Tchebycheff Approximations by General Spline Functions
by LEROY AMUNRUD

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Mathematics
Montana State University

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This thesis presents a development of the theory of Tchebycheff approximations by polynomials with imposed boundary conditions and the theory of Tchebycheff approximations by general spline functions. Existence and characterization theorems are given along with computational procedures and examples.

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Approved:

Louis C. Barrett
Head, Major Department

R.E. Lowney
Chairman, Examining Committee

K Goering
Graduate Dean

MONTANA STATE UNIVERSITY
Bozeman, Montana

August, 1968

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Abstract

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1. Introduction

A problem encountered repeatedly in scientific research is the following:

> Given a function $f(x)$, a norm $||\cdot||$, and a class S of admissible approximating functions, find a function $P(x) \in S$ such that $||f(x) - P(x)||$ is a minimum.

The function which is to be approximated may be given in many different ways. For example, the function may be a continuous function defined by a slowly converging power series or it may be a discrete function defined as the numerical solution of a differential equation. The characteristics of the function $f(x)$ and the intended use of the approximation influence the choice of the norm and the class of admissible approximating functions.

A desirable norm in many applications is the Tchebycheff norm (called the ℓ_∞ norm in the discrete case, the uniform norm in the continuous case) described by

$$||X^*|| = \sup_{x \in I} \{|X^*(x)|\}$$

where I is some given set of points. Tchebycheff [12] and de la Vallée Poussin [8] developed much of the early theory associated with this norm. In the past few years E. K. Blum [3], P. C. Curtis, Jr. [7], E. W. Cheney [5], A. A. Goldstein [5], C. W. Clenshaw [6], J. C. C. Nitsche [10], and many others have made contributions to the advancement of this theory. Also in a recent publication [9] C. L. Lawson presents characterization theorems and solution procedures associated with the problem of partitioning an interval such that the largest error incurred in approximating a continuous function by separate polynomials or rational forms on each subinterval is minimized. This type of approximation, i.e. one in which different approximating functions are used on different subintervals of the argument domain, is called a segmented approximation. If the end points of the subintervals are not specified, the problem of finding the "best" segmented approximating function is more complicated than the standard fixed interval approximation problem due to the added difficulty of finding the optimum set of end points for the subintervals. However, the increased precision of a segmented approximation often justifies the additional work.

A logical extension of Lawson's type of approximation is to require that the approximating function be continuous or have derivatives up through some order at the end points of the subintervals. One class of functions of this type is the class of general spline functions.

Definition 1.

Let $\nu_1, \nu_2, \ldots, \nu_k$ be non-negative integers and $\delta_1 \leqq \delta_2 \leqq \cdots \leqq \delta_k$ be a set of points in $[\alpha, \beta]$.

A function $s_k(x)$ which satisfies the conditions:

i) $s_k(x)$ is a polynomial in x on each subinterval $[\alpha,\delta_1]$, $[\delta_r,\delta_{r+1}]$, $r = 1,2,\ldots,k-1$, and $[\delta_k,\beta]$;

ii) $s_k(x)$ is continuous on $[\alpha,\beta]$,

iii) $s_k(x)$ has derivatives up through order ν_r at δ_r, $r = 1,2,\ldots,k$,

is called a general spline function. The points $\delta_1,\delta_2,\ldots,\delta_k$ are called the join points of $s_k(x)$.

For a definition of a spline function and other associated definitions see [1], [2] and [11].

The purpose of this thesis is to present characterization theorems and solution procedures associated with Tchebycheff approximations where the class of approximating functions is the class of general spline functions. Basic to the theory of Tchebycheff approximations by general spline functions is the theory of Tchebycheff approximations by polynomials with imposed boundary conditions. Consequently this theory is developed first. It should be noted that all polynomials used in the following discussions are real polynomials.

2. Tchebycheff Theory With Imposed Boundary Conditions

In order to simplify the notation, the zero derivative of a function shall be used to designate the function itself.

