A continuum mixture theory with an application to turbulent snow, air flows and sedimentation
by Rand Alan Decker

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Civil Engineering
Montana State University

Abstract:
A theory, based on the Newtonian balance equations is developed for a generalized mixture of n constituents which respond as a whole in a "fluid" manner. These constituents are capable of inter-constituent exchanges of mass, linear momentum and moment of momentum. The physical requirement that the mixture behavior be the sum of its constituent behaviors leads to a set of restriction on each of the constituent balance equations.

This theory is specialized for the case of a snow and air mixture flow. A constitutive assumption is made concerning the transfer of linear momentum between the air and snow phases of the flow. The resulting equations of motion for the snow phase are expanded to include the effects of a turbulent mixture flow. A constitutive assumption is made for the turbulent variables of the snow phase in terms of the intensity of shearing in the airflow. The resulting turbulent momentum balance equation for the snow phase contains a set of terms which could be characterized as apparent or turbulent buoyancies. As a consequence of the constitutive assumption the magnitude of this set of terms is large where the shearing of gradients of the airflow are large.

The system of non-linear partial differential equations resulting from the turbulent equations of motion for the snow phase are approximated by finite difference techniques. Solutions for the snow phase velocity and density fields are investigated for a variety of one and two dimensional airflow regimes.

The snow phase velocity and density field solutions are then compared with the observed phenomena of snow and air mixture flows over flat surfaces and over the crest of a mountain ridge. Lastly the accumulation rates of deposited snow on the immediate lee slope of a mountain ridge are compared with observations.

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APPROVAL

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Rand Alan Decker

This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citation, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

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Date Feb 27, 1986

This work is dedicated to the "first" professor in the author's life, his father: Dr. R. W. Decker whose belief in the power of the intellect inspired this effort more than any other.

VITA

Rand Alan Decker was born April 18, 1954 in Urbana, Illinois. He is the son of Mrs. and Dr. R. W. Decker. Mr. Decker graduated from Hanover High School, Hanover, New Hampshire in 1972. He received his Bachelor of Science degree in Geological Engineering from the College of Mines at the University of Utah in 1977.

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ABSTRACT

A theory, based on the Newtonian balance equations is developed for a generalized mixture of n constituents which respond as a whole in a "fluid" manner. These constituents are capable of inter-constituent exchanges of mass, linear momentum and moment of momentum. The physical requirement that the mixture behavior be the sum of its constituent behaviors leads to a set of restriction on each of the constituent balance equations.

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CHAPTER 1

INTRODUCTION

The desire to create physical theories which rationally describe the mechanical behavior of the observed environment is a prime motivation in the study of Newtonian or classical mechanics.

The evolution of mechanics has evolved to the point where the systems under theoretical consideration can be represented as mixtures of constituents which are capable of inter-constituent transfers of mass, linear momentum and moment of momentum. The range of mechanical behavior displayed by various types of mixtures is large. One limiting case is that of a mixture or composite of elastic solids. At the other extreme is the case of a mixture of constituents which respond as a whole in a "fluid" like manner, even though some of the constituents may not be fluid elements. There is a continuous variation of mixtures between these limiting cases.

The present state of the art in mixture mechanics is the theoretical investigation of these limiting cases. These investigations involve the development of internally consistent theories which arise from the balance principles and thermodynamic principles of classical Newtonian mechanics. These theories may then be investigated, usually by numerical approximation techniques, for solutions. If solutions do exist then their application or degree of correlation with an observed phenomenon may be affected.

This investigation contains these three elements. A mixture theory is developed for a system of constituents which respond as a whole in a fluid manner. This theory is then applied to the specific phenomenon of turbulent sedimentation of snow in an atmospheric flow. The systems of partial differential equations which arise from the turbulent equations of motion for the snow phase are then solved by finite difference numerical techniques. The resulting solutions for the snow phase density fields and velocity fields for one dimensional and two dimension mixture flows are then compared qualitatively with observations. Lastly, theoretical snow phase accumulation distributions and rates for the two dimensional mixture flow are compared qualitatively with observations of wind-aided snow accumulation on the immediate lee side of a transverse mountain ridge.

Historical Perspectives: Mixture Theories

J. Clerk Maxwell made the first in-depth investigation of the Newtonian mechanics of a system of particles. Maxwell's mixture was a system of variable sized spheres in a frictionless medium. These spheres are allowed to interact with each other via conservative or elastic collisions. Arguing from the necessary probabilistic forms of a functional for this system of discrete particles Maxwell derived the Kinetic Theory of Gases (Maxwell, 1860). This very eloquent theory has since been verified to be an excellent physical analogy of gas behavior.

During the first half of the twentieth century the concepts of the Newtonian mechanics for continuous systems have been highly refined, verified and summarized (Truesdell & Toupin, 1960). Initially mixture theories were developed for elastic composites and consolidating porous solids. These works have been summarized (Boit, 1963). From these foundations the theories of continuous systems of flowing and reacting constituents were argued (Bowen, 1976; Eringen & Ingram, 1965; Green & Naghdi, 1965). The development of internally consistent mixture theories continues presently. A large portion of recent research efforts is aimed at developing constitutive relationships for chemically reacting thermo-mechanical mixtures. These efforts are inhibited by the form and restrictions which a constitutive relationship must take to satisfy the entropy inequality (deGroot & Mazur, 1962; Green & Naghdi, 1971; Prigogine & Defay, 1954).

Investigations into the existence of solutions to the accepted continuum mixture theories for multi-phase flows and comparison of these solutions to observed mixture flow phenomena are all fairly recent (Decker & Brown, 1983, 1985; Drew, 1975; Hill, Bedford & Drumheller, 1980; McTique, 1981, 1983).

Historical Perspective: Wind-Aided Snow and Air Mixture Flows

The investigation of wind-aided snow sedimentation, known variously in the literature as blowing snow or drifting snow was initiated as an element of the original polar regions geophysical studies. In this early literature there is no distinction in the definition between blowing snow and drifting snow, the latter also being a snow and air mixture flow. However, in lay terms drifting snow would imply the accumulation or sedimentation of snow in the lee of structures or land forms. The bulk of this pioneering research was directed at gauging and deriving emperical expressions for snow and air mixture flows over flat terrain (Budd, Dingle & Radok, 1966; Dyunin, 1954, 1954, 1959; Kobayashi, 1972; Mellor, 1965; Radok, 1968, 1977). These early works were often shown to compare favorably with the emperical expressions derived for sand and air mixture flows (Bagnold, 1941).

It was recognized that, if the snow and air mixture flow over flat terrain is to be one dimensional, the mass fraction of the snow phase in transport must be equilibrated with respect to changes in horizontal distance. The horizontal distances required for a snow and air mixture flow to equilibrate have been gauged (Takeuchi, 1980).

The salient material of this early work in one dimension snow and air mixture flows has been summarized (Schmidt, 1982).

The initial research into two dimensional snow and air mixture flows was aimed at determining emperical expressions for the spatial patterns of deposited snow in natural environments and adjacent to structures (Alford, 1980; Tabler, 1975, 1980). Further, rational theories using the trajectories of discrete snow particles have been derived and solutions for the resulting depositional patterns presented (Schmidt & Randloph, 1981).

Two dimensional snow and air mixture flows in the immediate lee of a transverse mountain ridge have been investigated emperically and rationally only recently (Decker & Brown, 1983, 1985; Föhn & Meister, 1983). These investigations have considered the snow phase density and velocity fields in the mixture flow and the spatial patterns of deposited or accumulated snow on the lee slope.

Lastly, it should be noted that successful efforts at scale modeling of snow and air mixture flows have been made for one, two and three dimensional flows (Anno & Konishi, 1981; Anno, 1984, 1984; Iversen, 1980; Wuebben, 1978).

CHAPTER 2

MIXTURE THEORY: THE THEORETICAL DEVELOPMENT

Definitions

Consider a continuous material composed of n constituents. This continuum mixture is in shear (viscous) flow. Due to the possibility of phase changes or inter-constituent reactions the mass density of any single constituent: a may vary with time. However we define the instantaneous mixture density ρ at a point as the sum of the instantaneous constituent densities at that point, where ρ_a is the mass per unit mixture volume for the a constituent.

$$\rho = \sum_{a=1}^{n} \rho_a \qquad (1)$$

We define the mixture velocity to be the material time derivative of a "parcel" of mixture as:

$$\frac{d}{dt}\underline{x} = \dot{\underline{x}} \qquad (2)$$

The diffusion velocity is defined as the velocity difference between any single constituent a with respect to the mixture velocity.

$$\underline{u}_a = \grave{\underline{x}}_a - \dot{\underline{x}} \qquad (3)$$

Where $\grave{\underline{x}}_a$ is the velocity of the a constituent. In this development both the overdot and back-facing overscore represent the material derivative with respect to either the mixture or a specific constituent, respectively.

We impose the condition that the sum of all constituent linear momenta must be equal to the linear momentum of the mixture.

$$\sum_{a=1}^{n} \rho_a \dot{\underline{x}}_a = \rho \dot{x} \tag{4}$$

Note that this implies that the sum of all constituent diffusion linear momenta must equal zero.

$$\sum_{a=1}^{n} \rho_a \dot{\underline{x}}_a = \sum_{a=1}^{n} \rho_a (\underline{u}_a + \dot{\underline{x}}) = \rho \dot{\underline{x}} \tag{5}$$

or

$$\sum_{a=1}^{n} \rho_a \underline{u}_a = \rho \dot{\underline{x}} - \sum_{a=1}^{n} \rho_a \dot{\underline{x}} = \underline{0}$$

At this point it is of interest to examine a set of relationships with respect to any differentiable function of the mixture, Γ. Consider the time derivative of Γ with respect to the motion of the mixture.

$$\dot{\Gamma} = \frac{\partial}{\partial t} \Gamma + \dot{\underline{x}} \cdot \underline{\vec{\nabla}} \Gamma \tag{6}$$

Unless otherwise stated the gradient operator is with respect to the spatial coordinate system. Likewise the time derivative of Γ with respect to the motion of the a constituent is:

$$_a\acute{\Gamma} = \frac{\partial}{\partial t} \Gamma + \dot{\underline{x}}_a \cdot \underline{\vec{\nabla}} \Gamma \tag{7}$$

Note that Equations 6 and 7 imply that

$$\dot{\Gamma} = {_a\acute{\Gamma}} \text{ if and only if } \dot{\underline{x}}_a = \dot{x} \tag{8}$$

This is a physically consistent result.

In a similar manner let Γ_a be any differentiable function of the a constituent. Then the substantive derivative of Γ_a with respect to the motion of the mixture is:

$$\dot{\Gamma}_a = \frac{\partial}{\partial t} \Gamma_a + \dot{\underline{x}} \cdot \underline{\vec{\nabla}} \Gamma_a \tag{9}$$

Likewise the substantive derivative of Γ_a with respect to the motion of the a constituent is:

$$ {}_{a}\grave{\Gamma}_{a} = \frac{\partial}{\partial t}\Gamma_{a} + \grave{\underline{x}}_{a} \cdot \underline{\vec{\nabla}}\,\Gamma_{a} \tag{10}$$

Subtracting Equations 6 from 7 results in:

$$ \dot{\Gamma}_{a} - \dot{\Gamma} = (\grave{\underline{x}}_{a} - \dot{\underline{x}}) \cdot \underline{\vec{\nabla}}\,\Gamma = \underline{u}_{a} \cdot \underline{\vec{\nabla}}\,\Gamma \tag{11}$$

Similarly, subtracting Equations 9 from 10 results in:

$$ {}_{a}\grave{\Gamma}_{a} - \dot{\Gamma}_{a} = (\grave{\underline{x}}_{a} - \dot{\underline{x}}) \cdot \underline{\vec{\nabla}}\,\Gamma_{a} = \underline{u}_{a} \cdot \underline{\vec{\nabla}}\,\Gamma_{a} \tag{12}$$

That is, the difference in derivatives for any mixture (or for any a constituent) function with respect to mixture motion or a constituent motion is due solely to the diffusion velocity.

