Adjunction spaces and k-spaces
by Dale W Behrens

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY . Mathematics
Montana State University

Abstract:
Let X □f Y denote the adjunction space of X and Y via f:A → Y where A is a closed subset of X and let p:X + Y → X □f Y be the identification from the disjoint union of X and Y to the adjunction space. Let kX be the k-space associated with Hausdorff X. The question of when k distributes through the adjunction is settled for the following cases. An example is given for which k(X □f Y) ≠ kX □kf kY where X □f Y = X/A (Theorem 2.2). It is proven that if p is compact covering (Theorem 2.4), for example when X is paracompact (Corollary 2,8), then k(X □ Y) = kX □kf kY. Let βX be the Stone-Cech compactification of completely regular X. Using the fact that β distributes over the adjunction, (β (X □f Y) □ βX □βf βY), whenever X is normal and whenever Y and X □f Y are completely regular, it is shown that k will distribute through the adjunction by enlarging the space X, that is, there exists a space X1 where X □ X1 □ βX such that k(X □f Y) □ kX1 □kg kY where g extends f from A to A1 □ X1.

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Approved:

Robert D. Engle
Head, Major Department

Richard M. Gillette
Chairman, Examining Committee

Henry L Parsons
Graduate Dean

MONTANA STATE UNIVERSITY
Bozeman, Montana

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ABSTRACT

Let $X \cup_f Y$ denote the adjunction space of X and Y via $f:A \longrightarrow Y$ where A is a closed subset of X and let $p:X + Y \longrightarrow X \cup_f Y$ be the identification from the disjoint union of X and Y to the adjunction space. Let kX be the k-space associated with Hausdorff X. The question of when k distributes through the adjunction is settled for the following cases. An example is given for which $k(X \cup_f Y) \neq kX \cup_{kf} kY$ where $X \cup_f Y = X/A$ (Theorem 2.2). It is proven that if p is compact covering (Theorem 2.4), for example when X is paracompact (Corollary 2.8), then $k(X \cup_f Y) = kX \cup_{kf} kY$. Let βX be the Stone-Cech compactification of completely regular X. Using the fact that β distributes over the adjunction, $(\beta(X \cup_f Y) \cong \beta X \cup_{\beta f} \beta Y)$, whenever X is normal and whenever Y and $X \cup_f Y$ are completely regular, it is shown that k will distribute through the adjunction by enlarging the space X, that is, there exists a space X_1 where $X \subset X_1 \subset \beta X$ such that $k(X \cup_f Y) \cong kX_1 \cup_{kg} kY$ where g extends f from A to $A_1 \subset X_1$.

INTRODUCTION

Historically, J.H.C. Whitehead [13] systematized the concept of adjunction spaces and now the concept is used extensively in Algebraic Topology. The adjunction space is an identification space and hence inherites the problems of all identification spaces in that many properties of the original spaces are not preserved. Hu [5] for example lists several properties that hold and provides some counterexamples. We are interested in Hausdorff spaces and unfortunately identifications do not preserve this property, so we will assume through out this paper that all spaces are Hausdorff, including the adjunction space. Whitehead [13] does have a theorem stating sufficient conditions for an adjunction space to be Hausdorff which requires X to be normal and Y to be Urysohn (distinct points have neighborhoods whose closures are disjoint). For the convenience of the reader, we will include notation and some known facts of adjunction spaces.

Let A be a closed subset of X, let $f:A \longrightarrow Y$ be a continuous map, and let X + Y denote the <u>disjoint union</u> of X and Y. The <u>adjunction space</u> of X and Y via f, $X \cup_f Y$, is formed by identifying $a \in A$ with $f(a) \in Y$. Let $p:X + Y \longrightarrow X \cup_f Y$ be the identification. Our terminology will follow that of R. Brown [1].

The commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\bar{f}} & X \cup_f Y \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \bar{i}} \\ A & \xrightarrow[f]{} & Y \end{array}$$

where i and $\bar{i}$ are embeddings follows from the diagram

$$\begin{array}{ccc} X & \searrow^{\bar{f}} & \\ \downarrow & & \\ X + Y & \xrightarrow[p]{} & X \cup_f Y. \\ \uparrow & & \\ Y & \nearrow_{\bar{i}} & \end{array}$$

A subset U of $X \cup_f Y$ is open if and only if $\bar{f}^{-1}(U)$ and $\bar{i}^{-1}(U)$ are open. Therefore, $\bar{f}(X - A)$ is open in $X \cup_f Y$, $\bar{f}|(X - A)$ is a homeomorphism into, and $\bar{i}(Y)$ is closed.

The adjunction space is a pushout in the sense that if

$$\begin{array}{ccc} X & \xrightarrow{f'} & Z \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i'} \\ A & \xrightarrow[f]{} & Y \end{array}$$

commutes, then there is a unique g mapping $X \cup_f Y$ to Z such that $g \circ \bar{i} = i'$ and $g \circ \bar{f} = f'$. The pushout property is an important feature of adjunction spaces.

If Y is a single point, then $X \cup_f Y = X/A$ where X/A is the space X with A identified to a point.

In Algebraic Topology, one is interested in working with a certain category of spaces in which several standard operations are closed. The concept of k-spaces appears in Kelley [7]. Steenrod [12] and many others have demonstrated that the category of k-spaces is convenient and it includes all locally compact and all first countable spaces. There are certain drawbacks to the category of k-spaces in that subspaces of a k-space need not be a k-space, products of k-spaces need not be k-spaces, and function spaces of k-spaces with compact-open topology need not be k-spaces, but suitable modifications of these topologies allows the category of k-spaces and continuous maps to be convenient. For the convenience of the reader, we will include notation and some known facts of k-spaces.

