Part I, strong uniqueness in the Lp-spaces ; Part II, questions on polynomial product approximation and an application
by James Roy Angelos

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics
Montana State University


Abstract:
PART I: STRONG UNIQUENESS IN THE L^P-SPACES, 1< p < 00 Let (X,E,~) be a positive measure space and L^P(X,?,µ) =L^P, 1 < p < 00, be the Banach space of all equivalence classes of real valued p-integrable functions defined on X. Let M be a finite dimensional subspace of L^P, f?L^P\M, and m* the best L^P- approximation to f from M. It is shown that under certain conditions that m* is strongly unique of order a ˜ 1/2 or a ˜ 1/p and in some cases these orders are shown to be best possible. In the case when p ˜ 1 it is shown that the set {f?LI: L^1: f has a strongly unique best approximation from M} is dense in L^1, provided the measure is nonatomic. If the measure is allowed to have atoms, the above set is dense in {f?L^1: f has a unique best approximation from M}. PART II: QUESTIONS ON POLYNOMIAL PRODUCT APPROXIMATION AND AN APPLICATION Let D ˜ [a,b] x [c,d] and p_n and p_m be the sets of algebraic polynomials of degree at most n and m respectively. Let a<•••

Part I

# STRONG UNIQUENESS IN THE $L^p$-SPACES, $1 \leq p < \infty$

Part II

# QUESTIONS ON POLYNOMIAL PRODUCT APPROXIMATION AND AN APPLICATION

by

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A thesis submitted in partial fulfillment
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in

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## ABSTRACT


### PART I: STRONG UNIQUENESS IN THE $L^p$-SPACES, $1 \leq p < \infty$

Let $(X,\Sigma,\mu)$ be a positive measure space and $L^p(X,\Sigma,\mu) \equiv L^p$, $1 < p < \infty$, be the Banach space of all equivalence classes of real-valued p-integrable functions defined on X. Let M be a finite dimensional subspace of $L^p$, $f \in L^p \setminus M$, and m* the best $L^p$- approximation to f from M. It is shown that under certain conditions that m* is strongly unique of order $\alpha = 1/2$ or $\alpha = 1/p$ and in some cases these orders are shown to be best possible.

In the case when $p = 1$ it is shown that the set $\{f \in L^1$: f has a strongly unique best approximation from M$\}$ is dense in $L^1$, provided the measure is nonatomic. If the measure is allowed to have atoms, the above set is dense in $\{f \in L^1$: f has a unique best approximation from M$\}$.

### PART II: QUESTIONS ON POLYNOMIAL PRODUCT APPROXIMATION AND AN APPLICATION

Let $D = [a,b] \times [c,d]$ and $\Pi_n$ and $\Pi_m$ be the sets of algebraic polynomials of degree at most n and m respectively. Let $a \leq x_0 < x_1 < \ldots < x_n \leq b$ be n+1 distinct points and $\ell_0(x), \ell_1(x), \ldots, \ell_n(x)$ be the Lagrange polynomial basis for $\Pi_n$ defined on $x_0, x_1, \ldots, x_n$. Let $\psi_0(x), \ldots, \psi_m(x)$ be any basis for $\Pi_m$. The product Chebyshev approximation of $F \in C(D)$ is the function $(P_{n,m}F)(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{m} c_{ij}\psi_j(y)\ell_i(x)$ where $\sum_{i=0}^{n} a_i(y)\ell_i(x)$ is the best approximation to $F_y(x) = F(x,y)$ over $[a,b]$ from $\Pi_n$ and $\sum_{j=0}^{m} c_{ij}\psi_j(y)$ is the best approximation to $a_i(y)$ over $[c,d]$ from $\Pi_m$.

Error bounds are obtained for this and another method of multivariate approximation. These bounds are in terms of univariate error bounds and are shown to be sharp in an asymptotic. It is also shown that in some situations the product Chebyshev approximant of a continuous function may fail to converge.

A method of approximating the solutions of linear integral and integrodifferential equations is developed by utilizing the product approximant to the kernel function, the theory of degenerate kernels, and compact operator approximation theory. In all cases, the approximate solutions are shown to converge to the true solutions.

# Part I

## STRONG UNIQUENESS

## THE $L^p$-SPACES, $1 \leq p < \infty$

## INTRODUCTION

Given a real Banach space X with norm denoted by $\|\cdot\|$, a closed subset M of X, and $x \in X \setminus M$, $m^* \in M$ is called a strongly unique best approximation to x from M if there is a positive constant $\gamma$, depending on x, such that

$$\| x - m\| \geq \| x - m^*\| + \gamma \| m-m^* \|$$

for all $m \in M$.

The concept of strong uniqueness has been extensively studied by many people. The space C(T) where T is a compact metric space and M is a finite dimensional Chebyshev (Haar) subspace is the first setting known to enjoy the strong uniqueness property [14]. When T is a compact subset of [a,b], and M is a finite dimensional Chebyshev (Haar) subspace of C(T), strong uniqueness plays a vital role in the computation of a best approximation by use of the Remes exchange algorithm. In the event that strong uniqueness occurs at some point $x \in X \setminus M$, then it can be easily seen [4 , p. 82] that the best approximation operation is Lipschitz continuous at x.

In some nonlinear approximating sets, for example, the sets of rational functions on C[a,b] and reciprocals of polynomials on $C[0,\infty)$, strong uniqueness is known to occur. Again, because of this, the computation of a best approximation can be done by use of the Remes exchange algorithm and the best approximation operator is point Lipschitz continuous.

There are, however, many important cases where strong uniqueness fails. Wulbert [22] has shown that if X is a smooth Banach space and M is a nontrivial subspace of X, then no $x \in X \setminus M$ has a strongly unique best approximation from M. This class of smooth Banach spaces includes such spaces as the $L^p$-spaces, $1 < p < \infty$, and Hilbert space. Strong uniqueness also fails for monotone approximation in C[a,b] [ 8].

In examining the way that strong uniqueness failed for monotone approximation, Schmidt [18] was prompted to provide the following alternative to strong uniqueness. For $x \in X \setminus M$, $m^* \in M$ is called a strongly unique best approximation of order $\alpha$, $0 < \alpha \leq 1$, if for all $r > 0$ there exists a positive constant $\gamma$, depending on x and r, such that

$$\| x-m \| \geq \| x-m^* \| + \gamma \| m-m^* \|^{1/\alpha}$$

for all $m \in M$ with $\| m^*-m \| \leq r$. Using this concept, Schmidt [18] showed that strong uniqueness of order 1/2 holds for monotone approximation. Of course, if strong uniqueness of order $\alpha$ holds at $x \in X \setminus M$, then the best approximation operator is Lipschitz continuous of order $\alpha$ at x.

The purpose of this report is to study strong uniqueness in the $L^p$-spaces, $1 \leq p < \infty$. It is shown that all finite dimensional subspaces of $L^p$, $1 < p < \infty$, admit strongly unique best approximations of order 1/2 or 1/p to functions not in the subspace. For the case of

$2 \leq p < \infty$, the orders of 1/2 and 1/p are shown to be the largest possible. For the $L^1$ space with an arbitrary measure strong uniqueness is shown to be a somewhat prevalent property in the following sense. Given a finite dimensional subspace M of $L^1$ the set $\{f \in L^1$: f has a strongly unique best approximation from M$\}$ is dense in the set $\{f \in L^1$: f has a unique best approximation from M$\}$.

Chapter 1 provides background results on the differentiation of Banach space valued functions and on smooth Banach spaces with differentiable norms. Also included in this chapter are the properties of strong uniqueness of order $\alpha$. The chapter concludes with a general result on the strong uniqueness of order $\alpha$ of best approximations from finite dimensional subspaces of certain smooth Banach spaces.

Chapter 2 contains a study of the strong uniqueness of order $\alpha$ of best approximation from finite dimensional subspaces of $L^p$, $1 < p < \infty$.

