountably infinite set – corresponding to the infinitely-dense and infinitely-extended real number line – and the countably infinite set, of which the set of integers is one example. The all-important insight that allowed him this result was the set-theoretic notion of a one-to-one correspondence—two sets are said to be equivalent if the members of one can be made to correspond in a one-to-one relation with the members of the other. This fact allowed Cantor a means of determining that two *infinite* sets might be equal, or they might differ in size.

The singularity of Cantor's insight shook the world of mathematics, and caused Bertrand Russell to rhapsodize: "The mathematical theory of infinity may almost be said to begin with Cantor. The infinitesimal Calculus, though it cannot wholly dispense with infinity, has as few dealings with it as possible, and contrives to hide it away before facing the world. Cantor has abandoned this cowardly policy, and has brought the skeleton out of its cupboard."[1] Cantor's discoveries were apparently so without precedent that Russell said, of Zeno's paradoxes, that "all intervening attempts to deal with the problem are futile and negligible", before Cantor, that is.[2]

It is true that most thinkers prior to Newton and Leibniz, who separately developed the infinitesimal calculus in the 17th century, shied away from positing the existence of actual infinite quantities. Their response to Zeno's paradox, in general, was just some version of Aristotle's—*viz.* that the infinite exists in potentiality, but never in actuality. Yet there were certain medieval philosophers working in relative obscurity who cultivated conceptions of

[1] *The Principles of Mathematics*, p. 304.
[2] *Our Knowledge of the External World,* p. 169.

infinity that could be said to have anticipated some of Cantor's insights, namely the existence of actual infinities of different sizes, and the analytical power of the one-to-one correspondence. I speak particularly of Robert Grosseteste, the 13th-century English scholastic, Gersonides, a Jewish philosopher of the same time period, and Gregory of Romini, a scholastic of the 14th-century. Whether or not Cantor could have been influenced by their writings I do not wish to speculate, but it is surely worth our while to recognize the ways in which these thinkers advanced the conception of infinity closer to our modern notion. In the following I will aim to situate the work of these philosophers in their time and into ours by briefly surveying the attitude of the ancients toward the actual infinite and considering how that attitude persisted into the middle ages. At last I will sketch out the significant ideas of these thinkers and show how they stand apart from the ideas of their contemporaries, and explain in what ways I see them as anticipating the modern conception of infinity, as it's illustrated in Cantor's theories and, to a lesser extent, in Newton and Leibniz's calculus.

II. The ancients – Aristotle and Zeno

For the vast majority of ancient and medieval thinkers, the concept of "infinity" was troubled by paradox and inconsistency. Both the infinitely large and the infinitely small, when analyzed in earnest, were found to engender contradictions that, for a long time, seemed all but impenetrable. The inscrutability of the infinitely small was famously illustrated by Zeno in his series of paradoxes, allegedly conceived in defense of Parmenides' doctrine that all is One—the real world is a unified whole, and any change and differentiation that we perceive in it is illusion. In the best-known version of the paradox Achilles, the fleetest of foot, is chasing the tortoise, who is notoriously slow. In order for Achilles to overtake the tortoise it is clear that he must first reach the point where the tortoise was when Achilles began his chase. In the time it takes Achilles to get there however, the tortoise will have advanced some distance *x*. To reach the Tortoise now Achilles must first traverse that distance *x*, in which time the Tortoise will have advanced some more—say, a distance of *x'*. Now in the time it takes Achilles to travel the distance *x'* the Tortoise will have traveled even farther, *x*'', and while Achilles travels *this* distance the Tortoise will have advanced by *x*''', and Achilles must travel *x''',* but meanwhile the tortoise has advanced…and so on *ad infinitum*. Achilles will have to traverse an infinite number of distances x^n and since this is obviously impossible, Achilles can never reach the tortoise! Most of Zeno's paradoxes are in some way analogous to this one. Each shows that any continuous and finite length, be it mathematical or physical, can be divided up *infinitely many* times. And if the length is comprised of infinitely many intervals whose lengths, we must assume, are greater than 0, how can the larger length still be finite? Zeno's solution, evidently, is to assume that neither plurality, motion nor change are real. Few are really convinced of this, however, and so an explanation of how Zeno's paradox can be solved or dissolved is desired to get around the problem.

