Dynamic constraints on Feynmans path integral
by Bung-Ning Teng

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of
MASTER OF SCIENCE in Physics
Montana State University

Abstract:
The Feynman's path integral in space-time is closely related, via classical Lagrangian theory, to Dirac's formulation of quantum mechanics. The essential point of this approach is in making use of the Lagrangian function in space-time and Hamilton's first action principle to construct the probability amplitude in conventional quantum mechanics. This formulation has, as mentioned by Feynman, mathematical and physical incompleteness. We have investigated a part of the physical incompleteness, which can be treated by using the classical technique of eliminating ignorable. coordinates.

The corresponding conjugate momenta will then provide constraint relations, which can be used to reduce the Lagrangian function and the path integral to a modified form. The paths are now restricted, via the constraint relations, to limited configuration spaces. A general modified formulation is constructed for paths in a limited space-time region. An important special case is demonstrated. In the special case, Jacobi's action principle is employed to replace Hamilton's action principle, therefore, the paths thus formulated are in space-energy rather than in space-time.

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ABSTRACT

The Feynman's path integral in space-time is closely related, via classical Lagrangian theory, to Dirac's formulation of quantum mechanics. The essential point of this approach is in making use of the Lagrangian function in space-time and Hamilton's first action principle to construct the probability amplitude in conventional quantum mechanics. This formulation has, as mentioned by Feynman, mathematical and physical incompleteness. We have investigated a part of the physical incompleteness, which can be treated by using the classical technique of eliminating ignorable coordinates. The corresponding conjugate momenta will then provide constraint relations, which can be used to reduce the Lagrangian function and the path integral to a modified form. The paths are now restricted, via the constraint relations, to limited configuration spaces. A general modified formulation is constructed for paths in a limited space-time region. An important special case is demonstrated. In the special case, Jacobi's action principle is employed to replace Hamilton's action principle, therefore, the paths thus formulated are in space-energy rather than in space-time.

I. INTRODUCTION

The path integral approach to quantum mechanics which has been developed by Dirac[1,2,3] and Feynman[4,5] represents a point of view which has proven useful for the solution of important physical problems.[6] Of equal importance, however, is the conceptual picture it provides of quantum mechanics. In conventional quantum mechanics one customarily starts with stationary states and obtains the dynamical behavior only through the application of time-dependent perturbation theory. By contrast, the primitive concept of Feynman's approach is the probability amplitude (or propagator) for the dynamical evolution of a system. Moreover, the quantum dynamics of Feynman is closely related to the dynamics of the corresponding classical system, since the classical Lagrangian is employed to construct the probability amplitude in its path integral form.

The solution of problems in classical mechanics is often facilitated by transforming the Lagrangian to eliminate the so-called ignorable coordinates which do not appear explicitly in the Lagrangian. The modified Lagrangian, thus obtained, can be used to obtain equations of motion for the reduced problem. It is the purpose of this work to discuss the manner in which this procedure of classical mechanics can be related to the corresponding quantum mechanical problem. Our method

employs the modified classical Lagrangian to construct a path integral for the probability amplitude. The discussion is restricted throughout to non-relativistic quantum mechanics and to systems with a classical analogue.

Our development proceeds in complete analogy with the logic of Dirac and Feynman. For this reason we have included in this introduction several logical outlines by means of which the analogy may be compared, step by step.

In Section II we review and summarize the relevant parts of the work of Dirac and of Feynman, in order to make clear the relation of classical mechanics to the original (or standard) space-time path integral method.

The correspondence relationship between the probability amplitude for infinitesimal times and the classical action was first derived by Dirac in 1933. For a finite time interval $t_b - t_a$, the correspondence relation can be extended to calculate the probability amplitude by means of the path integral method, which was formulated by Feynman in 1948. This probability amplitude (or kernel or propagator) in space-time is nothing but the Green's function of the time-dependent Schroedinger's equation in conventional quantum mechanics. We summarize in Outline I the logical relations which lead from classical mechanics to the probability amplitude in quantum mechanics.

