New solution generating transformations for the stationary axially-symmetric Einstein-Maxwell equations

by Terry Lee Lemley

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Physics
Montana State University

Abstract:
Exact solution generating methods for the stationary axially-symmetric Einstein-Maxwell equations are investigated in this thesis. In particular, the attempt has been made to extend Cosgrove's vacuum symmetry groups, Q and Q~, to the electrovac case. The primary method used is that of invariant groups, but due to the close relationship to Backlund transformations, a brief review of the latter is given.

The symmetry groups Q and Q~ are found to exist for the electrovac case, but as of yet their application has met with minimal success.

It is still felt by the author that more of the vacuum results pertaining to Q and Q~ can be extended to the electrovac case.

NEW SOLUTION GENERATING TRANSFORMATIONS
FOR THE STATIONARY AXIALLY-SYMMETRIC
EINSTEIN-MAXWELL EQUATIONS

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ABSTRACT

Exact solution generating methods for the stationary axially-symmetric Einstein-Maxwell equations are investigated in this thesis. In particular, the attempt has been made to extend Cosgrove's vacuum symmetry groups, Q and $\tilde{Q}$, to the electrovac case. The primary method used is that of invariant groups, but due to the close relationship to Bäcklund transformations, a brief review of the latter is given. The symmetry groups Q and $\tilde{Q}$ are found to exist for the electrovac case, but as of yet their application has met with minimal success. It is still felt by the author that more of the vacuum results pertaining to Q and $\tilde{Q}$ can be extended to the electrovac case.

CHAPTER 1

INTRODUCTION

This thesis investigates a method of finding exact solutions for the stationary axially-symmetric Einstein-Maxwell equations. The adjectives preceding the words Einstein-Maxwell imply that a system with axial-symmetry, uniformly rotating about its axis of symmetry, is being investigated. The words Einstein-Maxwell reveal that this thesis belongs to the subject areas of gravitation and electrodynamics with their related equations.

Being nonlinear and coupled, the Einstein-Maxwell equations developed in Chapter 2 are very difficult to solve. Before special methods were introduced for finding new solutions, only a handful of solutions were known. Two methods which have been found useful in finding new solutions are the methods of group transformations and Bäcklund transformations. These are reviewed in Chapters 3 and 4, respectively.

The case where no exterior electromagnetic fields are present has been investigated by numerous workers in recent years. There is a growing consensus among these workers that this problem is nearly solved. A brief review of this vacuum case is given in Chapter 5.

Allowance for electromagnetic fields with the given symmetries introduces a source term into the Einstein equations. Chapter 6 reviews results found previous to this thesis for this case.

Chapter 7 extends the Q and $\tilde{Q}$ transformations found by Cosgrove (1979a) for vacuum to the case where stationary axially-symmetric electromagnetic fields are present. This is the first chapter in which original material is presented.

Chapter 8 uses the new transformations in an attempt to generate a new solution. The results of this thesis are given in Chapter 9.

CHAPTER 2

THE FIELD EQUATIONS

At the turn of the century only two forces in nature were known, the gravitational force and the electromagnetic force. Both were beautifully portrayed by two physical theories, the former by Newton's Law of Universal Gravitation and the latter by Maxwell's field equations. Einstein founded the theory of special relativity on what has amounted to the choosing of Maxwell's field equations over Newton's laws of motion. Including the gravitational field eventually led Einstein to the theory of general relativity. Thus, both classical fields played important roles in the development of general relativity.

General relativity consists partially in the solving of the equations governing the gravitational field, the Einstein equations. If electromagnetic fields are present, then general relativity modifies the Maxwell equations to include the effects of curved spacetime. The field equations are written in the deceptively simple tensor form:

$$R_{ik}-1/2g_{ik}R = (-8\pi G/c^4)T_{ik} \qquad (2.1)$$

are the Einstein equations and

$$F^{ik}{}_{;k} = -4\pi s^i \quad ; \quad F_{ik,l}+F_{li,k}+F_{kl,i} = 0 \qquad (2.2,3)$$

are the Maxwell equations. Together they are called the Einstein-Maxwell equations.

The Einstein-Maxwell equations are usually thought of in the following manner. On the left-hand side of Equations (2.1) are the terms related to the curvature of spacetime. This curvature has as its source the energy-momentum tensor T_{ik} which stands on the right-hand side of Equations (2.1). This source term includes the effects of all the matter fields, excluding gravity, and is basically the statement of mass-energy equivalence in the generation of the gravitational field. In particular, it will contain the energy and momentum of the electromagnetic fields present.

The Maxwell equations are modified directly by the covariant derivative which contains the effects of the gravitational field. (As is usual in the literature, a comma represents partial differentiation, and the semi-colon represents covariant differentiation.) Indirect effects of the gravitational field are contained in the current density term s^i on the right-hand side of Equations (2.2).

Taken together, the Einstein-Maxwell equations are a very difficult set of equations to solve. They are highly coupled and nonlinear.

In this thesis, several simplifying conditions will be added. First, the regions of space considered will be exterior to the sources; that is, the current density s^i will be zero, and the energy-momentum tensor T_{ik} will contain terms which depend only on the free electromagnetic fields. This situation is called electrovac. Second, the fields will be assumed stationary. Third, the fields will be assumed axially-symmetric. Finally, the fields will be assumed invariant under the simultaneous transformation of the coordinates, $(t,\phi)\rightarrow(-t,-\phi)$.

The first condition simplifies the field equations to

$$R_{ik} = (-8\pi G/c^4)T_{ik} \qquad (2.1a)$$

and

$$F^{ik}{}_{;k} = 0 \;; \quad F_{ik,l}+F_{li,k}+F_{kl,i} = 0. \tag{2.2a,3}$$

Axial-symmetry means the existence of an axis of symmetry around which any rotation leaves the fields invariant. Mathematically, this defines an azimuthal coordinate ϕ which measures the degree of rotation about the axis.

Stationary means that all of the sources have reached a uniform state of motion such that all the fields are independent of time. Mathematically, this defines the time coordinate t of which all the fields are independent.

Finally, the symmetry transformation $(t,\phi)\to(-t,-\phi)$ has the following physical meaning. If the object producing the fields is rotating about its axis, then the above transformation brings the velocity distribution back into itself. This means that the energy-momentum tensor and current density go into themselves. Since nothing has changed within the source under the transformation, the external fields produced by the source should be the same.

This last condition greatly simplifies the form of the metric tensor g_{ik}. Before adding the last condition, the line element had the form

$$\begin{aligned} ds^2 &= g_{ik}dx^i dx^k \\ &= g_{11}dtdt+2g_{12}dtd\phi+2g_{13}dtdx^3+2g_{14}dtdx^4+ \\ &\quad g_{22}d\phi d\phi+2g_{23}d\phi dx^3+2g_{24}d\phi dx^4+g_{33}dx^3dx^3+ \\ &\quad 2g_{34}dx^3dx^4+g_{44}dx^4dx^4. \end{aligned} \tag{2.4}$$

(Here $t = x^1$, $\phi = x^2$, and $g_{ik} = g_{ik}(x^3,x^4)$.)

Imposing the last condition rids the g_{ik} of the 13, 14, 23, and 24 components. The line element can now be written in the box diagonal form

$$ds^2 = f_{AB}dx^A dx^B - h_{MN}dx^M dx^N; \quad A,B = 1,2; \quad M,N = 3,4. \tag{2.5}$$

Notation

A brief review of the notation to be used in the development of the field equations will now be given. This notation follows Kinnersley (1977a).

To raise indices in a two-dimensional space with metric f_{AB}, either (f_{AB})'s inverse matrix or ε^{AB} may be used. ($\varepsilon^{12} = \varepsilon_{12} = -\varepsilon^{21}$ $-\varepsilon_{21} = 1$; $\varepsilon^{11} = \varepsilon_{22} = \varepsilon^{22} = \varepsilon_{11} = 0$.) This can be seen by the following argument.

For a general two by two matrix,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

The inverse is easily shown to be

$$A^{-1} = \frac{1}{(ad-bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{(ad-bc)} \varepsilon A^{\dagger} \varepsilon^{\dagger}, \tag{2.6}$$

with

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and the superscript $\dagger$ indicating matrix transposition. In the case where the matrix (f_{AB}) is symmetric, the inverse matrix is given by

$$(f_{AB})^{-1} = \frac{1}{\det(f_{AB})} \varepsilon (f_{AB}) \varepsilon^{\dagger}, \tag{2.7}$$

or in index notation by

$$f^{\tilde{A}\tilde{B}} \equiv (f_{AB})^{-1} = -\rho^{-2}\varepsilon^{AC}\varepsilon^{BD}f_{CD}; \; -\rho^{2} = \det(f_{AB}). \tag{2.7a}$$

Therefore, defining f^{AB} by

$$f^{AB} = \varepsilon^{AC}\varepsilon^{BD}f_{CD} = -\rho^{2}f^{\tilde{A}\tilde{B}}, \tag{2.8}$$

it is seen that

$$f^{AB}f_{BC} = -\rho^{2}f^{\tilde{A}\tilde{B}}f_{BC} = -\rho^{2}\delta^{A}{}_{C}. \tag{2.9}$$

Strictly speaking, since ε^{AC} is a tensor density of weight minus one, f^{AB} is a tensor density of weight minus two (Weinberg, 1972: 99). For coordinate transformations which have the Jacobian equal to one, they are strict tensors. This detail will be ignored as it does not affect the following development.

Given the vector V_A, raising the index is defined by

$$V^{B} \equiv \varepsilon^{BA}V_{A}. \tag{2.10}$$

It is important to note that the contracted index is the second index in the ε^{BA}. With this convention and the identity

$$\varepsilon_{AC}\varepsilon^{CB} = \varepsilon^{BC}\varepsilon_{CA} = -\delta_{A}{}^{B} = -\delta^{B}{}_{A}, \tag{2.11}$$

it is seen that to lower the index on V^B requires multiplying by ε_{BD}.

That is,

$$V_D = \varepsilon_{BD} V^B. \tag{2.12}$$

This is consistent with the raising operation leading back to the original vector.

Return to the stationary axially-symmetric metric of Equation (2.5). The indices will be raised using ε^{AB} in the (t,ϕ) coordinate block and the inverse of (h_{MN}) in the (x^3,x^4) coordinate block. If, instead of raising indices in this manner, the inverse of (f_{AB}) and ε^{MN} are used, then the raised indices will be marked with a superscript tilde. For example,

$$V^{\tilde{M}} = \varepsilon^{MN} V_N, \tag{2.13}$$

with $\varepsilon^{34} = \varepsilon_{34} = -\varepsilon^{43} = -\varepsilon_{43} = 1$ and $\varepsilon^{33} = \varepsilon_{33} = \varepsilon^{44} = \varepsilon_{44} = 0$.