Lemma 1. (Existence Lemma)

Let f be a continuous real valued function defined on the non-degenerate interval $[\alpha,\beta]$, and let f have derivatives up through some order $\nu \geqq 0$ at some fixed point δ in $[\alpha,\beta]$. Let n be a non-negative integer such that $n \geqq \nu$. Then there exists a polynomial $P_n(x)$ which has the same values for its derivatives up through order ν as f has at δ, is of degree less than or equal to n, and is a best approximation in the sense of the Tchebycheff norm relative to the conditions imposed on the derivatives at δ.

Proof:

Let $P_n(x)$ be given by

$$P_n(x) = \sum_{j=0}^{n} \lambda_j x^j$$

and let

$$\varphi(\lambda_1,\lambda_2,\ldots,\lambda_n) = ||f - P_n|| = \max_{\alpha \leqq x \leqq \beta} \left| f(x) - P_n(x) \right|$$

The polynomial $P_n(x)$ must satisfy the conditions

$$P_n^{(m)}(\delta) = f^{(m)}(\delta), \qquad m = 0,1,2,\ldots,\nu$$

where in general $f^{(m)}(x)$ denotes the m^{th} derivative of $f(x)$. If one sets $\delta^0 = 1$ even when δ is zero, and $f^{(m)}(\delta) = C_m$, then this system of equations can be written in the form

$$\sum_{j=\alpha}^{n} \frac{j!}{(j-\alpha)!} \lambda_j \delta^{j-\alpha} = C_\alpha, \qquad \alpha = 0,1,\ldots,\nu$$

In this system of equations, the coefficient matrix A of $\lambda_0,\lambda_1,\ldots,\lambda_\nu$ is upper triangular and has non-zero elements on its diagonal. Such a matrix is non-singular. Consequently this system of $(\nu + 1)$ equations can be used to express $\lambda_0,\lambda_1,\ldots,\lambda_\nu$ as continuous functions of the remaining λ_j's. That is

$$P_n(x) = a_0 + a_1x + \ldots + a_\nu x^\nu + \lambda_{\nu+1}x^{\nu+1}+\ldots+\lambda_n x^n$$

where from the form of A it follows that each a_j can be written in the form

$$a_j = \sum_{\ell=0}^{\nu} k_{j\ell}c_\ell + \sum_{\ell=\nu+1}^{n} k_{j\ell}\lambda_\ell, \quad j = 0,1,\ldots,\nu$$

Here each $k_{j\ell}$ is a constant. Thus each a_j is also of the form

$$a_j = K_j + \sum_{\ell=\nu+1}^{n} k_{j\ell}\lambda_\ell, \quad j = 0,1,\ldots,\nu$$

where each K_j is a constant.

Consider the function ψ defined by

$$\psi(\lambda_{\nu+1},\lambda_{\nu+2},\ldots,\lambda_n)=\left|\left|f(x)-\sum_{j=0}^{\nu} a_jx^j-\sum_{j=\nu+1}^{n}\lambda_jx^j\right|\right| \quad (1)$$

First it will be shown that ψ is a continuous function of the vector argument $\lambda = (\lambda_{\nu+1},\lambda_{\nu+2},\ldots,\lambda_n)$.

$$|\psi(\lambda')-\psi(\lambda)| = \Bigg|\,\left|\left|f(x)-\sum_{j=0}^{\nu} a_j'x^j-\sum_{j=\nu+1}^{n}\lambda_j'x^j\right|\right|$$

$$-\left|\left|f(x)-\sum_{j=0}^{\nu} a_jx^j-\sum_{j=\nu+1}^{n}\lambda_jx^j\right|\right|\,\Bigg|$$