A subset A of X is <u>compactly closed</u> provided it meets each compact subset of X in a closed set, and X is a <u>k-space</u> if X is Hausdorff and all compactly closed sets are closed. There is a unique k-space kX associated with each Hausdorff space X. The spaces X and kX have the same underlying set, and the closed sets of kX are the compactly closed subsets of X. The identity map from kX to X is continuous, kX and X have the same compact sets and if X is a k-space, then kX = X. If $f:X \longrightarrow Y$, then the same function $kf:kX \longrightarrow kY$ is continuous and the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow & & \uparrow \\ kX & \xrightarrow[kf]{} & kY \end{array}$$

commutes. One views k as a functor from the category of Hausdorff spaces and continuous maps into the category of k-spaces and continuous maps.

In relating adjunction spaces and k-spaces, the question arises as to when k distributes through the adjunction. An example is given for which $k(X \cup_f Y) \neq kX \cup_{kf} kY$ where $X \cup_f Y = X/A$ (Theorem 2.2). It is proven that if p is compact covering (Theorem 2.4), for example when X is paracompact (Corollary 2.8), then $k(X \cup_f Y) = kX \cup_{kf} kY$. Let βX be the Stone-Cech compactification of completely regular X. Using the fact that β distributes over the adjunction ($\beta(X \cup_f Y) \cong \beta X \cup_{\beta f} \beta Y$) whenever X is normal and whenever Y and $X \cup_f Y$ are completely regular, it is shown that k will distribute through the adjunction by enlarging the space X, that is, there exists a space X_1 where $X \subset X_1 \subset \beta X$ such that $k(X \cup_f Y) \cong kX_1 \cup_{kg} kY$ where g extends f from A to $A_1 \subset X_1$.

For notation, $Cl_X A$ will denote the closure of a subset A of X. Bdry A will denote the boundary of A and Int A will denote the interior of A. If a map is denoted by a letter under a bar, we will always assume the pushout diagram

$$\begin{array}{ccc} X & \xrightarrow{\bar{f}} & X \cup_f Y \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \bar{i}} \\ A & \xrightarrow[f]{} & Y \end{array}.$$

We emphasize again that all spaces considered are Hausdorff and all maps are continuous.

A mapping $f: X \longrightarrow Y$ is <u>compact covering</u> if each compact subset of Y is the image of some compact subset of X, f is a <u>compact mapping</u> if the preimage of each compact subset of Y is compact.

CHAPTER 1

In Chapter 1, we relate the compactness of X, A, f, Y, $X \cup_f Y$, and subsets of $X \cup_f Y$, and establish a useful necessary and sufficient condition for the identification $p: X + Y \longrightarrow X \cup_f Y$ to be compact covering. Using the necessary condition, we then show any compact subset of an adjunction space is itself an adjunction space.

It is clear that if X and Y are compact, then so is $X \cup_f Y$. If $X \cup_f Y$ is compact, then X need not be compact. However, Y is compact because $\bar{i}: Y \longrightarrow X \cup_f Y$ is an embedding and $\bar{i}(Y)$ is a closed subset of $X \cup_f Y$. In fact, $\bar{i}$ is a closed compact map.

<u>Proposition 1.1</u>. If A is a compact subset of X, then $p: X + Y \longrightarrow X \cup_f Y$ is a closed compact map.

Proof. If A is compact, then p is closed because f is closed [2, p128], and p is compact because $p^{-1}(z)$ is compact for each $z \in X \cup_f Y$ [4].

<u>Corollary 1.2</u>. If $X \cup_f Y$ and A are compact, then X and Y are compact.

<u>Proposition 1.3</u>. Suppose Int $A = \phi$ or A is not open. Then necessary and sufficient conditions for $X \cup_f Y$ to be compact is that Y be compact and closed subsets of X missing A be compact.

Proof. Let $\{U_a\}$ be an open cover of $X \cup_f Y$. Since $\overline{i}(Y)$ is compact, a finite number of U_a's cover $\overline{i}(Y)$, so the cover can be reduced to $\{U_a\} \cup \{U_i\}_{i=1}^{n}$ where the U_a's cover $\overline{f}(X - A)$. Since $\bigcup_{i=1}^{n} U_i \supset \overline{f}(A)$, $A \subset U = \bigcup_{i=1}^{n} \overline{f}^{-1}(U_i)$ and since Int $A = \phi$ or A is not open, $U \cap (X - A) \neq \phi$. Now $X - U$ is closed and misses A, so $X - U$ is compact. Therefore, there is a finite number of U_a's, say U_j, $j = 1, \dots, m$, such that $\bigcup_{j=1}^{m} \overline{f}^{-1}(U_j) \supset X - U$. Thus $\{U_j\}_{j=1}^{m} \cup \{U_i\}_{i=1}^{n}$ covers $X \cup_f Y$.

<u>Proposition 1.4</u>. Let K be a compact subset of $X \cup_f Y$. If $\overline{f}^{-1}(K) \cap A$ is compact, then $\overline{f}^{-1}(K)$ is compact.

Proof. Let $X_o = \overline{f}^{-1}(K)$, $Y_o = \overline{i}^{-1}(K)$, and $A_o = \overline{f}^{-1}(K) \cap A$ and apply Corollary 1.2 using the fact that $K = X_o \cup_g Y_o$ where $g = f|A_o$ [8].