The two-dimensional covariant derivative associated with h_{MN} will be designated ∇. The divergence of a vector field $V = (V_3,V_4)$ is given by

$$\nabla\cdot V = h^{-1/2} (h^{1/2} h^{MN} V_M)_{,N}, \tag{2.14}$$

h being the determinant of h_{MN}. The expression $h^{1/2} h^{MN}$ in two-dimensions is conformally invariant. That is, if the metric is transformed to $\bar{h}^{MN}$, with $\bar{h}_{MN} = e^{-2\Gamma} h_{MN}$, then $\bar{h}^{1/2} \bar{h}^{MN} = h^{1/2} h^{MN}$. This transformation is not a coordinate transformation; under this transformation $V_M \to V_M$ and $V^M \to e^{2\Gamma} V^M$. Therefore, if $\nabla\cdot V = 0 = (h^{1/2} h^{MN} V_M)_{,N}$, then $\bar{\nabla}\cdot\bar{V} = \bar{h}^{-1/2} (\bar{h}^{1/2} \bar{h}^{MN} \bar{V}_M)_{,N} =$

$\bar{h}^{-1/2}(h^{1/2}h^{MN}V_M)_{,N} = 0$. Since every two-dimensional metric space is conformally related to the Euclidean two-dimensional space, (Eisenhart, 1909:92-93), it is possible to pick $\bar{h}_{MN} = \delta_{MN}$. In this case, the covariant derivative ∇ reduces to the ordinary gradient in rectangular coordinates. The line element is now written

$$ds^2 = f_{AB}dx^A dx^B - e^{2\Gamma}\delta_{MN}dx^M dx^N \; ; \quad A,B = 1,2; \quad M,N = 3,4. \tag{2.5a}$$

Another common form of the line element, Lewis (1932), is related to the above by the relations

$$f_{11} = f, \; f_{12} = -f\omega, \; f_{22} = f\omega^2 - \rho^2 f^{-1}, \quad e^{2\Gamma} = f^{-1}e^{2\gamma}. \tag{2.15}$$

In this parameterization, the line element is written

$$ds^2 = f(dt - \omega d\phi)^2 - f^{-1}e^{2\gamma}((dx^3)^2 + (dx^4)^2) + f^{-1}\rho^2 d\phi^2. \tag{2.16}$$

Again, $-\rho^2 = \det(f_{AB})$ and is a function of x^3 and x^4.

The Maxwell Equations

Using the line element of Equation (2.5a), the field equations will now be written out, beginning with the Maxwell equations (2.2a,3).

Introducing the electromagnetic potential A_i by the equation $F_{ik} = A_{k,i} - A_{i,k}$, Equations (2.3) are automaticaly satisfied. The F_{ik} are independent of (t,ϕ), and the A_i will also be assumed independent of (t,ϕ). Hence, the only surviving components left are $F_{AM} = -A_{A,M}$ and $F_{MN} = A_{N,M} - A_{M,N}$. The Maxwell equations (2.2a) may be written, (Weinberg, 1972:125),

$$(\sqrt{-g}\, F^{ik})_{,k} = 0. \tag{2.2b}$$

These fall into two sets of equations. The first set is

$$0 = (\sqrt{-g}\, F^{MN})_{,N} = (\rho e^{2\Gamma} F^{MN})_{,N} = (\rho F^{M}{}_{N})_{,N}, \tag{2.17}$$

which has solutions

$$F^{MN} = C\rho^{-1}\varepsilon^{MN}. \tag{2.18}$$

This corresponds to a magnetic field in the azimuthal direction falling off as ρ^{-1}, which would be produced by an infinite line current along the symmetry axis. Since this solution is not physical for a bounded source the constant C will be set to zero.

The final set of equations is

$$0 = (\sqrt{-g}\, F^{AM})_{,M} = \nabla\cdot(\rho f^{\tilde{A}\tilde{B}}\nabla A_B) = \nabla\cdot(\rho^{-1} f^{AB}\nabla A_B). \tag{2.19}$$

Define $\tilde{\nabla}$ by $\tilde{\nabla} = (\partial/\partial x^4, -\partial/\partial x^3)$, then one has the following identity. For all scalar functions U, $\nabla\cdot(\tilde{\nabla}U) = 0$. Conversely, if $\nabla\cdot V = 0$, then there exists a scalar potential U such that $V = \tilde{\nabla}U$. Therefore, Equations (2.19) indicate the existence of scalar fields B_A such that

$$\tilde{\nabla}B_A = \rho^{-1} f_A{}^{B}\nabla A_B. \tag{2.20}$$

Using the relations $\tilde{\tilde{V}} = -V$ and $f^{AB}f_{BC} = -\rho^2\delta^A{}_C$, an inverse relation can be derived yielding

$$\tilde{\nabla}A_A = -\rho^{-1} f_A{}^{B}\nabla B_B. \tag{2.20a}$$

Forming the complex combination $\Phi_A = A_A + iB_A$, the Φ_A are seen to satisfy

$$\nabla\Phi_A = -i\rho^{-1} f_A{}^{B}\tilde{\nabla}\Phi_B. \tag{2.21}$$

The Einstein Equations

The Einstein equations (2.1a) are next considered. It can be shown that

$$2R^A{}_C = \rho^{-1}\nabla\cdot(\rho^{-1} f^{AD}\nabla f_{DC}) \tag{2.22}$$

and

$$4\pi T^A{}_B = \rho^{-1}\nabla\cdot(\rho^{-1}(f^{AC}A_B\nabla A_C - 1/2\,\delta^A{}_B f^{CD}A_C\nabla A_D)) \quad (2.23)$$

Hence, the Einstein equations are, setting G = c = 1,

$$\nabla\cdot(\rho^{-1}(f^{AC}\nabla f_{BC} - 4f^{AC}A_B\nabla A_C + 2\delta^A{}_B f^{CD}A_C\nabla A_D)) = 0. \quad (2.24)$$

Using the identity

$$V^A{}_B - V_B{}^A = \delta^A{}_B V^C{}_C, \quad (2.25)$$

this can be written

$$\nabla\cdot(\rho^{-1}(f^{AC}\nabla f_{BC} - 2f^{AC}A_B\nabla A_C - 2f_B{}^C A^A\nabla A_C)) = 0. \quad (2.26)$$

This, in turn, implies the existence of the scalar potentials ψ_{AC} such that

$$\tilde{\nabla}\psi_{AC} = \rho^{-1}(f_A{}^B\nabla f_{BC} - 2f_A{}^B A_C\nabla A_B - 2f_C{}^B A_A\nabla A_B) \,. \quad (2.27)$$

One notes that

$$\tilde{\nabla}\psi^A{}_A = \rho^{-1}(f^{AB}\nabla f_{AB}) = 1/2\,\rho^{-1}\nabla(f^{AB}f_{AB})$$
$$= -\rho^{-1}\nabla\rho^2 = -2\nabla\rho \quad (2.28)$$

or

$$\tilde{\nabla}\psi^A{}_A = -2\nabla\rho \,. \quad (2.28a)$$

This implies that

$$\nabla\cdot\nabla\rho = 0. \quad (2.29)$$

As a consequence of Equation (2.29), one can choose new coordinates $x^{3'} = \rho$, $x^{4'} = z$, where z is defined by

$$\nabla\rho = \tilde{\nabla}z \,, \quad (2.30)$$

and then the line element (2.16) becomes

$$ds^2 = f(dt-\omega d\phi)^2 - f^{-1}e^{2\gamma'}(d\rho^2+dz^2) + f^{-1}\rho^2 d\phi^2 \quad (2.31)$$

with $e^{2\gamma'} = e^{2\gamma}/(\rho_{,3}{}^2+\rho_{,4}{}^2)$. This form of the line element is known as the line element in Weyl canonical coordinates.

An inverse relation can be found to Equation (2.27) and is

$$\nabla f_{AC} = \rho^{-1} f_A{}^B(\tilde{\nabla}\psi_{BC}+2A_C\tilde{\nabla}B_B+2A_B\tilde{\nabla}B_C) \qquad (2.32)$$

with the symmetry condition on the f_{AC} requiring

$$f^{AB}(\nabla\psi_{BA}+2A_A\nabla B_B+2A_B\nabla B_A) = 0. \qquad (2.33)$$

Let

$$\Omega_{AC} = \psi_{AC}+2A_AB_C \qquad (2.34)$$

and define

$$H_{AC} = f_{AC}+i\Omega_{AC}-\Phi_A{}^*\Phi_C+\varepsilon_{AC}K, \qquad (2.35)$$

with

$$\nabla K = \Phi_X{}^*\nabla\Phi^X. \qquad (2.36)$$

Then it has been shown, Kinnersley (1977a), that

$$\nabla H_{AC} = -i\rho^{-1} f_A{}^B\tilde{\nabla}H_{BC}\ , \qquad (2.37)$$

which is the same form as Equations (2.21).

Defining $G = H_{11}$ and $\Phi = \Phi_1$, after some algebra, the following equations result.

$$f_{11}\nabla^2\Phi = (\nabla G+2\Phi^*\nabla\Phi)\cdot\nabla\Phi, \qquad (2.38)$$

$$f_{11}\nabla^2 G = (\nabla G+2\Phi^*\nabla\Phi)\cdot\nabla G. \qquad (2.39)$$

These equations are the Ernst equations, Ernst (1968b), and contain all the information necessary for the solution of the Einstein-Maxwell equations. (∇^2 is the Laplacian operator associated with the three-dimensional metric $(dx^3)^2+(dx^4)^2+\rho^2 d\phi^2$; that is, $\partial^2/\partial\rho^2+\partial^2/\partial z^2+1/\rho\,\partial/\partial\rho$.) Equation (2.29) must also be added to Equations (2.38,39) if noncanonical coordinates are chosen.

Nothing thus far has been said about the R_{AM} and R_{MN} field equations. The first set of these are identically zero. The R_{MN} equations allow for the γ in Equation (2.16) to be found by integrating the following two equations.

$$\rho^{-1}\gamma_{,3} = \frac{f^{-2}}{4}\left[G_{,3}G^*_{,4}+G_{,4}G^*_{,3}+2\Phi(G_{,3}\Phi^*_{,4}+G_{,4}\Phi^*_{,3}) + 2\Phi^*(G^*_{,3}\Phi_{,4}+G^*_{,4}\Phi_{,3})+4\Phi^*\Phi(\Phi_{,3}\Phi^*_{,4}+\Phi_{,4}\Phi^*_{,3})\right] - f^{-1}(\Phi_{,3}\Phi^*_{,4}+\Phi^*_{,3}\Phi_{,4}), \tag{2.40}$$

and

$$\rho^{-1}\gamma_{,4} = -\frac{f^{-2}}{4}\left[G_{,4}G^*_{,4}-G_{,3}G^*_{,3}+2\Phi(G_{,4}\Phi^*_{,4}-G_{,3}\Phi^*_{,3}) + 2\Phi^*(G^*_{,4}\Phi_{,4}-G^*_{,3}\Phi_{,3})+4\Phi^*\Phi(\Phi_{,4}\Phi^*_{,4}-\Phi_{,3}\Phi^*_{,3})\right] + f^{-1}(\Phi_{,4}\Phi^*_{,4}-\Phi^*_{,3}\Phi_{,3}). \tag{2.41}$$

If solutions to Equations (2.21,37) or (2.38,39) are known, then the right-hand sides of Equations (2.40,41) are known. Hence, Equations (2.21,37) or (2.38,39) are solved first, and then Equations (2.40,41) are integrated.