Outline I

The Classical Motion can be considered as the continuous development of a contact transformation. Dirac's search for a quantum analogue of this led to a correspondence relation between the probability amplitude and the classical action, which is:

$$\langle q_b\ t_b | q_a\ t_a \rangle \sim \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} L(q,\dot{q};t)dt\right]$$

For an infinitesimal time interval, ε, this becomes

$$\langle q_b\ t_a + \varepsilon | q_a\ t_a \rangle \simeq \exp\left[\frac{i}{\hbar} L(q,\dot{q};t_a)\varepsilon\right]$$

For a finite time interval, $t_b - t_a$, the probability amplitude may also be calculated using the classical Lagrangian function by means of:

Feynman's Path Integral, which gives the probability amplitude as a sum over all paths from q_a, t_a to q_b, t_b. That is,

$$\langle q_b\ t_b | q_a\ t_a \rangle = \sum_{\text{all paths}} k \frac{1}{A^k} \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} L(q,\dot{q};t)dt\right]$$

The probability amplitude has the physical meaning of a propagator (or kernel) for the time development of the wave function:

$$\psi(q_b\ t_b) = \int \langle q_b\ t_b | q_a\ t_a \rangle\ \psi(q_a\ t_a)dq_a \ .$$

The probability amplitude is identical in:

Conventional Quantum Mechanics to the Green's function of the time-dependent Schroedinger's equation

$$(-\frac{\hbar}{i}\frac{\partial}{\partial t_b} - H_{q_b})\langle q_b\ t_b | q_a\ t_a\rangle = -\frac{\hbar}{i}\delta(q_b - q_a)\delta(t_b - t_a) \ .$$

This outline forms the basic theoretical formulation of the space-time path integral in dealing with quantum mechanics and other fields of physics. Section II is devoted to a more complete review of this standard approach.

There is an "incompleteness" of this formulation as mentioned in Feynman's original paper.[4] We quote his comments about the path integral method as follows:

> It is also incomplete from the physical standpoint. ...The present formulation, can be mathematically demonstrated to be invariant under the unitary transformation. However, it has not been formulated in such a way that it is physically obvious that it is invariant. No direct procedure has been outlined to describe measurements of quantities other than position. ... It is to be expected that the postulates can be generalized by the replacement of the idea 'paths in a region of space-time R' to 'paths of class R', or 'paths having property R.' But which properties correspond to which physical measurements has not been formulated in a general way.

This comment motivated the present work, which we hope may contribute something to completing and generalizing the path integral method.

Relating to another effort in this direction, an additional historical remark will be made. In recent years, Davies and Garrod[7,8] (1962, 1966) have followed the same ideas as Dirac and Feynman, but have used Hamilton's principle of the second kind to formulate the path integral in space-momentum rather than in space-time. This may be seen as one kind of generalization of the path integral to paths in "space-momentum" or phase space.

It is a well known fact in classical mechanics, that when a conjugate momentum is a constant of the motion, the corresponding coordinate is missing in the Lagrangian function. This missing coordinate is called an ignorable (or cyclic) coordinate. The physical meaning of this for the path integral method is that the conserved quantity, say momentum of some sort, will provide a constraint relation to limit "paths to possible regions which conserve that quantity."

In section III, we exploit this fact and show how this procedure may be carried out. The analysis is general in that we consider a general constant of the motion and do not specialize to a particular dynamical variable such as energy, angular momentum, etc.

Following the general method of eliminating ignorable coordinates and their conjugate momenta, the Lagrangian function and Hamilton's action principle are reduced to

modified forms which contain the non-ignorable coordinates and the corresponding momenta only. The conserved quantities provide constraint relations. The logical development of this procedure is summarized in Outline II, which may be compared with Outline I.

Outline II

The Modified Classical Motion can be considered as the continuous development of a contact transformation in a limited region (restricted by the constraint relations) of space-time. The quantum analogue of this leads to a correspondence relation between the modified probability amplitude and the modified action, which is

$$\langle \bar{q}_b\ t_b | \bar{q}_a\ t_a \rangle \sim \exp\left[\frac{i}{\hbar} \int_{t_a}^{t_b} \bar{L}(\bar{q},\dot{\bar{q}};t)dt\right]$$

where $\bar{q}$ are non-ignorable coordinates only.