Lemma 1.5. If F is a closed subset of $X \cup_f Y$, then $Cl_X G - G$ is a subset of A where $G = \overline{f}^{-1}(F - \overline{f}(A))$.

Proof. Let $x \in Cl_X G - G$. If $x \notin A$, then $\overline{f}(x)$ is in $\overline{f}(X - A)$. Also, $\overline{f}(x) \in \overline{f}(Cl_X G) \subset Cl\,(F - \overline{f}(A)) \subset F$. Therefore, $x \in G$ since $\overline{f}|(X - A)$ is a homeomorphism; a contradiction. Thus, $x \in A$.

Proposition 1.6. The identification $p: X + Y \longrightarrow X \cup_f Y$ is compact covering if and only if $Cl_X \overline{f}^{-1}(K - \overline{f}(A))$ is compact for all compact subsets K of $X \cup_f Y$.

Proof. Let K be a compact subset of $X \cup_f Y$ and let $G = \overline{f}^{-1}(K - \overline{f}(A))$. If p is compact covering, there exists a compact subset C of $X + Y$ such that $p(C) = K$. Now $C \cap X$ and $C \cap Y$ are compact and $\overline{f}(C \cap X) \cup \overline{i}(C \cap Y) = K$. Since $G = (C \cap X) - A$, $Cl_X G$ is compact.

Suppose K is a compact subset of $X \cup_f Y$ and $Cl_X G$ is compact. Now $Cl_X G \cup \overline{i}^{-1}(K)$ is compact and by Lemma 1.5, $Cl_X G - G \subset A$, so $p(Cl_X G \cup \overline{i}^{-1}(K)) = K$.

<u>Corollary 1.7</u>. If K is a compact subset of $X \cup_f Y$ and p is compact covering, then K is an adjunction space formed from two compact sets.

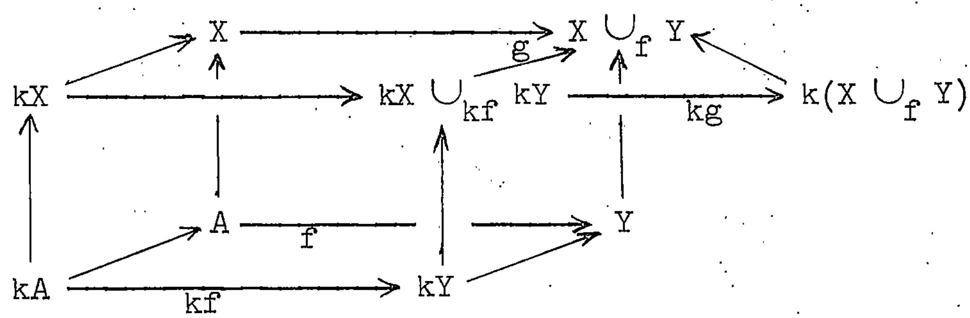
Proof. If $K \cap \overline{f}(A) = \phi$, there is nothing to show. If $K \cap \overline{f}(A) \neq \phi$, let $X_0 = \overline{f}^{-1}(K)$, $A_0 = X_0 \cap A$, $Y_0 = \overline{I}^{-1}(K)$, and $g = f|A_0$. Then $K = X_0 \cup_g Y_0$ where Y_0 is compact. Let $X_1 = Cl_X G_K$ where $G_K = \overline{f}^{-1}(K - \overline{f}(A))$. Then X_1 is compact since p is compact covering. We also know that $X_1 - G_K$ is contained in A. If $X_1 = G_K$, then $K = X_1 + Y_0$. Suppose $X_1 \neq G_K$, then let $A_1 = X_1 \cap A_0$ and $h = f|A_1$. Then $X_1 \cup_h Y_0 = K$.

The hypothesis of Corollary 1.7 cannot be weakened by removing the condition that p is compact covering. For an example, let X be the example used in the proof of Theorem 2.2 and consider X/A.

CHAPTER 2

In Chapter 2, we give an example when k does not distribute through the adjunction. We prove that if the identification $p: X + Y \longrightarrow X \cup_f Y$ is compact covering, for example when X is paracompact, then k will distribute.

Consider the diagram



Because f and kf are the same point set map, the unique continuous map g which is given by the pushout property of $kX \cup_{kf} kY$ is the identity. Therefore, $kX \cup_{kf} kY$ is Hausdorff provided $X \cup_f Y$ is Hausdorff and $kX \cup_{kf} kY$ is a k-space because it is the identification image of the k-space $kX + kY$. It now follows that kg is the identity. The problem is to show when $k(X \cup_f Y) = kX \cup_{kf} kY$.

Proposition 2.1. If $kX \cup_{kf} kY$ and $X \cup_f Y$ have the same compact sets, then $k(X \cup_f Y) = kX \cup_{kf} kY$.

Proof. The space $kX \cup_{kf} kY$ is a Hausdorff k-space with the same compact sets as $X \cup_f Y$.

<u>Theorem 2.2</u>. In general, $k(X \cup_f Y) \neq kX \cup_{kf} kY$.

The proof is based upon the following lemma where $X \cup_f Y = X/A$.

<u>Lemma 2.3</u>. There is a space X and a closed subset A such that kX/kA is not compact and X/A is compact if and only if there is a space Y and a closed subset B with empty interior such that (a) each closed subset of Y missing B is compact and (b) Y - B is compactly closed but not closed.