This development will be an exploration of the various constituent and mixture field equations which arise as a consequence of Newtonian balance laws.

Balance of Mass: Continuity

Consider Figure 1. There exists an arbitrary fixed closed region R characterized by surface S totally bounded within the continuous mixture body B. A surface element, dS_R, is characterized by surface normal $\underline{n}$, and there exists at dS_R an instantaneous a constituent velocity $\grave{\underline{x}}_{a}$ and an instantaneous mixture velocity $\dot{\underline{x}}$ with respect to a spatial Cartesian coordinate frame. The velocity difference is the diffusion velocity of the a constituent: $\underline{u}_{a}$.

It is possible to write an expression for the balance of mixture mass ρ within region R

$$ \int_{R} \frac{\partial}{\partial t}\rho\, dV = -\int_{S} \underline{n} \cdot \rho \dot{\underline{x}}\, dS_R \tag{13}$$

The leftside term is the time rate of change of mixture mass in region R. The rightside term is the convective flux of mixture mass from region R through the surface of the region S and by convention is negative. By an application of divergence (Gauss') theorem the rightside term may be rewritten as an integral over the volume of region R.

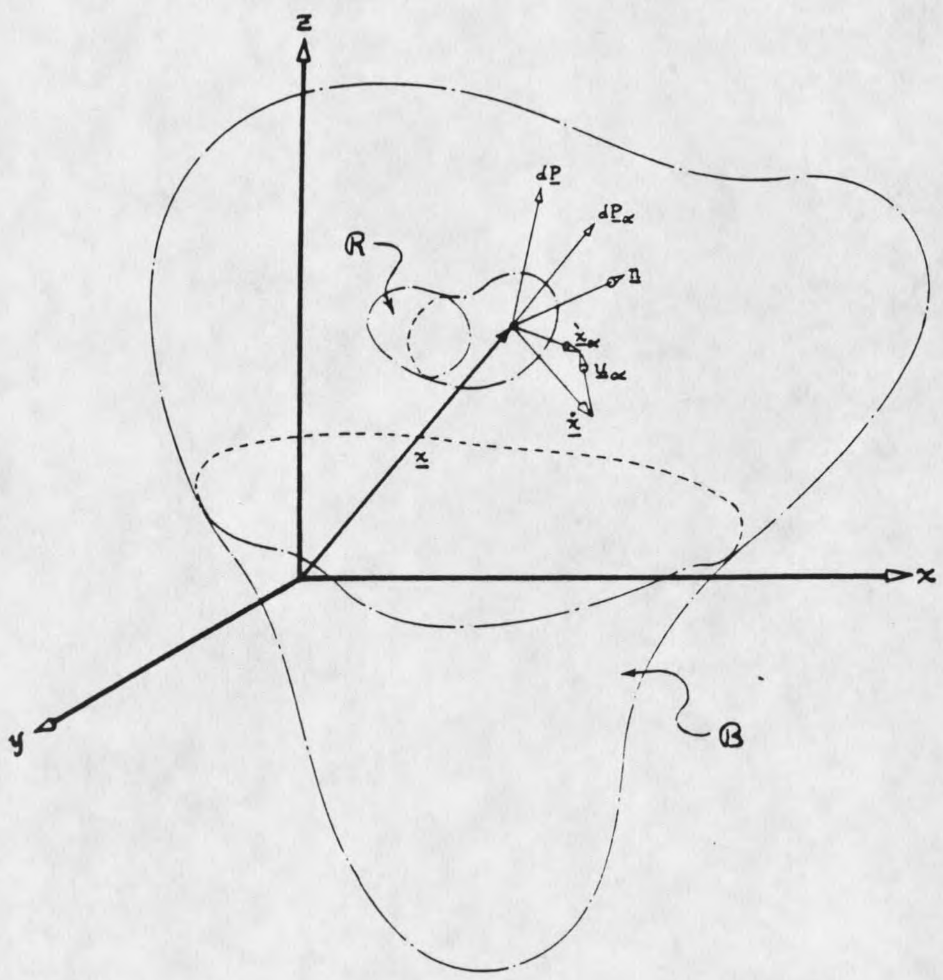

Figure 1. An arbitrary region R totally bounded by the continuous body of mixture B where a surface element dS_R is characterized by normal $\underline{n}$, a constituent velocity $\dot{\underline{x}}_a$, diffusion velocity $\underline{u}_a$ mixture velocity $\dot{\underline{x}}$, a constituent pressure $d\underline{P}_a$ and mixture pressure $d\underline{P}$.

$$\int_R \frac{\partial}{\partial t} \rho \, dV = -\int_R \underline{\vec{\nabla}} \cdot \rho \underline{\dot{x}} \, dV \tag{14}$$

or, rearranging

$$\int_R \left\{ \frac{\partial}{\partial t} \rho + \underline{\vec{\nabla}} \cdot \rho \underline{\dot{x}} \right\} dV = 0 \tag{15}$$

Sense region R is totally arbitrary, as long as it remains bounded in the body of mixture B this requires that for the integral Equation 15 to be equal to zero that the integrand itself must be identically equal to zero.

$$\frac{\partial}{\partial t} \rho + \underline{\vec{\nabla}} \cdot \rho \underline{\dot{x}} = 0 \tag{16}$$

This is the balance of mass or continuity equation for the mixture and is recognizable as being identical to the continuity equation for single phase flow.

Similarly it is possible to write an expression for the balance of a constituent mass ρ_a within region R.

$$\int_R \frac{\partial}{\partial t} \rho_a dV = -\int_S \underline{n} \cdot \rho_a \underline{\dot{x}}_a dS_R + \int_R \hat{c}_a \, dV \tag{17}$$

The additional, second rightside term characterizes the mass supplied to or taken from the a constituent by the n-1 other constituents comprising the mixture. This positive or negative mass supply to the a constituent may be due to phase changes and/or chemical reaction with the n-1 other constituents of the mixture. An application of divergence theorem and rearranging of Equation 17 results in:

$$\int_R \left\{ \frac{\partial}{\partial t} \rho_a + \underline{\vec{\nabla}} \cdot \rho_a \underline{\dot{x}}_a - \hat{c}_a \right\} dV = 0 \tag{18}$$

Again, based on the arbitrary nature of region R it follows that:

$$\frac{\partial}{\partial t} \rho_a + \underline{\vec{\nabla}} \cdot \rho_a \underline{\dot{x}}_a = \hat{c}_a \tag{19}$$

This is the continuity equation for the a constituent of an n constituent mixture.

Clearly Equation 19 represents a set of n equations, each one being the continuity equation for a specific constituent.

If we impose the physical requirement that the mixture behavior be the sum of the constituent behaviors and apply this requirement to the continuity equations then:

$$\frac{\partial}{\partial t}\rho + \underline{\vec{\nabla}} \cdot \rho \underline{\dot{x}} = \frac{\partial}{\partial t}\sum_{a=1}^{n} \rho_a + \underline{\vec{\nabla}} \cdot \sum_{a=1}^{n} \rho_a \underline{\dot{x}}_a - \sum_{a=1}^{n} \hat{c}_a \tag{20}$$

Recalling Equations 1 and 4, the definitions of mixture mass and mixture linear momentum, respectively, then Equation 20 reduces to:

$$\sum_{a=1}^{n} \hat{c}_a = 0 \tag{21}$$

That is, the sum of the inter-constituent mass supplies must be zero.

It is of interest to examine a relationship between the mass of a specific constituent: a and the motion of the mixture. Recall Equations 9 and 10 and consider the difference:

$$\begin{aligned} \dot{\rho}_a - {}_a\acute{\rho}_a &= \underline{\dot{x}} \cdot \underline{\vec{\nabla}} \rho_a - \underline{\dot{x}}_a \cdot \underline{\vec{\nabla}} \rho_a \\ &= -\underline{u}_a \cdot \underline{\vec{\nabla}} \rho_a \end{aligned} \tag{22}$$

Note that the transient terms have cancelled. Rearranging Equation 19, the continuity equation for the a constituent in light of Equation 10 allows for the substitution of

$${}_a\acute{\rho}_a = -\rho_a \underline{\vec{\nabla}} \cdot \underline{\dot{x}}_a + \hat{c}_a \tag{23}$$

into Equation 22, which after rearranging and the identically zero addition and subtraction of the term: $\rho_a \underline{\vec{\nabla}} \cdot \underline{\dot{x}}$ results in:

$$\begin{aligned} \dot{\rho}_a &= -\rho_a \underline{\vec{\nabla}} \cdot \underline{\dot{x}}_a + \hat{c}_a - \underline{u}_a \cdot \underline{\vec{\nabla}} \rho_a + \rho_a \underline{\vec{\nabla}} \cdot \underline{\dot{x}} - \rho_a \underline{\vec{\nabla}} \cdot \underline{\dot{x}} \\ &= -\rho_a \underline{\vec{\nabla}} \cdot (\underline{\dot{x}}_a - \underline{\dot{x}}) - \underline{u}_a \cdot \underline{\vec{\nabla}} \rho_a - \rho_a \underline{\vec{\nabla}} \cdot \underline{\dot{x}} + \hat{c}_a \\ &= -\rho_a \underline{\vec{\nabla}} \cdot \underline{u}_a - \underline{u}_a \cdot \underline{\vec{\nabla}} \rho_a - \rho_a \underline{\vec{\nabla}} \cdot \underline{\dot{x}} + \hat{c}_a \\ &= -\underline{\vec{\nabla}} \cdot \rho_a \underline{u}_a - \rho_a \underline{\vec{\nabla}} \cdot \underline{\dot{x}} + \hat{c}_a \end{aligned} \tag{24}$$

or

$$\dot{\rho}_a + \rho_a \vec{\underline{\nabla}} \cdot \dot{\underline{x}} = -\vec{\underline{\nabla}} \cdot \rho_a \underline{u}_a + \hat{c}_a$$

In light of Equation 16, the continuity equation for the mixture we can form the following quotient about the second leftside term of Equation 24

$$\dot{\rho}_a + \frac{\dot{\rho}\, \rho_a \vec{\underline{\nabla}} \cdot \dot{\underline{x}}}{-\rho\, \vec{\underline{\nabla}} \cdot \dot{\underline{x}}} = -\vec{\underline{\nabla}} \cdot \rho_a \underline{u}_a + \hat{c}_a \tag{25}$$

or

$$\dot{\rho}_a - \frac{\dot{\rho}\, \rho_a}{\rho} = -\vec{\underline{\nabla}} \cdot \rho_a \underline{u}_a + \hat{c}_a$$

or

$$\rho\, \overline{\left(\frac{\rho_a}{\rho}\right)}^{\bullet} = -\vec{\underline{\nabla}} \cdot \rho_a \underline{u}_a + \hat{c}_a$$

where the overbar implies differentiation of both factors in the quotient. Note that the quotient is an expression for the instantaneous dimensionless concentration at a point for the a constituent. Also take note that if there is no diffusion of the a constituent with respect to the mixture ($\underline{u}_a = 0$) and no mass exchange with the a constituent ($\hat{c}_a = 0$) then the instantaneous concentration at a point of the a constituent must be constant. This result is physically consistent and intuitively satisfying.