The field equations have now been introduced. Other forms of the equations exist, but these shall be introduced only where necessary. Equations (2.21,37) or (2.38,39) are very elegant and suggest that the gravitational field and electromagnetic field have much in common.

The next two chapters will develop some mathematical machinery useful in finding new solutions to these equations.

CHAPTER 3

TRANSFORMATION GROUPS

In this chapter, transformation groups will be reviewed and a method presented by example to find them for a given set of partial differential equations. The presentation given in this chapter follows closely that given by Hamermesh (1962: 279-321) and Cosgrove (1979a: 37-64) and is sufficient for the understanding of this thesis. A more detailed exposition concerning transformation groups is given in Eisenhart (1933).

Properties of Transformation Groups

Consider the transformation on the variables x^i, $i = 1,2,\ldots,n$, given by

$$x^i \to \bar{x}^i = f^i(x^1,x^2,\ldots,x^n;\ a^1,a^2,\ldots,a^t) \tag{3.1}$$

or symbolically by

$$x^i \to \bar{x}^i = f^i(x;a). \tag{3.1a}$$

The functions f^i are analytic in the parameters a^α, with $\alpha = 1,2,\ldots,t$; that is, they are expandable in a convergent Taylor series. In order for the transformations (3.1) to form a group, the f^i must satisfy certain conditions.

Group closure requires that if two transformations with parameters a^α and b^α are performed in succession, then there exists parameters c^α such that

$$f^i(x;c) = f^i(\bar{x};b) = f^i(f(x;a);b). \tag{3.2}$$

This means that the c^α may be expressed in terms of the a^α and b^α as

$$c^\alpha = \Phi^\alpha(a;b). \tag{3.3}$$

The Φ^α will be assumed analytic in its arguments. Using Equations (3.3), Equations (3.2) may be written as

$$f^i(x,\Phi(a;b)) = f^i(f(x;a);b). \tag{3.4}$$

Existence of the group identity implies the existence of the unique parameter set e^α such that

$$\bar{x}^i = f^i(x;e) = x^i \tag{3.5}$$

for all x.

Each transformation must have an inverse. This implies that for each parameter set a^α, there is a parameter set $\bar{a}^\alpha$ such that

$$\bar{\bar{x}}^i = f^i(\bar{x};\bar{a}) = f^i(f(x;a);\bar{a}) = x^i. \tag{3.6}$$

This implies that the transformations (3.1) are solvable for the x^i in terms of the $\bar{x}^i$; the condition on the f^i being that the Jacobian of the transformation must be different from zero. Taken with Equations (3.3), Equations (3.6) implies that

$$e^\alpha = \Phi^\alpha(a;\bar{a}), \tag{3.7}$$

which in turn implies that the $\bar{a}^\alpha$ can be written in terms of the a^α, since the e^α are fixed; that is,

$$\bar{a}^\alpha = \Omega^\alpha(a). \tag{3.8}$$

The $\Omega^\alpha(a)$ will be assumed analytic. The analyticity requirements are used to make the group a Lie group.

Using Equations (3.3,8), it is possible to show that the a^α may be expressed in terms of the a^α and c^α. In other words, the

Jacobians $|\partial\Phi^\alpha/\partial a^\beta|$ and $|\partial\Phi^\alpha/\partial b^\beta|$ are nonzero, implying the matrices $\partial\Phi^\alpha/\partial a^\beta$ and $\partial\Phi^\alpha/\partial b^\beta$ are invertible matrices.

Attention will now be shifted from the finite transformations (3.1) to the infinitesimal transformations associated with Equations (3.1). Infinitesimal transformations are transformations close to the identity; their parameters are close to the identity parameters e^α.

Let a^α be the parameters that take the points x^i into $\bar{x}^i$. The neighboring parameters $a^\alpha+da^\alpha$ will take the points x^i into the points $\bar{x}^i+d\bar{x}^i$, since the f^i are analytic in the a^α. Parameters that are close to the identity, $e^\alpha+\delta a^\alpha$, can be found which will take the points $\bar{x}^i$ into the points $\bar{x}^i+d\bar{x}^i$. Two different paths from x^i to $\bar{x}^i+d\bar{x}^i$ are thus possible and are given by

$$\bar{x}^i+d\bar{x}^i = f^i(x;a+da) \tag{3.9}$$

or the path

$$\bar{x}^i = f^i(x;a), \text{ and } \bar{x}^i+d\bar{x}^i = f^i(\bar{x},e+\delta a). \tag{3.10}$$

Expanding the last equation in a Taylor series yields

$$d\bar{x}^i = (\partial f^i(\bar{x};a)/\partial a^\beta)_{a\,=\,0}\delta a^\beta \equiv F^i{}_\beta(\bar{x})\delta a^\beta. \tag{3.11}$$

Equation (3.3) yields

$$a^\alpha+da^\alpha = \Phi^\alpha(a;e+\delta a) \tag{3.12}$$

so that by expanding the last equation yields

$$da^\alpha = (\partial\Phi^\alpha(a;b)/\partial b^\beta)_{b\,=\,e}\delta a^\beta \equiv \Theta^\alpha{}_\beta(a)\delta a^\beta. \tag{3.13}$$

At $a^\alpha = e^\alpha$, $\Theta^\alpha{}_\beta = \delta^\alpha{}_\beta$.

Since $\Theta^\alpha{}_\beta(a)$ is invertible, Equations (3.13) are solvable for the δa^α in terms of the da^α.

$$\delta a^\alpha = \psi^\alpha{}_\beta(a)da^\beta, \tag{3.14}$$

with the $\psi^{\alpha}{}_{\beta}(a)$ defined by

$$\psi^{\alpha}{}_{\eta}\Theta^{\eta}{}_{\beta} = \delta^{\alpha}{}_{\beta} \text{ and } \psi^{\alpha}{}_{\eta}(e) = \delta^{\alpha}{}_{\eta}. \quad (3.15)$$

Substituting Equations (3.14) into Equations (3.11) yields

$$d\bar{x}^{i} = F^{i}{}_{\beta}(\bar{x})\psi^{\beta}{}_{\alpha}(a)da^{\alpha} \quad (3.16)$$

or, equivalently,

$$\partial\bar{x}^{i}/\partial a^{\alpha} = F^{i}{}_{\beta}(\bar{x})\psi^{\beta}{}_{\alpha}(a). \quad (3.17)$$

If the $F^{i}{}_{\beta}(\bar{x})$ and $\psi^{\beta}{}_{\alpha}(a)$ are known, then the transformations (3.1) may be found.

The infinitesimal transformations (3.10) change the function $A(x)$ as follows,

$$\begin{aligned} dA &= (\partial A/\partial x^{i})dx^{i} = (\partial A/\partial x^{i})F^{i}{}_{\beta}(x)\delta a^{\beta} \\ &= \delta a^{\beta}F^{i}{}_{\beta}(x)(\partial A/\partial x^{i}) = \delta a^{\beta}X_{\beta}A \end{aligned} \quad (3.18)$$

with the operator X_{β} defined by

$$X_{\beta} = F^{i}{}_{\beta}(x)(\partial/\partial x^{i}) . \quad (3.19)$$

The X_{β} are called infinitesimal operators of the group because the operator $(1+X_{\beta}\delta a^{\beta})$ is close to the identity. Using $(1+X_{\beta}\delta a^{\beta})$ on the x^{i} yields Equations (3.11).

Defining the commutator of the infinitesimal operators by

$$[X_{\alpha}, X_{\beta}] = X_{\alpha}X_{\beta}-X_{\beta}X_{\alpha}, \quad (3.20)$$

it is shown in Hamermesh (1962: 299-301) that the commutators satisfy

$$[X_{\alpha}, X_{\beta}] = c^{\kappa}{}_{\alpha\beta}X_{\kappa}, \quad (3.21)$$

with the $c^{\kappa}{}_{\alpha\beta}$ independent of the a^{α}; and they are called the structure constants. It has been shown, see Cohn (1957: 94-106), that if the structure constants are known, then the group transformations can be found via the equations

$$F^j{}_\sigma(\partial F^i{}_\tau/\partial x^j)-F^j{}_\tau(\partial F^i{}_\sigma/\partial x^j) = c^\kappa{}_{\tau\sigma}F^i{}_\kappa, \qquad (3.22)$$

and

$$(\partial\psi^\kappa{}_\mu/\partial a^\lambda)-(\partial\psi^\kappa{}_\lambda/\partial a^\mu) = c^\kappa{}_{\tau\sigma}\psi^\tau{}_\mu\psi^\sigma{}_\lambda. \qquad (3.23)$$

Needless to say, the equations above are difficult to solve. If they can be solved, then Equations (3.17) are then used to find f^i. The important point is that if the $c^\kappa{}_{\tau\sigma}$ are found and correspond to those of a known group, the transformations are determined near the identity.

Applications

Transformation groups are used to generate new solutions from a given solution of the Einstein-Maxwell equations. For example, starting with the flat space metric, it is possible to generate the Schwarzschild metric; that is, a transformation exists which has the physical property of producing mass. Although in the classical theory this is only a mathematical curiosity, in a quantum theory it could have major importance.

In its most general sense, the symmetry transformation changes both the dependent and the independent variables while leaving the field equations invariant. Clarification is needed as to the meaning of invariance and it is worthwhile to review this before examining methods of finding symmetry transformations.

Take the Laplace equation in three dimensions as the first example. It is written in manifestly covariant form

$$\nabla^2\Phi = 0. \qquad (3.24)$$

This equation holds true in all coordinates; however, the partial differential equation it represents has different forms in different coordinates. In Cartesian coordinates, the partial differential equation is

$$(\partial^2\Phi/\partial x^2) + (\partial^2\Phi/\partial y^2) + (\partial^2\Phi/\partial z^2) = 0, \tag{3.25}$$

in spherical coordinates,

$$\frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)\right) + \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0\ , \tag{3.26}$$

and in cylindrical coordinates, its form is

$$\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}\left(\frac{\rho\partial\Phi}{\partial\rho}\right)\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0. \tag{3.27}$$

It is clear that the form of the partial differential equation depends on the coordinates chosen, even though Equation (3.24) is covariant. Invariance of form is precisely what is meant when the equations are said to be invariant under a given symmetry transformation.

The transformation of coordinates which do leave Equations (3.25) invariant are the rotations and inversions. For instance, under the rotation of coordinates, (x,y,z) transforms into $(\bar{x},\bar{y},\bar{z})$ by an orthogonal transformation, and Equation (3.25) becomes

$$(\partial^2\Phi/\partial\bar{x}^2) + (\partial^2\Phi/\partial\bar{y}^2) + (\partial^2\Phi/\partial\bar{z}^2) = 0. \tag{3.28}$$

This means that if $\Phi(x,y,z)$ is a solution to Equation (3.25), then $\Phi(\bar{x},\bar{y},\bar{z})$ is also a solution to Equation (3.25) when $(\bar{x},\bar{y},\bar{z})$ are expressed in terms of (x,y,z).