Proof. Suppose X/A is compact and kX/kA is not compact. Let $(kX - kA)^\infty$ be the one point compactification of kX - kA. Following Michael [9], $(kX - kA)^\infty \cong kX/kA$ if and only if closed subsets of kX missing kA are compact. Since $kX/kA \not\cong (kX - kA)^\infty$, there is a subset M of X such that M misses A and is closed in kX and is not compact. Let $Y = M \cup B$ be a subspace of X where B = Bdry A. Clearly, Y - B = M is compactly closed in Y. To show Y - B is not closed in Y, we observe that M is not closed in X. Then $Cl_X M \cap B \neq \phi$ for if it were, then $Cl_X M$ would be compact and $M \cap Cl_X M = M$ would be closed in X. Thus $Cl_Y M \cap B \neq \phi$. Now suppose F is a closed subset of Y missing B. Then $Cl_X F \cap A = \phi$. Therefore, $Cl_X F$ is compact and we have

$Cl_X F \cap M$ closed, but $Cl_X F \cap M = Cl_X F \cap Y = F$ is closed in X. Thus F is compact in X and hence in Y since F misses A.

For the converse, since closed subsets of Y missing B are compact, $Y/B \cong (Y - B)^\infty$ and hence Y/B is compact. For notation, consider the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\ p\ } & Y/B \\ \uparrow & & \uparrow \\ kY & \xrightarrow[\ q\ ]{} & kY/kB. \end{array}$$

Suppose kY/kB is compact. Since Y - B is compactly closed, we have Y - B closed in kY. Hence q(Y - B) is compact in kY/kB, which implies p(Y - B) is compact in Y/B. Thus Y - B is closed in Y, a contradiction.

Proof of Theorem 2.2. Let $X = N \cup A$ be a subspace of βN, the Stone-Cech compactification of the positive integers, where A is a follows. Let $\mathcal{E} = \{E \subset N \mid E$ is infinite$\}$. Fix $x_E \in Cl_{\beta N} E - E$ for each $E \in \mathcal{E}$ and let $A = \{x_E \mid E \in \mathcal{E}\}$. The cardinality of A and X is c [3, p97]. Any infinite closed subset of βN has cardinality 2^c [2, p244], so if K were an infinite compact subset of X, then K, also compact in βN, would have cardinality 2^c. Therefore, compact subsets of X must be finite and we conclude that X is not a k-space because kX is discrete and X is not discrete. Moreover, X - A is compactly closed but not closed, and by the definition of A, closed subsets of X missing A are finite. By Lemma 2.3, X/A is compact and kX/kA is not compact.

Our example provides a completely regular non k-space X, a compact metric space Y = X/A, and a continuous, closed, onto map $f: X \longrightarrow Y$ which is not compact covering (follows from Theorem 2.4 which is below). Moreover, Bdry $f^{-1}(f(A))$ which is A is not compact.

Any example where $k(X \cup_f Y) \neq kX \cup_{kf} kY$ contains an example where $Z/B \neq kZ/kB$, Z a subspace of X, and Int $B = \phi$, for suppose $k(X \cup_f Y) \neq kX \cup_{kf} kY$. Then there is a compact subset K of $X \cup_f Y$ such that $g^{-1}(K)$ is not compact in $kX \cup_{kf} kY$ (refer to the diagram at the beginning of Chapter 2). Note that $K \cap \overline{f}(A) \neq \phi$. Let $X_0 = \overline{f}^{-1}(K)$, $A_0 = X_0 \cap A$, $Y_0 = \overline{i}^{-1}(K)$, and $h = f|A_0$. Then $K = X_0 \cup_h Y_0$ is compact and $g^{-1}(K) = kX_0 \cup_{kh} Y_0$ is not compact. Note that Y_0 is compact.

We now have a space X, a closed subset A of X, and a compact space Y such that $X \cup_f Y$ is compact, $kX \cup_{kf} Y$ is not compact. Note that X - A cannot be closed for if it were, then the identification $X + Y \longrightarrow X \cup_f Y$ would be compact covering. Suppose $X - A$ is compactly closed. Then the conditions of Lemma 2.3 are met. Suppose $X - A$ is not compactly closed. Then kA is not open in kX. By Proposition 1.3 and because $kX \cup_{kf} Y$ is not compact, there is a subset M of X such that M misses A and is closed in kX and is not compact. Let $Z = M \cup \text{Bdry } A$ be a subspace of X. Then Z and Bdry A satisfy the conditions of Lemma 2.3.

<u>Theorem 2.4</u>. If $p: X + Y \longrightarrow X \cup_f Y$ is compact covering, then $k(X \cup_f Y) = kX \cup_{kf} kY$.

Proof. Consider the diagram

$$\begin{array}{ccc} X + Y & \xrightarrow{\ p\ } & X \cup_f Y \\ \uparrow & & \uparrow \\ kX + kY & \xrightarrow[\ q\ ]{} & kX \cup_{kf} kY. \end{array}$$

We show $X \cup_f Y$ and $kX \cup_{kf} kY$ have the same compact sets. Let K be a compact subset of $kX \cup_{kf} kY$. Now K is compact in $X \cup_f Y$ because the identity is continuous. Conversely, suppose K is compact in $X \cup_f Y$. Since p is compact covering, there is a compact subset C of $X + Y$ such that $p(C) = K$. Thus C is compact in $kX + kY$ which implies $q(C)$

is compact. Therefore, $q(C) = p(C) = K$ because p and q are the same point set map.

Corollary 2.5. If A is a compact subset of X, then $k(X \cup_f Y) = kX \cup_{kf} kY$.

Proof. If A is compact, then p is compact covering by Proposition 1.1.

Corollary 2.6. If $f:A \longrightarrow Y$ is a compact mapping, then $k(X \cup_f Y) = kX \cup_{kf} kY$.