Chapter 3 provides some background material on approximation in $L^1$ space which is then used to prove that strong uniqueness occurs quite often when approximating from finite dimensional subspaces.

# Chapter 1

# PRELIMINARY RESULTS

## 1.0 Introduction

In this chapter we shall discuss some preliminary results about the differentiation of functions between Banach spaces. These results will then be applied to differentiation of a norm on a Banach space. In addition, a discussion of strong uniqueness of order $\alpha$, $0 < \alpha \leq 1$, of unique best approximations from subspaces of Banach spaces is given.

## 1.1 General Differentiation

Let X and Y be real Banach spaces with norms denoted by $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. The unit sphere of X and Y will be denoted by S(X) and S(Y), respectively, and their normed duals by X* and Y*, respectively. The set of all continuous linear operators from X to Y will be denoted by B(X,Y). $B_n(X,Y)$ will be the set of all continuous n-linear operators from X to Y, i.e., if $L \in B_n(X, Y)$ we have

(a) $L(x_1, \ldots, \alpha x_i' + \beta x_i'', \ldots, x_n)$

$= \alpha L(x_1, \ldots, x_i', \ldots, x_n) + \beta L(x_1, \ldots, x_i'', \ldots, x_n)$

for $i = 1, 2, \ldots, n$ and all real numbers $\alpha$ and $\beta$, and

(b) there is a number $M > 0$ such that

$$\| L(x_1, x_2, \ldots, x_n) \|_Y \leq M \| x_1\|_X \| x_2 \|_X \cdots \| x_n \|_X$$

for all $x_1, x_2, \ldots x_n \in X$.

The following definitions and results can be found in various forms in [7], [13] and [21].

<u>Definition 1.1.1.</u> Let F be a function from X to Y and $x \in X$. If there exists an element $DF(x)$ in $B(X, Y)$ such that

$$\lim_{t \to 0} \left\| \frac{F(x + t z) - F(x)}{t} - DF(x)(z) \right\|_Y = 0$$

uniformly for all $z \in S(X)$, then F is said to be <u>Fréchet differentiable</u> at x. The operator $DF(x)$ is called the <u>Fréchet derivative</u> of F at x and the operator $DF: X \to B(X, Y)$ which assigns to x the operator $DF(x)$ is called the Fréchet derivative of F.

By applying the above definition one can easily prove

<u>Proposition 1.1.2.</u> If $F: X \to Y$ is Fréchet differentiable at x, then F is continuous at x.

We shall have need of

<u>Proposition 1.1.3.</u> If the function $F: X \to R$ (the real line) has a local minimum or local maximum at $x_0 \in X$ and $DF(x_0)$ exists, then $DF(x) = 0$.

Higher derivatives $D^n F(x) \in B_n(X, Y)$ are defined recursively as follows.

<u>Definition 1.1.4.</u> Suppose $F: X \to Y$ is (n-1)- times continuously Fréchet differentiable in a neighborhood of a point $x \in X$. If there exists an element $D^n F(x)(\cdot, \cdot, \ldots, \cdot)$ in $B_n(X, Y)$ such that

$$\lim_{t \to 0} \left\| \frac{1}{t} (D^{n-1} F(x + tz_1)(z_2, \ldots, z_n) - D^{n-1} F(x)(z_2, \ldots, z_n)) - D^n F(x)(z_1, z_2, \ldots, z_n) \right\|_Y = 0$$

uniformly for all $z_1, z_2, \ldots, z_n \in S(X)$, then F is said to be n-times continuously Fréchet differentiable at x. The operator $D^nF: X \to B_n(X,Y)$ is called the nth Fréchet derivative of F, for n = 1, 2, ....

In the next section we shall need the following generalization of Taylor's Theorem.

<u>Proposition 1.1.5.</u> If $F: X \to R$ is n-times continuously Fréchet differentiable then

$$F(x_1) = F(x_0) + \sum_{k=1}^{n-1} \frac{1}{k!} D^k F(x_0)(x_1 - x_0, \ldots, x_1 - x_0) + \int_0^1 D^nF(x_0 + t(x_1 - x_0))(x_1 - x_0, \ldots, x_1 - x_0) \frac{(1-t)^{n-1}}{(n-1)!} dt$$

By applying the mean-value theorem for integrals we have, for some $t_0$, $0 \le t_0 \le 1$,

$$\int_0^1 D^nF(x_0 + t(x_1 - x_0))(x_1 - x_0, \ldots, x_1 - x_0) \frac{(1-t)^{n-1}}{(n-1)!} dt = D^nF(x_0 + t_0(x_1 - x_0))(x_1 - x_0, \ldots, x_1 - x_0) \times \int_0^t \frac{(1-t)^{n-1}}{(n-1)!} dt$$

$$= \frac{1}{n!} D^nF(x_0 + t_0(x_1 - x_0))(x_1 - x_0, \ldots, x_1 - x_0)$$

Therefore

$$F(x_1) = F(x_0) + \sum_{k=1}^{n-1} \frac{1}{k!} D^kF(x_0)(x_1 - x_0, \ldots, x_1 - x_0)$$

$$+ \frac{1}{n!} D^nF(x_0 + t_0(x_1 - x_0))(x_1 - x_0, \ldots, x_1 - x_0).$$

## 1.2 Smooth Banach Spaces and Norm Differentiation

Let $x \in S(X)$. Then the Hahn-Banach Theorem implies that there is an element $f_x \in S(X^*)$ such that $f_x(x) = \|x\| = \|f_x\| = 1$. Such a functional $f_x$ is called a support functional and the mapping $x \mapsto f_x$ from $S(X)$ to $S(X^*)$ is called a support mapping. We can extend this mapping to all of $X\setminus\{0\}$ by noting that if $\lambda > 0$ then $f_{\lambda x} = \lambda f_x$.

Definition 1.2.1 [6]. The Banach space X is said to be smooth at $x_0 \in S(X)$ if there exists a unique $f \in S(X^*)$ such that $f(x_0) = 1$, i.e., there exists a unique support functional for $x_0$. If X is smooth at each point of S(X) then we say X is smooth.

Definition 1.2.2. The norm of a Banach space X is said to be Gâteaux differentiable at $x_0 \in S(X)$ if for $z \in S(X)$

$$\lim_{t \to 0} \frac{\|x_0 + tz\| - \|x_0\|}{t} \equiv G(x_0; z)$$

exists. If this limit exists at each point of S(X) we say X has a Gâteaux differentiable norm.

We note some properties of $G(\cdot;\cdot)$.

1. $G(x;\cdot)$ is a mapping that assigns to each $x\in S(X)$ a real number.

2. $G(\cdot;\cdot)$ is a mapping that assigns to each $x\in S(X)$ the functional $G(x;\cdot)$.

3. If $\lambda > 0$, then $G(\lambda x;\cdot) = \lambda G(x;\cdot)$.

4. $G(x;x) = \|x\| = 1$.