Aristotle, whose writings are the source of our knowledge of Zeno's paradoxes, advances a theory of infinity that will dissolve the problems Zeno presents. This conception of infinity provides the template for about 1500 years of philosophical approaches to the subject, as well. In his *Physics* and *Metaphysics,* Aristotle asserts that infinity cannot exist in actuality, but only in potentiality. We can continually divide the parts of the finite magnitude, potentially infinitely many times, but the *infinitely* small will never be actualized, for the smallest part can always be divided once more.[3] A similar principle precludes the actualization of the infinitely large. We may add finite quantity to finite quantity potentially infinitely many times, but there will never

[3] "But the infinite does not exist potentially in the sense that it will ever actually have separate existence; it exists potentially only for knowledge. For the fact that the process of dividing never comes to an end ensures that this activity exists potentially, but not that the infinite exists separately." (*Metaphysics* 1048b 13-17.)

be a point when we cannot add something to the quantity, and if the magnitude can yet be increased it must therefore be finite. An actually infinitely large magnitude will never be reached, nor will an indivisibly small – and thus actually infinitely small – magnitude ever be obtained. This notion of infinity, Aristotle holds, is sufficient: the universe, after all, is finite in size, and the mathematician has no need for infinities except in potentiality.

III. Infinity in the middle ages

As is the case with many philosophical topics, in their approaches to infinity most medieval thinkers follow the Philosopher. Medieval philosophy is often characterized, indeed, as a synthesis of Aristotelianism and Christianity. Aristotle's conception of the actually infinite as impossible, however, could have been seen by some as suggesting a limitation to the omnipotence of the Christian Creator. Most, though, stuck with Aristotle in asserting that the existence of an infinite magnitude entailed contradiction, and since God would not mar the world with a paradox, an actual infinite could never be created.

Christian cosmologies often followed from Aristotle's and so were finite in extension, wherein the outer-most sphere was that of Heaven. Walter Burley, the English scholastic, summarizes the common view: "Certain theologians allow that God can increase the volume of heaven, that He can, for example, make heaven be twice as large, three times as large, and so forth, indefinitely […] These theologians however would deny that God can create an actual infinite magnitude, for this latter proposition may hold a contradiction."[4] The positions of Aquinas, Averroes, Henry of Ghent and many others followed more or less from Aristotle's argument: the actual infinite is contradictory, and therefore God could not create it.[5] Likewise God could be said to be have infinite *capabilities*, in keeping with His omniscience, but this is nothing more than to say that God's ability is *potentially* infinite.

[4] *Super octo libros Physicorum*, lib. II, tract. II, cap. V, fol. 75, cols. b, c.
[5] See Pierre Duhem's *Medieval Cosmology,* Ch. 2

Could God Himself have been conceived as an example of an actual infinite? This thought would have been problematic due to the association of "endless" with the concept of infinity (Aristotle's word for infinity, *apeiron,* means, quite literally, *without end*): God, as a perfect being, could not be unfinished in any sense, for to be perfect is to be *complete*. Interestingly, Augustine, in his Manichean period, had conceived of God as corporeal and surrounding the finite world, "infinite in all directions."[6] As a Christian, though, he came to adopt a more Platonic conception which was free from that association, wherein God is like Truth, immaterial and, if infinite, only so in potentiality.

It is perhaps not before the thinking of Robert Grosseteste that the actual infinite is seen associated with God Himself, rather than just the potential infinite, as it was associated with His agency. It is suggested by James Ginther[7], at least, that Grosseteste was responsible for the conception of God that gained currency after the 13th century: that actual infinitude was a metaphysical property of God's being. The theological implications of Grosseteste's work do not concern us here, but that Grosseteste believed in the actual infinite and had an informed conception of it does. This is illustrated in his work of naturalist philosophy, *On Light*. In this succinct piece Grosseteste puts forth the theory that light was the first corporeal form created by God. As a simple being (or an indivisibly small magnitude, a monad, in Leibniz's sense) put into existence by God, light "multiplied itself by its very nature an infinite number of times on all sides and spread itself out uniformly in every direction. In this way it proceeded in the beginning of time to extend matter which it could not leave behind, by drawing it out along with itself into a mass the size of the material universe." Instantaneously, then, light, as an infinitely small particle, multiplied itself infinitely many times to fill the three dimensions of the finite universe. He continues: "This extension of matter could not be brought about through a finite multiplication of light, because the multiplication of a simple being a finite number of times does