Proof. Let K be a compact subset of $X \cup_f Y$. Then $\bar{i}^{-1}(K)$ is compact, and $\bar{f}^{-1}(K)$ is compact by Proposition 1.4. Hence p is compact covering.

Michael [9, 10] proves that if X is paracompact and $f:X \longrightarrow Y$ is closed and onto, then f is compact covering, and if X and Y are paracompact, then $p:X + Y \longrightarrow X \cup_f Y$ is compact covering. It is evident from the proof given in [10] that X paracompact is enough for the latter. For convenience, we include a proof.

<u>Proposition 2.7</u>. Consider the diagram

$$\begin{array}{ccccc} X & \longrightarrow & X + Y & \xrightarrow{\ p\ } & X \cup_f Y \\ {\scriptstyle p_a}\downarrow & & {\scriptstyle q}\downarrow & \swarrow{\scriptstyle g} & \\ X/A & \xrightarrow[h]{} & (X + Y)/(A + Y). & & \end{array}$$

Then (i) $X/A = (X + Y)/(A + Y)$,

(ii) g is closed where $p \circ g = q$,

(iii) if p_a is compact covering, then so is p.

Proof. Parts (i) and (ii) are evident. For (iii), let p_a be compact covering, and let K be a compact subset of $X \cup_f Y$. Then $h^{-1} \circ g(K)$ is a compact subset of X/A. By Proposition 1.6, since $Cl_X p_a^{-1}(h^{-1} \circ g(K) - p(A)) = Cl_X \overline{f}^{-1}(K - \overline{f}(A))$ is compact, p is compact covering.

<u>Corollary 2.8</u>. If X is paracompact, then $k(X \cup_f Y) = kX \cup_{kf} kY$.

Proof. If X is paracompact, then p_a is compact covering [9].

The converse of Proposition 2.7 (iii) does not hold. For example, let A be a closed subset of X such that $p_a : X \longrightarrow X/A$ is not compact covering ($N \cup A$ in the proof of

Theorem 2.2). Let $f:A \longrightarrow A$ be the identity. Then $p:X + A \longrightarrow X \cup_f A = X$ is compact covering (in fact, closed and compact) and p_a is not compact covering.

However, there is a partial converse as follows. Refer to the diagram of Proposition 2.7.

Proposition 2.9. If q is compact covering, then so is p_a.

Proof. Let K be a compact subset of X/A. Since q is compact covering, there is a compact subset C of X + Y such that $q(C) = K$. By Lemma 1.5 and since $(C \cap X) - A = p_a^{-1}(K - p(A))$, $Cl_X p_a^{-1}(K - p(A)) \subset C$ is compact.

Corollary 2.10. If p and g are compact covering, then so is p_a.

Corollary 2.11. If p is compact covering and Y or $X \cup_f Y$ is compact, then p_a is compact covering.

Proof. If Y or $X \cup_f Y$ is compact, then g is closed and compact.

CHAPTER 3

Let X be completely regular and let βX denote the Stone-Cech compactification of X. For convenience, we will use Cl A to denote the Stone-Cech compactification closure of a subset A and $Cl_X A$ will denote the X-closure of A. If X and Y are completely regular, then one easily shows that $\beta X + \beta Y = \beta(X + Y)$ and that $Cl\, F - F \subset \beta X - X$ whenever F is a closed subset of X.

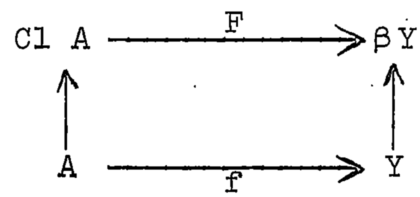
Proposition 3.1. Let X and kX be completely regular. Then in general $\beta(kX) \neq k(\beta X)$.

Proof. Because βX is compact, we have $k(\beta X) = \beta X$. Let $X = N \cup A$ as in the example in the proof of Theorem 2.2. Since $\beta X = \beta N$, $|\beta X| = 2^c$. However, kX is discrete and $|kX| = c$ and therefore, $|\beta(kX)| = 2^{2^c}$ [3, p130].

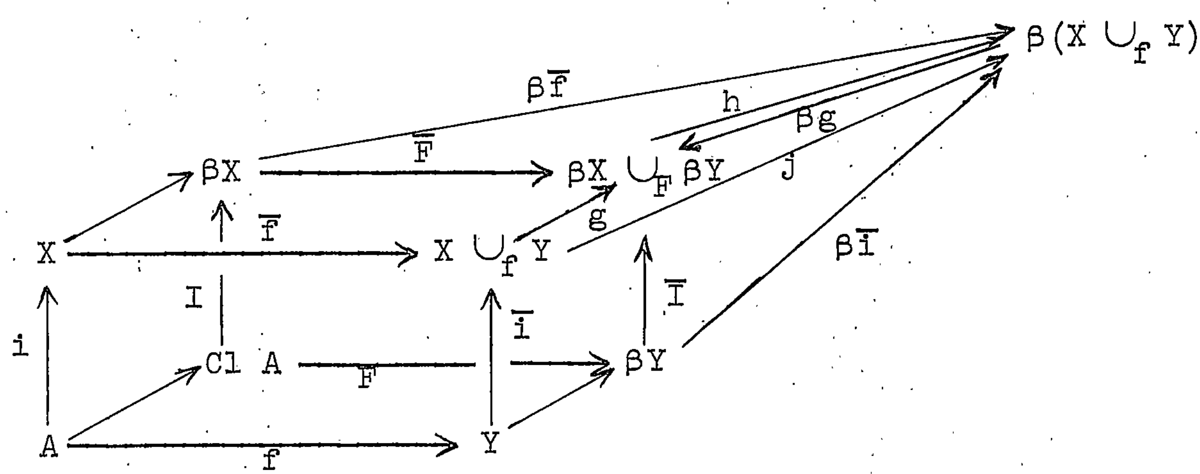
It should be noted that even though X may be completely regular, kX need not be [11].

