Bilinear forms on vector spaces of countable dimensions in the case characteristic of k equals two
by Robert Dean Engle

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Mathematics
Montana State University

Abstract:
For k-vector spaces of a denumerable dimension supplied with a symmetric bilinear form, Kaplansky proved that in the case where characteristic k is 2 and O(ge)=1 there are, up to orthagonal isomorphism, precisely four non-isomorphic types. Kaplansky has raised the question of how this result is altered when characteristic k is2 and O(gk)>1 . The question is answered in this thesis for the case O(gn)=2.

This thesis is divided into two parts, the first part is an investigation of properties of fields of characteristic 2. In the second part the classification of non-isomorphic spaces is carried out. The major tool used in the classification is a certain theorem of automorphisms (2.4) for fields of characteristic 2.

BILINEAR FORMS ON VECTOR SPACES OF COUNTABLE DIMENSION
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A thesis submitted to the Graduate Faculty in partial
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The author, Robert Dean Engle, was born in Billings, Montana, October 31, 1935 to Mr. and Mrs. Ross Engle. He received his secondary education at Billings Senior High School, Billings, Montana graduating in 1953. He attended Montana State University at Missoula, Montana and received a Bachelor of Arts degrees in Mathematics and Physics in 1957. In 1961 he received a Master of Arts degree in Mathematics from Montana State University at Missoula.

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ABSTRACT

For k-vector spaces of a denumerable dimension supplied with a symmetric bilinear form, Kaplansky proved that in the case where characteristic k is 2 and $O(g_k) = 1$ there are, up to orthagonal isomorphism, precisely four non-isomorphic types. Kaplansky has raised the question of how this result is altered when characteristic k is 2 and $O(g_k) > 1$. The question is answered in this thesis for the case $O(g_k) = 2$.

This thesis is divided into two parts, the first part is an investigation of properties of fields of characteristic 2. In the second part the classification of non-isomorphic spaces is carried out. The major tool used in the classification is a certain theorem of automorphisms (2.4) for fields of characteristic 2.

CHAPTER I

INTRODUCTION

The classification, up to metric (orthogonal) isomorphism, of finite dimensional k-vector spaces F, supplied with a bilinear form $\phi: F \times F \to k$ is a very difficult problem. It has been solved for particular fields k, such as the field of rationals, reals, p-adic numbers and function fields of one variable over a finite constant field (see, e.g. (4) and (2)). For vector spaces E of a denumerable (algebraic) dimension, the difficulties vanish in a large number of cases by showing that E admits an orthonormal basis for a large class of underlying fields (Char $k \neq 2$) (2). In the denumerable case, an exceptional role is once more played by the underlying fields of characteristic 2. Even under the assumption that k be perfect, i.e., every element of k is a square, Kaplansky proved that there are, up to orthogonal isomorphism, precisely four non-isomorphic types of denumerable vector spaces supplied with a symmetric bilinear form. Kaplansky has raised the question of how this result is altered when k is not perfect. In the following, we will answer this question for a large number of non-perfect fields.

CHAPTER II

NOTATIONS AND RESULTS

2.1: Let E be a k-vector space over the commutative field k, supplied with a symmetric, non-degenerate bilinear form $\phi: E \times E \to k$. Because ϕ is non-degenerate, i.e., if $\phi(x,y) = 0$ for all $x \in E$, then $y = 0$, is also called semi-simple. E may thus contain isotropic vectors, i.e., vectors $x \neq 0$ with $\phi(x,x) = 0$, but no vectors $x \neq 0$ with $x \perp E$. As usual we write $\|x\|$ for the "length" $\phi(x,x)$ of a vector $x \in E$. $\|x\|$ is also called the norm and we will occasionally refer to the map $x \longrightarrow \|x\|$ as the norm form. The values $\phi(x,y)$ of ϕ will often be denoted by (x,y) if there is no risk of confusion. If we wish to indicate clearly the space E and the form "carried" by E we refer to the space (E,ϕ). Thus, if (E,ϕ) and $(\bar{E},\bar{\Phi})$ are k spaces, then they are said to be orthogonally isomorphic ($(E,\phi) \cong (\bar{E},\bar{\Phi})$ or simply $E \cong \bar{E}$ if there is no risk of confusion) if and only if there exists a vector space isomorphism $\psi: E \longrightarrow \bar{E}$ such that $\phi(x,y) = \bar{\Phi}(\psi(x), \psi(y))$ for all $x, y \in E$. For brevity, we will use isomorphism as synonymous with orthogonal isomorphism. If F is a subspace of E, then $F^{\perp}$ is the space of all vectors $x \in E$ such that $(x,y) = 0$ for all $y \in F$. $F \cap F^{\perp}$ is called the radical ($\mathrm{rad}(F)$) of F, and F is semi-simple if and only if $\mathrm{rad}(F) = 0$. If F and G are subspaces of E, then $F \perp G$ means that for all $x \in F$ and for all $y \in G$, $(x,y) = 0$. In this case we say that F is orthogonal to G. We always have $F \subset F^{\perp\perp}$ a subspace F of E is said to be closed if $F = F^{\perp\perp}$. If F is closed and G is a finite dimensional subspace of E, then $F + G$ is closed.