As an example of a noncoordinate transformation, take the equation to be the vacuum Ernst equation, see Equation (2.39),

$$(G+G^*)\nabla^2 G = 2\nabla G\cdot\nabla G\ . \tag{3.28}$$

The transformation, called the Ehler's or gravitational duality transformation, is given by

$$G \to \bar{G} = G/(1+i\gamma G); \qquad (3.29)$$

γ is a real parameter. It is simple to verify that $\bar{G}$ satisfies Equation (3.28) and is a new solution.

How are transformations like the rotations and the Ehler's transformation (3.29) found? Obviously, the rotations are encountered early in a physicist's career and have a clear physical meaning. Many other transformations in physics are modeled after the rotations and have a similar intuitive feeling. Ehler's transformation (3.29), however, is not intuitive at first encounter. A method is needed independent of intuition whereby such transformations can be found. Fortunately, such a method exists and consists first in finding the infinitesimal transformation which leaves the partial differential equations invariant, and then the integration of Equations (3.22,23,17) may be performed to yield the finite transformation. Although this method of finding symmetry transformations is complex, it again has the advantage of not relying on intuition. (Intuition is good to have, but seldom is it proved without effort.)

A Method

The vacuum Ernst equation (3.28) will be used to demonstrate this method. In particular, the Ehler's transformation (3.29) will be one of the symmetry transformations found.

Consider a finite transformation taking G into $\bar{G}$ with the transformation parameter s. Restrict $\bar{G}$ to be an explicit function of only G, G*, and s, written out as

$$G \to \bar{G} = \bar{G}(G,G^*;s). \tag{3.30}$$

Similarly for G*,

$$G^* \to \overline{G^*} = \overline{G^*}(G,G^*;s) . \tag{3.31}$$

$\bar{G}$ is not assumed to be the complex conjugate of $\overline{G^*}$. Expand Equations (3.30,31) in a McClaurin's series with the boundary conditions $\bar{G}(s = 0)$ and $\overline{G^*}(s = 0) = G^*$. To first order in s,

$$\bar{G} = G+sA(G,G^*), \quad A = \left.\frac{d\bar{G}}{ds}\right|_{s = 0}, \tag{3.32}$$

and

$$\overline{G^*} = G^*+sB(G,G^*), \quad B = \left.\frac{d\overline{G^*}}{ds}\right|_{s = 0}. \tag{3.33}$$

It is necessary for $\bar{G}$ to satisfy Equation (3.28) and for $\overline{G^*}$ to satisfy the complex conjugate equation

$$(G+G^*)\nabla^2 G^* = 2\nabla G^* \cdot \nabla G^*. \tag{3.34}$$

Using the chain rule of calculus,

$$\nabla\bar{G} = \nabla G+s(A_G\nabla G+A_{G^*}\nabla G^*) \tag{3.35}$$

and

$$\nabla^2\bar{G} = \nabla^2 G+s(A_G\nabla^2 G+A_{G^*}\nabla^2 G^*+A_{GG}\nabla G\cdot\nabla G+ 2A_{GG^*}\nabla G\cdot\nabla G^*+A_{G^*G^*}\nabla G^*\cdot\nabla G^*), \tag{3.36}$$

with $A_G = \partial A/\partial G$, $A_{GG^*} = \partial^2 A/\partial G\partial G^*$, etc. Also,

$$(\bar{G}+\overline{G^*}) = (G+G^*)+s(A+B). \tag{3.37}$$

Similar expressions for ∇G^* and $\nabla^2 G^*$ can also be found. Putting these results into Equations (3.28,34) and using the fact that G and G* already satisfy these equations, the first order terms in s are left equal to zero. The resulting equations are

$$(2(A+B)/(G+G^*) - 2A_G + (G+G^*)A_{GG})\nabla G\cdot\nabla G +$$
$$(2(G+G^*)A_{GG^*} - 4A_{G^*})\nabla G\cdot\nabla G^* +$$
$$((G+G^*)A_{G^*G^*} + 2A_{G^*})\nabla G^*\cdot\nabla G^* = 0, \tag{3.38}$$

and

$$(2(A+B)/(G+G^*) - 2B_{G^*} + (G+G^*)B_{G^*G^*})\nabla G^*\cdot\nabla G^* +$$
$$(2(G+G^*)B_{GG^*} - 4B_G)\nabla G\cdot\nabla G^* + ((G+G^*)B_{GG} + 2B_G)\nabla G\cdot\nabla G = 0. \tag{3.39}$$

The coefficients of $\nabla G\cdot\nabla G$, $\nabla G^*\cdot\nabla G^*$, and $\nabla G\cdot\nabla G^*$ are set equal to zero. (G and G* are assumed functionally independent; therefore, (G+G*) is not a function of (G-G*).) The following coupled set of linear partial differential equations results:

$$(G+G^*)^2A_{GG}-2(G+G^*)A_G+2A = -2B,$$
$$(G+G^*)A_{GG^*}-2A_{G^*} = 0,$$
$$(G+G^*)A_{G^*G^*}+2A_{G^*} = 0,$$
$$(G+G^*)^2B_{G^*G^*}-2(G+G^*)B_{G^*}+2B = -2A,$$
$$(G+G^*)B_{G^*G}-2B_G = 0,$$
$$(G+G^*)B_{GG}+2B_G = 0. \tag{3.40a-f}$$

First solve Equation (3.40c) using the integrating factor (G+G*). This yields the solution

$$A = -f(G)/(G+G^*)+g(G).$$

From Equation (3.40b) it is found that

$$\frac{d}{dG}(\ln f) = 4/(G+G^*) .$$

Since f is a function of G only, the above implies that f = 0. (Do not confuse the f here with the metric function introduced in Chapter 2.) The only solution possible is then A = g(G). In a similar manner, using Equations (3.17f,e), it is shown that B = h(G*). Two coupled ordinary differential equations now result from Equations (3.40a,d). They are

$$(G+G^*)^2 g_{GG} - 2(G+G^*)g_G + 2g = -2h$$

and

$$(G+G^*)^2 h_{G^*G^*} - 2(G+G^*)h_{G^*} + 2h = -2g.$$

To solve these two equations, use the power series expansions

$$g = \sum_{i=0}^{\infty} g_i G^i \quad \text{and} \quad h = \sum_{i=0}^{\infty} h_i (G^*)^i,$$

with g_i, h_i constants. The solution is

$$A = g_0 + g_1 G + g_2 G^2, \tag{3.41a}$$

$$B = -g_0 + g_1 G^* - g_2 (G^*)^2, \tag{3.41b}$$

with g_0, g_1, g_2 arbitrary complex constants. If the added requirement of $\overline{G^*} = (\bar{G})^*$ is given, then the g_0 and g_2 are purely imaginary, and g_1 is real. The infinitesimal transformations are now written

$$G \to \bar{G} = G + s(g_0 + g_1 G + g_2 G^2), \tag{3.42a}$$

$$G^* \to \overline{G^*} = G^* + s(-g_0 + g_1 G^* - g_2 (G^*)^2). \tag{3.42b}$$

To work only with real parameters, define $a^1 = isg_0$, $a^2 = sg_1$, and $a^3 = isg_2$. Then the infinitesimal transformations may be written in the form of Equation (3.11) with

$$\bar{G} = G - ia^1 + a^2 G - ia^3 G^2, \tag{3.43a}$$

$$\overline{G^*} = G^* + ia^1 + a^2 G^* + ia^3 (G^*)^2. \tag{3.43b}$$

The infinitesimal operators are, comparing Equations (3.43a,b) with Equations (3.19),

$$X_3 = i\partial/\partial G^* - i\partial/\partial G \ , \quad X_2 = G\partial/\partial G + G^*\partial/\partial G^* \ ,$$

$$X_3 = i(G^*)^2\partial/\partial G^* - iG^2\partial/\partial G \ , \tag{3.44}$$

with the commutators

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = -2X_2, \quad [X_2, X_3] = X_3. \tag{3.45}$$

Equations (3.45) yield the structure constants, and these may be used to integrate the group. Instead, define the following operators by

$$J_1 = 1/2(X_1+X_3), \quad J_2 = X_2, \quad J_3 = 1/2(X_1 - X_3). \tag{3.46}$$

Then the new commutators are given by

$$[J_1,J_2] = J_3, \quad [J_2,J_3] = -J_1, \quad [J_3,J_1] = -J_2. \tag{3.47}$$

These are the same commutation relations as for the Lorentz transformations in three dimensions, Hamermesh (1962: 307), and define the group SO(2,1). Thus, the symmetry group found above is locally isomorphic to SO(2,1).

One Parameter Subgroups

One parameter subgroups of Equations (3.43a,b) will now be examined. An important point to note on one parameter groups is that they are all locally isomorphic to the translation group in one dimension, Hamermesh (1962: 295). This means that for one parameter groups, the Φ^α of Equations (3.3) satisfy

$$\Phi(a;b) = a+b. \tag{3.48}$$

This implies that the $\psi(a)$ in Equations (3.17) is equal to one, meaning Equation (3.17) can be written

$$\frac{d\bar{x}^i}{da} = F^i(\bar{x}), \tag{3.49}$$

and to get the transformation functions f^i, it is only necessary to solve Equation (3.49).

The one parameter subgroups of Equations (3.43a,b) satisfy the following differential equations:

$$\frac{d\bar{G}}{da^1} = -i, \quad \frac{d\overline{G^*}}{da^1} = i, \quad a^1 = 1,\ a^2 = a^3 = 0; \tag{3.50a}$$

$$\frac{d\bar{G}}{da^2} = \bar{G}, \quad \frac{d\overline{G^*}}{da^2} = \overline{G^*}, \quad a^2 = 1,\ a^1 = a^3 = 0; \tag{3.50b}$$

$$\frac{d\bar{G}}{da^3} = -i\bar{G}^2, \quad \frac{d\overline{G^*}}{da^3} = i(\overline{G^*})^2, \quad a^3 = 1,\ a^1 = a^2 = 0. \tag{3.50c}$$

These equations are easily solved and have solutions

$$\bar{G} = G - ia^1, \tag{3.51a}$$

$$\bar{G} = e^{a^2}G, \tag{3.51b}$$

$$\bar{G} = G/(1+ia^3G). \tag{3.51c}$$

As promised, the Ehler's transformation has been reproduced in Equation (3.51c) as a one parameter subgroup of the symmetry group of form given by Equations (3.30,31). The other two subgroups are explained as follows. Equation (3.51a) results from the definition of ψ_{AB} in Equations (2.27) up to an additive constant, and Equation (3.51b) corresponds to the change of coordinates $(t,\phi)\rightarrow(e^{-1/2a^2}t, e^{1/2a^2}\phi)$.