Willard [14] showed that for a metric space X, $\beta(X/A) \cong \beta X/Cl\, A$. We generalize this result to the general adjunction space with suitable restrictions and obtain the result that $\beta(X/A) \cong \beta X/Cl\, A$ whenever X and X/A are completely regular. The generalization is used to establish Theorem 3.5. Theorem 3.4 and Theorem 3.5 give characterizations of $k(X \cup_f Y)$ even though k may or may not distribute through the adjunction.

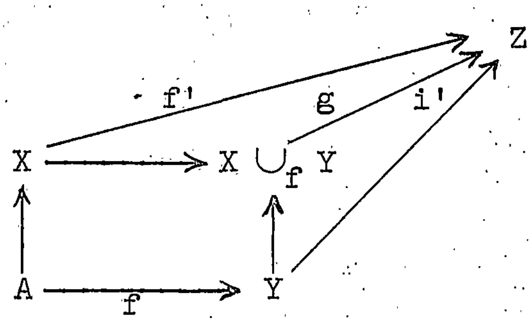
Let X, Y, and $X \cup_f Y$ be completely regular. Assume that F extends f so that the diagram



commutes. For example, when X is normal, $\beta A = \text{Cl } A$ and $F = \beta f$. Consider the diagram



where the unique continuous map g is given by the pushout property of $X \cup_f Y$. The map g is 1-1 because in general, if one has the commuting diagram

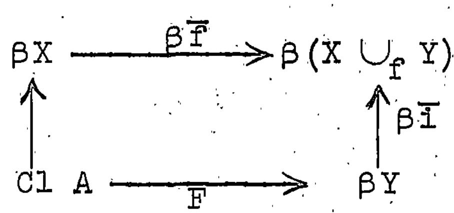


where $f'|(X - A)$ and i' are 1-1, then g is 1-1.

The image of $X \cup_f Y$ under g is dense because if U is open in $\beta X \cup_F \beta Y$, then $X \cap \overline{F}^{-1}(U) \neq \phi$ or $Y \cap \overline{I}^{-1}(U) \neq \phi$ since X and Y are dense in βX and βY respectively, hence $U \cap g(X \cup_f Y) \neq \phi$.

The map βg is continuous and if the diagram

(*)



commutes (for example, if X is normal), then h is the unique continuous map given by the pushout property of $\beta X \cup_F \beta Y$.

We now show that $\beta g \circ h | g(X \cup_f Y)$ is the identity on $\beta X \cup_F \beta Y$ restricted to $g(X \cup_f Y)$ and that $h \circ \beta g | (X \cup_f Y)$ is the identity on $\beta(X \cup_f Y)$ restricted to $X \cup_f Y$ and conclude that $\beta g \circ h$ and $h \circ \beta g$ are identity maps because $g(X \cup_f Y)$ and $X \cup_f Y$ are dense. Note that $\beta g \circ h$ is the identity on $g(X \cup_f Y)$ if and only if $\beta g \circ h \circ g = g$ which in turn is equivalent to $\beta g \circ h \circ g \circ \overline{f} = \overline{F}$ and $\beta g \circ h \circ g \circ \overline{i} = \overline{I}$. Now $\beta g \circ h \circ g \circ \overline{f} = \beta g \circ \beta \overline{f} = \beta(g \circ \overline{f}) = \beta \overline{F} = \overline{F}$ and $\beta g \circ h \circ g \circ \overline{i} = \beta g \circ \beta \overline{i} = \beta(g \circ \overline{i}) = \beta \overline{I} = \overline{I}$. Also, $h \circ \beta g$ is the identity on $X \cup_f Y$ if and only if $h \circ \beta g \circ j = j$ which is equivalent to $h \circ \beta g \circ j \circ \overline{f} = j \circ \overline{f}$ and $h \circ \beta g \circ j \circ \overline{i} = j \circ \overline{i}$. Now $h \circ \beta g \circ j \circ \overline{f} = h \circ g \circ \overline{f} = j \circ \overline{f}$ and $h \circ \beta g \circ j \circ \overline{i} = h \circ g \circ \overline{i} = j \circ \overline{i}$.

Consequently, we have that h is a homeomorphism and we have established the following theorem.

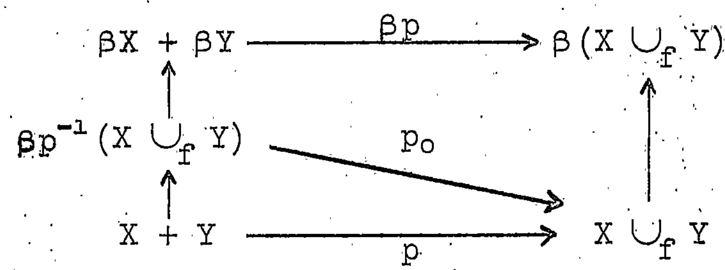
Theorem 3.2. If X, Y and $X \cup_f Y$ are completely regular, $F: \mathrm{Cl}\ A \longrightarrow Y$ extends f, and (*) commutes, then $\beta(X \cup_f Y) \cong \beta X \cup_F \beta Y$.