For an infinitesimal time interval, ε, this becomes

$$\langle \bar{q}_b\ t_a+\varepsilon | \bar{q}_a\ t_a \rangle \approx \exp\left[\frac{i}{\hbar} \bar{L}(\bar{q},\dot{\bar{q}};t_a)\varepsilon\right] .$$

For a finite time interval, $t_b - t_a$, the modified probability amplitude may also be calculated using the modified Lagrangian function by means of:

The Modified Feynman's Path Integral, which gives the modified probability amplitude as a sum over limited paths from $\bar{q}_a$, t_a to $\bar{q}_b$, t_b. That is

$$\langle \bar{q}_b\ t_b | \bar{q}_a\ t_a \rangle = \sum_{\substack{k \\ \text{all limited} \\ \text{paths}}} \frac{1}{B^k} \exp\left[\frac{i}{\hbar} \int_{t_a}^{t_b} \bar{L}(\bar{q},\dot{\bar{q}};t)dt\right] .$$

The modified probability amplitude has the physical meaning of a propagator (or kernel) for the time-development of the wave function in limited regions

$$\psi(\bar{q}_b\ t_b) = \int \langle \bar{q}_b\ t_b | \bar{q}_a\ t_a \rangle\ \psi(\bar{q}_a\ t_a) d\bar{q}_a \quad .$$

The modified probability amplitude is identical in:

Conventional Quantum Mechanics to the modified Green's function of the time-dependent Schroedinger's equation

$$(-\frac{h}{i}\frac{\partial}{\partial t_b} - H\bar{q}_b)\langle \bar{q}_b\ t_b | \bar{q}_a\ t_a \rangle = -\frac{\hbar}{i}\delta(\bar{q}_b - \bar{q}_a)\delta(t_b - t_a)$$

where the ignorable coordinates are missing, and their conjugate momenta are constant.

Outline II gives the modified formulation of a limited space-time path integral (in regions restricted by conserved quantities) which is developed in detail in Section III.

In Section IV we treat an important special case of the modified path integral method using a special modified Lagrangian.

When the Hamiltonian of the system is a constant of the motion, the time "t" is an ignorable coordinate. Therefore, "t" may be eliminated as an independent variable. After the elimination of "t", the classical Lagrangian function is reduced to a function of space and energy, where the energy is a constant. Then we must use as the independent variable, an arbitrary parameter ν, which parameterizes the paths that conserve energy; i.e., "paths having the property that they conserve energy." Moreover, this parameter can be chosen to

be one of the space coordinates, which are now free from "t", but are functions of the parameter. In this case the modified Hamilton's principle, due to Jacobi, is called Jacobi's action principle of the first kind.

We again employ an outline to display the logic of the path integral analysis of this special case of energy conservation.

Outline III

The Modified Classical Path can be considered as the continuous development of a special contact transformation in a limited region of space-energy. The quantum analogue of this leads to a correspondence relation between the modified probability amplitude and the modified action, which is

$$\langle q_b\ \nu_b \mid q_a\ \nu_a\rangle \sim \exp\left[\frac{i}{\hbar}\int_{\nu}^{\nu} F(q,\dot{q},-E;\nu)d\nu\right];\ \dot{q} \equiv \frac{dq}{d\nu}$$

The time does not appear, but the energy appears as a constant. For an infinitesimal parameter interval, σ, this becomes

$$\langle q_b\ \nu_a+\sigma \mid q_a\ \nu_a\rangle \simeq \exp\left[\frac{i}{\hbar} F(q,\dot{q},-E;\nu_a)\sigma\right] .$$

For a finite parameter interval, $\nu_b-\nu_a$, the modified probability amplitude may also be calculated using the modified Lagrangian function by means of:

The Modified Feynman's Path Integral, which gives the energy conserving probability amplitude as a sum over limited paths (that conserve energy) from q_a, ν_a to q_b, ν_b. That is

$$\langle q_b\ \nu_b \mid q_a\ \nu_a\rangle = \sum_{\substack{k \\ \text{all limited} \\ \text{paths}}} \frac{1}{A^k} \exp\left[\frac{i}{\hbar}\int_{\nu_a}^{\nu_b} F(q,\dot{q},-E;\nu)d\nu\right].$$

The modified quantum amplitude has the physical meaning of the kernel for the spatial development of the wave function of constant energy,

$$\phi(q_b\ \nu_b) = \int \langle q_b\ \nu_b | q_a\ \nu_a \rangle\ \phi(q_a\ \nu_a) dq_a \quad .$$

The energy conserving probability amplitude is identical in:

Conventional Quantum Mechanics to the Green's function of the time-independent Schroedinger's equation:

$$(H - E_b)\langle q_b\ \nu_b | q_a\ \nu_a \rangle = \frac{\hbar}{i}\ \delta(q_b - q_a)\ \delta(E_b - E_a) \quad .$$

This special case of the energy-conserving spatial propagator is the subject of Section IV, and serves as a specific illustration of the use of a modified Lagrangian in the construction of a modified quantum amplitude.