The whole group of transformations is represented by a real two by two matrix with determinant equal to one;

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad-bc = 1, \tag{3.52}$$

such that

$$\bar{G} = (dG-ic)/(a+ibG). \tag{3.53}$$

See Cosgrove (1979b: Equation 2.3)) for example. If two transformations are performed in succession, the final transformed G will have the same form given in Equation (3.53), but the elements will be those of the two representation matrices multiplied together. This group is named H by Kinnersley (1977a:1531) and (P) by Cosgrove (1979a:56).

The Neugebauer-Kramer Mapping

Only continuous transformations have been considered up to this point. At this point, a very important discrete symmetry transformation will be reviewed.

Define

$$2f = G+G^* \text{ and } 2i\psi = G-G^*. \tag{3.54}$$

Using the vacuum Ernst equations yields

$$\nabla_3 \cdot (f^{-1}\nabla f+f^{-2}\psi\nabla\psi) = 0, \tag{3.55}$$

$$\nabla_3 \cdot (f^{-2}\nabla\psi) = 0, \tag{3.56}$$

and these are equivalent to the Ernst equation.

Equation (3.56) implies the existence of a potential ω such that

$$\nabla\omega = \rho f^{-2}\tilde{\nabla}\psi. \tag{3.57}$$

Eliminating ψ in favor of ω yields the following two equations,

$$\nabla_3 \cdot (f^{-1}\nabla f+\rho^{-2}f^2\omega\nabla\omega) = 0 , \tag{3.58}$$

$$\nabla_3 \cdot (\rho^{-2}f^2\nabla\omega) = 0. \tag{3.59}$$

(Equations (3.58,59) are restatements of Equations (2.26) for vacuum in the Lewis parameterization. Likewise, Equations (3.55,56) are restatements of Equations (2.27).)

It was noted that there is a mapping between Equations (3.55,56) and (3.58,59), given by

$$f \to \rho f^{-1},\ \omega \to i\psi,\ \psi \to -i\omega, \tag{3.60}$$

which sends one set of equations into the other and vice versa. This mapping is the Neugebauer-Kramer mapping and will be used in latter chapters.

The concept of group transformations has now been reviewed. A method has been presented by which symmetry groups may be found for given partial differential equations. Using this method for the vacuum Ernst equation, a known symmetry group has been reproduced. This symmetry group will reappear in Chapter 5 where the vacuum case will be briefly reviewed.

CHAPTER 4

BÄCKLUND TRANSFORMATIONS

Another transformation which has proved useful in generating new solutions from old solutions is the Bäcklund transformation, see Eisenhart (1909:284). The variables to be transformed are now the first partial derivatives. These, in turn, may be integrated to yield new solutions. In this chapter, Bäcklund transformations are introduced and then demonstrated by an example. The presentation in this chapter follows closely that given by Lamb (1974) and Neugebauer (1979).

Properties of the Bäcklund Transformation

The notation used in this section is: $z = z(x,y)$, $\bar{z} = \bar{z}(\bar{x},\bar{y})$, $p = \partial z/\partial x$, $\bar{p} = \partial\bar{z}/\partial\bar{x}$, $q = \partial z/\partial y$, $\bar{q} = \partial\bar{z}/\partial\bar{y}$, $r = \partial^2 z/\partial x^2$, $\bar{r} = \partial^2\bar{z}/\partial\bar{x}^2$, $s = \partial^2 z/\partial x\partial y$, $\bar{s} = \partial^2\bar{z}/\partial\bar{x}\partial\bar{y}$, $t = \partial^2 z/\partial y^2$, $\bar{t} = \partial^2\bar{z}/\partial\bar{y}^2$. Subscripts indicate the partial derivative with respect to the subscripted variable.

The Bäcklund transformation to be considered is written

$$p = f(\bar{x},\bar{y},z,\bar{z},\bar{p},\bar{q}),\quad q = h(\bar{x},\bar{y},z,\bar{z},\bar{p},\bar{q}), \tag{4.1}$$

and

$$\bar{x} = x,\quad \bar{y} = y. \tag{4.1a}$$

In order that the integrability condition $p_y = q_x$ holds, f and h must satisfy

$$W = p_y - q_x$$
$$= f_{\bar{y}} - h_{\bar{x}} + f_{\bar{z}}q - h_{\bar{z}}p + f_z q - h_z p +$$
$$(f_{\bar{p}} - h_{\bar{q}})s + f_{\bar{q}}t = h_{\bar{p}}r = 0. \qquad (4.2)$$

This equation may be satisfied identically in which case $f_{\bar{p}} - h_{\bar{q}} = f_{\bar{q}} = h_{\bar{p}} = 0$ and $f_{\bar{y}} - h_{\bar{x}} + f_{\bar{z}}q - h_{\bar{z}}p + f_z q - h_z p - 0$. Equation (4.2) may also be considered as a second-order partial differential equation for $\bar{z}$. If this equation is of a form known as Monge-Ampere, Forsythe (1959:200), then the transformations (4.1) are known as Bäcklund transformations.

In practice, it is the partial differential equations (4.2) which are started with and the Bäcklund transformations (4.1) which are sought.

Neugebauer's Bäcklund Transformation

The example will again be the vacuum Ernst equation (3.28). In the case of noncanonical coordinates, the field equations also include Equation (2.29). Written out, the field equations are

$$(G+G^*)[G_{,3,3}+G_{,4,4}+\frac{1}{\rho}(\rho_{,3}G_{,3}+\rho_{,4}G_{,4})] = 2[(G_{,3})^2 + (G_{,4})^2], \qquad (4.3)$$

$$(G+G^*)[G^*_{,3,3}+G^*_{,4,4}+\frac{1}{\rho}(\rho_{,3}G^*_{,3}+\rho_{,4}G^*_{,4})] = 2[(G^*_{,3})^2+(G^*_{,4})^2], \qquad (4.4)$$

and

$$\rho_{,3,3}+\rho_{,4,4} = 0. \qquad (4.5)$$

It is convenient to change variables. First, the independent variables are changed to $x_1 = x^3+ix^4$ and $x_2 = x^3-ix^4$. Equations (4.3-5) become

$$(G+G^*)[G_{,1,2} + \frac{1}{2\rho}(\rho_{,1}G_{,2}+\rho_{,2}G_{,1})] = 2G_{,1}G_{,2}, \qquad (4.3a)$$

$$(G+G^*)[G^*_{,1,2} + \frac{1}{2\rho}(\rho_{,1}G^*_{,2}+\rho_{,2}G^*_{,1})] = 2G^*_{,1}G^*_{,2}, \qquad (4.4a)$$

and

$$\rho_{,1,2} = 0. \qquad (4.5a)$$

At this point, Neugebauer (1979) changes the order of the equations by introducing the following dependent variables:

$$M_1 = G_{,1}/(G+G^*)\ ,\ M_2 = G^*_{,1}/(G+G^*)\ ,\ M_3 = \rho_{,1}/\rho, \qquad (4.6a)$$

$$N_1 = G^*_{,2}/(G+G^*)\ ,\ N_2 = G_{,2}/(G+G^*)\ ,\ N_3 = \rho_{,2}/\rho. \qquad (4.6b)$$

The Equations (4.3a-5a) then become equivalent to the system of first order coupled partial differential equations

$$M_{i,2} = c_i^{\ kl}M_kN_l\ ,\ N_{i,1} = c_i^{\ kl}N_kM_l\ , \qquad (4.7)$$

with $c_1^{\ 11} = c_2^{\ 22} = c_3^{\ 33} = -c_1^{\ 12} = -c_2^{\ 21} = -1$, and $c_1^{\ 32} = c_1^{\ 13} = c_2^{\ 31} = c_2^{\ 23} = -1/2$. These equations already contain the integrability conditions $G_{,1,2} = G_{,2,1}$ and $G^*_{,1,2} = G^*_{,2,1}$. Therefore, Equation (4.2) will be satisfied automatically if the transformed $\bar{M}_i$'s and $\bar{N}_i$'s satisfy Equations (4.7).

The Bäcklund transformation is assumed to have the following form.

$$\bar{M}_i = A_i^{\ k}M_k\ ,\ \bar{N}_i = (A^{-1})_i^{\ k}N_k\ , \qquad (4.8)$$

with

$$(A_i^{\ k}) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}\ ,\ \alpha\beta = \gamma. \qquad (4.9)$$

The requirement that $\bar{M}_i$ and $\bar{N}_i$ satisfy Equations (4.7) results in the following equations for α and γ.

$$d\gamma = \gamma(\gamma-1)M_3 dx_1 + (\gamma-1)N_3 dx_2, \tag{4.10}$$

$$d(\alpha\gamma^{-1/2}) = [-\gamma^{1/2}M_2 + (\alpha\gamma^{-1/2})(M_2-M_1) + (\alpha\gamma^{-1/2})^2 M_1 \gamma^{1/2}]dx_1 +$$

$$[-\gamma^{-1/2}N_1 + (\alpha\gamma^{-1/2})(N_1-N_2) + (\alpha\gamma^{-1/2})^2 N_2 \gamma^{-1/2}]dx_2. \tag{4.11}$$

The solution of these equations introduces integration constants. For example, the solution to Equation (4.10) is

$$\gamma = \frac{b-(\rho-iz)}{b+(\rho+iz)}, \tag{4.10a}$$

where b is an integration constant, ρ satisfies Equation (4.5a), and z is ρ's harmonic conjugate. Each set of integration constants represents a different transformation and will be labelled $(I_1)_a$. The I_1 represents all the transformations of Equation (4.8,9), and the a represents the integration constants. For example, the matrix representing $(I_1)_a$ will be written in the same form as the matrix of Equation (4.9) with elements $\alpha_a, \beta_a, \gamma_a$. The α_a and γ_a must satisfy Equations (4.10,11). It will now be shown that the I_1 form a group.

Given two transformations $(I_1)_b$ and $(I_1)_a$, it must be shown that the product $(I_1)_a(I_1)_b$ is also an I_1 transformation. By definition, the transformation $(I_1)_a(I_1)_b$ is represented by the matrix

$$\begin{pmatrix} \alpha_b\alpha_a & 0 & 0 \\ 0 & \beta_b\beta_a & 0 \\ 0 & 0 & \gamma_b\gamma_a \end{pmatrix}, \quad (\alpha_b\alpha_a)(\beta_b\beta_a) = (\gamma_b\gamma_a). \tag{4.13}$$

Then it can be shown that $\alpha_b\alpha_a$ and $\gamma_b\gamma_a$ satisfy Equations (4.10,11), remembering that the $(I_1)_a$ transformation acts on the $(I_1)_b M_i$, $(I_1)_b N_i$.

Hence, the I_1 transformations exhibit the group closure property. Since the matrices representing the group transformations are diagonal, the transformations I_1 are also commutative.

Associativity, existence of the identity, and inverses for each transformation are then shown to follow. The major step in each of the above proofs is the step showing that the respective matrix element satisfies Equations (4.10,11).