At this point in the theoretical development it is of interest to generate a generalized identity which is a consequence of the physical requirement that the mixture behavior be sum of the constituent behaviors. Then for any function of the mixture, Γ

$$\rho \Gamma = \sum_{a=1}^{n} \rho_a \Gamma_a \tag{26}$$

or

$$\Gamma = \sum_{a=1}^{n} \frac{\rho_a}{\rho} \Gamma_a$$

where the derivatives with respect to the mixture of both sides of Equation 26:

$$\dot{\Gamma} = \sum_{a=1}^{n} \overline{\left(\frac{\rho_a}{\rho}\Gamma_a\right)}^{\bullet} = \sum_{a=1}^{n} \left\{ \overline{\left(\frac{\rho_a}{\rho}\right)}^{\bullet}\Gamma_a + \frac{\rho_a}{\rho}\dot{\Gamma}_a \right\} \tag{27}$$

or

$$\rho\dot{\Gamma} = \sum_{a=1}^{n} \left\{ \rho\, \overline{\left(\frac{\rho_a}{\rho}\right)}^{\bullet}\Gamma_a + \rho_a\dot{\Gamma}_a \right\}$$

By substitution of Equation 25 and the identically zero addition and subtraction of the term: ${}_a\grave{\Gamma}_a$ into Equation 27 results in:

$$\rho\dot{\Gamma} = \sum_{a=1}^{n} \left\{ (-\vec{\underline{\nabla}} \cdot \rho\underline{u}_a + \hat{c}_a)\Gamma_a + \rho_a(\dot{\Gamma}_a + {}_a\grave{\Gamma}_a - {}_a\grave{\Gamma}_a) \right\} \tag{28}$$

or, in light of Equations 9 and 10 and noting that the transient terms arising from Equations 9 and 10 cancel:

$$\rho\dot{\Gamma} = \sum_{a=1}^{n} \left\{ (-\vec{\underline{\nabla}} \cdot \rho\underline{u}_a + \hat{c}_a)\Gamma_a + \rho_a\left({}_a\grave{\Gamma}_a - \dot{\underline{x}}_a \cdot \vec{\underline{\nabla}}\,\Gamma_a + \dot{\underline{x}} \cdot \vec{\underline{\nabla}}\,\Gamma_a\right) \right\}$$

$$= \sum_{a=1}^{n} \left\{ -\Gamma_a \vec{\underline{\nabla}} \cdot \rho\underline{u}_a + \Gamma_a\hat{c}_a + \rho_a\,{}_a\grave{\Gamma}_a - \rho_a\underline{u}_a \cdot \vec{\underline{\nabla}}\,\Gamma_a \right\}$$

$$= \sum_{a=1}^{n} \left\{ \rho_a\,{}_a\grave{\Gamma}_a + \Gamma_a\hat{c}_a - \vec{\underline{\nabla}} \cdot \Gamma_a\rho_a\underline{u}_a \right\}$$

Balance of Linear Momentum

Consider Figure 1. It is possible to write an expression for the balance of mixture linear momentum $\rho\dot{\underline{x}}$ within region R.

$$\int_R \frac{\partial}{\partial t}\rho\dot{\underline{x}}\,dV = -\int_{S_R}(\underline{n}\cdot\dot{\underline{x}})(\rho\dot{\underline{x}})\,dS + \int_{S_R} d\underline{P} + \int_R \rho\underline{b}\,dV \tag{29}$$

The leftside term is the time rate of change of mixture linear momentum in region R. The rightside terms are, respectively: the convective flux of mixture linear momentum from region R through the surface of the region: S, the addition or subtraction of mixture linear momentum in region R due to mixture surface forces on the surface of the region and the addition or subtraction of mixture linear momentum in region R due to body forces in the region.

Note that the term characterizing the mixture surface forces can be expressed as a mixture stress or traction vector, $\underline{t}^{(n)}$, and then recall that any mixture stress vector on a surface with surface normal $\underline{n}$ can be expressed as the product of the surface normal $\underline{n}$ and the local mixture stress tensor $\underset{\sim}{T}$.

$$\int_{S_R} d\underline{P} = \int_{S_R} \underline{t}^{(n)} \, ds = \int_{S_R} n\underset{\sim}{T} \, ds \tag{30}$$

Applying divergences theorem to Equation 30 results in:

$$\int_{S_R} n\underset{\sim}{T} \, ds = \int_R \underline{\vec{\nabla}} \cdot \underset{\sim}{T} \, dV \tag{31}$$

The term characterizing the convective flux of mixture linear mometum may be rewritten about a dyadic product of the mixture velocities and divergences theorem may then be applied to the resulting expression.

$$\int_{S_R} (n \cdot \underline{\dot{x}})(\rho\underline{\dot{x}}) \, ds = \int_{S_R} \underline{n}(\rho\underline{\dot{x}}\,\underline{\dot{x}}) ds = \int_R \underline{\vec{\nabla}} \cdot (\rho\underline{\dot{x}}\,\underline{\dot{x}}) \, dV \tag{32}$$

Substitution of Equations 31 and 32 into 29 and collecting all terms under a single integral results in:

$$\int_R \left\{ \frac{\partial}{\partial t} \rho\underline{\dot{x}} + \underline{\vec{\nabla}} \cdot (\rho\underline{\dot{x}}\,\underline{\dot{x}}) - \underline{\vec{\nabla}} \cdot \underset{\sim}{T} - \rho\underline{b} \right\} dV = 0 \tag{33}$$

Since region R is totally arbitrary this requires that:

$$\frac{\partial}{\partial t} \rho\underline{\dot{x}} + \underline{\vec{\nabla}} \cdot (\rho\underline{\dot{x}}\,\underline{\dot{x}}) - \underline{\vec{\nabla}} \cdot \underset{\sim}{T} - \rho\underline{b} = 0 \tag{34}$$

Consider the following reorganization of the first and second terms of Equation 34:

$$\frac{\partial}{\partial t}\rho\underline{\dot{x}} + \vec{\underline{\nabla}}\cdot(\rho\underline{\dot{x}}\underline{\dot{x}}) = \frac{\partial}{\partial t}\rho\underline{\dot{x}} + \underline{\dot{x}}(\vec{\underline{\nabla}}\cdot\rho\underline{\dot{x}}) + \rho\underline{\dot{x}}(\vec{\underline{\nabla}}\cdot\underline{\dot{x}})$$

$$= \overline{(\rho\underline{\dot{x}})}^{\bullet} + \rho\underline{\dot{x}}(\vec{\underline{\nabla}}\cdot\underline{\dot{x}}) \tag{35}$$

$$= \rho\underline{\ddot{x}} + \dot{\rho}\underline{\dot{x}} + \rho\underline{\dot{x}}(\vec{\underline{\nabla}}\cdot\underline{\dot{x}})$$

$$= \rho\underline{\ddot{x}} + \dot{x}(\dot{\rho} + \rho\vec{\underline{\nabla}}\cdot\underline{\dot{x}})$$

$$= \rho\underline{\ddot{x}}$$

Where the factor in parenthesis is recognizable as Equation 16, the continuity equation for the mixture and is identically zero. Substituting this result into Equation 34 leads to:

$$\rho\underline{\ddot{x}} = \vec{\underline{\nabla}}\cdot\underset{\sim}{T} + \rho\underline{b} \tag{36}$$

This is the momentum balance equation for the mixture and is recognizable as being identical to the momentum balance equation for single phase flow.

Similarly it is possible to write an expression for the balance of a constituent linear momentum $\rho_a\underline{\dot{x}}_a$ within region R.

$$\int_R \frac{\partial}{\partial t}\rho_a\underline{\dot{x}}_a = -\int_{S_R}(\underline{n}\cdot\underline{\dot{x}}_a)(\rho\underline{\dot{x}}_a)\,ds + \int_{S_R} d\underline{P}_a + \int_R\left\{\rho_a\underline{b}_a + \hat{c}_a\underline{\dot{x}}_a + \hat{\underline{p}}_a\right\}dV \tag{37}$$

The additional terms: $\hat{c}_a\underline{\dot{x}}_a$ and $\hat{\underline{p}}_a$ characterize respectively the a constituent linear momentum addition or subtraction in region R due to mass supplied to or from the a constituent by the n-1 other constituents of the mixture and a constituent linear momentum supplied to or from the other constituents by means other than mass exchange, such as particulate collisions or fluid drag.

Note that the term characterizing the a constituent surface forces on the surface of region R can be expressed:

$$\int_{S_R} d\underline{P}_a = \int_{S_R}\underline{t}_a^{(n)}\,ds = \int_{S_R}\underline{n}\,\underset{\sim}{T}_a\,ds \tag{38}$$

where $\underline{t}_\alpha^{(n)}$ and $\underset{\sim}{T}_\alpha$ are the α constituent traction vector and α constituent stress tensor, respectively. Applying divergence theorem to Equation 38 results in:

$$\int_{S_R} \underline{n}\, \underset{\sim}{T}_\alpha \, ds = \int_R \vec{\underline{\nabla}} \cdot \underset{\sim}{T}_\alpha \, dV \tag{39}$$

The term characterizing the convective flux of α constituent linear momentum may be rewritten about a dyadic product of the α constituent velocities and divergence theorem may then be applied to the resulting expression.

$$\int_{S_R} (\underline{n} \cdot \dot{\underline{x}}_\alpha)(\dot{\underline{x}}_\alpha \rho_\alpha) ds = \int_{S_R} \underline{n}(\rho_\alpha \dot{\underline{x}}_\alpha \dot{\underline{x}}_\alpha) ds = \int_R \vec{\underline{\nabla}} \cdot (\rho_\alpha \dot{\underline{x}}_\alpha \dot{\underline{x}}_\alpha) dV \tag{40}$$

Substitution of Equations 39 and 40 into 37 and collecting all terms under a single integral results in:

$$\int_R \left\{ \frac{\partial}{\partial t} \rho_\alpha \dot{\underline{x}}_\alpha + \vec{\underline{\nabla}} \cdot (\rho_\alpha \dot{\underline{x}}_\alpha \dot{\underline{x}}_\alpha) - \vec{\underline{\nabla}} \cdot \underset{\sim}{T}_\alpha - \rho_\alpha \underline{b}_\alpha - \hat{c}_\alpha \dot{\underline{x}}_\alpha - \hat{\underline{p}}_\alpha \right\} dV = 0 \tag{41}$$

Since region R is totally arbitrary this requires that:

$$\frac{\partial}{\partial t} \rho_\alpha \dot{\underline{x}}_\alpha + \vec{\underline{\nabla}} \cdot (\rho_\alpha \dot{\underline{x}}_\alpha \dot{\underline{x}}_\alpha) - \vec{\underline{\nabla}} \cdot \underset{\sim}{T}_\alpha - \rho_\alpha b_\alpha - \hat{c}_\alpha \dot{\underline{x}}_\alpha - \hat{\underline{p}}_\alpha = 0 \tag{42}$$

Consider the following reorganization of the first, second and fifth terms of Equation 42:

$$\begin{aligned}
\frac{\partial}{\partial t} \rho_\alpha \dot{\underline{x}}_\alpha + \vec{\underline{\nabla}} \cdot (\rho_\alpha \dot{\underline{x}}_\alpha \dot{\underline{x}}_\alpha) - \hat{c}_\alpha \dot{\underline{x}}_\alpha &= \frac{\partial}{\partial t} \rho_\alpha \dot{\underline{x}}_\alpha + \dot{\underline{x}}_\alpha (\vec{\underline{\nabla}} \cdot \rho_\alpha \dot{\underline{x}}_\alpha) + \rho_\alpha \dot{\underline{x}}_\alpha (\vec{\underline{\nabla}} \cdot \dot{\underline{x}}_\alpha) - \hat{c}_\alpha \dot{\underline{x}}_\alpha \\
&= \overline{\rho_\alpha \dot{\underline{x}}_\alpha}^{\,\prime} + \rho_\alpha \dot{\underline{x}}_\alpha (\vec{\underline{\nabla}} \cdot \dot{\underline{x}}_\alpha) - \hat{c}_\alpha \dot{\underline{x}}_\alpha \\
&= \rho_\alpha \ddot{\underline{x}}_\alpha + \dot{\rho}_\alpha \dot{\underline{x}}_\alpha + \rho_\alpha \dot{\underline{x}}_\alpha (\vec{\underline{\nabla}} \cdot \dot{\underline{x}}_\alpha) - \hat{c}_\alpha \dot{\underline{x}}_\alpha \\
&= \rho_\alpha \ddot{\underline{x}}_\alpha + \dot{\underline{x}}_\alpha (\dot{\rho}_\alpha + \rho_\alpha \vec{\underline{\nabla}} \cdot \dot{\underline{x}}_\alpha - \hat{c}_\alpha) \\
&= \rho_\alpha \ddot{\underline{x}}_\alpha
\end{aligned} \tag{43}$$

The factor in parenthesis is recognizable as Equation 19, the continuity equation for the α constituent and written in this form is identically zero. Substitution of this result into Equation 42 leads to:

$$\rho_a \ddot{\underline{x}}_a = \vec{\underline{\nabla}} \cdot \underset{\sim}{T}_a + \rho_a \underline{b}_a + \hat{\underline{p}}_a \tag{44}$$

This is the balance of linear momentum equation for the a constituent.