[6] *Confessions*, v (7)
[7] See Ginther's *Master of the Sacred Page: a Study of the Theology of Robert Grosseteste,* p. 107

not produce a quantity, as Aristotle shows in *De Caelo et Mundo*. However, the multiplication of a simple being an infinite number of times must produce a finite quantity."[8] Grosseteste here traverses the same territory as Zeno's paradoxes, and he seems to suggest that the infinitely small presents none of the problems that Zeno supposes. The infinitely small, when added together an *infinite number of times*, produces a quantity that is finite. The parallel is that though the distances that Achilles must traverse may be infinite in number, because they are becoming infinitely small their addition sums to a finite distance, and a distance that Achilles can and will travel. In mathematical terms, one can divide infinity by infinity – an operation which would be impossible until the development of Calculus – and come up with, as Grosseteste anticipated, a result that is finite.

In *On Light* Grosseteste goes on to assert, in a rudimentary way, the existence of infinities of different sizes: "Now one simple being cannot exceed another simple being infinitely, but only a finite quantity infinitely exceeds a simple being. For an infinite quantity exceeds a simple being by infinity times infinity."[9] The infinite quantity as compared to the simple being, and the infinite quantity as compared to the finite, parallel perfectly Cantor's two transfinite numbers – $\aleph_1$, the size of the set of real numbers, or the set of all points on the real number line, which is infinitely dense (as simple beings in a quantity) and infinitely extended (as an infinite quantity), and $\aleph_0$, the size of the set of the infinitely extended integers. Grosseteste goes on to make a move that modern mathematics would discourage, wherein he says essentially that the set of integers is twice the size of the set of even numbers. According to Cantor's concept of the one-to-one correspondence, because the sets of integers and evens are each infinite and can be put into a one-to-one relation, they must be equal in magnitude. The reason $\aleph_1$ and $\aleph_0$ cannot be made to correspond one-to-one is because the density of the real number line precludes us from putting the real numbers into a coherent order. This mistake should not be

[8] Grosseteste, *On Light*.
[9] Ibid.

to Grosseteste's discredit, for the fact that he countenanced actual infinites of different sizes, including those of the kind that Cantor conceived, is rather remarkable.

There *was* a medieval thinker who apparently recognized the capacity of the one-to-one correspondence as an analytical tool for measuring infinites. This was Gersonides, the French-born Jewish philosopher. He considered the question: given eternal time, how will the number of rotations of the sun compare to the number of rotations of the moon (granted, of course, a geocentric cosmology)? For each rotation of the moon about the earth the sun rotates a fixed number of times. Gersonides, however, believed that as infinites, the number of rotations of one will be equal to the number of rotations to the other because, not withstanding the ratio between them, each number can be brought into a one-to-one correspondence with the natural numbers – each is countably infinite, in other words.[10] I will refrain from saying more about Gersonides, but rather refer my reader to a recent article of George Kohler's, entitled "Medieval Infinites in Mathematics and the Contributions of Gersonides", which describes Gersonides' ideas in great detail. I will simply add my claim to his that Gersonides, like Robert Grosseteste, advanced the concept of infinity substantially in the middle ages.

Finally, I come to Gregory of Rimini, the Italian-born scholastic of the 14th century. Gregory conceived of actual infinite magnitude not, as other medieval thinkers did, as a magnitude such that no greater exists. Rather he likened it to a magnitude greater than any finite quantity, no matter how large it is.[11] For this reason the prolific French philosopher of science Pierre Duhem asserts, in one of the ten volumes of his historical survey of science, *Le système du monde*, that, "for Gregory of Rimini, [actually] infinite magnitude is *transfinite magnitude*."[12] While to consider infinity as an amount to which nothing can be added yields absurdity (for as Aristotle and his followers realized, one can always add more to such a quantity), Gregory's

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[10] Kohler, "Medieval Infinities in Mathematics and the Contribution of Gersonides" *History of Philosophy Quarterly* 23.2, April 2006, p. 110

[11] Duhem, p. 109

[12] Ibid.