In Section V, a conclusion is given.

II. THE FEYNMAN (OR ORIGINAL) PATH INTEGRAL IN SPACE-TIME

In 1933, Dirac took up the question of what corresponds in quantum mechanics to the Lagrange equations of classical mechanics. A little consideration shows, however, that one cannot expect to be able to take over the classical Lagrange equations in any very direct way. These equations involve partial derivatives of the Lagrangian with respect to the coordinates and velocities and no meaning can be given to such derivatives in quantum mechanics.

We must therefore, seek the quantum analogue of the Lagrangian theory in an indirect way. We must try to take over the ideas, but not the equations of the classical theory. Dirac's main point of view is briefly reviewed as follows:*

In classical mechanics, the action function S and the Lagrangian function L are related by the expression,

$$S = \int_{t_a}^{t_b} L(q_i, \dot{q}_i; t)\,dt \tag{2.1}$$

where q_i are the generalized coordinates and $\dot{q}_i$ are the generalized velocities, which are all functions of the time t.

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*For detail, see Dirac's original work cited in Ref. (1).

Hamilton's first action principle states that the system follows a path such that the variation of the action function (2.1) for fixed t_b and t_a is zero. In accordance with the method of the calculus of variations, with the variations $\delta q_i(t_b) = \delta q_i(t_a) = 0$ at the ends points, we then obtain from

$$\delta S = \delta \int_{t_a}^{t_b} L(q_i, \dot{q}_i; t)dt = 0 \qquad (2.2)$$

the Lagrange equations of motion in the familiar form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad . \qquad (2.3)$$

For an infinitesimal time interval, Lagrangian theory is closely related to the theory of contact transformations in classical mechanics. We shall therefore begin with a brief discussion of the analogue between classical and quantum contact transformations.

Let two sets of variables be q_i, p_i and Q_i, P_i* ($i=1,2,\ldots n$), and suppose the q's and Q's to be all independent, so that any function of the dynamical variables can be expressed in terms of them. In classical mechanics, the contact transformation equations for this case can be put in

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*In classical mechanics, q_i, Q_i represent usually two different coordinate systems, say rectangular and spherical, but here, we consider them as the same coordinate at different times. This will be seen more clearly as we follow the development .

the form

$$p_i = \frac{\partial S}{\partial q_i}, \quad P_i = -\frac{\partial S}{\partial Q_i} \tag{2.4}$$

where q_i, p_i are coordinates and momenta at time t_a say, and Q_i, P_i are coordinates and momenta at another time, say t_b. S is some function of the q's and Q's as defined in (2.1).

This connection of the action function with the contact transformation in classical mechanics can be carried over into quantum mechanics in the following way.

We take the dynamical variables in classical mechanics to correspond with the operators q, Q in quantum mechanics. In addition we consider two representations in which the q's and Q's are diagonal respectively. There will be a transformation function $\langle q' | Q' \rangle$ (or matrix element of a unitary matrix) connecting the two representations, which in the present case are actually the same representation taken at different times. The q', Q' are numerical values (eigenvalues) of the corresponding operators q, Q.

We shall now show that this unitary transformation in quantum mechanics is the quantum analogue of $\exp[\frac{iS}{\hbar}]$; where S as defined in (2.1) is the classical action function.

If α is any function of the dynamical variables in quantum theory, it will have a "mixed" matrix representation $\langle q' | \alpha | Q' \rangle$ which may be defined in terms of the usual

representation of either $\langle q'|\alpha|q''\rangle$ or $\langle Q''|\alpha|Q'\rangle$ by

$$\langle q'|\alpha|Q'\rangle = \int\langle q'|\alpha|q''\rangle dq''\langle q''|Q'\rangle = \int\langle q'|Q''\rangle dQ''\langle Q''|\alpha|Q'\rangle .$$

From the first of these definitions we obtain,

$$\langle q'|q_i|Q'\rangle = q_i'\langle q'|Q'\rangle \tag{2.5}$$

$$\langle q'|p_i|Q'\rangle = -i\hbar\frac{\partial}{\partial q_i'}\langle q'|Q'\rangle \tag{2.6}$$

and from the second:

$$\langle q'|Q_i|Q'\rangle = Q_i'\langle q'|Q'\rangle \tag{2.7}$$

$$\langle q'|P_i|Q'\rangle = +i\hbar\frac{\partial}{\partial Q'}\langle q'|Q'\rangle \tag{2.8}$$

Note the difference in sign in (2.6) and 2.8).