5. For $\lambda\in R$ we have $G(x;\lambda z) = \lambda G(x;z)$.

6. Suppose $x_0\in S(X)$, $f_{x_0}$ is a support functional at $x_0$, and the Gâteaux derivative of the norm exists at $x_0$. If $t > 0$, then for $y\in S(X)$

$$f_{x_0}(y) = \frac{f_{x_0}(ty)}{t\|x_0\|} = \frac{f_{x_0}(x_0) - 1 + f_{x_0}(ty)}{t\|x_0\|}$$

$$= \frac{f_{x_0}(x_0 + ty) - \|x_0\|^2}{t\|x_0\|}$$

$$\leq \frac{|f_{x_0}(x_0 + ty)| - \|x_0\|^2}{t\|x_0\|}$$

$$\leq \frac{\|f_{x_0}\| \ \|x_0 + ty\| - \|x_0\|^2}{t\|x_0\|} .$$

$$= \frac{\|x_0 + ty\| - \|x_0\|^2}{t}$$

For $t < 0$ and $y\in S(X)$

$$f_{x_0}(y) = \frac{f_{x_0}(-ty)}{-t\|x_0\|} = \frac{f_{x_0}(-ty) - f_{x_0}(x_0) + 1}{-t\|x_0\|}$$

$$= \frac{\|x_0\|^2 - f_{x_0}(x_0 + ty)}{-t\|x_0\|}$$

$$\geq \frac{\|x_0\|^2 - |f_{x_0}(x_0 + ty)|}{-t\|x_0\|}$$

$$\geq \frac{\|x_0\|^2 - \|f_{x_0}\| \; \|x_0 + ty\|}{-t\|x_0\|}$$

$$= \frac{\|x_0\| - \|x_0 + ty\|}{-t}$$

$$= \frac{\|x_0 + ty\| - \|x_0\|}{t} .$$

Therefore $G(x_0; y) = f_{x_0}(y)$ and so the mapping $G(x_0;\cdot)$ is a support functional which implies the mapping $G(\cdot;\cdot)$ is a support mapping. However, the mapping $G(\cdot;\cdot)$ may not be a bounded linear operator as can be seen from the following.

Theorem 1.2.3.[6]. Let $x_0 \varepsilon S(X)$. The following are equivalent:

(i) $X$ is smooth at $x_0$;

(ii) every support mapping is norm to weak-star continuous from $S(X)$ to $S(X^*)$ at $x_0$.

(iii) there exists a support mapping that is norm to weak-star continuous from $S(X)$ to $S(X^*)$ at $x_0$;

(iv) the norm of $X$ is Gâteaux differentiable at $x_0$ .

In the next chapter we shall require the first and higher derivatives of the norm to be continuous operators in order to apply the results of the previous section. It therefore becomes necessary to define a norm derivative that insures this.

Definition 1.2.4. The norm of a Banach space X is said to be Fréchet differentiable at $x_0 \in S(X)$ if

$$\lim_{t \to 0} \frac{\| x_0 + t z \| - \| x_0 \|}{t} \equiv D(x_0;z)$$

exists uniformly for all $z \in S(X)$. If this limit exists at each point of S(X), we say X has a Fréchet differentiable norm.

The Fréchet derivative has all the same properties as the Gâteaux derivative and in addition the operator $D(\cdot;\cdot)$ is a bounded linear operator from S(X) to S(X*). This can be seen by noting that the limit in the above definition is uniform for all $z \in S(X)$ and that $D(x_0;\cdot) = f_{x_0}(\cdot)$. With these properties we can then define higher derivatives in the same way as in Definition 1.1.4. Some examples of spaces with Fréchet differentiable norms will now be given.

Example 1: Hilbert Space.

Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and $x, y \in H \setminus \{0\}$. The first Fréchet derivative of the norm on H at x is then given by

$$D(x;\, y) = \lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

$$= \lim_{t \to 0} \frac{\|x + ty\|^2 - \|x\|^2}{t(\|x + ty\| + \|x\|)}$$

$$= \lim_{t \to 0} \frac{2\langle x, ty\rangle + t^2 \langle y,y\rangle}{t(\|x + ty\| + \|x\|)}$$

$$= \lim_{t \to 0} \frac{2\langle x,y\rangle + t\,\langle y,y\rangle}{\|x + ty\| + \|x\|}.$$

$$= \frac{1}{\|x\|} \langle x,y\rangle .$$

For the second Fréchet derivative, let x, y, $z \in H$. Then

$$D^2(x;y,z) = \lim_{t \to 0} \frac{1}{t}\Big[D(x + ty;\, z) - D(x\,;\, z)\Big]$$

$$= \lim_{t \to 0} \frac{1}{t}\left[\frac{\langle x + ty, z\rangle}{\|x + ty\|} - \frac{\langle x,z\rangle}{\|x\|}\right]$$

$$= \lim_{t \to 0} \frac{1}{t}\left[\frac{\langle x,z\rangle}{\|x + ty\|} + \frac{t\langle y,z\rangle}{\|x + ty\|} - \frac{\langle x,z\rangle}{\|x\|}\right]$$

$$= \lim_{t \to 0} \frac{1}{t}\left[\frac{\|x\| - \|x + ty\|}{\|x\|\;\|x + ty\|} \langle x,z\rangle\right] + \frac{\langle y,z\rangle}{\|x\|}$$

$$= \lim_{t \to 0} \frac{1}{t}\left[\frac{\|x\|^2 - \|x + ty\|^2}{\|x\|\;\|x + ty\|\,(\|x\| + \|x + ty\|)} \langle x,z\rangle\right] + \frac{\langle y,z\rangle}{\|x\|}$$

$$= \lim_{t \to 0} \left[\frac{-2\langle x,y\rangle - t\,\langle y,y\rangle}{\|x\|\;\|x + ty\|\,(\|x\| + \|x + ty\|)} \langle x,z\rangle\right] + \frac{\langle y,z\rangle}{\|x\|}$$

$$= \frac{\langle y,z\rangle}{\|x\|} - \frac{\langle x,y\rangle\ \langle x,z\rangle}{\|x\|^3} .$$

Note that

1. $D^2(x;\cdot,\cdot) \in B_2(H,R)$ and is symmetric.

2. $D^2(x;y,y) \geq 0$.

3. $D^2(x;x,x) = 0$.

4. If $\lambda \neq 0$ then $D^2(\lambda x;\cdot,\cdot)$ exists and $D^2(\lambda x;\cdot,\cdot) = \frac{1}{|\lambda|} D^2(x;\cdot,\cdot)$.

It is known [20] that the above properties are always true for the second Frêchet derivative of a norm, when it exists.

<u>Example 2: $L^p$ - space, $1 < p < \infty$.</u>

In this example the derivatives of $\|\cdot\|_p^p$ are given, where we have $L^p \equiv L^p(X, \Sigma, \mu)$, $(X, \Sigma, \mu)$ a positive measure space. In order to do this the following lemma is given which can be found in [20] and is due to Banach and Saks.

<u>Lemma:</u> If a and b are any real numbers and $1 < p < \infty$, then there exists a positive constant M, independent of a, b, and signa such that

$$\left| |a + b|^p - |a|^p - \sum_{i=1}^{e(p)} \binom{p}{i} |a|^{p-i} (\text{signa})^i b^i \right| \leq M |b|^p.$$

Here e(p) denotes the greatest integer less than or equal to p and

$$\text{signa} = \begin{cases} 1 & a > 0 \\ 0; & a = 0 \\ -1; & a < 0 \end{cases}.$$

By applying this lemma Sundaresan [20] proves

<u>Theorem.</u> If $1 \leq p < \infty$, then the norms of $l^p$ and $L^p$ are of class

$C^\infty$ $(C^{p-1})$ if p is an even (odd) integer. They are of class $C^{e(p)}$ if p is not an integer.