Recall that as early as Equation 19, the continuity equation for the a constituent we have imposed the physical requirement that the mixture behavior must be the linear or unweighted sum of the constituent behaviors. We have applied this requirement to specific behaviors or quantities such as mass, linear momentum and the mass balance equations, the latter which resulted in the physically consistent result that the sum of the constituent mass supplies will be zero.

In light of these successful impositions of the "summed behavior" requirement we logically presume to impose this requirement on the balance of linear momentum.

Consider the mixture external body force term from Equation 36 and the a constituent external body force term from Equation 44 and let:

$$\rho \underline{b} = \sum_{a=1}^{n} \rho_a \underline{b}_a \tag{45}$$

Consider now that if the sum of the balance of linear momentums of the constituents is equal to the balance of linear momentums of the mixture then, via Equations 36 and 44

$$\sum_{a=1}^{n} \vec{\underline{\nabla}} \cdot \underset{\sim}{T}_a + \sum_{a=1}^{n} \rho_a \underline{b}_a + \sum_{a=1}^{n} \hat{\underline{p}}_a - \sum_{a=1}^{n} \rho_a \ddot{\underline{x}}_a = \vec{\underline{\nabla}} \cdot \underset{\sim}{T} + \rho \underline{b} - \rho \ddot{\underline{x}} \tag{46}$$

Via Equation 28, if we let $\Gamma = \dot{\underline{x}}$ and ${}_a\Gamma_a = \dot{\underline{x}}_a$ we obtain:

${}_{\dot{a}}\Gamma_a = \ddot{\underline{x}}_a$ results in:

$$\rho \ddot{\underline{x}} = \sum_{a=1}^{n} \rho_a \ddot{\underline{x}}_a + \sum_{a=1}^{n} \dot{\underline{x}}_a \hat{c}_a - \sum_{a=1}^{n} \vec{\underline{\nabla}} \cdot (\rho_a \dot{\underline{x}}_a \underline{u}_a) \tag{47}$$

Substitution of Equation 47 into 46 in light of Equation 45 results in:

$$\sum_{a=1}^{n} \vec{\underline{\nabla}} \cdot \underset{\sim}{T}_a + \sum_{a=1}^{n} \hat{\underline{p}}_a = \vec{\underline{\nabla}} \cdot \underset{\sim}{T} - \sum_{a=1}^{n} \dot{\underline{x}}_a \hat{c}_a + \sum_{a=1}^{n} \vec{\underline{\nabla}} \cdot (\rho_a \dot{\underline{x}}_a \underline{u}_a) \tag{48}$$

Recalling Equation 3 and substituting into Equation 48 leads to:

$$\sum_{a=1}^{n} \underline{\vec{\nabla}} \cdot \underset{\sim}{T}_a + \sum_{a=1}^{n} \hat{\underline{p}}_a = \underline{\vec{\nabla}} \cdot \underset{\sim}{T} - \underline{\dot{x}} \sum_{a=1}^{n} \hat{c}_a - \sum_{a=1}^{n} \hat{c}_a \underline{u}_a$$
$$+ \underline{\dot{x}} \cdot \underline{\vec{\nabla}} \sum_{a=1}^{n} \rho_a \underline{u}_a + \sum_{a=1}^{n} \underline{\vec{\nabla}} \cdot \rho \underline{u}_a \underline{u}_a \tag{49}$$

Note that by Equations 6 and 21 Equation 49 further reduces to

$$\sum_{a=1}^{n} \underline{\vec{\nabla}} \cdot \underset{\sim}{T}_a + \sum_{a=1}^{n} \hat{\underline{p}}_a = \underline{\vec{\nabla}} \cdot \underset{\sim}{T} + \sum_{a=1}^{n} \underline{\vec{\nabla}} \cdot \rho_a \underline{u}_a \underline{u}_a - \sum_{a=1}^{n} \hat{c}_a \underline{u}_a \tag{50}$$

We may deduce from the order of the various terms that:

$$\sum_{a=1}^{n} (\hat{\underline{p}}_a + \hat{c}_a \underline{u}_a) = 0 \tag{51}$$

If there is no mass exchange between constituents ($\sum_{a=1}^{n} \hat{c}_a = 0$) then we are left with the physically consistent result that the sum of the interconstituent linear momentum supplies must be zero ($\sum_{a=1}^{n} \hat{\underline{p}}_a = 0$). Further, also note that the mixture stress tensor is the sum of the partial stress tensors of the constituents and the sum of the convective flux of constituent linear momenta.

$$\underset{\sim}{T} = \sum_{a=1}^{n} (\underset{\sim}{T}_a - \rho_a \underline{u}_a \underline{u}_a) \tag{52}$$

By definition of the dyadic product the term $\rho_a \underline{u}_a \underline{u}_a$, the sum of terms $\sum_{a=1}^{n} \rho_a \underline{u}_a \underline{u}_a$ must be symmetric.

Balance of Moment of Momentum

Again, consider Figure 1. It is possible to write an expression for the balance of moment of momentum, about the origin of the spatial frame for the a constituent with region R.

$$\int_R \frac{\partial}{\partial t}(\underline{x} \times \rho_a \dot{\underline{x}}_a) dV$$

$$= -\int_{S_R} (\underline{x} \times \rho_a \dot{\underline{x}}_a)(\dot{\underline{x}}_a \cdot \underline{n}) ds + \int_{S_R} \underline{x} \times d\underline{P}_a + \int_R \Big\{ \underline{x} \times (\rho_a \underline{b}_a + \hat{c}_a \dot{\underline{x}}_a + \hat{\underline{p}}_a)$$

$$+ \hat{\underline{m}}_a \Big\} dV \quad (53)$$

The additional term of $\hat{\underline{m}}_a$ characterizes the moment supply to and from the a constituent via couple interaction with the n–1 other constituents of the mixture.

Consider the first rightside term of Equation 53 (the flux of a constituent moment of momentum through the surface of R). Application of the divergence theorem can result with:

$$\int_{S_R} (\underline{x} \times \rho_a \dot{\underline{x}}_a)(\dot{\underline{x}}_a \cdot n) ds = \int_{S_R} ((\dot{\underline{x}}_a \times \rho_a \dot{\underline{x}}_a) \dot{\underline{x}}_a) \underline{n}\, ds$$

$$= \int_R ((\underline{x} \times \rho_a \dot{\underline{x}}_a) \dot{\underline{x}}_a) \cdot \overleftarrow{\underline{\nabla}}\, dV \quad (54)$$

Consider now the second rightside term of Equation 53 (the moment of momentum from the surface of R due to the surface forces of the a constituent) and note that by Equations 38:

$$\int_{S_R} \underline{x} \times d\underline{P}_a = \int_{S_R} \underline{x} \times \underline{t}_a^{(n)} ds = \int_{S_R} \underline{x} \times \underline{n}\, \underset{\sim}{T}_a ds = \int_{S_R} \underline{x} \times \underset{\sim}{T}_a^T \underline{n}\, ds \quad (55)$$

By application of the identity $\underline{a} \times (\underset{\sim}{Q}\, \underline{b}) = (\underline{a} \times \underset{\sim}{Q}) \underline{b}$ for all vectors $\underline{a}$ and second order tensors $\underset{\sim}{Q}$ and the divergence theorem, Equation 55 may be written:

$$\int_{S_R} \underline{x} \times \underset{\sim}{T}_a^T \underline{n}\, ds = \int_{S_R} (\underline{x} \times \underset{\sim}{T}_a^T) \underline{n} = \int_R (\underline{x} \times \underset{\sim}{T}_a^T) \cdot \overleftarrow{\underline{\nabla}}\, dV \quad (57)$$

Hence, the balance of moment of momentum for the a constituent can be reorganized under a single volume integral:

$$\int_R \Big\{ \frac{\partial}{\partial t}(\underline{x} \times \rho_a \dot{\underline{x}}_a) + ((\underline{x} \times \rho_a \dot{\underline{x}}_a) \dot{\underline{x}}_a) \cdot \overleftarrow{\underline{\nabla}} - (\underline{x} \times \underset{\sim}{T}_a^T) \cdot \overleftarrow{\underline{\nabla}}$$

$$- \underline{x} \times (\rho_a \underline{b}_a + \hat{c}_a \dot{\underline{x}}_a + \hat{\underline{p}}_a) - \hat{\underline{m}}_a \Big\} dV = 0 \quad (58)$$

Since region R is totally arbitrary this requires that:

$$\frac{\partial}{\partial t}(\underline{x} \times \rho_a \underline{\dot{x}}_a) + ((\underline{x} \times \rho_a \underline{\dot{x}}_a)\,\underline{\dot{x}}_a) \cdot \overleftarrow{\underline{\nabla}} - (\underline{x} \times \underset{\sim}{T}{}_a^{\mathrm{T}}) \cdot \overleftarrow{\underline{\nabla}}$$

$$- \underline{x} \times (\rho_a \underline{b}_a + \hat{c}_a \underline{\dot{x}}_a + \hat{\underline{p}}_a) - \hat{\underline{m}}_a = 0 \qquad (59)$$

Now, by application of the identity

$$(\underline{a} \times \underset{\sim}{Q}^{\mathrm{T}}) \cdot \overleftarrow{\underline{\nabla}} = \underline{a} \times (\underset{\sim}{Q}^{\mathrm{T}} \cdot \overleftarrow{\underline{\nabla}}) + \underline{Q}_{\mathrm{A}} \qquad (60)$$

for all vectors $\underline{a}$ and tensors $\underset{\sim}{Q}$. $\underline{Q}_{\mathrm{A}}$ is the axial vector of tensor $\underset{\sim}{Q}$ and has the components:

$$\begin{aligned} Q_{A_1} &= Q_{23} - Q_{32} \\ Q_{A_2} &= Q_{31} - Q_{13} \\ Q_{A_3} &= Q_{12} - Q_{21} \end{aligned} \qquad (61)$$

the third term of Equation 59 can be rewritten

$$(\underline{x} \times \underset{\sim}{T}{}_a^{\mathrm{T}}) \cdot \overleftarrow{\underline{\nabla}} = \underline{x} \times (\underset{\sim}{T}{}_a^{\mathrm{T}} \cdot \overleftarrow{\underline{\nabla}}) + \underline{T}_{a_{\mathrm{A}}} \qquad (62)$$