Hence

$$|\psi(\lambda')-\lambda(\lambda)| \leq \left|\left| \sum_{j=0}^{\nu} (a_j'-a_j)x^j + \sum_{j=\nu+1}^{n} (\lambda_j'-\lambda_j)x^j \right|\right|$$

$$\leq \max_{\substack{0 \leq j \leq \nu \\ \nu+1 \leq i \leq n}} \left\{ |a_j'-a_j|, |\lambda_i'-\lambda_i| \right\} \sum_{j=0}^{n} ||x^j||$$

Each a_j' is a continuous function of $\lambda_{\nu+1}', \lambda_{\nu+2}', \ldots, \lambda_n'$. Thus $a_j' \to a_j$, $j = 0,1,\ldots,\nu$, as $\lambda' \to \lambda$. Furthermore each $||x^j||$, $j = 0,1,\ldots,n$ is bounded. This completes the proof that $\psi(\lambda)$ is a continuous function of λ. Compare this proof with that on page 130 of [13].

Now either ν is equal to n or ν is less than n. Consider first the case where ν is equal to n. Then

$$P_n(x) = \sum_{j=0}^{\nu} a_j x^j$$

where each a_j is uniquely determined. Consequently there is one and only one polynomial which satisfies the given conditions, and for this case the lemma is seen to be true.

Next let ν be less than n. It follows from the continuity proof given above that for the continuous function

$$\sum_{j=0}^{\nu} K_j x^j$$

$$\varphi = \left|\left| \sum_{j=0}^{\nu} K_j x^j - \sum_{j=0}^{\nu} a_j x^j - \sum_{j=\nu+1}^{n} \lambda_j x^j \right|\right|$$

$$= \left|\left| - \sum_{j=0}^{\nu} \sum_{\ell=\nu+1}^{n} k_{j\ell}\lambda_\ell x^j - \sum_{j=\nu+1}^{n} \lambda_j x^j \right|\right|$$

is a continuous function of λ. The shell

$$\lambda_{\nu+1}^2 + \ldots + \lambda_n^2 = 1$$

is a bounded, closed (compact) set in ordinary $(n-\nu)$ dimensional space, and on it the continuous function φ must assume a minimum σ. Since a norm by definition is always greater than or equal to zero, $\sigma \geqq 0$. The functions $1, x, x^2, \ldots$ are linearly independent. Thus, if at least one of the λ_j's, $j = \nu+1, \ldots, n$ is not zero, then $\sigma \neq 0$. It follows, by the homogeneity of φ, that for any $(\lambda_{\nu+1}, \lambda_{\nu+2}, \ldots, \lambda_n)$ with at least one non-zero component,

$$\varphi \geqq \sigma \sqrt{\lambda_{\nu+1}^2 + \ldots + \lambda_n^2} > 0$$

Let ρ be the lower bound of $\psi(\lambda)$. Then it is true that $\rho \geqq 0$. Now one only needs to show that this bound is attained. That is, one only needs to show that there exists a λ^* such that

$$\psi(\lambda^*) = \rho \tag{2}$$

Assume that

$$\sqrt{\sum_{j=\nu+1}^{n} \lambda_j^2} > R = (\rho + 1 + ||f(x) - \sum_{j=0}^{\nu} K_j x^j||)/\sigma$$

Then

$$\psi(\lambda) = ||f(x) - \sum_{j=0}^{\nu} a_j x^j - \sum_{j=\nu+1}^{n} \lambda_j x^j||$$

$$\psi(\lambda) = ||f(x) - \sum_{j=0}^{\nu} (K_j + \sum_{\ell=\nu+1}^{n} k_{j\ell}\lambda_\ell)x^j - \sum_{j=\nu+1}^{n} \lambda_j x^j||$$

$$\geqq ||-\sum_{j=0}^{\nu} \sum_{\ell=\nu+1}^{n} k_{j\ell}\lambda_\ell x^j - \sum_{j=\nu+1}^{n} \lambda_j x^j|| - ||f(x) - \sum_{j=0}^{\nu} K_j x^j||$$

$$\geqq \sigma(\rho + 1 + ||f(x) - \sum_{j=0}^{\nu} K_j x^j||)/\sigma - ||f(x) - \sum_{j=0}^{\nu} K_j x^j||$$

$$= \rho + 1$$

Hence the lower bound of $\psi(\lambda)$, for all λ, is the same as the lower bound when λ is restricted by $\sqrt{\sum_{j=\nu+1}^{n} \lambda_j^2} \leqq R$. Since this sphere is closed and bounded, the lower bound is attained, and the existence of λ^* is established.