Equation (2.5) and (2.7) may be generalized as follows: Let $f(q)$ be any function of the q's and $g(Q)$ any function of the Q's. Then

$$\langle q'|f(q)g(Q)|Q'\rangle = \iint\langle q'|f(q)|q''\rangle dq''\langle q''|Q''\rangle dQ''\langle Q''|g(Q)|Q'\rangle$$

$$= f(q')g(Q')\langle q'|Q'\rangle .$$

Further, if $f_k(q)$ and $g_k(Q)$, $(k=1,2,\ldots m)$ denote two sets of functions of the q's and Q's respectively,

$$\langle q'|\Sigma_k f_k(q)g_k(Q)|Q'\rangle = \Sigma_k f_k(q')g_k(Q')\langle q'|Q'\rangle .$$

Thus if α is any function of the dynamical variables and we suppose it to be expressed as a function $\alpha(q,Q)$ of the q's and Q's in a "well-ordered"* way, that is, so that it consists of a sum of all terms of the form $f(q)g(Q)$, we have:

$$\langle q' | \alpha(q,Q) | Q' \rangle = \alpha(q',Q') \langle q' | Q' \rangle \ . \qquad (2.9)$$

This is a rather remarkable equation, giving us a connection between $\alpha(q,Q)$, which is a function of operators, and $\alpha(q',Q')$, which is a function of numerical variables. Moreover, it provides the basis for carrying over the contact transformation between dynamical variables in classical mechanics into a contact transformation between operators in quantum mechanics. We demonstrate this immediately below.

In conventional quantum mechanics, $\langle q' | Q' \rangle$ is the matrix element of a unitary transformation between two representations. In this analysis, we postulate that

$$\langle q' | Q' \rangle = \exp \left[\frac{iU(q',Q')}{\hbar} \right] \qquad (2.10)$$

since the right-hand side also amounts to a unitary transformation. Later we prove that the function U defined by (2.10) is the analogue of the classical action function S,

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*See Ref. (2,3) for a discussion of the importance of "well-ordered" functions in quantum mechanics.

thus linking our unitary transformation with the classical action.

If in equation (2.9) we choose $\alpha = p_i$, and apply (2.10) we get from (2.6)

$$\langle q' | p_i | Q' \rangle = \frac{\partial U(q',Q')}{\partial q_i} \langle q' | Q' \rangle .$$

By comparing this with (2.9) we obtain

$$p_i = \frac{\partial U(q,Q)}{\partial q_i}$$

as an equation between operators or dynamical variables, which holds provided $\partial U(q,Q)/\partial q_i$ is well ordered. Similarly, we get

$$P_i = - \frac{\partial U(q,Q)}{\partial Q_i}$$

which holds provided $\partial U(q,Q)/\partial Q_i$ is well-ordered. These equations are of the same form as (2.4) and show that U defined by (2.10) is the analogue of the classical action S, which is what we had to prove. We proceed to examine this analogy in more detail.

The equations of motion of classical mechanics cause the dynamical variables to vary in such a way that their values q_b, p_a at any time t_b are connected with their values q_a, p_a at another time t_a by a contact transformation, which may be put into the form

$$p_i = \frac{\partial S}{\partial q_i} \qquad P_i = -\frac{\partial S}{\partial Q_i} \tag{2.4}$$

where q, p = q_b, p_b; Q, P = q_a, p_a and S is the time integral of the Lagrangian over the range t_a to t_b along the classical path,

$$S = \int_{t_a}^{t_b} L(q_i, \dot{q}_i; t)dt \ . \tag{2.1}$$

In quantum mechanics, the q_b, p_b will still be connected with the q_a, p_a by a contact transformation (or unitary transformation) and there will be a transformation function $\langle q_b\ t_b | q_a\ t_a \rangle$ connecting the two representations in which q_b at t_b and q_a at t_a are diagonal respectively. The work of the preceding section now shows that the contact transformations in classical mechanics and in quantum mechanics have the relationship,

$$\langle q_a\ t_b | q_a\ t_a \rangle \text{ corresponds to } \exp\left[\frac{i}{\hbar}\int_{t_a}^{t_b} L(q_i, \dot{q}_i; t)dt\right]^* \tag{2.11}$$

where L is the Lagrangian function. If we take t_a to differ by the infinitesimal ε from t_b, we get

$$\langle q_b\ t_a+\varepsilon | q_a\ t_a \rangle \text{ corresponds to } \exp\left[\frac{i}{\hbar} L\varepsilon\right] \tag{2.12}$$

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*Dirac's original work uses h instead of $\hbar$.