Also since $\underline{x}$ is a time independent position variable, the first and second terms of Equation 59 can be reorganized such that:

$$\frac{\partial}{\partial t}(\underline{x} \times \rho_a \underline{\dot{x}}_a) + ((\underline{x} \times \rho_a \underline{\dot{x}}_a)\underline{\dot{x}}_a) \cdot \overleftarrow{\underline{\nabla}} = \underline{x} \times \left\{ \frac{\partial}{\partial t} \rho_a \underline{\dot{x}}_a + \rho_a \underline{\dot{x}}_a \cdot \overleftarrow{\underline{\nabla}}\, \underline{\dot{x}}_a + \underline{\dot{x}}_a \cdot \overleftarrow{\underline{\nabla}}\, \rho_a \underline{\dot{x}}_a \right\}$$

$$= \underline{x} \times \left\{ \overline{\rho_a \underline{\dot{x}}_a}^{\,\prime} + \underline{\dot{x}}_a \cdot \overleftarrow{\underline{\nabla}}\, \rho_a \underline{\dot{x}}_a \right\} = \underline{x} \times \left\{ \rho_a \underline{\ddot{x}}_a + \dot{\rho}_a \underline{\dot{x}}_a + \rho_a \underline{\dot{x}}_a \overrightarrow{\underline{\nabla}} \cdot \underline{\dot{x}}_a \right\} \qquad (63)$$

Hence in light of Equations 62 and 63, Equation 59, the local form of the balance of moment of momentum equation, can be rewritten:

$$\underline{x} \times \Big\{ \rho_a \ddot{x}_a - \underset{\sim}{T}{}_a^{\mathrm{T}} \cdot \overleftarrow{\underline{\nabla}} - \rho_a \underline{b}_a - \hat{\underline{p}}_a$$

$$+ (\dot{\rho}_a + \rho_a \overrightarrow{\underline{\nabla}} \cdot \underline{\dot{x}}_a - \hat{c}_a)\,\underline{\dot{x}}_a \Big\} - \underline{T}_{a_{\mathrm{A}}} - \hat{\underline{m}}_a = 0 \qquad (64)$$

The first four terms of Equation 64 constitute the balance of linear momentum of the a constituent (Equation 44) while the fifth through seventh terms is the expression for continuity of mass for the a constituent (Equation 19). Both sets of terms, written in these forms sum identically to zero leaving the result:

$$\underline{T}_{a_A} + \underline{\hat{m}}_a = 0 \tag{65}$$

or in component form

$$T_{a_{23}} - T_{a_{32}} = \hat{m}_{a_1}$$

$$T_{a_{31}} - T_{a_{13}} = \hat{m}_{a_2}$$

$$T_{a_{12}} - T_{a_{21}} = \hat{m}_{a_3}$$

This result implies that any given constituent or partial stress tensor is not symmetric except in the case of no inter-constituent moment of momentum or couple interaction ($\underline{\hat{m}}_a = 0$).

If we accept the axiom of balance of moment of momentum for the mixture to be:

$$\underset{\sim}{T} = \underset{\sim}{T}^T \tag{66}$$

then, by Equation 52, the definition of the mixture stress tensor:

$$\sum_{a=1}^{n} (\underset{\sim}{T}_a - \rho_a \underline{u}_a \underline{u}_a) = \sum_{a=1}^{n} (\underset{\sim}{T}_a^T - \rho_a \underline{u}_a \underline{u}_a) \tag{67}$$

or

$$\sum_{a=1}^{n} \underset{\sim}{T}_a = \sum_{a=1}^{n} \underset{\sim}{T}_a^T$$

In words, the balance of moment of momentum principle shows that the per constituent or partial stress tensors may not be symmetric but that the sum of the partial stress tensors is symmetric.

In summary we can tabulate for the mixture and on a per constituent basis the results of this mixture theory formulation:

Continuity (Balance of Mass)

$$\frac{\partial}{\partial t}\rho + \vec{\underline{\nabla}} \cdot \rho \underline{\dot{x}} = 0 \qquad \text{(mixture)} \qquad (68)$$

$$\frac{\partial}{\partial t}\rho_a + \vec{\underline{\nabla}} \cdot \rho_a \underline{\dot{x}}_a = \hat{c}_a \qquad (a \text{ constituent}) \qquad (69)$$

$$\text{subject to } \sum_{a=1}^{n} \hat{c}_a = 0 \qquad (70)$$

Balance of Linear Momentum

$$\rho \underline{\ddot{x}} = \vec{\underline{\nabla}} \cdot \underset{\sim}{T} + \rho \underline{b} \qquad \text{(mixture)} \qquad (71)$$

$$\rho_a \underline{\ddot{x}}_a = \vec{\underline{\nabla}} \cdot \underset{\sim}{T}_a + \rho_a \underline{b}_a + \hat{\underline{p}}_a \qquad (a \text{ constituent}) \qquad (72)$$

$$\text{subject to } \sum_{a=1}^{n} (\hat{\underline{p}}_a + \hat{c}_a \underline{u}_a) = 0 \qquad (73)$$

and

$$\underset{\sim}{T} = \sum_{a=1}^{n} (\underset{\sim}{T}_a - \rho_a \underline{u}_a \underline{u}_a) \qquad (74)$$

Balance of Moment of Momentum

$$\underset{\sim}{T} = \underset{\sim}{T}^T \qquad \text{(mixture)} \qquad (75)$$

$$\underline{T}_{a_A} + \hat{\underline{m}}_a = 0 \qquad (a \text{ constituent}) \qquad (76)$$

$$\text{subject to: } \sum_{a=1}^{n} \underset{\sim}{T}_a = \sum_{a=1}^{n} \underset{\sim}{T}_a^T \qquad (77)$$

CHAPTER 3

THE THEORY OF TURBULENT, WIND-AIDED SNOW SEDIMENTATION

In the previous chapter a mixture theory is developed, through the momentum balance equations for a mixture of n constituents which respond as a whole in a fluid manner. It will be of interest to apply this theory to the well documented but mechanically complex phenomena of turbulent wind-aided snow sedimentation. This investigation is made for two purposes, one: to determine whether or not solutions to the theory exist and two: to detemine if these solutions model or approximate in a qualitative sense certain observed aspects of the phenomena. However, in this next chapter and preliminary to any applications it will be necessary to derive the turbulent equations of motion for the snow phase of the mixture flow.

The Equations of Motion for a Mixture of Snow and Air

The pertinent equations of motion for a mixture of snow entrained in an atmospheric flow are summarized below (Decker and Brown, 1983). The subscript s and a are indicative of the snow phase and air phase respectively of the mixture.

Continuity

$$\frac{\partial}{\partial t} \rho_s + \underline{\vec{\nabla}} \cdot \rho_s \underline{\dot{x}}_s = \hat{c}_s \tag{78}$$

$$\frac{\partial}{\partial t} \rho_s + \underline{\vec{\nabla}} \cdot \rho_s \underline{\dot{x}}_a = \hat{c}_a \tag{79}$$

and

Balances of Linear Momentum

$$\rho_s \ddot{\underline{x}}_s = \vec{\underline{\nabla}} \cdot \underset{\sim}{T}_s + \rho_s \underline{b}_s + \hat{\underline{p}}_s \tag{80}$$

$$\rho_a \ddot{\underline{x}}_a = \vec{\underline{\nabla}} \cdot \underset{\sim}{T}_a + \rho_a \underline{b}_a + \hat{\underline{p}}_a \tag{81}$$

Note that $\dot{\underline{x}}_s = \underline{u}_s$ and $\dot{\underline{x}}_a = \underline{u}_a$. These are the velocity fields for the snow phase and air phase respectively and should not be confused with diffusion velocities as previously defined.

Consider that if the intrinsic time scale of the snow sedimentation process (> 1.0 hr) is large compared to the inertial time scale (∿ 1.0 sec) of the process, then the transient effects may be neglected and the process could be considered steady. This assumption is consistent with other attempts to apply a mixture theory to sedimentation processes (Drew, 1975; McTique, 1981, 1983). Further, consider that if approximately 40% of the total entrained snow in transport over a flat surface sublimates into the air phase over a distance of 3 km (Schmidt, 1982), then for the transport distances of this investigation (10 m), it is reasonable to neglect the effects of inter-phase mass exchange. Further, it is assumed that the only body force acting on either the snow or air phase is the gravitational potential. However, it should be noted that the formation of snow cornices does occur in blowing snow environments on the immediate lee of mountain slopes and there is good evidence to support the hypothesis that snow cornices form as a consequence of very large electrostatic potentials (Latham & Montagne, 1970). Lastly, it is assumed that if the mass fraction of snow in transport remains small (< 10%) compared to the total mass of the mixture. Therefore the components of the partial stress tensor of the snow phase are negligible relative to those of the air phase. This assumption is consistent with those postulated for dispersed multi-phase flows (McTique, 1983) and is analogous to the statement that as long as the mass fraction of snow in transport remains small the streamlines of the airflow will remain essentially unchanged from those of single phase airflow, i.e., the snow phase does not have a significant effect upon the air phase flow.

In light of these assumptions the equations of motions for the snow phase when the motion of the air phase is known are:

$$\underline{\vec{\nabla}} \cdot \rho_s \underline{u}_s = 0 \qquad \text{Continuity} \tag{82}$$

and

$$\rho_s \underline{u}_s \cdot \underline{\vec{\nabla}}\, \underline{u}_s = \rho_s \underline{g} + \hat{\underline{p}}_s \qquad \text{Balance of Linear Momentum} \tag{83}$$

<u>Momentum Supply Between the Air and Snow Phases</u>

At this point it is necessary to make a constitutive assumption for the momentum supply or transfer terms: $\hat{\underline{p}}_a$ and $\hat{\underline{p}}_s$ between the air phase and snow phase of the mixture flow. Let:

$$\hat{\underline{p}}_a = -\hat{\underline{p}}_s = \rho_s D\,(\underline{u}_s - \underline{u}_a) \tag{84}$$

That is, the momentum supply between the air and snow phases is equal and opposite in sign and is dependent on the velocity difference between the phases and a drag coefficient: D with dimensions 1/time. This satisfies the restriction of Equation 51 on the balance of linear momentum when mass supply: $\hat{c}_s = \hat{c}_a = 0$. Further, momentum supply between the phases is dependent on the local mass density of the snow phase, which is physically consistent inasmuch as mass density of the snow phase approaches zero so does the momentum supply between the phases.

Equation 84 characterizes the transfer of momentum between the snow and air phase of the mixture flow. It is effectively a description of the drag between the air and snow phases. It would be possible to describe this term as the difference of some polynomial value of these velocities. The difference of the squares of the snow phase and air phase velocities would be a logical increase in the degree of theoretical complexity. This would lead to an increase in the degree of non-linearity of the resulting systems of partial differential equations to be solved for the snow phase velocity field. Since this system of partial differential equations will be solved by numerical approximation techniques the instability of the subsequent algebraic equation system may also increase. Also, a velocity squared drag or momentum transfer term may not be an objective or frame indifferent constitutive assumption. Lastly, it will be shown that the theory in fact oversuspends the snow phase within the mixture flow. Increasing the magnitude of the momentum transfer between the phases via a velocity squared drag term would increase the oversuspension of the snow phase in the mixture flow.

For any constitutive assumption the magnitude and direction of the term must remain objective or retains its values regardless of the frame of reference, i.e., a valid constitutive assumption must be indifferent to any time dependent orthogonal change of reference frame. Therefore now consider that for any time dependent orthogonal transformation of coordinate frame $\underset{\sim}{Q}$ where $\dot{\underline{x}}_s$ and $\dot{\underline{x}}_a$ are the snow and air velocities with respect to the original coordinate frame and $\dot{\underline{x}}_s{}^*$ and $\dot{\underline{x}}_a{}^*$ are these velocities with respect to the new coordinate frame such that:

$$\underline{x}_s{}^* = \underset{\sim}{Q}\,\underline{x}_s + \underline{c} \tag{85}$$

and

$$\underline{x}_a{}^* = \underset{\sim}{Q}\,\underline{x}_a + \underline{c} \tag{86}$$

Where $\underline{c}$ is a time dependent translation of the coordinate frame.