Thus if f, $g_1, \ldots, g_k \in L^p\backslash\{0\}$, k is a positive integer, and $p > k$ we have

$$D(f;g_1) = p\int_X g_1 \, |f|^{p-1} \text{ sign } f \, d\mu$$

$$D^2(f;g_1,g_2) = \frac{p(p-1)}{2}\int_X g_1 g_2 \, |f|^{p-2} (\text{sign } f)^2 d\mu$$

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$$D^k(f;\, g_1,\, g_2,\, \ldots,\, g_k) = \frac{p(p-1)\ldots(p-k+1)}{k!}\int_X g_1 g_2 \cdots g_k$$

$$\times \, |f|^{p-k} (\text{sign } f)^k \, d\mu$$

and the mappings $D^k(\cdot;\cdot;\,\ldots,\,\cdot)\colon L^p \to B_k(L^p,R)$ which assign to each $f \in L^p\backslash\{0\}$ the k-linear operator $D^k(f;\cdot,\ldots,\cdot)$ are continuous. Recall that the above expressions are the derivatives of $\|\cdot\|_p^p$. If p is an even integer the (p-1)th derivative for f, $g_1, \ldots, g_p \in L^p\backslash\{0\}$ is given by

$$D^{p-1}(f;\, g_1,\ldots,g_{p-1}) = \frac{p}{(p-1)!}\int_X g_1 \, g_2 \cdots g_{p-1} \, f \, d\mu$$

and so

$$\lim_{t\to 0} \frac{1}{t}\left(D^{p-1}(f+tg_p;\, g_1,\ldots,g_{p-1}) - D^{p-1}(f;\, g_1,\ldots,g_{p-1})\right)$$

$$= \lim_{t\to 0} \frac{1}{t} \frac{p}{(p-1)!}\left(\int_X g_1\, g_2 \cdots g_{p-1}(f + t\, g_p)\, d\mu - \int g_1\, g_2 \cdots g_{p-1} f\, d\mu\right)$$

$$= \frac{p}{(p-1)!} \int_X g_1\, g_2 \cdots g_p\, d\mu.$$

Therefore $D^{p-1}(\cdot;\cdot, \ldots, \cdot): L^p \to B_{p-1}(L^p, F)$ is differentiable and its derivative is a constant mapping. Hence $\|\cdot\|_p^p$ is of class $C^\infty$. If $p$ is an odd integer the $(p-1)$th derivative for $f, g_1, \ldots, g_p \in L^p \setminus \{0\}$ is given by

$$D^{p-1}(f; g_1, \ldots, g_{p-1}) = \frac{p}{(p-1)!} \int_X g_1 g_2 \cdots g_{p-1}\, |f|\, d\mu.$$

and so

$$\lim_{t\to 0} \frac{1}{t} (D^{p-1}(f+tg_p; g_1, \ldots, g_{p-1}) - D^{p-1}(f; g_1, \ldots, g_{p-1}))$$

$$= \lim_{t\to 0} \frac{1}{t} \frac{p}{(p-1)!} \int_X g_1 g_2 \cdots g_{p-1} |f+tg_p| d\mu \int_X g_1 g_2 \cdots g_{p-1} |f| d\mu$$

$$= \frac{p}{(p-1)!} \lim_{t\to 0} \int_X g_1 g_2 \cdots g_{p-1} \frac{|f+tg_p| - |f|}{t}\, d\mu.$$

This limit may fail to exist for some $f, g_p \in L^p \setminus \{0\}$. For example, if $X = [0,1]$ and $\mu$ is Lebesque measure, let $f = \chi_{[0,1/2]}$ and $g_p = \chi_{(1/2,1]}$ where $\chi_A$ denotes the characteristic function for A. Then

$$\frac{p}{(p-1)!} \lim_{t\to 0} \int_X g_1 g_2 \cdots g_{p-1} \left(\frac{|f+tg_p| - |f|}{t}\right) d\mu$$

$$= \frac{p}{(p-1)!} \lim_{t \to 0} \int_{1/2}^{1} g_1 g_2 \cdots g_{p-1} \frac{|t|}{t} \, d\mu$$

which does not exist. Therefore, $\| \cdot \|_p^p$ is of class $C^{p-1}$.

A similar argument shows that if p is not an integer then the norm is of class $C^{e(p)}$. If $p = 1$ the norm is not even once Gâteaux differentiable. Indeed consider $R^n$ with the $l^1$ norm. The point (1, 0, ..., 0) on the unit sphere has more than one support functional. Thus by Theorem 1.2.1, the norm cannot be Gâteaux differentiable.

## 1.3 Strong Uniqueness of order $\alpha$, $0 < \alpha \leq 1$.

Let X be a real Banach space with norm denoted by $\| \cdot \|$, M a closed subspace of X, and $x \in X \setminus M$. The set

$$P_M(x) = \{m^* \in M : \| x - m^* \| = \inf_{m \in M} \| x - m \| \}$$

is called the set of best approximations to x from M. If $P_M(x) = \{m^*\}$, then $m^*$ is the unique best approximation to x from M.

<u>Definition 1.3.1.</u> If $P_M(x) = \{m^*\}$, then $m^*$ is called <u>strongly unique</u> if there is a constant $r > 0$ such that

$$(1.3.1) \quad \| x - m \| \geq \| x - m^* \| + r \, \| m - m^* \|$$

for all $m \in M$. The largest constant $r = \gamma(x)$ for which (1.3.1) holds is called the <u>strong unicity constant</u>.

The concept of strong uniqueness has been extensively studied in the space C(X), X a compact metric space, and is used in the study of the Remes algorithm. The properties of the strong unicity constant have also been studied, particularly in the space C[a,b] with polynomial and rational function approximation. However, there are some situations where strong uniqueness fails to hold. Fletcher and Roulier [8] have shown that strong uniqueness does not hold in the setting of monotone approximation in C[a,b]. Another situation is when X is a smooth Banach space. The following is due to Wulbert [22] and Bartelt [2]; a new proof is given based on norm differentiation.

<u>Theorem 1.3.2.</u> Let x and m* be as in Definition 1.3.1 and suppose X is smooth at x-m*. Then m* is not strongly unique.

<u>Proof.</u> Since X is smooth at x-m* the Gâteaux derivative of the norm exists there. Now m = 0 minimizes $\| x-m^* + m \|$ over M and Proposition 1.1.2 can be shown to apply in this situation, see [13], so that $G(x-m^*;m) = 0$, for $m \in M$, i.e., for $m \in M \setminus \{0\}$

$$\lim_{t \to 0} \frac{\| x - m^* + tm\| - \| x - m^*\|}{t} = 0.$$

Suppose m* is strongly unique, then there is a constant $r > 0$ such that for all t and $m \in M$ with $\|m\| = 1$ we have

$$\| x - (m^* - tm)\| \geq \| x - m^*\| + r \| tm \|$$

thus

$$0 < r \leq \lim_{t \to 0} \frac{\| x - (m^* - tm)\| - \| x - m^*\|}{\| tm \|} = 0$$

which is a contradiction and the theorem is established.

The failure of strong uniqueness for monotone approximation and in smooth spaces or more precisely the way strong uniqueness fails in these settings provided the motivation for the following definition due to Schmidt [18].

Definition 1.3.3. Let x and m* be as in Definition 1.3.1. We say m* is strongly unique of order $\alpha$, $0 < \alpha \leq 1$, if for all $r > 0$ there exists $K(r) > 0$ such that

(1.3.2) $\| m - m^*\| \leq K(r)(\| x - m\| - \| x - m^*\| )^{\alpha}$

for all $m \in M$ with $\|m - m^*\| \leq r$, or, equivalently, if for all $r > 0$ there exists $\gamma(r) > 0$ such that

(1.3.3) $\| x - m\| \geq \| x - m^*\| + \gamma(r) \| m - m^*\|^{1/\alpha}$

for all $m \in M$ with $\| m - m^*\| \leq r$.

Remark. Strong uniqueness of order $\alpha$, $\alpha > 1$, is impossible. Indeed, if m* is strongly unique of order $\alpha$, $\alpha > 1$, then by (1.3.3) for $r > 0$ we have

$$\| m - m^*\| \geq \| x - m\| - \| x - m^* \| \geq \gamma (r) \| m - m^*\|^{1/\alpha}$$

for all $m \in M$ with $\|m - m^*\| \leq r$. Then $0 < \gamma (r) \leq \| m - m^*\|^{1-(1/\alpha)} \to 0$ as $\| m - m^* \| \to 0$, a contradiction.

The following lemma states that if strong uniqueness of order $\alpha$ holds in some relative neighborhood of m*, it holds in all relative neighborhoods of m*.

<u>Lemma 1.3.4.</u> Let x, m*, and $\alpha$ be as in Definition 1.3.3. The following are equivalent:

(i) m* is strongly unique of order $\alpha$;

(ii) there exists a $\rho > 0$ and $K > 0$ such that $\| m - m^* \| \leq K (\| x - m\| - \|x-m^*\| )^{\alpha}$ for all $m \epsilon M$ with $\| m - m^*\| \leq \rho$;

(iii) There exists a $\rho > 0$ and $\gamma > 0$ such that $\| x - m\| \geq \| x-m^* \| + \gamma \| m - m^* \|^{1/\alpha}$ for all $m \epsilon M$ with $\| m - m^*\| \leq \rho$.