The transformation functions, i.e., $\langle q_b\ t_b | q_a\ t_a \rangle$, are very fundamental things in quantum mechanics, and it is satisfactory to find that they have their classical analogues, expressible simply in terms of the Lagrangian.

We have here the natural extension of the well-known result that the phase of the wave function corresponds to Hamilton's principle function (i.e., $S = \int_{t_a}^{t_b} L dt$) in classical mechanics.

The analogy (2.12) suggests that we ought to consider the classical Lagrangian, not as a function of the coordinates and velocities, but rather as a function of the coordinates at t_a and the coordinates at $t_a+\varepsilon$, where ε is an infinitesimal time interval.

For simplicity, we shall take the case of a single degree of freedom, although the argument applies to the general case. We shall use the notation,

$$\exp[\tfrac{i}{\hbar}\int_{t_a}^{t_b} L dt] = A(t_b, t_a)$$

so that $A(t_b, t_a)$ is the classical analogue of $\langle q_b\ t_b | q_a\ t_a \rangle$.

Suppose we divide the time interval t_a-t_b into a large number of small intervals t_a-t_1, t_1-t_2, ...t_{n-1}-t_n, t_n-t_b, by the introduction of a sequence of intermediate times t_1, t_2, ...t_n. Then

$$A(t_b, t_a) = A(t_b, t_n)A(t_n, t_{n-1}) \ldots A(t_2, t_1)A(t_1, t_a). \quad (2.13)$$

Now in the quantum theory we have

$$\langle q_b\, t_b | q_a\, t_a\rangle = \int\!\!\int \langle q_b | q_n\rangle dq_n \langle q_n | q_{n-1}\rangle dq_{n-1} \cdots$$
$$\cdots \int\!\!\int \langle q_2 | q_1\rangle dq_1 \langle q_1 | q_a\rangle \qquad (2.14)$$

where q_k denotes q at the intermediate time t_k (k=1,2,...n). (This is exactly the probability amplitude in Feynman's path integral method which will be seen later.)

Equation (2.14) at first sight does not seem to correspond properly to the equation (2.13), since on the right-hand side of (2.14) we must integrate after doing the multiplication while on the right-hand side of (2.13) there is no integration.

Let us examine this discrepancy by seeing what becomes of (2.14) when we regard t as extremely small. From the results of (2.11) and (2.12) we see that the integrand in (2.14) must be of the form $\exp[\frac{iG}{\hbar}]$ where G is a function of q_a, q_1, q_2, ...q_n, q_b which remains finite as $\hbar$ tends to zero. Let us now picture one of the intermediate q's, say q_k, as varying continuously while the others are fixed. Owing to the smallness of $\hbar$, we shall then in general have $G/\hbar$ varying extremely rapidly. This means that $\exp[\frac{iG}{\hbar}]$ will oscillate with a very high frequency about the value zero, as a result of which its integral will be practically zero. The only important part in the domain of integration of q_k is thus

that for which a comparatively large variation of q_k produces only a very small variation in G. This part is the neighborhood of a point for which G is stationary with respect to small variations in q_k.

We can apply this argument to each of the variables of integration on the right-hand side of (2.14) and obtain the result that the only important part in the domain of integration is that for which G is stationary for small variations in all the intermediate q's. But, by applying (2.11) to each of the small time sections, we see that G has for its classical analogue

$$\int_{t_n}^{t_a} L\,dt + \int_{t_{n-1}}^{t_n} L\,dt + \ldots \int_{t_1}^{t_2} L\,dt + \int_{t_a}^{t_1} L\,dt = \int_{t_a}^{t_b} L\,dt,$$

which is just the action function S, that classical mechanics requires to be stationary for small variations in all the intermediate q's. (Here Hamilton's action principle plays its role.) This shows the way in which (2.14) goes over into the classical result (2.13) when $\hbar$ becomes extremely small.