2.2: Let F be a finite dimensional subspace of E. For a fixed basis $\{s_1, \ldots, s_n\}$ of F, let $D = (-1)^{\frac{n(n-1)}{2}} \det \| \phi(s_i, s_j) \|$. If $\{s_1', \cdots, s_n'\}$ is any other basis of F and $D' = (-1)^{\frac{n(n-1)}{2}} \det \| \phi(s_i', s_j') \|$, then there exists $\delta \neq 0$ such that $D = \delta^2 D'$. Hence D defines a class modulo squares in k, which is an invariant of F, i.e., invariant under orthogonal isomorphisms of F. This class is called the discriminant of F (for example,

F is semi-simple if and only if its discriminant is not the zero class).

2.3: $\dim E \leq \aleph_0$ means E has a denumerable algebraic basis. If $\mathrm{char}\, k \neq 2$ then E has an orthogonal basis (E always semi-simple), i.e., a basis $\{e_1, e_2, \cdots\}$ such that $(e_i, e_j) = 0$ if $i \neq j$ and $(e_i, e_i) \neq 0$. If, however, $\mathrm{char}\, k = 2$, then E does not necessarily possess an orthogonal basis. But as in the finite dimensional case E is an orthogonal sum of subspaces of dimension at most 2. The irreducible 2-dimensional summands are hyperbolic planes P_i , i.e., planes spanned by a basis e_i , e_i' with $\|e_i\| = \|e_i'\| = 0$ and $\phi(e_i, e_i') = 1$. In other words, E decomposes into an orthogonal sum $E = F \oplus \bigoplus_{i \in I} P_i$, where F has an orthogonal basis and the P_i's are hyperbolic planes. We also say that the subspace $\bigoplus_{i \in I} P_i$ of E is spanned by a symplectic basis. In particular, if the form ϕ on E is alternate, i.e., $\phi(x,x) = 0$ for all $x \in E$, then $E = \bigoplus_{i \in I} P_i$ (for a proof of these theorems in the countable case, see e.g. (3)). On the other hand we see that if E contains no isotropic vectors, then E possesses an orthogonal basis.

2.4: Let E be a k-vector space over a commutative field of any characteristic supplied with a non-degenerate alternate form ϕ . Furthermore, let V and $\overline{V}$ be subspaces of E such that $V^{\perp} = \overline{V}^{\perp} = (0)$ and $\dim (E/V) = \dim (\overline{E}/\overline{V})$. If F and $\overline{F}$ are finite dimensional isomorphic subspaces of E with $F \cap V = \overline{F} \cap \overline{V} = (0)$, then every orthogonal isomorphism T_0 of $F \to \overline{F}$ can be extended to an orthogonal automorphism T of E with $T(V) = \overline{V}$. This theorem will play an important role in our classification below (section 4). It was proved in (1). A proof can also be found in (2). The above formulation is more general than the formulations in (1) and (2). However, inspection of the proof yields the more general result quoted.

2.5: In the following investigations we will assume that (E, ϕ) is a k -vector space with $\mathrm{char}\, k = 2$ and $\dim E = \aleph_0$. The following notations will facilitate the reading: " $E_{(o)}$" will stand for a space spanned by a denumerable sympletic basis, i.e , $E_{(0)} = \overset{\infty}{\underset{i=1}{\bigoplus}} P_i$, P_i a hyperbolic plane. Similarly, " $E_{(\alpha)}$" will stand for a space spanned by a denumerably infinite orthogonal basis, each basis vector of which has the same length $\alpha \neq 0$. Finally a 1-dimensional space $k(e)$ with $\|e\| = \alpha (\neq 0)$ will be referred to as $\Delta_{(\alpha)}$ and similarly $k(e_1, e_2)$ with $(e_1, e_2) = 0$, $\|e_1\| = \alpha (\neq 0)$, $\|e_2\| = \beta$ $(\neq 0)$ as $\Delta_{(\alpha\beta)}$.