This lemma can be generalized to include the case where the derivatives up through order ν_1 are given at one point $\delta_1 \in [\alpha,\beta]$ and up through order ν_2 at a second point $\delta_2 \in [\alpha,\beta]$, provided the degree n of the polynomial satisfies the relation $n \geqq \nu_1 + \nu_2 + 1$. The only modification required is that the argument associated with the matrix A must be applied twice, first at δ_1 (if zero is involved, set $\delta_1 = 0$) and secondly at δ_2, where the form of the matrix is slightly different. In fact this lemma can be generalized to include any finite number of points at which conditions are imposed. Also one could allow the constants, $C_0, C_1, \ldots, C_\nu$, to be chosen arbitrarily rather than being chosen equal to derivatives of the function f. This last generalization would require no changes in the proof of the lemma.

When the conditions on the derivatives are imposed at an end point of the interval $[\alpha,\beta]$, then the imposed conditions shall be called boundary conditions. In the case of boundary conditions, the next two lemmas give a characterization of the solution whose existence was established in lemma 1.

Definition 2.

Let $f(x)$ be a continuous function on $[\alpha,\beta]$, n be a non-negative integer, $Q_n(x)$ be a polynomial of degree less than or equal to n and let

$$\max_{\alpha \leqq x \leqq \beta} \left| f(x) - Q_n(x) \right| = \rho_{Q_n}(f)$$

Consider a set of points

$$\alpha \leqq x_1 < x_2 < \dots < x_V \leqq \beta$$

subject to the conditions that

$$\text{i)} \quad |f(x_i) - Q_n(x_i)| = \rho_{Q_n}(f), \quad i = 1,2,\dots,V$$

and

$$\text{ii)} \quad f(x_i)-Q_n(x_i)=-[f(x_{i-1})-Q_n(x_{i-1})], \quad i = 2,3,\dots,V.$$

The maximum number of points x_i which can be made to satisfy these two conditions is called the <u>oscillation number</u> of $(f(x) - Q_n(x))$ and is designated $V(f - Q_n)$.

If the number of x_i can be made large without bound, then one writes

$$V(f - Q_n) = \infty$$

Lemma 2.

Let n, ν_1 and ν_2 be non-negative integers with $n \geqq \nu_1+\nu_2+1$. Let $P_n(x)$ be a polynomial of degree n or less with prescribed values for its derivatives of order less than or equal to ν_1 at α and ν_2 at β. If no values are prescribed at $\alpha(\beta)$ set $\nu_1(\nu_2)$ equal to -1. Let $\nu = \nu_1 + \nu_2$ and let

$$\alpha < x_1 \leqq x_2 \leqq x_3 \leqq \dots \leqq x_{n-\nu} < \beta$$

be $(n-\nu)$ points in the interval (α,β). (When $\nu_1 = -1$ include α in the interval and when $\nu_2 = -1$ include β.) If the difference

$$g(x) = f(x) - P_n(x)$$

has the values

$$g(x_i) = \epsilon(-1)^{i-1}\gamma_i, \ \gamma_i > 0, \ i = 1,2,\dots,n-\nu, \text{ and}$$
$$\epsilon = 1 \text{ if } g(x_1) > 0, \text{ or } \epsilon = -1 \text{ if } g(x_1) < 0,$$

then for any polynomial $Q_n(x)$ of degree n or less with prescribed values for all derivatives up through ν_1 at α and up through ν_2 at β,

$$\rho_{Q_n}(f) \geqq \min\,(\gamma_1,\gamma_2,\dots,\gamma_{n-\nu})$$

Proof:

Assume the lemma is false. Then there exists a polynomial $r_n(x)$ of degree n or less which satisfies the given boundary conditions and such that

$$\max_{x_1 \leqq x \leqq x_{n-\nu}} \left| f(x)-r_n(x) \right| < \min\, (\gamma_1,\gamma_2,\ldots,\gamma_{n-\nu})$$

Consider the polynomial

$$\Delta(x) = r_n(x)-P_n(x) = (f(x)-P_n(x)) - (f(x)-r_n(x))$$

At the $(n-\nu)$ points, $x_1,x_2,\ldots,x_{n-\nu}$, the sign of $\Delta(x)$ is the same as the sign of $f(x) - P_n(x)$. Thus $\Delta(x)$ has $(n-\nu-1)$ zeros in the interval (α,β) (possibly including α and/or β). Furthermore $\Delta(x)$ has a zero of order (ν_1+1) at α and (ν_2+1) at β. Consequently $\Delta(x)$ must have $(n+1)$ zeros in the interval $[\alpha,\beta]$, which is a contradiction.

Lemma 3.

Using the notation of (1) and (2) introduced in the proof of lemma 1, let

$$\psi(\lambda^*) = \rho$$

and let n, ν_1 and ν_2 be non-negative integers such that $\nu_1 + \nu_2 = \nu \leqq n-1$. Assume there exists a polynomial $\mathbb{P}_n(x)$ of best approximation of degree less than or equal to n with prescribed values for its derivatives of each order less than or equal to ν_1 at α and less than or equal to ν_2 at β, and such that if $\rho > 0$ and $\nu_1 \geqq 0$, it is true that

$$|f(\alpha) - \mathbb{P}_n(\alpha)| < \rho \tag{3}$$

and if $\rho > 0$ and $\nu_2 \geqq 0$, it is true that

$$|f(\beta) - \mathbb{P}_n(\beta)| < \rho \tag{4}$$

Then $\mathbb{P}_n(x)$ is unique and is completely characterized by the property that the oscillation number satisfies the inequality

$$V(f(x) - \mathbb{P}_n(x)) \geqq n - \nu .$$

Proof:

Assume $V(f(x) - \mathbb{P}_n(x)) \geqq n - \nu$. If $\rho_{\mathbb{P}_n}(f)$ is zero, then $\mathbb{P}_n(x)$ is certainly a polynomial of best approximation. If $\rho_{\mathbb{P}_n}(f)$ is not zero, then in lemma 2 let

$$\gamma_1 = \rho_{\mathbb{P}_n}(f)$$

$$\gamma_2 = \rho_{\mathbb{P}_n}(f)$$

$$\vdots$$

$$\gamma_{n-\nu} = \rho_{\mathbb{P}_n}(f)$$

Then it follows that for any polynomial $Q_n(x)$

$$\rho_{Q_n}(f) \geqq \rho_{\mathbb{P}_n}(f)$$

Thus

$$\rho_{\mathbb{P}_n}(f) = \rho$$

Assume $V(f(x) - \mathbb{P}_n(x)) \leqq n-\nu-1$ and that $\mathbb{P}_n(x)$ is a polynomial of best approximation relative to the conditions imposed on the derivatives. There are only two possible cases: either $f(x)$ is a polynomial of degree less than or equal to n with the prescribed values for its derivatives or it is not.

Case I

Let $f(x)$ be a polynomial of degree less than or equal to n having the prescribed values for its derivatives. For this function $\rho = 0$. In order for $V(f(x) - \mathbb{P}_n(x))$ to be less than $n-\nu$, $\mathbb{P}_n(x)$ cannot be identically $f(x)$. Thus

$$\rho_{\mathbb{P}_n}(f) > \rho$$

and $\mathbb{P}_n(x)$ is not a polynomial of best approximation.