The Neugebauer-Kramer mapping introduced in Chapter 3 now plays an important role. In the (M_i, N_i) realization, this mapping is represented by the matrices $U_i^{\ k}$ and $V_i^{\ k}$, where

$$(U_i^{\ k}) = \begin{pmatrix} 0 & -1 & 1/2 \\ -1 & 0 & 1/2 \\ 0 & 0 & 1 \end{pmatrix} \text{, and } (V_i^{\ k}) = \begin{pmatrix} -1 & 0 & 1/2 \\ 0 & -1 & 1/2 \\ 0 & 0 & 1 \end{pmatrix} . \qquad (4.14)$$

With this discrete group, to be called S, define a new group of transformations I_2 by SI_1S. I_2 is also a group of Bäcklund transformations given in the (M_i, N_i) realization by the matrices $W_i^{\ k}$ and $Z_i^{\ k}$ where

$$(W_i^{\ k}) = \begin{pmatrix} \beta & 0 & (\gamma-\beta)/2 \\ 0 & \alpha & (\gamma-\alpha)/2 \\ 0 & 0 & \gamma \end{pmatrix} , \qquad (4.15)$$

and

$$(Z_i^{\ k}) = \begin{pmatrix} -\alpha^{-1} & 0 & (2\alpha\gamma)^{-1}(\alpha-\gamma) \\ 0 & \beta^{-1} & (2\beta\gamma)^{-1}(\beta-\gamma) \\ 0 & 0 & \gamma^{-1} \end{pmatrix} . \qquad (4.16)$$

From these, for I_2 transformations, γ still satisfies Equation (4.10), but now

$$d(\alpha\gamma^{-1/2}) = [-\gamma^{1/2}(M_3/2-M_1)+(\alpha\gamma^{-1/2})(M_2-M_1)+$$
$$(\alpha\gamma^{-1/2})^2(M_3/2-M_2)\gamma^{1/2}]dx_1+$$
$$[-\gamma^{-1/2}(N_3/2-N_1)+(\alpha\gamma^{-1/2})(N_2-N_1)+$$
$$(\alpha\gamma^{-1/2})^2(N_3/2-N_2)\gamma^{1/2}]dx_2 \; . \qquad (4.17)$$

It would seem that after each transformation, first order partial differential equations would have to be solved since the M_i, N_i are first derivatives. But Neugebauer has produced a very important theorem.

Neugebauer's Commutation Theorem

Given an initial solution M_i, N_i and an arbitrary Bäcklund transformation $(I_1)_a$, then there are always three nontrivial Bäcklund transformations $(I_2)_b$, $(I_1)_c$, $(I_2)_d$, such that the product transformation $(I_2)_d(I_1)_c(I_2)_b(I_1)_a$ is the identity transformation. The corresponding α's and γ's are

$$\alpha_b = \frac{(\alpha_a-\gamma_a)}{(\gamma_a(\alpha_a-1))} \; , \; \alpha_c = \frac{\gamma_a}{\alpha_a} \; , \; \alpha_d = \frac{(\alpha_a-1)}{(\alpha_a-\gamma_a)} \; ,$$

$$\gamma_b = \frac{1}{\gamma_a} \; , \; \gamma_c = \gamma_a \; , \; \gamma_d = \frac{1}{\gamma_a} \; . \qquad (4.18)$$

Here the α_a and α_c satisfy Equation (4.11) and the α_b and α_d satisfy Equation (4.17). All the γ's satisfy Equation (4.10).

This is a very important theorem. It considerably simplifies generating new solutions from old solutions without having to integrate partial differential equations at each step.

Now that two major methods of solution generation have been reviewed, their use with the vacuum equations will be summarized in the next chapter.

CHAPTER 5

REVIEW OF THE VACUUM CASE

In the last two chapters, two methods were presented for generating new solutions of the vacuum Ernst equations. In Chapter 2, both the electromagnetic and gravitational fields satisfy similar equations. This similarity has been used to extend vacuum results to electrovac; therefore, it is natural to review the vacuum case first. The notation to be used in this chapter follows that given by Kinnersley and Chitre (1977a,b; 1978a,b).

Fields, Potentials, and Generating Functions

Defining N_{AB} by

$$\nabla N_{AB} = H_{XA}{}^{*}\nabla H^{X}{}_{B}, \tag{5.1}$$

it is possible to show that

$$\overset{2}{H}_{AB} = i(N_{AB}+H_{AX}H^{X}{}_{B}) \tag{5.2}$$

satisfies the field Equations (2.37). Continuing in this way, define

$$\nabla \overset{mn}{N}_{AB} = \overset{m}{H}_{XA}{}^{*}\nabla \overset{n}{H}{}^{X}{}_{B}; \tag{5.3}$$

then it is possible to show that

$$\overset{n+1}{H}_{AB} = i(\overset{1n}{N}_{AB}+H_{AX}\overset{n}{H}{}^{X}{}_{B}) \tag{5.4}$$

satisfies Equations (2.37). All of the $\overset{n}{H}_{AB}$ satisfy

$$\nabla \overset{n}{H}_{AB} = -i\rho^{-1} f_{A}{}^{X}\tilde{\nabla}\overset{n}{H}_{XB}. \tag{5.5}$$

Defining

$$H_{AB} = i\epsilon_{AB}, \tag{5.6}$$

implies that

$$\overset{0n}{N}_{AB} = -i\overset{n}{H}_{AB} , \tag{5.7}$$

and the fields $\overset{n}{H}_{AB}$ are incorporated into the hierarchy of potentials $\overset{mn}{N}_{AB}$. This extends the range of m and n to all integer values. Kinnersley and Chitre (1977b) have given certain relations satisfied by the $\overset{mn}{N}_{AB}$. These are

$$\overset{mn}{N}_{AB}-\overset{nm}{N}_{BA}{}^{*} = \overset{m}{H}_{XA}{}^{*}\overset{n}{H}{}^{X}{}_{B}+\epsilon_{AB}\delta^{m}{}_{0}\delta^{n}{}_{0} \tag{5.8}$$

and

$$\overset{m,n+1}{N}_{AB} - \overset{m+1,n}{N}_{AB} = i\overset{m1}{N}_{AX}\overset{n}{H}{}^{X}{}_{B}-3\epsilon_{AB}\delta^{m}{}_{-1}\delta^{n}{}_{0}. \tag{5.9}$$

It has proven useful to define generating functions in terms of the potentials. The two generating functions used in the vacuum case are, somewhat changing the notation of Kinnersley and Chitre (1978b),

$$N_{AB}(s,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} s^{m}t^{n}\overset{mn}{N}_{AB} \tag{5.10}$$

and

$$F_{AB}(t) = \sum_{n=0}^{\infty} t^{n}\overset{n}{H}_{AB} = iN_{AB}(0,t). \tag{5.11}$$

There are algebraic relationships between these two generating functions, such that when the $F_{AB}(t)$ are known, the $N_{AB}(s,t)$ can be found.

The Symmetry Groups

In Chapter 3, the group H was found starting with the vacuum Ernst equation. There are other groups beside H and these are now reviewed.

The Groups G and H

Kinnersley (1977a) starts with the (t,ϕ) coordinate transformation group G, isomorphic to the group of 2 by 2 real matrices with unit determinant SL(2,R). G's three generators are chosen as

$$t' = t+a\phi,\ \phi' = \phi, \tag{5.12}$$

$$t' = t,\ \phi' = \phi+bt, \tag{5.13}$$

$$t' = ct,\ \phi' = c^{-1}\phi, \tag{5.14}$$

named $(G_1)_a$, $(G_2)_b$, $(G_3)_c$, respectively. The finite action of these three generators on the f_{AB} is given by

$$f_{11}' = f_{11},\ f_{12}' = f_{12}-af_{11},$$
$$f_{22}' = f_{22}-2af_{12}+a^2f_{11}; \tag{5.15}$$

$$f_{22}' = f_{22},\ f_{12}' = f_{12}-bf_{22},$$
$$f_{11}' = f_{11}-2bf_{12}+b^2f_{22}; \tag{5.16}$$

$$f_{11}' = c^{-2}f_{11},\ f_{12}' = f_{12},\ f_{22}' = c^2f_{22}. \tag{5.17}$$

Being a group of coordinate transformations, G does not physically alter the spacetime. However, by defining the group $H \equiv SGS$ with S being the Neugebauer-Kramer map, H is found to be the group for Equations (3.55,56). Its three generators, $(H_1)_a = S(G_1)_{-ia}S$,

$(H_2)_b = S(G_2)_{ib}S$, $(H_3)_c = S(G_3)_{c^{-1}}S$, produce the following action on f and ψ respectively.

$$f' = f,\ \psi' = \psi - a; \tag{5.18}$$

$$f' = \frac{1}{(1-b\psi^2+b^2f^2)},\ \psi' = \frac{\psi - b(f^2+\psi^2)}{(1-b\psi^2+b^2f^2)}; \tag{5.19}$$

$$f' = c^2 f,\ \psi' = c^2\psi. \tag{5.20}$$

These three finite transformations are recognized as the one parameter subgroups found in Chapter 3.

The Group K

Next, the infinitesimal action of G (H) is found on the ψ_{AB} (f_{AB}). This allows the infinitesimal action of G and H to be found on the H_{AB}, and then on the $\overset{mn}{N}_{AB}$.

Forming commutators with the different infinitesimal operators of G and H, an infinite-dimensional Lie algebra is generated. The operators thus defined will be named $\overset{k}{T}_{XY}$ and define the group K. Their action on the $\overset{mn}{N}_{AB}$ is given by

$$\overset{k}{T}_{XY}\overset{mn}{N}_{AB} = \varepsilon_{A(X}\overset{m+k,n}{N}_{Y)B} - \overset{m,n+k}{N}_{A(X}\varepsilon_{Y)B} + \sum_{s=1}^{k} \overset{ms}{N}_{A(X}\overset{k-s,n}{N}_{Y)B}, \tag{5.21}$$

where the parentheses indicate symmetrization with respect to the enclosed indices. The operators $\overset{0}{T}_{XY}$ are the infinitesimal operators of G, and the ($\overset{-1}{T}_{11}$, $\overset{0}{T}_{12}$, $\overset{1}{T}_{22}$) are the infinitesimal operators of H. The commutators by which the $\overset{k}{T}_{XY}$ were originally defined are

$$[\overset{k}{T}_{XY}, \overset{1}{T}_{AB}] = \varepsilon_{B(X}\overset{k+1}{T}_{Y)A} + \varepsilon_{A(X}\overset{k+1}{T}_{Y)B}. \tag{5.22}$$

For example, the commutator of $\overset{0}{T}_{11}$ and $\overset{1}{T}_{22}$ defines $\overset{1}{T}_{12}$.