Corollary 3.3. If X and X/A are completely regular, then $\beta X/\mathrm{Cl}\ A \cong \beta(X/A)$.

Proof. Let Y be a singleton space and apply Theorem 3.2.

Theorem 3.4. Let X, Y, and $X \cup_f Y$ be completely regular, and let $p: X + Y \longrightarrow X \cup_f Y$ be closed. Then $k(X \cup_f Y) \cong kX_0 \cup_{kg} kY$ where $g: A_0 \longrightarrow Y$ extends f, $A_0 \subset \mathrm{Cl}\, A$, and $X_0 = X \cup A_0$.

Proof. Consider the diagram



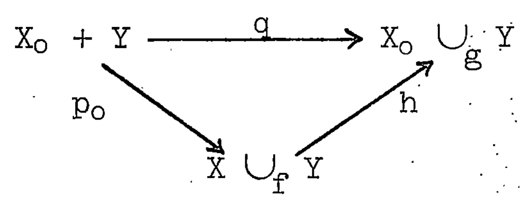
where $p_0 = \beta p | \beta p^{-1}(X \cup_f Y)$. We have

$A \subset \bigcup_{y \in f(A)} \mathrm{Cl}\, f^{-1}(y) \subset \mathrm{Cl} \bigcup_{y \in f(A)} f^{-1}(y) = \mathrm{Cl}\, A$. Let

$A_0 = \bigcup_{y \in f(A)} \mathrm{Cl}\, f^{-1}(y)$ and let $X_0 = X \cup A_0$ be a subspace of βX. Note that $\mathrm{Cl}\, A \cap X_0 = A_0$ because in general, if F is a closed subset of X, then $\mathrm{Cl}\, F - F \subset \beta X - X$. Therefore, A_0 is a closed subset of X_0. With p closed, we have $\beta p^{-1}(z) = \mathrm{Cl}\, p^{-1}(z) = \mathrm{Cl}\, \overline{f}^{-1}(z) \cup \overline{i}^{-1}(z)$ for each $z \in X \cup_f Y$ [6]. Thus $\beta p^{-1}(X \cup_f Y) = X_0 + Y$ because if $w \in X - A$, then $\beta p^{-1}(p(w)) = \mathrm{Cl}\, p^{-1}(p(w)) = w$; if $w \in Y$, then $\beta p^{-1}(p(w)) = \mathrm{Cl}\, \overline{f}^{-1}(p(w)) \cup \overline{i}^{-1}(p(w)) =$ $\mathrm{Cl}\, \overline{f}^{-1}(p(w)) \cup w$; if $w \in A_0$, then $w \in \mathrm{Cl}\, f^{-1}(y) =$

$\mathrm{Cl}\ \overline{f}^{-1}(\overline{i}(y))$ for some $y \in f(A)$, hence $w \in \beta p^{-1}(\overline{i}(y))$. For the reverse inclusion, if $w \in \beta p^{-1}(X \cup_f Y)$, then $w \in \beta p^{-1}(z) = \mathrm{Cl}\ \overline{f}^{-1}(z) \cup \overline{i}^{-1}(z)$ for some $z \in X \cup_f Y$, so $w \in X_o + Y$.

Now let $g = \overline{i}^{-1} \circ p_o | A_o : A_o \longrightarrow Y$. Then $g|A = f$ and $g(A_o) = f(A)$ since $p_o(\mathrm{Cl}\ f^{-1}(y)) = \overline{i}(y)$ for each $y \in f(A)$ ($f^{-1}(y) \subset \beta p^{-1}(\overline{i}(y))$ implies that $\mathrm{Cl}\ f^{-1}(y) \subset \beta p^{-1}(\overline{i}(y))$). Let q be the identification mapping $X_o + Y \longrightarrow X_o \cup_g Y$. We have p_o an identification because it is closed [3,p147], and also p_o is compact [3,p147]. Hence there is a homeomorphism h such that the diagram

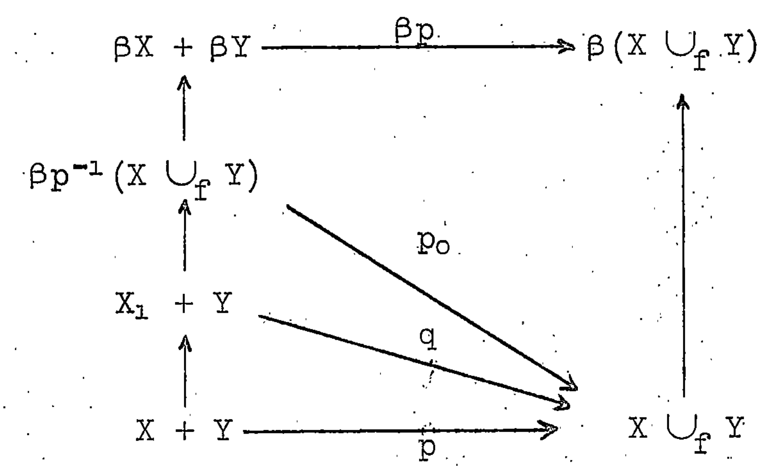


commutes [2,p123]. Therefore, $k(X \cup_f Y) \cong k(X_o \cup_g Y) = kX_o \cup_{kg} kY$ by Theorem 2.4. ||

From [3,p147], $\beta p^{-1}(X \cup_f Y)$ is the largest subspace of $\beta X + \beta Y$ to which p has a continuous extension into $X \cup_f Y$ and this extension is the only closed and compact extension. It would be desirable to have $A_o = A$ whenever p is compact covering but this is not always true. For example, if p is a compact map, then $A_0 = A$ where as if X = Reals, A = Integers, and $p:X \longrightarrow X/A$, then $A_0 = \mathrm{Cl}\ A$ which properly contains A. For a characterization agreeing with compact covering, we have the following theorem.