conception apparently avoids such troubles, and furthermore allows for infinite magnitudes of different sizes. In response to those who would say that one infinity being *larger* than another is absurd, Gregory writes: "These words [*larger* and *smaller*] can be taken in their proper meaning; […] These words can also be taken in an improper sense; if a multitude contains all the units of another multitude and also some different units than the former, one says that it is larger than the former multiple, even though it does not contain a *larger number* of units […] If one adopts the first definition, the words *larger* and *smaller* must not be used in the comparison of infinites to one another […] According to the second definition, on the other hand, an infinite can be larger than another infinite, in the same way that it can be a *whole* with respect to the second infinite."[13] In his abstruse way, Gregory is apparently arguing that though infinite magnitudes are not *greater* than each other in the finite, numerical sense, the fact that one magnitude can contain more elements than another allows us to see that it is greater in some other sense and is, therefore, evidently of a larger magnitude. Duhem champions Gregory's insight in this arena, writing that: "There is a clear affinity between the thoughts of the two powerful logicians [Gregory and Georg Cantor] even though five-and-a-half centuries separate the times during which they were writing. Gregory of Rimini certainly glimpsed the possibility of the system Cantor constructed; he deemed that there was room for a mathematics of infinite magnitudes and multitudes next to the mathematics of finite numbers and magnitudes. He thought that the two doctrines were two divisions of a more general science."[14] Not all of this is obvious given what Duhem tells us of Gregory's work in the rest of the passage, but it is certainly clear that, in Gregory's definition of the actual infinite and his arguments for the existence of infinites of varying sizes, a part of what Cantor accomplished is contained in a primitive way in Gregory's thinking.

[13] *In primum Sententiarum*, lib. I, dists. XLII, XLIII, XLIV, quaest. IV, art. 2, fol. 173, col. d; fol. 174, col. a.
[14] Duhem, p. 112

IV. Conclusion – "Dominus regnabit in infinitum"

There may be many other unsung middle age thinkers who had similarly forward-thinking notions of the infinite. I've sketched briefly the ideas of just three. Gersonides' and Gregory's accomplishments are perhaps better described elsewhere, in Kohler's piece and in Pierre Duhem's study, the volume I cited which catalogues medieval thinking on infinity, place, time, and other similar topics in an elaborate and encyclopedic fashion. But neither Duhem nor any of the other sources I found (with the exception of the book-length study of Grosseteste's theology) cite Grosseteste as advancing the idea of infinity in any significant way. Perhaps the above will rectify this somewhat, for in his analysis of continuity, as it comes into play with the proliferation of monad-like light particles in *On Light*, we can see Grosseteste as anticipating the solutions of the Calculus to dividing infinity by infinity, and dividing by zero. Similarly, the suggestion that infinite magnitude relates to the magnitude of a point-particle to a degree of "infinity times infinity" suggests Cantor's uncountable infinity $\aleph_1$, which Cantor had seen as analogous with the Absolute Infinite, and God.

Modern mathematics, and to a lesser extent philosophy, is perhaps less interested in the historical developments in the idea of infinity than it's modern conception, as put forth by Georg Cantor. Thus I will leave off by citing some of Cantor's own philosophical thoughts on infinity, which we'll see are strangely consonant, even in their religious bent, with the work of the thinkers mentioned above. Like Grosseteste and Gregory of Rimini, Cantor conceived of the actual infinite as *finished* and yet larger than any quantity of the same kind: *vollendet-undendlich*, or "finished infinite", he called it.[15] He was motivated in large part by the idea that infinitude was a property of God. He wrote: "Accordingly I distinguish an eternal uncreated infinity or absolutum which is due to God and his attributes, and a created infinity or

[15] Cantor, *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts.* p. 175.

transfinitum, which is to be used wherever in the created nature an actual infinity has to be noticed, for example, with respect to, according to my firm conviction, the actually infinite number of created individuals, in the universe as well as on our earth and, most probably, even in every arbitrarily small extended piece of space."[16] In one last quotation, Cantor cites a line from the Bible – "Dominus regnabit in infinitum (aeternum) *et ultra*"[17] – in reference to his concept of ω, the first infinite ordinal number. He writes: "'The Lord rules in infinity (eternity) and beyond.' I think this 'and beyond' is a hint that ω is not the end of the tale."[18] Whether it was, in fact, Grosseteste, Gregory, Gersonides or God Himself who clued Cantor in to the possibility of multiple actual infinites, these former mortal thinkers should be recognized, at least, for advancing the concept of the infinite to a stage that foreshadowed Cantor's definitive work.

[16] Ibid. p. 399
[17] Exodus, 18:15
[18] Cantor, *Georg Cantor Briefe*, p. 148