<u>Proof.</u> The proofs of (ii) $\rightarrow$ (iii) and (i) $\rightarrow$ (ii) are evident from Definition 1.3.3. To prove (iii) $\rightarrow$ (i), assume (iii) holds and let $r > 0$. Define $\gamma(r) = \gamma$ for $0 < r \leq \rho$;then (1.3.2) holds for $r \leq p$. For $r > 0$ and $\rho < \| m - m^*\| < r$, (iii) yields

$$\left\| x - (m^* + \frac{\rho}{\| m - m^*\|} (m-m^*)) \right\| \geq \| x - m^* \| + \gamma (\rho^{1/\alpha}).$$

But,

$$\left\| x- ( m^* + \frac{\rho}{\| m-m^*\|} (m - m^*)) \right\| = \left\| (x - m^*) (1 - \frac{\rho}{\| m-m^*\|}) + \frac{\rho(x-m)}{\| m-m^*\|} \right\| \leq (1 - \frac{\rho}{\| m-m^* \|}) \| x-m^*\| + \frac{\rho}{\| m-m^*\|} \| x-m \|.$$

Therefore

$$\frac{\rho}{\| m-m^*\|} (\| x-m \| - \| x-m^* \|) \leq \gamma\rho^{1/\alpha}$$

which implies

$$\| x-m \| \geq \| x-m^* \| + \gamma_\rho^{(1/\alpha)-1} \| m-m^* \|$$

$$= \| x-m^* \| + \left(\frac{\gamma_\rho^{(1/\alpha)-1}}{\| m-m^* \|^{(1/\alpha)-1}}\right) \| m-m^* \|^{1/\alpha}$$

$$> \| x-m^* \| + \gamma\left(\frac{\rho}{r}\right)^{(1/\alpha)-1} \| m-m^* \|^{(1/\alpha)}$$

Defining $\gamma(r) = \gamma(\frac{\rho}{r})^{(1/\alpha)-1}$ for $r > \rho$, (1.3.2.) holds for $r > \rho$ and the proof is complete.

The next lemma provides a necessary and sufficient condition for strong uniqueness of order $\alpha$ for finite dimensional subspaces.

Lemma 1.3.5. Let x, m* and $\alpha$ be as in Definition 1.3.3. Assume that $\dim(M)<\infty$. Then m* is strongly unique of order $\alpha$ if and only if

$$(1.3.4) \quad \lim_{k\to\infty} \frac{\| x-m_k \| - \| x-m^* \|}{\| m_k - m^* \|^{1/\alpha}} > 0$$

for all sequences $\{m_k\}$ in $M\setminus\{m^*\}$ with $\lim_{k\to\infty} \| m_k - m^* \| = 0$

Proof. Clearly, if m* is strongly unique of order $\alpha$ then (1.3.4) holds for all sequences $\{m_k\}$ in $M\setminus\{m^*\}$ with $\lim_{k\to\infty} \| m_k - m^* \| = 0$. Note that the hypothesis, $\dim(M)<\infty$, is not needed here. Suppose then that m* is not strongly unique of order $\alpha$. It will be shown that, under this assumption, there exists a sequence $\{m_k\}$ in $M\setminus\{m^*\}$,

$\lim_{k\to\infty} \| m_k - m^* \| = 0$ and

$$\lim_{k\to\infty} \frac{\|x-m_k\| - \|x-m^*\|}{\|m_k - m^*\|^{1/\alpha}} = 0.$$

Two cases are considered.

Case 1: Suppose $m^*$ is not unique, then there exists $m_0 \varepsilon M$, $m_0 \neq m^*$, which is also in $P_M(x)$. Since $M$ is a linear subspace and $P_M(x)$ is convex, $m_k = m^* + (1/k)(m_0 - m^*)$ is also in $P_M(x)$ for $k \geq 1$. So $\{m_k\}$ is in $M\setminus\{m^*\}$, $\lim_{k\to\infty}\|m_k-m^*\| = 0$ and

$$\frac{\|x-m_k\| - \|x-m^*\|}{\|m_k-m^*\|^{1/\alpha}} = 0.$$

Case 2: Suppose $m^*$ is unique. Since $m^*$ is not strongly unique of order $\alpha$, there exists $r > 0$ and a sequence $\{m_k\}$ in $M \ \{m^*\}$ such that $\|m_k-m^*\| \leq r$ and

$$\gamma_k \equiv \frac{\|x - m_k\| - \|x - m^*\|}{\|m_k - m^*\|^{1/\alpha}} \to 0$$

as $k \to \infty$. Since $\{\|m_k\|\}$ is bounded and $\dim(M) < \infty$, we may assume that $\lim_{k\to\infty}\|m_k - m_0\| = 0$ for some $m_0 \varepsilon M$. But,

$$\begin{aligned} \|x - m_0\| - \|x - m^*\| &= \lim_{k\to\infty}(\|x - m_k\| - \|x - m^*\|) \\ &= \lim_{k\to\infty} \gamma_k \|m_k - m^*\|^{1/\alpha} = 0 \end{aligned}$$

since $\lim_{k\to\infty} \gamma_k = 0$ and $\|m_k-m^*\| \leq r$. So $\|x-m_0\| = \|x-m^*\|$ and since $m^*$ is unique it must be that $m_0 = m^*$. Thus $\lim_{k\to\infty}\|m_k-m^*\| = 0$ and the lemma is established.

Remarks.

1. The hypothesis that dim(M) $< \infty$ can be removed if

$$\lim_{k\to\infty} \frac{\| x-m_k \| - \|x-m^* \|}{\| m_k - m^*\|^{1/\alpha}} \geq \gamma > 0$$

for all sequences $\{m_k\}$ in $M\setminus\{m^*\}$ with $\lim_{k\to\infty} \| m_k-m^*\| = 0$ and where $\gamma$ depends only on x and m*.

2. If $\alpha = 1$, then in linear approximation the local strong unicity constant is equal to the global strong unicity constant, see [10]. However, if $0 < \alpha < 1$ the dependence of K(r) on r in (1.3.2) cannot be removed because

$$\lim_{\| m\| \to\infty} \frac{(\| x-m\| - \| x-m^* \|)^{\alpha}}{\| m-m^* \|} = 0.$$

When considering approximation in the $L^p$-spaces we shall require

Lemma 1.3.6. Let x, m*, and $\alpha$ be as in Definition 1.3.3 and $p > 1$. The following are equivalent:

(i) m* is strongly unique of order $\alpha$;

(ii) there exists a $\rho > 0$ and $\gamma > 0$ such that $\| x-m\|^p \geq \|x-m^* \|^p + \gamma \| x-m^*\|^{1/\alpha}$ for all $m\in M$ with $\| m-m^*\| \leq \rho$.

For the proof we will need

Lemma 1.3.7. If $0 < a \leq b \leq 2a$ then for $p > 1$, $b^p - a^p \geq pa^{p-1} (b-a)$ and $b - a \geq (1/p(2a)^{p-1}) (b^p - a^p)$.

Proof. By the mean-value theorem there is a number c between a and b such that $b^p - a^p = pc^{p-1}(b - a) \geq pa^{p-1}(b-a)$ and since $c \leq b \leq 2a$, $b^p - a^p \leq p(2a)^{p-1}(b-a)$.