<u>Theorem 3.5</u>. Let X be normal, Y and $X \cup_f Y$ completely regular. Then $k(X \cup_f Y) \cong kX_1 \cup_{kh} kY$ where $A_1 \subset \mathrm{Cl}\ A$, $X_1 = X \cup A_1$, and $h:A_1 \longrightarrow Y$ extends f.

Proof. For each compact subset K of $X \cup_f Y$, let $G_K = \overline{f}^{-1}(K - \overline{f}(A))$. Consider the diagram



where $p_0 = \beta p | \beta p^{-1}(X \cup_f Y)$, $X_1 = A \cup (\bigcup_K \mathrm{Cl}\ G_K)$, the union being over all compact subsets K of $X \cup_f Y$, and $q = p_0 | (X_1 + Y)$. Let $A_1 = X_1 \cap \mathrm{Cl}\ A$. Then A_1 is a closed subset of X_1. Because X is normal, $\beta(X \cup_f Y) \cong \beta X \cup_{\beta f} \beta Y$.

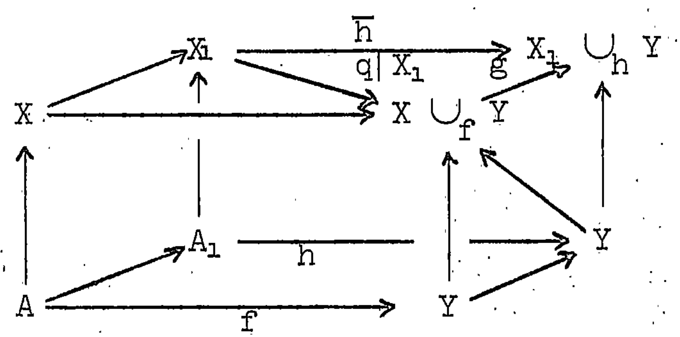
Thus $\beta p^{-1}(z) \in X - A$ if $z \in \overline{f}(X - A)$ and $\beta p^{-1}(z) \subset \mathrm{Cl}\, A + Y$ if $z \in \overline{i}(Y)$. Hence $\beta p^{-1}(X \cup_f Y) \subset (X \cup \mathrm{Cl}\, A) + Y$. If K is a compact subset of $X \cup_f Y$, then K is also compact in $\beta X \cup_{\beta f} \beta Y$. Therefore,

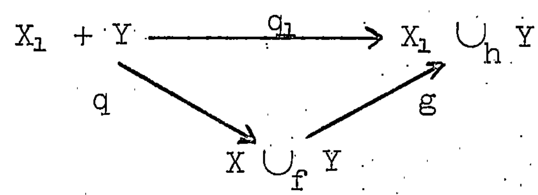
$\mathrm{Cl}\, \overline{\beta f}^{-1}(K - \overline{\beta f}(\mathrm{Cl}\, A)) - \overline{\beta f}^{-1}(K - \overline{\beta f}(\mathrm{Cl}\, A)) = \mathrm{Cl}\, G_K - G_K$ is contained in Cl A by Lemma 1.5. Therefore,

$X_1 = A_1 \cup (X - A)$.

We have that q is an identification because if $U \subset X \cup_f Y$ is such that $q^{-1}(U)$ is open, then $q^{-1}(U) \cap (X + Y) = p^{-1}(U)$ is open in X + Y. Hence U is open. Let $h = \overline{i}^{-1} \circ q| A_1 : A_1 \longrightarrow Y$; then $h| A = f$. Let $q_1 : X_1 + Y \longrightarrow X_1 \cup_h Y$ be the adjunction identification.

Consider the diagrams





where g is the unique continuous map given by the pushout property of $X \cup_f Y$. Also, $g = q_1 \circ q^{-1}$. Therefore, g is open [2,p123]. It is clear that g is 1-1 and since $X_1 - A_1 = X - A$, g is onto.

To show that q is compact covering, suppose K is a compact subset of $X \cup_f Y$, then $\overline{h}^{-1}(K - \overline{h}(A_1)) = G_K$. Therefore, $Cl_{X_1}\ \overline{h}^{-1}(K - \overline{h}(A_1)) = Cl\ G_K \subset X_1$ is compact. So by Proposition 2.10, q is compact covering.

Applying Theorem 2.4, $k(X \cup_f Y) \cong kX_1 \cup_{kh} kY$. ||

In Theorem 3.5, X normal can be replaced by the conditions that X is completely regular and diagram (*) commutes.

If p is compact covering, then by Proposition 2.9, $X_1 = X$. We also have that the largest space to which p has a continuous extension is no larger than $(X \cup Cl\ A) + Y$.

In general, A_0 of Theorem 3.4 and A_1 of Theorem 3.5 are not related. If f has compact point inverses (for example when f is 1-1), then $A_0 = A$, and if p is not compact covering (refer to the example following Corollary 1.7), then there exists a compact subset K of $X \cup_f Y$ such that $Cl_X \overline{f}^{-1}(K - \overline{f}(A))$ is not compact, so A_1 properly contains A. On the other hand, for X = Reals, A = Integers, we have that the identification $X \longrightarrow X/A$ is compact covering, so $A_1 = A$ but $A_0 = Cl\ A$ which properly contains A.

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