The Groups Q and $\tilde{Q}$

Another group, Q, lying outside the K group was found by Cosgrove (1979a). This group changes not only the potentials, but also the coordinates (ρ,z). The infinitesimal action of Q on the coordinates (ρ,z) is

$$Q\rho = \rho z, \quad Qz = 1/2(z^2-\rho^2), \tag{5.23}$$

and on the potentials $\overset{mn}{N}_{AB}$ is

$$Q\overset{mn}{N}_{AB} = 1/4[(m+A-1)\overset{m+1,n}{N}_{AB} + (n+B-1)\overset{m,n+1}{N}_{AB} + i\overset{m1}{N}_{A2}\overset{n}{H}_{1B}]. \tag{5.24}$$

Given the Neugebauer-Kramer map S, a conjugate group $\tilde{Q} \equiv SQS$ is defined. Its infinitesimal action on (ρ,z) is identical to Equation (5.23), and its infinitesimal action on N_{AB} is

$$\tilde{Q}\overset{mn}{N}_{AB} = 1/4[(m+1)\overset{m+1,n}{N}_{AB} + n\overset{m,n+1}{N}_{AB}]. \tag{5.25}$$

From Equations (5.24,25) it is seen that

$$Q-\tilde{Q} = -1/4\overset{1}{T}_{12} \tag{5.26}$$

holds for these infinitesimal operators.

Applications

The problem of generating new solutions from a given solution now separates into three parts. Given a solution; for example flat spacetime, the potentials $\overset{mn}{N}_{AB}$ must be found. Second, the integration of the infinitesimal operators must be performed. Third, only those transformations which produce physical solutions should be used. By physical solution, it is usually meant that as the quantity ρ^2+z^2 approaches infinity, the metric approaches the flat spacetime metric.

Finding the generating functions $N_{AB}(s,t)$ and $F_{AB}(t)$ has proved the most effective way in finding the potentials $\overset{mn}{N}_{AB}$. An example of a generating function which has been found is the flat spacetime generating function $F_{AB}(t)$, Kinnersley and Chitre (1978a),

$$F_{AB}(t) = \begin{pmatrix} \frac{t}{S(t)} & \frac{i}{S(t)} \\ \frac{-i(1-2tz+S(t))}{S(t)} & \frac{(1-2tz-S(t))}{2tS(t)} \end{pmatrix}, \tag{5.27}$$

with

$$S(t)^2 = (1-2tz)^2+(2t\rho)^2. \tag{5.28}$$

Other generating functions which have been found for a given spacetime are listed below.

1. Zipoy-Voorhees; Kinnersley and Chitre (1978b)
2. Weyl; Hoenselaers, Kinnersley, and Xanthopoulos (1979)
3. Kerr-NUT; Hauser and Ernst (1979a)

Generating functions have proved useful in the integration of infinitesimal transformations to finite transformations. For example, Cosgrove (1980a) has successfully integrated the infinitesimal Q and $\tilde{Q}$ transformations to the finite $(Q)_{4s}$ and $(\tilde{Q})_{4s}$ transformations using $N_{AB}(s,t)$ and $F_{AB}(t)$. Some of these finite transformations are

$$(\tilde{Q})_{4s}N_{AB}(t_1, t_2) = \frac{t_2}{s+t_2} N_{AB}(s+t_1, s+t_2) + \frac{s}{s+t_2}\varepsilon_{AB}, \tag{5.29}$$

$$(Q)_{4s}F_{11}(t) = \frac{itF_{11}(s+t)}{(s+t)F_{12}(s)}, \tag{5.30a}$$

$$(Q)_{4s}F_{12}(t) = \frac{iF_{12}(s+t)}{F_{12}(s)}. \tag{5.30b}$$

More relations of this type are found in the last reference cited. In these, it must be remembered that (ρ, z) also changes.

Other finite transformations which have been found are listed below.

1. The null parameter case; that is when the parameters $\overset{k}{q}_{xy}$ in the operator $\overset{k}{q}{}^{xy}\overset{k}{T}_{xy}$ satisfies $\overset{k}{q}{}^{xy}\underset{k}{\overset{k}{q}}{}_{xy} = 0$. Kinnersley and Chitre (1978a)

2. For static metrics, T_{12} has been integrated to yield all the static spacetimes. Kinnersley and Chitre (1978a)

3. The B subgroup, having infinitesimal operators $\overset{k}{\beta} = \overset{k}{T}_{11} + \overset{k+2}{T}_{22}$, k = 0,1,2,..., preserves asymptotic flatness. This subgroup has been used to generate new solutions. For example, Kerr-Nut was first generated from the Schwarzschild potentials using this subgroup. Its applicability does depend on the initial solution. Kinnersley and Chitre (1978b).

4. The HKX transformations may be used to generate asymptotically flat spacetime with arbitrary multipole moments. Hoenselaers, Kinnersley, and Xanthopoulos (1979).

Bäcklund Transformations

In Chapter 4, the Bäcklund transformation found by Neugebauer was described. Cosgrove (1979b) has shown that I_1 and I_2 are equivalent to the two groups HQ and $\tilde{G}Q$. (HQ simply means the group consisting of products of the elements in the groups H and Q. Since these two groups commute, order is no problem.) Neugebauer (1980a,b) has applied I_1 and I_2 and the commutation theorem to generate arbitrary multipole moments for asymptotically flat spacetime. Cosgrove (1980a), using both Harrison's Bäcklund transformation and Neugebauer's commutation theorem, has also been able to generate asymptotically flat spacetimes with arbitrary multipole moments.

Summary

Other methods exist besides the group transformation and the Bäcklund transformation. Belinsky and Sakharov (1978,1979) use an inverse-scattering technique. Hauser and Ernst (1979a,b; 1980a,b) use an integral equation method. Both are capable of generating asymptotically flat spacetimes with arbitrary multipole moments. With so many methods capable of generating equivalent results, see Cosgrove (1980a), it is no wonder that these workers feel the vacuum case to be essentially solved.

CHAPTER 6

REVIEW OF THE ELECTROVAC CASE

Many of the results from the vacuum case have been extended to the electrovac case. These electrovac results will be reviewed in this chapter. The notation again follows that given by Kinnersley and Chitre (1977a,b; 1978a,b).

Fields, Potentials, and Generating Functions

The fields $\overset{n}{H}_{AB}$ and the potentials $\overset{mn}{N}_{AB}$ were introduced in Chapter 5. In a like manner, the fields $\overset{n}{H}_{AB}$ and $\overset{n}{\phi}_A$ and the potentials $\overset{mn}{N}_{AB}$, $\overset{mn}{M}_A$, $\overset{mn}{L}_B$, $\overset{mn}{K}$ are introduced for electrovac. They are defined recursively by the following relationships:

$$\overset{1}{\Phi}_A = \Phi_A, \ \overset{1}{H}_{AB} = H_{AB}, \tag{6.1}$$

$$\nabla\overset{mn}{K} = \overset{m}{\Phi}_X * \nabla\overset{n}{\Phi}{}^X, \quad \nabla\overset{mn}{L}_B = \overset{m}{\Phi}_X * \nabla\overset{n}{H}{}^X{}_B, \tag{6.2}$$

$$\nabla\overset{mn}{M}_A = \overset{m}{H}_{XA} * \nabla\overset{n}{\Phi}{}^X, \ \nabla\overset{mn}{N}_{AB} = \overset{m}{H}_{XA} * \nabla\overset{n}{H}{}^X{}_B, \tag{6.3}$$

$$\overset{n+1}{\Phi}_A = i(\overset{1n}{M}_A + 2\Phi_A\overset{1n}{K} + H_{AX}\overset{n}{\Phi}{}^X), \tag{6.4}$$

$$\overset{n+1}{H}_{AB} = i(\overset{1n}{N}_{AB} + 2\Phi_A\overset{1n}{L}_B + H_{AX}\overset{n}{H}{}^X{}_B). \tag{6.5}$$

All the fields $\overset{n}{\Phi}_A$ and $\overset{n}{H}_{AB}$ satisfy the same field equations.

$$\nabla\overset{n}{\Phi}_A = -i\rho^{-1} f_A{}^X \tilde{\nabla}\overset{n}{\Phi}_X, \tag{6.6}$$

$$\nabla\overset{n}{H}_{AB} = -i\rho^{-1} f_A{}^X \tilde{\nabla}\overset{n}{H}_{XB}. \tag{6.7}$$

Defining

$$\overset{10}{K} = -1/2i, \quad \overset{0}{H}_{AB} = i\epsilon_{AB}, \tag{6.8}$$

implies

$$\overset{0n}{M}_A = -i\overset{n}{\Phi}_A, \quad \overset{0n}{N}_{AB} = -i\overset{n}{H}_{AB}, \tag{6.9}$$

and that the original fields are contained in the hierarchy of potentials. The relations found by Kinnersley and Chitre (1977b) are

$$\overset{mn}{K} - \overset{nm}{K}{}^* = \overset{m}{\Phi}_X{}^* \overset{n}{\Phi}{}^X, \tag{6.10}$$

$$\overset{mn}{L}_B - \overset{nm}{M}_B{}^* = \overset{m}{\Phi}_X{}^* \overset{n}{H}{}^X{}_B, \tag{6.11}$$

$$\overset{mn}{N}_{AB} - \overset{nm}{N}_{BA}{}^* = \overset{m}{H}_{XA}{}^* \overset{n}{H}{}^X{}_B + \epsilon_{AB}\delta^m{}_0\delta^n{}_0, \tag{6.12}$$

and

$$\overset{m,n+1}{K} - \overset{m+1,n}{K} = 2i\overset{m1}{K}\,\overset{1n}{K} + i\overset{m1}{L}_X\,\overset{n}{\Phi}{}^X + (i\delta^m{}_0\delta^n{}_0)/2, \tag{6.13}$$

$$\overset{m,n+1}{L}_B - \overset{m+1,n}{L}_B = 2i\overset{m1}{K}\,\overset{1n}{L}_B + i\overset{m1}{L}_X\,\overset{n}{H}{}^X{}_B, \tag{6.14}$$

$$\overset{m,n+1}{M}_A - \overset{m+1,n}{M}_A = 2i\overset{m1}{M}_A\,\overset{1n}{K} + i\overset{m1}{N}_{AX}\overset{n}{\Phi}{}^X, \tag{6.15}$$

$$\overset{m,n+1}{N}_{AB} - \overset{m+1,n}{N}_{AB} = 2i\overset{m1}{M}_A\,\overset{1n}{L}_B + i\overset{m1}{N}_{AX}\overset{n}{H}{}^X{}_B - 3\epsilon_{AB}\delta^m{}_{-1}\delta^n{}_0. \tag{6.16}$$

Generating functions are defined by

$$K(s,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} s^m t^n \overset{mn}{K}, \quad K(t) = \sum_{n=0}^{\infty} t^n \overset{1n}{K}, \tag{6.17}$$

$$L_A(s,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} s^m t^n \overset{mn}{L}_A, \quad L_A(t) = \sum_{n=0}^{\infty} t^n \overset{1n}{L}_A, \tag{6.18}$$

$$M_A(s,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} s^m t^n \overset{mn}{M}_A, \quad \Phi_A(t) = \sum_{n=0}^{\infty} t^n \overset{n}{\Phi}_A, \tag{6.19}$$

$$N_{AB}(s,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} s^m t^n \overset{mn}{N}_{AB}, \quad F_{AB}(t) = \sum_{n=0}^{\infty} t^n \overset{n}{H}_{AB}. \tag{6.20}$$