We now return to the general case when $\hbar$ is not small. We see that, for comparison with quantum theory, equation (2.13) must be interpreted in the following way. Each of the $A(t_n,t_{n-1})$ must be considered as a function of the q's at the two times to which it refers. The right-hand side of (2.13) is then a function, not only of q_b and q_a, but also of q_1, q_2, ... q_n, and in order to get from it a function of q_a

and q_b only, which we can equate to the left-hand side, we must substitute for $q_1, q_2, \ldots q_n$ their values given by Hamilton's first action principle. This process of substitution for the intermediate q's then corresponds to the process of integration over all values of q's in (2.14). (In Feynman's path integral language, this corresponds to the integration over all paths.)

Equation (2.14) contains the quantum analogue of the action principle, as may be seen more explicitly from the following argument. From (2.14) we can extract the statement (a rather trivial one) that, if we take specified values for q_a and q_b, then the importance of our considering any set of values for the intermediate q's is determined by the importance of this set of values in the integration on the right-hand side of (2.14). If we now make $\hbar$ tend to zero, this statement goes over into the classical statement that, if we take specified values for q_a and q_b, then the importance of our considering any set of values for the intermediate q's is zero unless these values make the action function stationary. This statement is one way of formulating the classical Hamilton's first action principle.

The above review gives the basic idea of the formulation of a quantum analogue of classical Lagrangian theory by Dirac.

In the following, we show the close relationship of Dirac's ideas and Feynman's path integral method.

Feynman's path integral method contains as its essential idea the concept of a probability amplitude with motion between two specified space-time points. This can be closely related to Dirac's idea in the following way.

From (2.14) we have

$$\langle q_b\ t_b|q_a\ t_a\rangle = \int..\int\langle q_b|q_{n+1}\rangle dq_{n+1}\langle q_{n+1}|q_n\rangle dq_n \\ .. \langle q_2|q_1\rangle dq_1\langle q_1|q\ \rangle, \qquad (2.14)$$

and this probability amplitude in quantum mechanics has been proved to be related to the classical action function by using the correspnndence relation.

$$\langle q_b\ t_b|q_a\ t_a\rangle \sim \exp[\frac{i}{\hbar}\int_{t_a}^{t_b} L(q,q;t)dt] \qquad (2.11)$$

What Feynman did in his work[4] was to show that for an infinitesimal time interval $t_{n+1}-t_n = \varepsilon$ this is an identity, if a proper normalization factor which depends on the system under consideration is included, that is:

$$\langle q_{n+1}\ t_{n+1}|q_n\ t_n\rangle = \frac{1}{A}\exp\ [\frac{i}{\hbar}\int_{t_n}^{t_{n+1}} L(q,\dot{q};t)dt] \qquad (2.15)$$

where A is the normalization factor.

The classical action function is expressed in the form

$$S = \sum_{n=-\infty}^{\infty} S(q_{n+1},q_n) \qquad (2.16)$$

where

$$S(q_{n+1},q_n) = \text{Min}\int_{t_n}^{t_{n+1}} L(q,q;t)dt \qquad (2.17)$$

If we put (2.15) into (2.14) and use Hamilton's first action principle as expressed in (2.17), which is the exact argument given by Dirac, we get

$$\langle q_b\ t_b | q_a\ t_a\rangle = \lim_{\epsilon\to 0}\int_R \exp\left[\frac{i}{\hbar}\sum_{n=-\infty}^{\infty} S(q_{n+1},q_n)\right]\frac{dq_{n+1}}{A}\frac{dq_n}{A}\cdots \qquad (2.18)$$

This is the probability amplitude in its path integral form, since the integration is taken over those values $q_n, q_{n+1}, \ldots$ which lie in a region R. Figure 1 shows this schematically for paths in space-time.

We proceed to show the equivalence of this path integral formulation and the conventional formulation of quantum mechanics.

How is the wave function defined from the new point of view? Does the wave function satisfy Schroedinger's equation? These two questions are answered in the following way.

We shall see that it is the possibility of expressing S as the sum as given in (2.16), and hence the amplitude as a product of contributions from successive sections of the path, which leads to the possibility of defining a quantity having the properties of a wave function.

In Figure 1, we may divide the region R into R' and R'',