Case II

Assume $f(x)$ is not a polynomial of degree less than or equal to n having the prescribed values for its derivatives. In this case $\rho \neq 0$.

Subdivide the interval from α to β into the intervals

$$[u_0,u_1],\ [u_1,u_2],\ \ldots$$

so small that the oscillation (maximum minus the minimum) of $[f(x) - \mathbb{P}_n(x)]$ in each subinterval is less than or equal to $\frac{1}{2}\rho_{\mathbb{P}_n}(f)$. (This is possible in that a function which is continuous on a closed, bounded set is uniformly continuous there.)

If $|f(x) - \mathbb{P}_n(x)| = \rho_{\mathbb{P}_n}(f)$ in the interval $[u_k,u_{k+1}]$, the interval $[u_k,u_{k+1}]$ is called a distinguished interval. Label each such interval + or - according as $(f(x) - \mathbb{P}_n(x))$ is positive or negative respectively.

Label the distinguished intervals D_1, D_2,..... in order. Without loss of generality assume D_1 is +. Starting with D_1 proceed through the distinguished intervals until the first - distinguished interval is encountered. Call this group of + distinguished intervals group 1. Starting with the first encountered - distinguished interval proceed until the next + distinguished interval is encountered. Call this group of - distinguished intervals group 2, and continue in this fashion. Since $V(f - \mathbb{P}_n)$ is $\leqq n-\nu-1$, the total number of groups, T, is less than or equal to $(n-\nu-1)$. Construct a polynomial $r(x)$ of degree n which is negative in the intervals of group 1, positive in the intervals of group 2, etc., and such that the derivatives of $r(x)$ are zero at α up to order ν_1 and at β up to order ν_2.

Consider the polynomial

$$Q_n(x) = \mathbb{P}_n(x) - \epsilon\ r(x)$$

for an arbitrary $\epsilon > 0$. If ϵ is chosen sufficiently small, $Q_n(x)$ is a better fitting polynomial than $\mathbb{P}_n(x)$, and also satisfies the conditions imposed at the boundaries. This contradicts the assumption that $\mathbb{P}_n(x)$ is a best fitting polynomial.

To prove the uniqueness of the best approximating polynomial, assume there exist two polynomials of best approximation $\mathbb{P}_n(x)$ and $Q_n^*(x)$. Consider the polynomial

$$\varphi(x) = \tfrac{1}{2}[\mathbb{P}_n(x) + Q_n^*(x)]$$

$\varphi(x)$ is also a polynomial of best approximation, and satisfies the boundary conditions. Thus from the first part of this lemma, it must have an oscillation number $\geqq (n-\nu)$. From this it follows that $\mathbb{P}_n(x) = Q_n^*(x)$ at $(n-\nu)$ interior points. Furthermore they have the same $(\nu+2)$ conditions imposed upon their derivatives at the boundary points. This implies that

$$\mathbb{P}_n(x) \equiv Q_n^*(x)$$

Thus the solution is unique.

If a boundary condition is imposed at a boundary point δ such that

$$|f(\delta) - \mathbb{P}_n(\delta)| = \rho$$

then the polynomial of best approximation $\mathbb{P}_n(x)$ may not be unique. For example if

$$f(x) = -x^2 , \qquad x \in [-1,0]$$

and if a first degree approximating polynomial must have the value 10 at $x = 0$, then any polynomial of the form

$$\mathbb{P}_1(x) = cx + 10, \qquad 1.0 \leqq c \leqq 21.0$$

is a polynomial of best approximation relative to the imposed conditions. See figure 1.

It should be noted that this non-uniqueness can never occur when the boundary conditions require the approximating polynomial to have the same value as the given function at the boundary points. For in this case either $\rho = 0$ or conditions (3) and (4) of lemma 3 are satisfied.