Taking the gradient of Equations (6.4,5) leads to the equations

$$\nabla F_{AB}(t) = \frac{it}{S^2(t)} [((1-2tz)\nabla H_{AX} - 2t\rho\tilde{\nabla} H_{AX})F^X{}_B(t) + 2((1-2tz)\nabla\Phi_A - 2t\rho\tilde{\nabla}\Phi_A)L_B(t)], \tag{6.21}$$

$$\nabla\Phi_A(t) = \frac{it}{S^2(t)} [((1-2tz)\nabla H_{AX} - 2t\rho\tilde{\nabla} H_{AX})\Phi^X(t) + 2((1-2tz)\nabla\Phi_A - 2t\rho\tilde{\nabla}\Phi_A)K(t)], \tag{6.22}$$

$$S^2(t) = (1-2tz)^2 + (2t\rho)^2. \tag{6.23}$$

The first integral of the above equations is

$$2K(t)\mathrm{Det}(F(t)) - 2\Phi^X(t)L^Y(t)F_{XY}(t) = \frac{i}{S(t)}. \tag{6.24}$$

Important algebraic relationships can be derived from Equations (6.10-16). Some of these are:

$$N_{AB}(s,t) = \frac{s}{s-t}\,\varepsilon_{AB} + \frac{2itS(s)}{(s-t)}[K(s)F_{XA}(s)F^X{}_B(t) + (L_A(s)F_{XB}(t) - L_B(t)F_{XA}(s))\Phi^X(s)], \tag{6.25}$$

$$M_A(s,t) = \frac{2itS(s)}{(s-t)}[K(s)\Phi^X(t)F_{XA}(s) - K(t)\Phi^X(s)F_{XA}(s) - L_A(s)\Phi_X(s)\Phi^X(t)], \tag{6.26}$$

$$L_B(s,t) = \frac{stS(s)}{(s-t)} [\mathrm{Det}(F(t))L_B(t) - L^Y(s)F_{XY}(s)F^X{}_B(t)], \quad (6.27)$$

$$K(s,t) = \frac{i(st-s^2-t^2)}{2(s-t)} + \frac{stS(s)}{(s-t)} [K(t)\mathrm{Det}(F(t)) - L^Y(s)F_{XY}(s)\Phi^X(t)], \quad (6.28)$$

$$K^*(s) = -\Phi^X\Phi_X{}^*(s) - iS(s)\mathrm{Det}(F(s))/2, \quad (6.29)$$

$$L_A(s) = -\Phi^X F_{XA}{}^*(s) + S(s)\Phi_X(s)F^X{}_A(s)/s, \quad (6.30)$$

$$F_{CA}{}^*(s) = -2iS^{-1}(s)((1-2sz)\delta_C{}^B + 2isf_C{}^B) \times (K(s)F_{BA}(s) - \Phi_B(s)L_A(s) + \Phi_B{}^*\Phi_X(s)F^X{}_A(s)), \quad (6.31)$$

$$\Phi_C{}^*(s) = sS^{-1}(s)((1-2sz)\delta_C{}^B + 2isf_C{}^B) \times (F_{BX}(s)L^X(s) - \Phi_B{}^*\mathrm{Det}(F(s)). \quad (6.32)$$

In the above, $\mathrm{Det}(F(s))$ is the determinant of $F_{AB}(s)$.

The Symmetry Groups

The vacuum groups can be enlarged to the electrovac case. The groups G' and H' are defined and used to find an enlarged symmetry group K'.

The Groups G' and H'

The group G' is comprised of the coordinate transformations given in Equations (5.12-14) and the electromagnetic and gravitational gage transformations

$$\Phi_A \to \Phi_A + a_A, \quad H_{AB} \to H_{AB} - \Phi_A{}^* a_B - \Phi_A a_b{}^* - a_A{}^* a_B + i\alpha_{AB}. \quad (6.33)$$

These gage transformations have a_B complex and α_{AB} real and symmetric; they do not change the metric f_{AB} or the electromagnetic field F_{AM}. This now defines G' which is an eight parameter group.

The group H' is also an eight parameter group leaving Equations (2.38,39) invariant. The generators of H' are the gravitational and electromagnetic gage transformations:

$$\Phi_1 \to \Phi_1, \ H_{11} \to H_{11} + i\alpha, \tag{6.34}$$

$$\Phi_1 \to \Phi_1 + a, \ H_{11} \to H_{11} - 2a^*\Phi_1 - a^*a; \tag{6.35}$$

the generalized Ehler's transformation

$$\Phi_1 \to \frac{\Phi_1}{(1+\gamma H_{11})}, \ H_{11} \to \frac{H_{11}}{(1+i\gamma H_{11})}; \tag{6.36}$$

the Harrison (1968) mixing transformation

$$\Phi_1 \to \frac{\Phi_1 + cH_{11}}{(1-2c^*\Phi_1 - c^*cH_{11})}, \tag{6.37}$$

$$H_{11} \to \frac{H_{11}}{(1-2c^*\Phi_1 - c^*cH_{11})}; \tag{6.38}$$

and the scaling and electromagnetic duality transformation

$$\Phi_1 \to \beta e^{i\varepsilon}\Phi_1, \ H_{11} \to \beta^2 H_{11}. \tag{6.39}$$

In the above, α, β, γ, ε are real constants while a and c are complex constants. There is no simple way to get H' from G' since the Neugebauer-Kramer mapping's action on the metric f_{AB} has not been found; however, the mapping's action on the group generators has been found and will be described later in this chapter.

The Group K'

Again, by forming all possible commutators of the infinitesimal operators of G' and H', an infinite-dimensional parameter group is generated. This group is called K' and is the electrovac generalization of K. The infinitesimal operators are ${}^{k}T_{xy}$, ${}^{k}C_{x}$, ${}^{k}C_{x}{}^{*}$, ${}^{k}\Sigma$, and their action is given in Kinnersley and Chitre (1977b:1540) and reproduced here. (In the above reference, the action is given with the group parameters already implied. For example, what Kinnersley and Chitre refer to as ${}^{k}c_{A}$ is the infinitesimal operator ${}^{k}c^{X}\,{}^{k}C_{X} + {}^{k}c^{X*}\,{}^{k}C_{X}{}^{*}$. To avoid confusion, ${}^{k}\gamma_{XY}$, ${}^{k}c_{X}$, ${}^{k}c_{X}{}^{*}$, ${}^{k}\sigma$ will be the parameters and ${}^{k}T_{XY}$, ${}^{k}C_{X}$, ${}^{k}C_{X}{}^{*}$, ${}^{k}\Sigma$ will be the operators.)

$$ {}^{k}T_{XY}\,{}^{mn}N_{AB} = \epsilon_{A(Y}\,{}^{m+k,n}N_{X)B} - {}^{m,n+k}N_{A(X}\,\epsilon_{Y)B} + \sum_{s=1}^{k} {}^{ms}N_{A(X}\,{}^{k-s,n}N_{Y)B}, \qquad (6.40) $$

$$ {}^{k}T_{XY}\,{}^{mn}M_{A} = \epsilon_{A(Y}\,{}^{m+k,n}M_{X)} + \sum_{s=1}^{k} {}^{ms}N_{A(X}\,{}^{k-s,n}M_{Y)}, \qquad (6.41) $$

$$ {}^{k}T_{XY}\,{}^{mn}L_{B} = \epsilon_{B(Y}\,{}^{m,n+k}L_{X)} + \sum_{s=1}^{k} {}^{ms}L_{(X}\,{}^{k-s,n}N_{Y)B}, \qquad (6.42) $$

$$ {}^{k}T_{XY}\,{}^{mn}K = \sum_{s=1}^{k} {}^{ms}L_{(X}\,{}^{k-s,n}M_{Y)}, \qquad (6.43) $$

$$ {}^{k}C_{X}\,{}^{mn}N_{AB} = 2i\epsilon_{XA}\,{}^{m+k,n}L_{B} - 2i\sum_{s=1}^{k} {}^{ms}N_{AX}\,{}^{k-s,n}L_{B}, \qquad (6.44) $$

$$ {}^{k}C_{X}\,{}^{mn}M_{A} = 2i\epsilon_{XA}\,{}^{m+k,n}K + {}^{m,n+k-1}N_{AX} - 2i\sum_{s=1}^{k-1} {}^{ms}N_{AX}\,{}^{k-s,n}K, \qquad (6.45) $$

$$ {}^{k}C_{X}\,{}^{mn}L_{B} = -2i\sum_{s=1}^{k} {}^{ms}L_{X}\,{}^{k-s,n}L_{B}, \qquad (6.46) $$

$$\overset{k}{C}_X \overset{mn}{K} = L_X^{m,n+k-1} - 2i \sum_{s=1}^{k-1} L_X^{ms} K^{k-s,n} , \tag{6.47}$$

$$\overset{k}{C}_X{}^{*} \overset{mn}{N}_{AB} = -2i\varepsilon_{XB} M_A^{m,n+k} + 2i \sum_{s=1}^{k} M_A^{ms} N_{XB}^{k-s,n} , \tag{6.48}$$

$$\overset{k}{C}_X{}^{*} \overset{mn}{M}_A = 2i \sum_{s=1}^{k} M_A^{ms} M_X^{k-s,n} , \tag{6.49}$$

$$\overset{k}{C}_X{}^{*} \overset{mn}{L}_B = -2i\varepsilon_{XB} K^{m,n+k} + N_{XN}^{m+k-1,n} + 2i \sum_{s=1}^{k} K^{ms} N_{XB}^{k-s,n} , \tag{6.50}$$

$$\overset{k}{C}_X{}^{*} \overset{mn}{K} = M_X^{m+k-1,n} + 2i \sum_{s=1}^{k} K^{ms} M_X^{k-s,n} , \tag{6.51}$$

$$\overset{k}{\Sigma} \overset{mn}{N}_{AB} = -2 \sum_{s=1}^{k} M_A^{ms} L_B^{k-s+1,n} , \tag{6.52}$$

$$\overset{k}{\Sigma} \overset{mn}{M}_A = -i M_A^{m,n+k} - 2 \sum_{s=1}^{k} M_A^{ms} K^{k-s+1,n} , \tag{6.53}$$

$$\overset{k}{\Sigma} \overset{mn}{L}_B = i L_B^{m+k,n} - 2 \sum_{s=1}^{k} K^{ms} L_B^{k-s+1,n} , \tag{6.54}$$

$$\overset{k}{\Sigma} \overset{mn}{K} = i K^{m+k,n} - i K^{m,n+k} - 2 \sum_{s=1} K^{ml} K^{k-s+1,n} - 1/2\delta^m{}_0 \delta^n{}_0 \delta^k{}_1 , \tag{6.55}$$

with k running over all integer values.

Kinnersley (1980) has found eight-parameter subgroups ($\overset{m,n}{G}$) of K' isomorphic to SU(2,1). They are

$$\overset{(m,n)}{G} = \{\gamma_{11}^{-m}, \gamma_{12}^{0}, \gamma_{22}^{m}, c_1^{n}, c_2^{(m+n)}, c_1^{*(1-m-n)}, c_2^{*(1-n)}, \sigma^{0}\} . \tag{6.56}$$

Also found were involutive automorphisms which map the subgroups into one another. These are