Proof of Lemma 1.3.6. Assume (i) holds, then by Lemma 1.3.4 there exist $\rho > 0$ and $\gamma^* > 0$ such that $\|x-m\| - \|x-m^*\| \geq \gamma^* \|m-m^*\|^{1/\alpha}$ for all $m \in M$ with $\|m-m^*\| \leq \rho$. By Lemma 1.3.7,

$$\begin{aligned} \|x-m\|^p - \|x-m^*\|^p &\geq p\|x-m^*\|^{p-1}(\|x-m\| - \|x-m^*\|) \\ &\geq (p\gamma^* \|x-m^*\|^{p-1})\|m-m^*\|^{1/\alpha} \end{aligned}$$

for all $m \in M$ with $\|m-m^*\| \leq \rho$ and setting $\gamma = p\gamma^* \|x-m^*\|^{p-1}$, (ii) holds.

Assume (ii) holds, then there exists $\rho > 0$ and $\gamma > 0$ such that $\|x-m\|^p - \|x-m^*\|^p \geq \gamma \|m-m^*\|^{1/\alpha}$ for all $m \in M$ with $\|m-m^*\| \leq \rho$. Let $\rho_1 = \min(\rho, \|x-m^*\|)$. Then if $\|m-m^*\| \leq \rho_1$, we have $\|x-m\| \leq \|x-m^*\| + \|m^*-m\| \leq 2\|x-m^*\|$. Therefore by Lemma 1.3.7,

$$\begin{aligned} \|x-m\| - \|x-m^*\| &\geq \frac{1}{p(2\|x-m^*\|^{p-1})}(\|x-m\|^p - \|x-m^*\|^p) \\ &\geq \left(\frac{\gamma}{p(2\|m-m^*\|^{p-1})}\right)\|m-m^*\|^{1/\alpha}. \end{aligned}$$

Thus, by Lemma 1.3.4, (i) holds and the proof is complete.

Remark. Expression (1.3.4) of Lemma 1.3.5 can be replaced with

$$\lim_{k\to\infty} \frac{\|x-m_k\|^p - \|x-m^*\|^p}{\|m_k-m^*\|^{1/\alpha}} > 0$$

where $p > 1$ and the lemma still holds.

A general situation where strong uniqueness of order $\alpha$ holds will now be presented. This situation will be shown later in Chapter 2 to be the case for the $L^p$ - spaces, $2 < p < \infty$, under certain conditions.

Theorem 1.3.8. Let $x$ and $m^*$ be as in Definition 1.3.1 and $X$ a Banach space with a twice continuously Fréchet differentiable norm in a neighborhood of $x - m^*$. If $\inf_{m\in S(M)} D^2(x-m^*; m,m) > 0$, where $S(M) = \{m\in M: \|m\| = 1\}$, then $m^*$ is strongly unique of order 1/2. Moreover, 1/2 is the largest possible order.

Proof. Let $2\gamma = \inf_{m\in S(M)} D^2(x-m^*; m,m) > 0$. Since $D^2(\cdot;\cdot,\cdot)$ is a continuous bilinear operator valued function at $x - m^*$, there exists $\varepsilon > 0$ such that

$$\| D^2(x-m^*;\cdot,\cdot) - D^2(x-m^* + m;\cdot,\cdot)\| < \gamma$$

for $m\in M$ with $\|m\| < \varepsilon$. Thus for $m_0 \in S(M)$ and $m\in M$ with $\|m\| < \varepsilon$

$$2\gamma - D^2(x-m^* + m; m_0, m_0) \le D^2(x-m^*; m_0, m_0) - D^2(x-m^* + m; m_0, m_0) < \gamma$$

and hence $D^2(x-m^* + m; m_0, m_0) \ge \gamma$. So if $\|m\| < \varepsilon$ and $0 \le t \le 1$ we have $D^2(x-m^* + tm; \frac{m}{\|m\|}, \frac{m}{\|m\|}) \ge \gamma$, and thus

$$(1.3.5)\quad D^2(x-m^* + tm; m,m) \ge \gamma \|m\|^2.$$

Now by Proposition 1.1.3 and the remarks following Proposition 1.1.5 there is a $t_0$, $0 \leq t_0 \leq 1$ such that

$$\|x-m^* + m\| - \|x-m^*\| = (1/2)\, D^2(x-m^* + t_0 m;m,m)$$
$$\leq (1/2)\|D^2(x-m^* + t_0 m)\|\,\|m\|^2.$$

Since $D^2(x-m^*;\cdot,\cdot)$ is a bounded operator, there is a number $L > 0$ such that, for $\|m\|$ sufficiently small, $\|D(x-m^* + tm)\| \leq L$, and so

$$0 \leq \frac{\|x-m^* + m\| - \|x-m^*\|}{\|m\|^{2-\delta}} \leq L\|m\|^{\delta} \to 0$$

as $\|m\| \to 0$. So by Lemma 1.3.5, $m^*$ cannot be strongly unique of order $1/(2-\delta)$ and since $\delta$ is arbitrary, $m^*$ is not strongly unique of order $\alpha$, $\alpha > 1/2$. To show that strong uniqueness of order 1/2 holds, let $m \varepsilon M$ with $\|m\| < \varepsilon$. Then by (1.3.5)

$$\|x-m^* + m\| - \|x-m^*\| = (1/2)\, D^2(x-m^* + tm;m,m)$$
$$\geq (1/2)\,\gamma\,\|m\|^2.$$

Therefore, by Lemma 1.3.4, $m^*$ is strongly unique of order 1/2.

Suppose M is a finite dimensional subspace of a Hilbert space H. Let $x \varepsilon H \setminus M$ and 0 the best approximation to x from M. Thus by example 1, section 2,

$$D^2(x;m,m) = \frac{\|m\|^2}{\|x\|} - \frac{<x,m>^2}{\|x\|^3} > 0$$

for all m S(M) provided strict inequality holds in the Cauchy-Schwarz inequality. This will occur provided $x \notin M$. Therefore, Theorem 1.3.8 applies to approximation in a Hilbert space.

Schmidt [18] has shown that in the setting of monotone approximation in C[a,b] strong uniqueness of order 1/2 holds and an example of Fletcher and Roulier [ 8] shows that, in general, 1/2 is the largest possible order.

# Chapter 2

## STRONG UNIQUENESS IN THE $L^p$-SPACES, $1 < p < \infty$

### 2.0 Introduction

Let $(X, \Sigma, \mu)$ be a complete positive measure space. $L^p(X, \Sigma, \mu) \equiv L^p$ will denote the Banach space of all equivalence spaces of real-valued p-integrable functions on X with norm given by

$$\| f \|_p = \left( \int |f|^p d\mu \right)^{1/p}.$$

The above integral will denote integration over the entire set X unless specified otherwise. Let M be a finite dimensional subspace of $L^p$ and $f \in L^p \setminus M$. Since M is finite dimensional, there exists an element $g^* \in M$ such that

$$\| f - g^* \|_p = \inf_{g \in M} \| f - g \|_p,$$

i.e., $g^*$ is a best $L^p$ - approximation to f from M, see [4, p. 20]. It is known that the $L^p$-spaces ($1 < p < \infty$) are strictly convex and so $g^*$ is unique, see [4, p. 23]. We require the following characterization theorem for $L^p$ - approximation. A more general form can be found in [19].

Theorem A. Let M be a finite dimensional subspace of $L^p$, $f \in L^p \setminus M$, and $g^* \in M$. Then $g^*$ is the best $L^p$ - approximation to f from M if and only if

$$\int g |f - g^*|^{p-1} \operatorname{sign}(f - g^*) \, d\mu = 0$$

for all $g \in M$.

Recall that the first Fréchet derivative of $\|\cdot\|_p^p$ at $f-g^*$ is given by

$$D(f-g^*;g) = p \int g|f-g^*|^{p-1} \operatorname{sign}(f-g^*)\, d\mu .$$

We know from Chapter 1, Section 2 that the $L^p$-spaces $(1 < p < \infty)$ are smooth and so strong uniqueness of order 1 does not hold. Therefore, if such a property holds at all it must be of some order $\alpha$, $0 < \alpha < 1$. It will be shown that under certain conditions Theorem 1.3.8 applies provided $2 < p < \infty$, otherwise the order $1/p$ holds. The case where $1 < p < 2$ must be treated differently since the second Fréchet derivative of the norm does not exist. Again, in this case, the order is either $1/2$ or $1/p$. The case where $p=2$ was discussed following Theorem 1.3.8.