CHAPTER III

FIELDS

In this short section we list a few elementary facts about fields of characteristic 2. First let k be any commutative field, k^* be the multiplicative group of k, and k^{*2} the subgroup of squares. g_k will denote the multiplicative group k^*/k^{*2} of non-zero elements modulo squares. If $\alpha, \beta \in k$, we write $\alpha \sim \beta$ if there exists $\gamma \in k$ such that $\alpha = \gamma^2\beta$. The order of g_k will be represented by $O(g_k)$. g_k can be finite or infinite, if it is finite, it is a power of 2 since every element of g_k is of order 2. Provided that $\mathrm{char}\ k \neq 2$ there exists for every natural n fields satisfying $O(g_k) = 2^n$. (Examples for $n = 0, 1, 2, 3$ are: The fields of complex numbers, real numbers, p-adic numbers with $p \neq 2$, 2-adic numbers respectively. For k the field of rations g_k is already infinite.) A detailed investigation of such fields is given in (1). If, on the other hand, $\mathrm{char}\ k = 2$, then the subset k^2 of squares (including zero) is a subfield of k; i.e., k is a k^2 vector space. The elements of g_k correspond to the straight lines through the origin of this vector space. More precisely, we have the

Lemma 3.1: If $\mathrm{char}\ k = 2$ then either $O(g_k) = 1$ or else $O(g_k) = \mathrm{Card}\ k$.

Proof: If $O(g_k) > 1$, there exists α and $\beta \in k^*$ such that $\alpha \not\sim \beta$. Let $S = \{\alpha + x^2\beta \mid x \in k\}$, then $\mathrm{Card}(S) = \mathrm{card}(k)$. For $\alpha + x^2\beta = \alpha + y^2\beta$ if and only if $(x-y)^2\beta = 0$; i.e., $x = y$ since $\mathrm{char}\ k = 2$. Suppose $\alpha + x^2\beta = \sigma^2(\alpha + y^2\beta)$ with $x \neq y$, then $\sigma \neq 1$ and $\alpha = \left(\frac{\sigma y + x}{1+\sigma}\right)^2\beta$ which contradicts the choice of α and β. Thus $\mathrm{card}(S) \leq O(g_k)$ and hence the assertion.

The fields with $O(g_k) = 1$ $(\mathrm{char}\ k = 2)$ are precisely the perfect fields. In particular, $O(g_k) = 1$ for finite k (if $\mathrm{char}\ k \neq 2$ then $O(g_k) = 2$ for finite k). The degree $[k : k^2]$ may be finite or infinite. If it is finite, it necessarily is a power of 2.

Lemma 3.2: Let $\mathrm{Char}\, k = 2$ then for some natural number n, $[k : k^2] = 2^n$ provided that $[k : k^2]$ is finite.

Proof: Let $S = \{\theta_1, \cdots, \theta_n\}$ be a finite set of independent algebraic elements over k^2 such that $k = k^2(\theta_1, \cdots, \theta_n)$. Since $x^2 - \theta_i^2$ is irreducible over k^2 ($x^2 - \theta_i^2 = (x-\theta_i)^2$ and $\theta_i \notin k^2$) we have $[k^2(\theta_i) : k^2] = 2$. Successive adjunctions give $[k : k^2] = 2^n$.

The fact that $[k : k^2]$ is invariant under a finite algebraic extension is given in:

Lemma 3.3: If $\mathrm{Char}\, k = 2$, $[k : k^2]$ finite, and $\bar{k}$ a finite extension of k, then $[\bar{k} : \bar{k}^2] = [k : k^2]$.

Proof: It is sufficient to give the proof for a simple extnesion $\bar{k} = k(\theta)$. Let $[k(\theta) : k] = n+1$, then $1, \theta, \theta^2, \cdots, \theta^n$ are linearly independent and form a basis of $k(\theta)$ over k. We have $k^2 \subset k(\theta)^2 \subset k(\theta)$ and $k^2 \subset k \subset k(\theta)$, hence $[k(\theta) : k(\theta)^2][k(\theta)^2 : k^2] = [k(\theta) : k][k : k^2]$. We will show that $[k(\theta)^2 : k^2] = [k(\theta) : k]$. If $x \in k(\theta)^2$, then $x = (\alpha_0 + \alpha_1\theta + \cdots + \alpha_n\theta^n)^2$ $= \alpha_0^2 + \alpha_1^2\theta^2 + \cdots + \alpha_n^2\theta^{2n}$, $\alpha_i \in k$. Thus $k(\theta)^2 \subset k^2(\theta^2)$ and hence $k(\theta)^2 = k^2(\theta^2)$. It is clear that $1, \theta^2, \cdots, \theta^{2n}$ are linearly independent over k^2: For suppose $\alpha_0^2 + \alpha_1^2\theta^2 + \cdots + \alpha_n^2\theta^{2n} = 0$, hence $(\alpha_0 + \alpha_1\theta + \cdots + \alpha_n\theta^n)^2 = 0$ and $\alpha_0 + \alpha_1\theta + \cdots \alpha_n\theta^n = 0$. Therefore $\alpha_i = 0$, $i = 1, 2, \cdots, n$.