Then:

$$\underline{u}_s{}^* = \dot{\underline{x}}_s{}^* = \overline{(\underset{\sim}{Q}\,\underline{x}_s + \underline{c})}^{\cdot} = \underset{\sim}{Q}\,\dot{\underline{x}}_s + \dot{\underset{\sim}{Q}}\,\underline{x}_s + \dot{\underline{c}} \tag{87}$$

and

$$\underline{u}_a{}^* = \dot{\underline{x}}_a{}^* = \overline{(\underset{\sim}{Q}\,\underline{x}_a + \underline{c})}^{\cdot} = \underset{\sim}{Q}\,\dot{\underline{x}}_a + \dot{\underset{\sim}{Q}}\,\underline{x}_a + \dot{\underline{c}} \tag{88}$$

Therefore

$$\underline{u}_s{}^* - \underline{u}_a{}^* = \underset{\sim}{Q}\,\dot{\underline{x}}_s + \dot{\underset{\sim}{Q}}\,\underline{x}_s + \dot{\underline{c}} - \underset{\sim}{Q}\,\dot{\underline{x}}_a - \dot{\underset{\sim}{Q}}\,\underline{x}_a - \dot{\underline{c}}$$

$$= \underset{\sim}{Q}\,(\dot{\underline{x}}_s - \dot{\underline{x}}_a) + \dot{\underset{\sim}{Q}}\,(\underline{x}_s - \underline{x}_a) \tag{89}$$

Since $\underline{x}_s = \underline{x}_a$ (i.e., the position vectors to the point where the momentum transfer is effected are the same) then

$$\underline{u}^*{}_s - \underline{u}^*{}_a = \underset{\sim}{Q}\,(\underline{u}_s - \underline{u}_a) \tag{90}$$

Hence, the constitutive assumptions on the supply or transfer of momentum between the air and snow phases is objective or frame indifferent (Malvern, 1969).

In summary, the equations of motion for the snow phase of the mixture flow are:

$$\underline{\vec{\nabla}} \cdot \rho_s \underline{u}_s = 0 \qquad \text{Continuity} \tag{91}$$

and

$$\rho_s \underline{u}_s \cdot \underline{\vec{\nabla}}\, \underline{u}_s = \rho_s \underline{g} - \rho_s D(\underline{u}_s - \underline{u}_a) \qquad \text{Balance of Linear Momentum} \tag{92}$$

The Turbulent Equations of Motion for the Snow Phase of the Mixture Flow

Consider the dimensionless Reynolds's number, classically a ratio or measure of the inertial forces to the viscous forces of a flow. For large Reynold's numbers, when inertia is the predominant force of a flow these flows are observed to be turbulent. For a variety of engineering applications a large amount of research has been done on determining the "critical" Reynold's number for which a laminar flow may become unstable and go through transition to turbulent. At Reynold's number much greater than the critical Reynold's number the flow will be fully turbulent.

By definition the Reynold's number is:

$$R = \frac{V_{ref}\, L_{ref}}{\nu} \tag{93}$$

where V_{ref} is a reference or characteristic velocity, L_{ref} is a characteristic length and ν is the kinematic viscosity of the fluid. Obviously the magnitude of the Reynold's number for any given flow is controlled in part by the researcher's perception of what constitutes a "characteristic" set of velocities and lengths.

In the specific case of atmospheric flows there is a large volume of research investigating the domains of these characteristic dimensions. However, irregardless of the V_{ref} and L_{ref} chosen for an atmospheric flow $\nu_{air} \simeq 1.3 \times 10^{-5}$ m²/s. Consequently, over the resulting range of Reynold's numbers, the magnitudes of these Reynold's numbers are quite large and these flows are considered fully turbulent (Britter, Hunt, & Richards, 1981; Bradley, 1980; Jackson & Hunt, 1975; Plate, 1971).

The equations of motion of the snow phase of the mixture flow can be decomposed and expanded to include the effect of a turbulent mixture flow.

Adopting the standard Reynold's description (Hinze, 1975) of a turbulent variable for the snow phase results in:

$$\underline{u}_s = \overline{\underline{u}}_s + \underline{u}'_s \qquad (94)$$

$$\underline{u}_a = \overline{\underline{u}}_a + \underline{u}'_a \qquad (95)$$

$$\rho_s = \overline{\rho}_s + \rho'_s \qquad (96)$$

That is, the instantaneous value of any turbulent variable is the sum of its mean or time averaged value and its turbulent fluctuations, denoted by the overbar and overscore, respectively. Any product of mean and fluctuating variables may be subject to additional time averaging and must conform to the following conditions, where for any time dependent variables g and f:

$$\overline{\overline{f}} = \overline{f} \tag{97}$$

$$\overline{f} \pm \overline{g} = \overline{f \pm g} \tag{98}$$

$$\overline{\overline{f}g} = \overline{f}\,\overline{g} \tag{99}$$

$$\overline{\vec{\underline{\nabla}} f} = \vec{\underline{\nabla}}\,\overline{f} \tag{100}$$

$$\overline{f'} = 0 \tag{101}$$

$$\overline{f'g'} \neq \overline{f'}\,\overline{g'} = 0 \tag{102}$$

$$\overline{(f')^{2n-1}} = 0 \qquad n = 1, 2, 3, \ldots \tag{103}$$

$$\overline{(f')^{2n}} \neq 0 \qquad n = 1, 2, 3, \ldots \tag{104}$$

Substituting Equations 94 and 96 into the continuity equation (Equation 91) for the snow phase of the mixture flow results in:

$$\vec{\underline{\nabla}} \cdot (\overline{\rho}_s + \rho'_s)(\overline{\underline{u}}_s + \underline{u}'_s) = 0 \tag{105}$$

or

$$\vec{\underline{\nabla}} \cdot \left\{ \overline{\rho}_s \overline{\underline{u}}_s + \overline{\rho}_s \underline{u}'_s + \rho'_s \overline{\underline{u}}_s + \rho'_s \underline{u}'_s \right\} = 0$$

When a subsequent time average of Equation 105 is taken the terms $\overline{\overline{\rho}_s \underline{u}'_s}$ and $\overline{\rho'_s \overline{\underline{u}}_s}$, by Equation 101 are identically zero, resulting in:

$$\vec{\underline{\nabla}} \circ \left\{ \overline{\rho}_s \overline{\underline{u}}_s + \overline{\rho'_s \underline{u}'_s} \right\} = 0 \tag{106}$$

Now consider that if the turbulent fluctuations of a variable are joint-normally distributed then the conservation of mass at a point will be independent of the turbulence and the turbulent continuity equations will be:

$$\vec{\underline{\nabla}} \cdot \rho_s \underline{u}_s = \vec{\underline{\nabla}} \cdot \overline{\rho}_s \overline{\underline{u}}_s = 0 \tag{107}$$

This requires that:

$$\vec{\underline{\nabla}} \cdot \overline{\rho' \underline{u}'} = 0 \tag{108}$$

In the absence of any information about the nature of a turbulent flow the assumption that the turbulence is joint-normally distributed is a simplication but is also the only logical description available (Oral communication, J. T. Oden). The resultant turbulent continuity equation (Equation 107) is identical with that derived by Drew (1975) and McTique (1981) for the particulate phase of a multiphase turbulent flow.

Substitution of Equations 94 through 96 into the linear momentum balance equation (Equation 92) for the snow phase of the mixture flow results in:

$$\left\{(\bar{\rho}_s + \rho'_s)(\underline{\bar{u}}_s + \underline{u}'_s)\right\} \cdot \underline{\vec{\nabla}}\,(\underline{\bar{u}}_s + \underline{u}'_s)$$

$$= (\bar{\rho}_s + \rho'_s)\underline{g} - (\bar{\rho}_s + \rho'_s)\, D\, (\underline{\bar{u}}_s + \underline{u}'_s - \underline{\bar{u}}_a - \underline{u}'_a) \qquad (109)$$

or

$$(\bar{\rho}_s \underline{\bar{u}}_s + \bar{\rho}_s \underline{u}'_s + \rho'_s \underline{\bar{u}}_s + \rho'_s \underline{u}'_s) \cdot (\underline{\vec{\nabla}}\,\underline{\bar{u}}_s + \underline{\vec{\nabla}}\,\underline{u}'_s)$$

$$= \bar{\rho}_s \underline{g} + \rho'_s \underline{g} - \bar{\rho}_s D(\underline{\bar{u}}_s - \underline{\bar{u}}_a) - \bar{\rho}_s D(\underline{u}'_s - \underline{u}'_a)$$

$$- \rho'_s D(\underline{\bar{u}}_s - \underline{\bar{u}}_a) - \rho'_s D(\underline{u}'_s - \underline{u}'_a)$$

When the left side is expanded and a subsequent time average of Equation 109 is taken, by Equations 101 and 103 the terms $\overline{\bar{\rho}_s \underline{\bar{u}}_s \cdot \underline{\vec{\nabla}}\,\underline{u}'_s}$, $\overline{\bar{\rho}_s \underline{u}'_s \cdot \underline{\vec{\nabla}}\,\underline{\bar{u}}_s}$, $\overline{\rho'_s \underline{\bar{u}}_s \cdot \underline{\vec{\nabla}}\,\underline{\bar{u}}_s}$, $\overline{\rho'_s \underline{u}'_s \cdot \underline{\vec{\nabla}}\,\underline{u}'_s}$, $\overline{\rho'_s \underline{g}}$, $\overline{\bar{\rho}_s D(\underline{u}'_s - \underline{u}'_a)}$ and $\overline{\rho'_s D(\underline{\bar{u}}_s - \underline{\bar{u}}_a)}$ are identically zero resulting in:

$$\bar{\rho}_s \underline{\bar{u}}_s \cdot \underline{\vec{\nabla}}\,\underline{\bar{u}}_s = \bar{\rho}_s \underline{g} - \rho_s D(\underline{\bar{u}}_s - \underline{\bar{u}}_a) - \underline{F}_T \qquad (110)$$

where

$$\underline{F}_T = \overline{\bar{\rho}_s \underline{u}'_s \cdot \underline{\vec{\nabla}}\,\underline{u}'_s} + \overline{\rho'_s \underline{\bar{u}}_s \cdot \underline{\vec{\nabla}}\,\underline{u}'_s} + \overline{\rho'_s \underline{u}'_s} \cdot \underline{\vec{\nabla}}\,\underline{\bar{u}}_s + \overline{\rho'_s D(\underline{u}'_s - \underline{u}'_a)}$$

Note that if $\rho'_s \ll \bar{\rho}_s$, $\underline{u}'_s \ll \underline{\bar{u}}_s$, $\underline{u}'_a \ll \underline{\bar{u}}_a$ and $\underline{u}'_s \simeq \underline{u}'_a$ then the fourth term of $\underline{F}_T$ may be assumed to be negligibly small relative to the other terms.

The Constitutive Assumption for Turbulent Fluctuating Variables in Terms of Mean Flow Variables

In order to render the system determinate, a constitutive assumption relating the turbulent fluctuation of a given variable in terms of a mean flow variable must be made. If the bulk of the mixture flow inertia and all the shear of the mixture flow are intrinsic to the air phase then it would be consistent to relate the turbulent fluctuations of the snow phase variables to a mean flow variable of the air. Consider then the following constitutive assumption for ρ'_s and $\underline{u}'_s$.

$$\rho'_s = \frac{\sqrt{2}\,\overline{\rho}_s\,\underline{\epsilon}}{3} \cdot \underline{1} \sqrt{II_{\overline{\underset{\sim}{\Delta}}_a}} \tag{111}$$

$$\underline{u}'_s = \frac{\underline{\gamma}}{\sqrt{2}} \sqrt{II_{\overline{\underset{\sim}{\Delta}}_a}} \tag{112}$$

where:

$$\overline{\underset{\sim}{\Delta}}_a = \begin{bmatrix} 0 & \dfrac{\partial \overline{u}_a}{\partial y} & \dfrac{\partial \overline{u}_a}{\partial z} \\ \dfrac{\partial \overline{v}_a}{\partial x} & 0 & \dfrac{\partial \overline{v}_a}{\partial z} \\ \dfrac{\partial \overline{\omega}_a}{\partial x} & \dfrac{\partial \overline{\omega}_a}{\partial y} & 0 \end{bmatrix} \tag{113}$$

$$II_{\overline{\underset{\sim}{\Delta}}_a} = \frac{1}{2}\, \overline{\underset{\sim}{\Delta}}_{a_{ij}}\, \overline{\underset{\sim}{\Delta}}_{a_{ij}}, \; i \neq j \tag{114}$$

$II_{\overline{\underset{\sim}{\Delta}}_a}$ is the second scalar invariant of the deviatoric mean airflow gradient tensor. $\overline{u}_a$, $\overline{v}_a$ and $\overline{\omega}_a$ are the components of the mean airflow velocities. $\underline{\epsilon}$ is a vector valued function with the dimensions of time and $\underline{\gamma}$ is a vector valued function with the dimensions of length.