## 2.1 Strong Uniqueness for the $L^p$-spaces

Throughout this section $M \subset L^p$ is a finite dimensional subspace, $f \in L^p \setminus M$, and $g^* \in M$ is the best $L^p$-approximation to $f$ from $M$.

Theorem 2.2.1. Let $2 < p < \infty$ and define $\operatorname{supp}(f-g^*) = \{x \in X: (f-g^*)(x) \neq 0\}$ and $\operatorname{supp}(g)$ similarly. If for all $g \in M \setminus \{0\}$, $\mu(\operatorname{supp}(f-g^*) \cap \operatorname{supp}(g)) > 0$, then $g^*$ is strongly unique of order $1/2$. Moreover, $1/2$ is the largest possible order.

Proof. In view of Theorem 1.3.8 and Lemma 1.3.6 we need only show

$$\inf_{g\epsilon S(M)} \left(\int g^2|f-g^*|^{p-2}\, d\mu\right) > 0.$$

Recall that the expression in the parentheses is a nonzero multiple of the second Fréchet derivative of $\| \cdot \|_p^p$ , and so we shall use Lemma 1.3.6. Since $\dim(M) < \infty$, $S(M)$ is compact so it suffices to prove

$$(2.2.1) \quad \int g^2|f-g^*|^{p-2}\, d\mu > 0$$

for all $g\epsilon S(M)$. Let $g\epsilon S(M)$. By hypothesis $\mu\{x \epsilon \text{supp}(f-g^*): g(x) \neq 0\} > 0$ so $g^2|f-g^*|^{p-2} > 0$ on a set of positive measure. Therefore (2.2.1) holds for all $g\epsilon S(M)$. The proof that 1/2 is the largest possible order for which $g^*$ is strongly unique follows from Theorem 1.3.8 and Lemma 1.3.6.

We also have the following corollaries.

Corollary 1. If $\mu\{x\epsilon X: f(x) = g^*(x) = 0\} = 0$, then $g^*$ is strongly unique of order 1/2.

Proof. In this case, $\text{supp}(f-g^*)$ differs from $X$ by a set of measure zero, hence for all $g\epsilon M\setminus\{0\}$, $\mu(\text{supp}(f-g^*) \cap \text{supp}(g)) > 0$.

Corollary 2. If $\mu\{x\epsilon X: g(x) = 0\} > 0$ implies $g = 0$ for all $g\epsilon M$, then $g^*$ is strongly unique of order 1/2.

Proof. Suppose there exists $g\epsilon M$ such that $\mu(\text{supp}(f-g^*) \cap \text{supp}(g)) = 0$, that is, $\mu\{x \epsilon \text{supp}(f-g^*): g(x) \neq 0\} = 0$ hence $\mu\{x\epsilon X \setminus \text{supp}(f-g^*): g(x) = 0\} > 0$. But $\{x\epsilon X\setminus \text{supp}(f-g^*): g(x) = 0\} \subset \{x\epsilon X: g(x) =$

$0\}$ so $\mu\{x\varepsilon X: g(x) = 0\} = 0$ which implies, by hypothesis, that $g = 0$. Thus if $g\varepsilon M\setminus\{0\}$, then $\mu(\text{supp}(f-g^*)\cap\text{supp}(g)) > 0$.

Remark. The hypothesis of Corollary 2 is a Haar type condition in that it places a restriction on the zeros of the elements of $M\setminus\{0\}$.

A more general situation than that of Theorem 2.2.1 is we allow $\mu(\text{supp }(f-g^*)\cap\text{supp}(g)) = 0$ for some $g\varepsilon M\setminus\{0\}$.

Theorem 2.2.2. Suppose there exists an element $g\varepsilon M\setminus\{0\}$ such that $\mu(\text{supp}(f-g^*)\cap\text{supp}(g)) = 0$. Then $g^*$ is strongly unique of order $1/p$. Moreover, $1/p$ is the largest possible order.

Proof. Without loss of generality assume $g^* = 0$. It is first shown that $1/p$ is the largest possible order for which $g^* = 0$ is strongly unique. Let $\alpha > 1/p$ and $g\varepsilon M\setminus\{0\}$ such that $\mu(\text{supp}(f)\cap\text{supp}(g)) = 0$. Denote by $Z(f)$ the set $\{x\varepsilon X: f(x) = 0\}$. Then $\mu(Z(f)) > 0$, for otherwise by Corollary 1 $\mu(\text{supp}(f)\cap\text{supp}(g)) > 0$ for all $g\varepsilon M\setminus\{0\}$ which is a contradiction. If $\lambda > 0$, then, since $g(x) = 0$ a.e. on $\text{supp}(f)$,

$$\frac{\|f-\lambda g\|_p^p - \|f\|_p^p}{\|\lambda g\|_p^{1/\alpha}} = \frac{\int_{\text{supp}(f)}|f|^p d\mu + \int_{Z(f)}|\lambda g|^p d\mu - \int_{\text{supp}(f)}|f|^p d\mu}{\lambda^{1/\alpha}\left(\int_{Z(f)}|g|^p d\mu\right)^{1/p\alpha}}$$

$$= \lambda^{p-(1/\alpha)}\left(\int_{Z(f)}|g|^p d\mu\right)^{1-(1/p\alpha)}$$

$$= \lambda^{p-(1/\alpha)} \| g\|_p^{p-(1/\alpha)}$$

But $\lambda^{p-(1/\alpha)}\| g\|_p^{p - (1/\alpha)} \to 0$ as $\lambda \to 0$, thus by Lemma 1.3.5 and Lemma 1.3.6 $g^* = 0$ is not strongly unique of order $\alpha > 1/p$.

Now we prove that $1/p$ is the correct order. Assume there exists $g \in M \setminus \{0\}$ such that $\mu(\text{supp}(f) \cap \text{supp}(g)) = 0$. Then $g(x) = 0$ a.e. on supp(f). Let $M_1 = \{g \in M: g(x) = 0$ a.e. on supp(f)$\}$. Now $M_1$ is a subspace of M so obtain a basis for $M_1$ and expand this basis to one for all of M. The additional basis elements span a subspace $M_2$, where if $g_2 \in M_2$ and $g_2(x) = 0$ a.e. on supp(f) then $g_2 = 0$. Thus $\mu(\text{supp}(f) \cap \text{supp}(g_2)) > 0$ for all $g_2 \in M_2 \setminus \{0\}$ and 0 is strongly unique of order 1/2 relative to $M_2$ by Theorem 2.2.1. Let $g \in M$. Then since M is the direct sum of $M_1$ and $M_2$, $g = g_1 + g_2$ where $g_1 \in M_1$ and $g_2 \in M_2$. Suppose $\| g \|_p \leq 1$. Define a new norm $\|g\|' = \|g_1\|_p + \|g_2\|_p$ which is equivalent to the $L^p$ norm on M since dim M $< \infty$. Since 0 is strongly unique of order 1/2 relative to $M_2$, given $r_1 > 0$ there is a constant $\gamma_1 > 0$ such that $\| f + g_2\|_p^p - \| f\|_p^p \geq \gamma_1 \| g_2\|_p^2$ for all $g_2 \in M_2$ with $\| g_2\|_p \leq r_1$. Now

$$\frac{\| f + (g_1 + g_2)\|_p^p - \| f \|_p^p}{\| g_1 + g_2\|_p^p} = \frac{\int_{\text{supp}(f)} |f + g_2|^p d\mu + \int_{Z(f)} |g_1 + g_2|^p d\mu - \int |f|^p d\mu}{\int |g_1 + g_2|^p d\mu}$$

$$= \frac{\int|f + g_2|^p d\mu - \int|f|^p d\mu - \int_{Z(f)}|g_2|^p d\mu + \int_{Z(f)}|g_1 + g_2|^p d\mu}{\int|g_1 + g_2|^p d\mu}$$

$$= \frac{\| f + g_2\|_p^p - \|f\|_p^p - \|g_2\|_p^p + \|g_1 + g_2\|_p^p}{\|g_1 + g_2\|_p^p}$$

$$\geq \frac{\gamma_1\|g_2\|_p^2 - \|g_2\|_p^p + \|g_1 + g_2\|_p^p}{\|g_1 + g_2\|_p^p} = 1 + \frac{\gamma_1\|g_2\|_p^2 - \|g_2\|_p^p}{\|g\|_p^p} .$$