Corollary 3.1: If $\mathrm{Char}\, k = 2$, S is an infinite set of algebraic elements over k, and $[k : k^2]$ is finite, then $[k(S) : k(S)^2] \leq [k : k^2]$.

Proof: Let $[k : k^2] = \ell$ and suppose $[k(S) : k(S)^2] > \ell$. Select any $\ell+1$ linearly independent elements of $k(S)$ over $k(S)^2$, say $S_0 = \{\theta_1, \cdots, \theta_{\ell+1}\}$. Consider $k(S_0)$. $\theta_1, \cdots, \theta_{\ell+1}$ are independent over $k(S)^2$, thus independent over $k(S_0)^2$. But by Lemma 2.3 $[k(S_0) : k(S_0)^2] = \ell$, whence a contradiction, therefore $[k(S) : k(S)^2] \leq \ell$.

For the above corollary, we remark that we cannot have equality in general as is witnessed by the transition from k to its algebraic closure $\bar{k}$ (for which $[\bar{k}:\bar{k}^2] = O(g_k) = 1$).

Lemma 3.2 yields $[k:k^2] = 2^n$ if $[k:k^2]$ is finite and $\operatorname{char} k = 2$. How does one actually obtain such a field? This question is answered in the following Lemma:

Lemma 3.4: If $[k:k^2]$ is finite, and $x_1, \cdots, x_n$ are independent transcendentals over k, then $[K:K^2] = [k:k^2]\, 2^n$ where $K = k(x_1, \cdots, x_n)$.