Note that the constitutive assumption for the turbulent fluctuation is, by definition of a scalar valued variable objective (Malvern, 1969). The form given by Equation 112 must be proven frame indifferent. For any orthogonal transformation of coordinate frame

$\underset{\sim}{Q}$ that $\underline{\gamma}^* = \underset{\sim}{Q}\,\underline{\gamma}$ where the starred quantity is the value of that variable in the transformed coordinate frame.

$$\underline{u}'_s{}^* = \frac{\underline{\gamma}^*}{\sqrt{2}} \sqrt{II_{\bar{\underset{\sim}{\Delta}}_a}}$$

$$= \underset{\sim}{Q} \frac{\underline{\gamma}}{\sqrt{2}} \sqrt{II_{\bar{\underset{\sim}{\Delta}}_a}} \tag{115}$$

Note that $\underline{u}'_s{}^*$ transforms like a frame indifferent vector value function and hence the constitutive assumption on $\underline{u}'_s$ is objective.

For the special case of one-dimensional mixture flow $(\frac{\partial}{\partial x} = \frac{\partial}{\partial z} = \bar{v}_a = \bar{\omega}_a = 0)$ the constitutive assumptions for ρ'_s and u'_s reduce to

$$\rho'_s = \frac{\rho_s(\epsilon_x + \epsilon_y + \epsilon_z)}{3} \sqrt{\left(\frac{\partial \bar{u}_a}{\partial y}\right)^2} = \frac{\rho_s(\epsilon_x + \epsilon_y + \epsilon_z)}{3} \left| \left(\frac{\partial \bar{u}_a}{\partial y}\right) \right| \tag{116}$$

$$u'_s = \frac{\gamma_x}{2} \sqrt{\left(\frac{\partial \bar{u}_a}{\partial y}\right)^2} = \frac{\gamma_x}{2} \left| \left(\frac{\partial \bar{u}_a}{\partial y}\right) \right| \tag{117}$$

These results are analogous to the one-dimensional phenomenologically derived Prandtl mixing length theory for turbulent single phase flow (Hinze, 1975; Schlicting, 1979).

By substitution of the constitutive assumptions for ρ'_s and $\underline{u}'_s$ into the terms of F_T the turbulent balance of linear momentum equation for the snow phase of the mixture flow results

$$\bar{\rho}_s \underline{\bar{u}}_s \cdot \underline{\vec{\nabla}}\, \underline{\bar{u}}_s = \rho_s \underline{g} - \rho_s D(\underline{\bar{u}}_s - \underline{\bar{u}}_a) - F_T$$

where

$$F_T = \bar{\rho}_s \Big\{ \frac{\underline{\gamma}}{\sqrt{2}} \sqrt{II_{\bar{\underset{\sim}{\Delta}}_a}} \cdot \underline{\vec{\nabla}} \frac{\underline{\gamma}}{\sqrt{2}} \sqrt{II_{\bar{\underset{\sim}{\Delta}}_a}} + \left(\frac{\sqrt{2}\underline{\epsilon}}{3} \cdot \underline{1} \sqrt{II_{\bar{\underset{\sim}{\Delta}}_a}}\right) \underline{\bar{u}}_s \cdot \underline{\vec{\nabla}} \frac{\underline{\gamma}}{\sqrt{2}} \sqrt{II_{\bar{\underset{\sim}{\Delta}}_a}}$$

$$+ \left(\left(\frac{\sqrt{2}\underline{\epsilon}}{3} \cdot \underline{1} \sqrt{II_{\bar{\underset{\sim}{\Delta}}_a}}\right) \frac{\underline{\gamma}}{\sqrt{2}} \sqrt{II_{\bar{\underset{\sim}{\Delta}}_a}}\right) \cdot \underline{\vec{\nabla}}\, \underline{\bar{u}}_s \Big\} \tag{117}$$

Equation 117, the turbulent balance of linear momentum equation along with the turbulent continuity equation below:

$$\underline{\vec{\nabla}} \cdot \overline{\rho}_s \underline{\overline{u}}_s = 0 \tag{118}$$

form the turbulent equations of motion for the snow phase of the mixture flow.

For the case of a two dimensional mixture flow ($\frac{\partial}{\partial z} = \omega_a = \omega_s = 0$) the constitutive assumptions on ρ'_s and $\underline{u}'_s$ reduce to

$$\rho'_s = \frac{\sqrt{2}\, \overline{\rho}_s \underline{\epsilon}}{3} \cdot \underline{1} \sqrt{\left\{\left(\frac{\partial \overline{u}_a}{\partial y}\right)^2 + \left(\frac{\partial \overline{v}_a}{\partial x}\right)^2\right\}} \tag{119}$$

and

$$\underline{u}'_s = \frac{\underline{\gamma}}{2} \sqrt{\left\{\left(\frac{\partial \overline{u}_a}{\partial y}\right)^2 + \left(\frac{\partial \overline{v}_a}{\partial x}\right)^2\right\}} \tag{120}$$

Consider the following upper bound approximation to the factor:

$$\sqrt{\left\{\left(\frac{\partial \overline{u}_a}{\partial y}\right)^2 + \left(\frac{\partial \overline{v}_a}{\partial x}\right)^2\right\}} = \left(\left|\frac{\partial \overline{u}_a}{\partial y}\right| + \left|\frac{\partial \overline{v}_a}{\partial x}\right| \right) \tag{121}$$

The error inherent in this approximation can be analyzed by considering the binomial expansion:

$$\sqrt{\left\{\left(\frac{\partial \overline{u}_a}{\partial y}\right)^2 + \left(\frac{\partial v_a}{\partial x}\right)^2\right\}} = \left(\frac{\partial \overline{u}_a}{\partial y} + \frac{\partial \overline{v}_a}{\partial x}\right) \sqrt{\left\{1 - \frac{2\left(\frac{\partial \overline{u}_a}{\partial y}\right)\left(\frac{\partial \overline{v}_a}{\partial x}\right)}{\left(\frac{\partial \overline{u}_a}{\partial y} + \frac{\partial \overline{v}_a}{\partial x}\right)^2}\right\}}$$

$$= \left(\frac{\partial \overline{u}_a}{\partial y} + \frac{\partial \overline{v}_a}{\partial x}\right) \left\{ 1 - \frac{\left(\frac{\partial \overline{u}_a}{\partial y}\right)\left(\frac{\partial \overline{v}_a}{\partial x}\right)}{\left(\frac{\partial \overline{u}_a}{\partial y} + \frac{\partial \overline{v}_a}{\partial x}\right)^2} + \frac{\left(\frac{\partial \overline{u}_a}{\partial y}\right)^2 \left(\frac{\partial \overline{v}_a}{\partial x}\right)^2}{2(2!)\left(\frac{\partial \overline{u}_a}{\partial y} + \frac{\partial \overline{v}_a}{\partial x}\right)^4} + \ldots \right\} \tag{122}$$

Inasmuch as all terms of the expansion are positive after the first sum in the expansion truncation at the second sum will result in a measure of the maximum error. Note that the error in this approximation is large when gradients of the airflow are small. However when gradients of the airflow are small the terms of $\underline{F}_T$ are also small and the error to ρ'_s and

$\underline{u}'_s$ will not be of consequence. Further, when gradients of the airflow are large the terms of $\underline{F}_T$ will be large but the error to ρ'_s and $\underline{u}'_s$ will be minimized.

By evoking the concept of a joint-normally distributed turbulence it is possible to write $\underline{\gamma} = \gamma \underline{\underline{1}}$ and $\underline{\epsilon} = \epsilon \underline{1}$.

Lastly, the turbulent balance of linear momentum equation (Equation 118) can have ρ_s divided out on a termwise basis. This results in the non-conservative form of the turbulent balance of linear momentum equation, which is uncoupled from the turbulent continuity equation.

In light of the above for a two dimensional mixture flow, the components of the non-conservative turbulent momentum balance equation are

$$u_s \frac{\partial u_s}{\partial x} + v_s \frac{\partial u_s}{\partial y} = g_x - D(u_s - u_a) - F_{T_x} \qquad \text{x-comp}$$

$$u_s \frac{\partial v_s}{\partial x} + v_s \frac{\partial v_s}{\partial y} = g_y - D(v_s - v_a) - F_{T_y} \qquad \text{y-comp} \tag{123}$$

where

$$F_{T_x} = \epsilon\gamma C \frac{\partial u_s}{\partial x} + \epsilon\gamma C \frac{\partial u_s}{\partial y} + \epsilon\gamma A u_s + \epsilon\gamma B v_s + \frac{\gamma^2}{4}(A+B)$$

$$F_{T_y} = \epsilon\gamma C \frac{\partial v_s}{\partial x} + \epsilon\gamma C \frac{\partial v_s}{\partial y} + \epsilon\gamma A u_s + \epsilon\gamma B v_s + \frac{\gamma^2}{4}(A+B) \tag{124}$$

and

$$A = \left|\frac{\partial v_a}{\partial x}\right| \left(\left|\frac{\partial^2 v_a}{\partial x^2}\right| + \left|\frac{\partial^2 u_a}{\partial x \partial y}\right| \right) + \left|\frac{\partial u_a}{\partial y}\right| \left(\left|\frac{\partial^2 v_a}{\partial x^2}\right| + \left|\frac{\partial^2 u_a}{\partial x \partial y}\right| \right)$$

$$B = \left|\frac{\partial v_a}{\partial x}\right| \left(\left|\frac{\partial^2 v_a}{\partial x \partial y}\right| + \left|\frac{\partial^2 u_a}{\partial y^2}\right| \right) + \left|\frac{\partial u_a}{\partial y}\right| \left(\left|\frac{\partial^2 v_a}{\partial x \partial y}\right| + \left|\frac{\partial^2 u_a}{\partial y^2}\right| \right) \qquad (125)$$

$$C = \left|\frac{\partial v_a}{\partial x}\right| \left(\left|\frac{\partial v_a}{\partial x}\right| + \left|\frac{\partial u_a}{\partial y}\right| \right) + \left|\frac{\partial u_a}{\partial y}\right| \left(\left|\frac{\partial v_a}{\partial x}\right| + \left|\frac{\partial u_a}{\partial x}\right| \right)$$

$u_s, v_s, \omega_s, u_a, v_a, \omega_a$ are the components, in rectangular cartesian coordinates of the snow and air phase velocity vectors: $\underline{u}_s$ and $\underline{u}_a$. The overscore has been dropped, since now all variables are in terms of the mean values.