Since $p > 2$ there exists $r_2 > 0$ such that if $\|g_2\|_p < r_2$ then $\gamma_1 \|g_2\|_p^2 - \|g_2\|_p^p \geq 0$. Since the norm $\|\cdot\|'$ is equivalent to the norm $\|\cdot\|_p$ on M, there exists a constant $K > 0$ such that $K\|g\|_p \geq \|g\|' \geq \|g_2\|_p$, thus there exists $r_3$, $0 < r_3 \leq 1$ such that if $\|g\|_p \leq r_3$ then $\|g_2\| \leq r = \min\{r_1, r_2\}$. Therefore if $\|g\| \leq r_3$, then

$$\frac{\|f + g\|_p^p - \|f\|_p^p}{\|g\|_p^p} \geq 1$$

and so by Lemma 1.3.5 and Lemma 1.3.6, 0 is strongly unique of order $1/p$. This completes the proof.

For the case $1 < p < 2$ the norm is only once continuously Fréchet differentiable so Theorem 1.3.8 does not apply. In this case, however, it will be shown that strong uniqueness of order 1/2 or 1/p holds.

Again, let M be a finite dimensional subspace of $L^p$, $1 < p < 2$, $f \varepsilon L^p \setminus M$, and suppose $g^* = 0$ is the best $L^p$-approximation to f from M. For $g \varepsilon M \setminus \{0\}$ let

$S_1 = \{x \varepsilon \text{ supp}(f): \text{sign } f(x) = \text{sign } g(x)\}$,

$S_2 = \{x \varepsilon \text{ supp}(f): \text{sign } f(x) \neq \text{sign } g(x) \text{ and } |g(x)| \leq |f(x)|\}$,

$S_3 = \{x \varepsilon \text{ supp}(f): \text{sign } f(x) \neq \text{sign } g(x) \text{ and } |g(x)| > |f(x)|\}$,

and $Z(f) = X \setminus \text{supp}(f)$. Then

$$\text{(2.2.2)} \quad \int |f + g|^p d\mu = \int_{Z(f)} |g|^p d\mu + \int_{S_1} (|f| + |g|)^p d\mu$$

$$+ \int_{S_2} |f| - |g|)^p d\mu + \int_{S_3} (|g| - |f|)^p d\mu$$

Consider now the following Taylor expansion. For $a \neq 0$, $b \neq 0$, and some t, $0 \leq t \leq 1$, we have

$$(a + b)^p = a^p + pa^{p-1} b + \frac{p(p-1)}{2} (a + tb)^{p-2} b^2.$$

For $b > -a$, this follow from the remainder formula for Taylor expansions. For $b = -a$, solve

$$a^p - pa^p + p(p - 1)/2 \ (1 - t)^{p-2} a^p = 0$$

or

$$1 - p + p(p - 1)/2 \ (1 - t)^{p-2} = 0$$

to get $t = 1 - (2/p)^{1/(p-2)}$ and since $1 < p < 2$, $0 < t < 1$. The above Taylor expansion is applied to $(|f| + |g|)^p$ on $S_1$, $(|f| - |g|)^p$ on

$S_2$, and $(|g| - |f|)^p$ on $S_3$. Hence, there exist functions $\theta_1$, $\theta_2$, and $\theta_3$ with $0 \leq \theta_i \leq 1$, $i = 1, 2, 3$, such that

(2.2.3) $$\int_{S_1} (|f| + |g|)^p d\mu$$

$$= \int_{S_1} (|f|^p + p|f|^{p-1} |g| + p(p-1)/2 \quad (|f| + \theta_1|g|)^{p-2} g^2) d\mu$$

(2.2.4) $$\int_{S_2} (|f| - |g|)^p d\mu$$

$$= \int_{S_2} (|f|^p - p|f|^{p-1} |g| + p(p-1)/2 \quad (|f| - \theta_2|g|)^{p-2} g^2) d\mu$$

(2.2.5) $$\int_{S_3} (|g| - |f|)^p d\mu$$

$$= \int_{S_3} (|g|^p - p|g|^{p-1}|f| + p(p-1)/2 \quad (|g| - \theta_3|f|)^{p-2} f^2) d\mu.$$

Combining (2.2.3), (2.2.4), and (2.2.5), (2.2.2) becomes

$$\int |f + g|^p d\mu = \int_{Z(f)} |g|^p d\mu$$

$$+ \int_{S_1} (|f|^p + p|f|^{p-1} |g| + p(p-1)/2 \ (|f| + \theta_1|g|)^{p-2} g^2) d\mu$$

$$+ \int_{S_2} (|f|^p - p|f|^{p-1} |g| + p(p-1)/2 \ (|f| - \theta_2|g|)^{p-2} g^2) d\mu$$

$$+ \int_{S_3} (|g|^p - p|g|^{p-1} |f| + p(p-1)/2 \ (|g| - \theta_3|f|)^{p-2} f^2) d\mu \ .$$

Now, if $|f| < |g|$, then $|f|^p < |g|^p$ and since $1 < p < 2$, $|g|^{p-2} < |f|^{p-2}$. So $|g|^{p-1} |f| < |f|^{p-1} |g|$ and $-p|g|^{p-1} |f| > -p|f|^{p-1} |g|$. Thus,

$$\int |f + g|^p d\mu > \int_{Z(f)} |g|^p d\mu$$

$$+ \int_{S_1} (|f|^p + p|f|^{p-1} |g| + p(p-1)/2 \ (|f| + \theta_1|g|)^{p-2} g^2) d\mu$$

$$+\int_{S_2} (|f|^p - p|f|^{p-1}\ |g| + p(p-1)/2\ (|f| - \theta_2\ |g|)^{p-2}\ g^2)d\mu$$

$$+\int_{S_3} (|f|^p - p|f|^{p-1}\ |g| + p(p-1)/2\ (|g| - \theta_3\ |f|)^{p-2}\ f^2)d\mu$$

$$= \int_{Z(f)} |g|^p\ d\mu + \int |f|^p + p \int g\ |f|^{p-1}\ \text{sign } f\ d\mu$$

$$+\ p(p-1)/2 \left( \int_{S_1} (|f| + \theta_1\ |g|)^{p-2}\ g^2 d\mu \right.$$

$$\left. + \int_{S_2} (|f| - \theta_2 |g|)^{p-2}\ g^2\ d\mu + \int_{S_3} (|g| - \theta_3 |f|)^{p-2}\ f^2\ d\mu \right).$$

By Theorem A, the third integral on the right is zero. Also, the first and final integrals are nonnegative so

$$\int |f + g|^p d\mu \geq \int |f|^p + p(p-1)/2 \left( \int_{S_1} (|f| + \theta_1\ |g|)^{p-2} g^2 d\mu \right.$$

$$\left. + \int_{S_2} (|f| - \theta_2 |g|)^{p-2} g^2 d\mu \right).$$

Define $\theta(x)$ by $\theta(x) = \theta_1(x)$ on $S_1$ and $\theta(x) = \theta_2(x)$ on $S_2$. Then

$$(2.2.6) \quad \int |f + g|^p d\mu \geq \int |f|^p d\mu + p(p-1)/2 \int_{U_g} (|f| + \theta |g|)^{p-2}\ g^2 d\mu$$