These component equations can be reorganized into the following system of non-linear partial differential equations:

$$\begin{bmatrix} (u_s + \epsilon\gamma C) & 0 \\ 0 & (u_s + \epsilon\gamma C) \end{bmatrix} \frac{\partial}{\partial x} \begin{Bmatrix} u_s \\ v_s \end{Bmatrix} + \begin{bmatrix} (v_s + \epsilon\gamma C) & 0 \\ 0 & (v_s + \epsilon\gamma C) \end{bmatrix} \frac{\partial}{\partial y} \begin{Bmatrix} u_s \\ v_s \end{Bmatrix} +$$

$$\begin{bmatrix} (D + \epsilon\gamma A) & B \\ A & (D + \epsilon\gamma B) \end{bmatrix} \begin{Bmatrix} u_s \\ v_s \end{Bmatrix} = \begin{Bmatrix} g_x + Du_a - \frac{\gamma^2}{4} (A+B) \\ g_y + Dv_a - \frac{\gamma^2}{4} (A+B) \end{Bmatrix} \qquad (126)$$

The two dimensional turbulent continuity equation is summarized below:

$$\frac{\partial}{\partial x} \rho_s u_s + \frac{\partial}{\partial y} \rho_s v_s = 0 \qquad (127)$$

For the case of one dimensional mixture flow $\left(\frac{\partial}{\partial x} = \frac{\partial}{\partial z} = \omega_a = v_a = \omega_s = 0\right)$ the component form of the momentum balance equation for the snow phase reduces to the following system of non-linear partial differential equations:

$$(v_s + \epsilon\gamma c) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial}{\partial y} \begin{Bmatrix} u_s \\ v_s \end{Bmatrix} + \begin{bmatrix} D & \epsilon\gamma B \\ 0 & (D + \epsilon\gamma B) \end{bmatrix} \begin{Bmatrix} u_s \\ v_s \end{Bmatrix} = \begin{Bmatrix} g_x + Du_a - \frac{\gamma^2}{4} B \\ g_y - \frac{\gamma^2}{4} B \end{Bmatrix} \quad (128)$$

The turbulent continuity equation for the snow phase reduces to:

$$\frac{\partial}{\partial y} \rho_s v_s = 0 \quad (129)$$

Lastly, by examining the nonconservative form of the turbulent balance of linear momentum equation for the case of still air ($\omega_a = v_a = u_a = 0$) mixture motion the coefficient: D of the momentum supply or transfer term may be analyzed. Consider:

$$v_s \frac{\partial v_s}{\partial y} + Dv_s = g_y \quad (130)$$

or

$$\frac{\partial v_s}{\partial y} = \frac{g_y}{v_s} - D \quad (131)$$

If g_y is the gravitational acceleration (-9.81 m/sec^2) and the snow phase mixture motion has a domain of *constant* still air fall velocities of -1.0 to -0.5 m/sec then D will have a range from 9.81 to 19.6 1/sec. For all following calculations D is set at 13.0 1/sec. This corresponds to a still air fall velocity of -0.75 m/sec.

CHAPTER 4

SOLUTIONS OF THE SNOW PHASE EQUATIONS OF MOTION FOR A ONE DIMENSIONAL AIRFLOW

In the previous chapter the turbulent equations of motion for the snow phase of an atmospheric mixture flow of snow and air were derived.

It is of interest to solve these equations of motion for the snow phase velocity and density profiles as function of height above the solid surface for a one dimensional atmospheric boundary layer flow. These solutions can be compared with observations concerning the nature of the snow phase velocity and density profiles for snow sedimentation flows over flat surfaces (Budd, Dingle & Radok, 1965; Mellor, 1965; Kobayashi, 1972; Schmidt, 1977; Takeuchi, 1980).

Consider a one dimensional turbulent atmospheric boundary layer airflow of the form (Plate, 1971):

$$\underline{u}_a = u_a = \frac{u^*}{k} \ln \frac{y}{y_o} \qquad (132)$$

with the following gradients

$$\frac{\partial u_a}{\partial y} = \frac{u^*}{ky}$$

$$\frac{\partial^2 u_a}{\partial y^2} = -\frac{u^*}{ky^2} \qquad (133)$$

where u^*, k and y_o are the friction velocity, Von Karman's constant and the surface roughness height respectively.

For this specific case of a one dimensional airflow the system of partial differential equations (Equation 128) for the snow phase may be solved by finite difference techniques for the component snow phase velocities as functions of height above the solid boundary.

Consider the forward difference approximation to $\frac{\partial}{\partial y}$ (Ames, 1977; Smith, 1978) applied to the snow phase momentum balance partial differential equation system. j is the cell location description on a vertical column solution domain of m cells and G is the cell height. The snow phase subscript, s, has been dropped.

$$(v_j^{n-1} + \epsilon\gamma c_j) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} (u_{j+1}^n - u_j^n)/G \\ (v_{j+1}^n - v_j^n)/G \end{Bmatrix} + \begin{bmatrix} D & \epsilon\gamma B_j \\ 0 & (D + \epsilon\gamma B_j) \end{bmatrix} \begin{Bmatrix} u_j^n \\ v_j^n \end{Bmatrix}$$

$$= \begin{Bmatrix} Du_{a_j} - \frac{\gamma^2}{4} B_j \\ g_y - \frac{\gamma^2}{4} B_j \end{Bmatrix} \qquad \begin{matrix} n = 1, 2, 3 \ldots \\ j = 1, 2, 3, \ldots, m \end{matrix} \tag{134}$$

The non-linear term is treated quasi-linearly by retaining its value from the n−1 iteration during the nth solution iteration procedure. Further we have defined the y coordinate axis to be parallel to the gravitational potential. The above discrete system of linear algebraic equations can be reorganized:

$$\frac{v_j^{n-1} + \epsilon\gamma C_j}{G} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} u_{j+1}^n \\ v_{j+1}^n \end{Bmatrix} + \begin{bmatrix} D - (v_j^{n-1} + \epsilon\gamma C_j) & \epsilon\gamma B_j \\ 0 & (D + \epsilon\gamma B_j) - \dfrac{(v_j^{n-1} + \epsilon\gamma C_j)}{G} \end{bmatrix} \begin{Bmatrix} u_j^n \\ v_j^n \end{Bmatrix}$$

$$= \begin{Bmatrix} Du_{a_j} - \frac{\gamma^2}{4} B_j \\ g_y - \frac{\gamma^2}{4} B_j \end{Bmatrix} \qquad \begin{matrix} n = 1, 2, 3 \ldots \\ j = 1, 2, 3, \ldots, m \end{matrix} \tag{135}$$

By using the one dimensional airflow velocity description of Equations 132 and 133, defining $g_y = -9.81$ m/sec², D = 13.0 1/sec, setting boundary condition values for u_{m+1} and N_{m+1} and defining ϵ and γ, primarily to maintain computational stability renders Equa- 134 a determinant system of 2 m algebraic equations in 2 m unknown snow phase velocity components.

ONEDEE is a Fortran code (see Appendix for listing) designed to solve Equation 134 by Gauss-Seidel iteration (Ames, 1977; Smith, 1978).

The ONEDEE solution for the snow phase is depicted in Figure 2. Note that the snow phase contains sufficient inertia to have a positive horizontal impact velocity at the snow surface, where the air velocity goes to zero. This is consistent with the mechanisms of saltation flow or impact induced restitution of the snow phase from the solid surface back into the mixture flow (Kobayashi, 1972).

In a naturally occurring snow-air mixture flow the bulk of the snow phase in transport over a flat surface is a product of saltation flow. However, this theory requires that some component of the snow in transport be a product of the apparent or turbulent buoyancy of the snow phase in regions of large airflow gradients. In other words, even when there is no snow phase restitution from the solid surface there should still be a discernibly, variable snow phase density profile vs. height. Snow density profiles for snow-air mixture flows adjacent open water (i.e., a non-restituting lower boundary) have been measured (Takeuchi, 1980). There is a non-uniform snow phase density profile where the difference in density maximum and minimum for these profiles are approximately one order of magnitude less than that for saltation flows (Oral communication, M. Takeuchi).

The turbulent continuity equation for the snow phase for a one-dimensional airflow (Equation 129) has the solution:

$$\rho_s v_s = \text{constant} \tag{136}$$

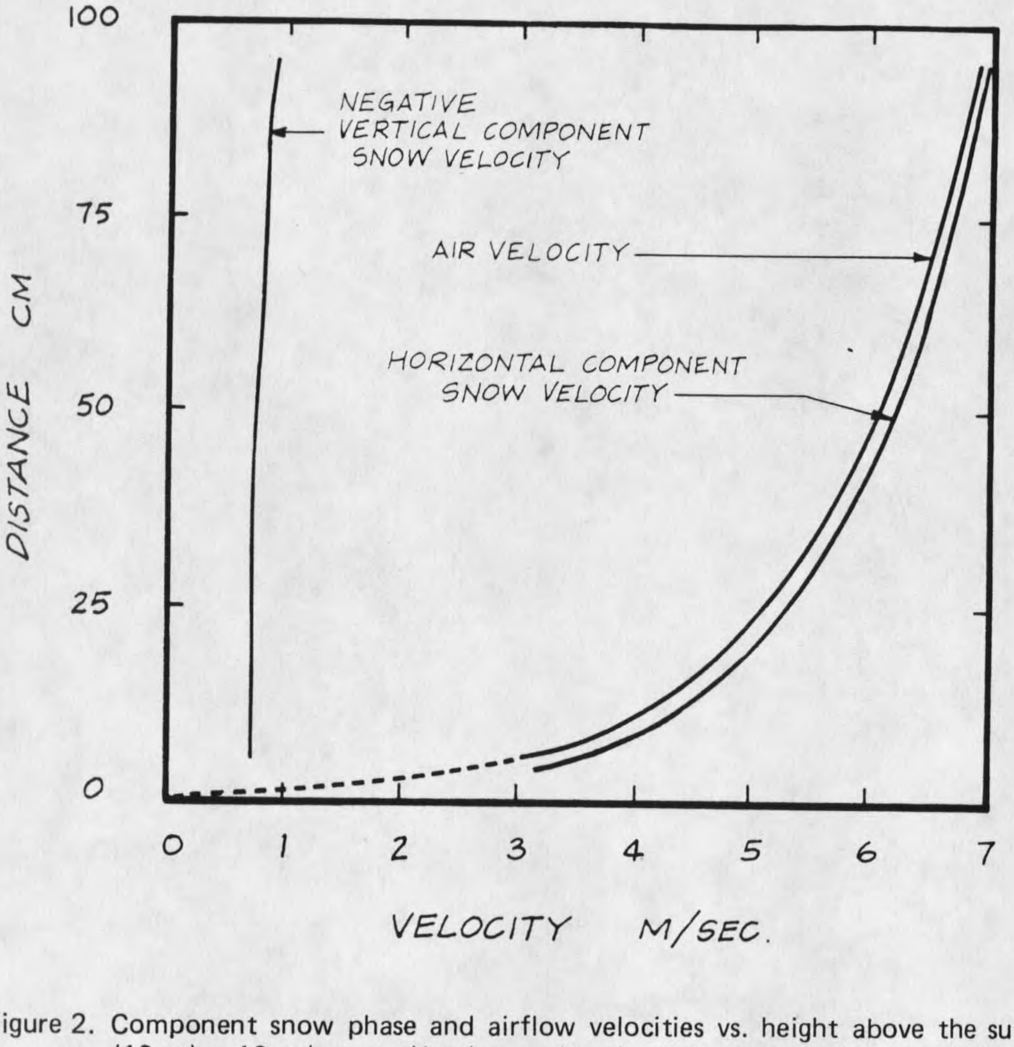

Figure 2. Component snow phase and airflow velocities vs. height above the surface u_a (10 m) = 10 m/sec, u_s (1 m) = u_a (1 m) = 7.0 m/sec, v_s (1 m) = -1.0 m/sec, γ = 0.6 m, ϵ = 0.6 sec, u^* = 0.33 m/sec, k = 0.25, y_o = 0.005 m.

If we define the snow phase mass density at the top of the boundary layer flow to be 1.0×10^{-4} kg/m^3 then the snow phase density profile corresponding to the snow phase vertical or fall velocity components of Figure 2 can be calculated. These data are presented in Figure 3. It is satisfying to note that a theoretical snow phase density profile does exist and has a range of 1.5×10^{-5} kg/m^3.