Proof: Let $\{\beta_1, \cdots, \beta_m\}$ be a basis of k over k^2. Define $x_j^0 = 1$, we will show that $\{\beta_i x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}\}$, $i = 1, 2, \cdots, m$, $\epsilon_i = 0$ or 1, is a basis of K over K^2. First we note that the above set has $m\,2^n$ elements. Secondly, we must show that the elements are linearly independent over K^2. This will be effected by induction on the number of transcendental variables $x_1, \cdots, x_n$. For zero variables we have by assumption $\{\beta_1, \cdots, \beta_m\}$ as a basis, thus they are linearly independent. Assume the assertion true for the number of variables less than n. Suppose $\sum f_i(x_1, \cdots, x_n)^2 \beta_j x_1^{\epsilon_1} \cdots x_n^{\epsilon_n} = 0$ and that some $f_i(x_1, \cdots, x_n)^2 \neq 0$. If $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_n = 0$, then we have $\sum f_i(x_1, \cdots, x_n)^2 = 0$. If all $f_i \in k$, then the zero variable case is contradicted. Let $f_i \neq 0$ be such that $f_i \notin k$, then some positive power of an x_j must occur in f_i, say x_n. Without loss of generality we may assume not all $f_i \neq 0$ have a factor x_n. Let $x_n = 0$, then $\sum f_i(x_1, \cdots, x_{n-1}, 0)^2 \beta_i = 0$ and by the induction assumption each $f_i(x_1, \cdots, x_{n-1}, 0)^2 \beta_i = 0$. This implies that x_n is a root of each $f_i \neq 0$ and thus a common factor of each f_i which is a contradiction. Now suppose that for some term $f_i(x_1, \cdots, x_n)^2 \beta_j x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$, $f_i \neq 0$ and an $\epsilon_i \neq 0$, say $\epsilon_n \neq 0$. Again we assume that not all f_i for which $f_i \neq 0$ have a common factor x_n. Split the sum into two parts; i.e., $\sum f_i(x_1, \cdots, x_n)^2 \beta_j x_1^{\epsilon_1} \cdots x_n^{\epsilon_n} = \sum f_1(x_1, \cdots x_n)^2 \beta_j x_1^{\epsilon_1} \cdots x_{n-1}^{\epsilon_{n-1}} x_n + \sum f_2(x_1, \cdots, x_n) \beta_j x_1^{\epsilon_1} \cdots x_{n-1}^{\epsilon_{n-1}}$. If all $f_2 = 0$, then dividing by x_n we have $\sum f_1(x_1, \cdots, x_n)^2 \beta_j x_1^{\epsilon_1} \cdots x_{n-1}^{\epsilon_{n-1}} = 0$. Let $x_n = 0$, by the induction hypothesis all $f_i(x_1, \cdots, x_{n-1}, 0) = 0$, and thus x_n is a factor of all f_1 which is a contradiction. If some $f_2 \neq 0$, then let $x_n = 0$ and we obtain $\sum f_2(x_1, \dots, x_{n-1}, 0)^2 \beta_j x_1^{\epsilon_1} \cdots x_{n-1}^{\epsilon_{n-1}} = 0$. By the induction assumptions $f_2(x_1, \dots, x_{n-1}, 0) = 0$ for all f_2 which implies x_n is a factor of each f_2. Let $f_2(x_1, \dots, x_n) = x_n f_2'(x_1, \dots, x_n)$, then $\sum f_i(x_1, \dots, x_n)^2 \beta_j x_1^{\epsilon_1} \cdots x_{n-1}^{\epsilon_{n-1}} x_n + \sum x_n^2 f_2'(x_1, \dots, x_n)^2 \beta_j x_1^{\epsilon_1} \cdots x_{n-1}^{\epsilon_{n-1}} = 0$. If all $f_1 = 0$ then we have a contradiction. If some $f_1 \neq 0$, let $x_n = 0$, then $\sum f_1(x_1, \dots, x_{n-1}, 0)^2 \beta_j x_1^{\epsilon_1} \cdots x_{n-1}^{\epsilon_{n-1}} = 0$. By the same reasoning as above we have x_n a factor of all f_1. But this contradicts the assumption that the f_i's do not have a common factor x_n, thus all $f_i = 0$ and we have linear independence. Finally it must be shown that the set $\{\beta_i x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}\}$ spans the K^2-space K. Every element of K is of the form $h(x_1, \dots, x_n)/g(x_1, \dots, x_n)$ where h and g are polynomials with coefficients in k, but $h/g = hg(g^{-1})^2$ thus we only need to consider polynomials. Let $P(x_1, \dots, x_n) = \sum a_{\alpha_1 \dots \alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be an arbitrary element of K. Consider $a_{\alpha_1 \dots \alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Let $a_{\alpha_1 \dots \alpha_n} = d_1^2 \beta_1 + \dots + d_m^2 \beta_m$ and decompose $x_1^{\alpha_1} \cdots x_n^{\alpha_n} = x_1^{\epsilon_1} \cdots x_n^{\epsilon_n} (x_1^{2r_1} \cdots x_n^{2r_n})$ such that $0 \leq \epsilon_i \leq 1$ and $\epsilon_i + 2r_i = \alpha_i$. It follows that $a_{\alpha_1 \dots \alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n} = \sum_{i=1}^{m} \beta_i x_1^{\epsilon_1} \cdots x_n^{\epsilon_n} (d_i x_1^{r_1} \cdots x_n^{r_n})^2$.

As an example we mention algebraic function fields K of finitely many variables, i.e., finite extensions of rational function fields $k(x_1, \dots, x_n)$ (the x_i's independent transcendentals over k), $K = k(x_1, \dots, x_n)(\omega_1, \dots, \omega_n)$. Let k^* denote the algebraic closure of k in K (the "field of constants"). From general field theory we know that any intermediate field K_1 between k and K is also finitely generated over k. Since $k \subset k^* \subset K$ it follows that k^* is finitely generated over k, but each element of k^* is algebraic over k, hence $[k^* : k]$ is finite. By Lemma 3.3 we have $[k : k^2] = [k^* : k^{*2}]$. This gives us:

Corollary 3.2: Let $[k : k^2]$ be finite, K an algebraic function field of n variables over k and k^* the field of constants in K. Then $[K : K^2] = [k : k^2]2^n = [k^* : k^{*2}]2^n$.

CHAPTER IV

CLASSIFICATION OF k-SPACES (E,ϕ) IN THE CASE OF COUNTABLE DIMENSION

Let ϕ be a non-degenerate bilinear form, $\phi: E\times E \longrightarrow k$, where E is a k-vector space of denumerable dimension. In the following we will classify all such spaces up to orthogonal isomorphisms for certain fields k. In the special case with $O(g_k) = [k:k^2] = 1$, this has been carried out by Kaplansky in (1): For such k we have up to orthogonal isomorphisms precisely 4 non-isomorphic spaces, namely $E_{(0)}$, $E_{(\alpha)}$, $E_{(0)} \oplus \Delta_{(1,1)}$, and $E_{(0)} \oplus \Delta_{(1)}$.

The fields for which we shall carry out the classification are those with $[k:k^2] \leq 2$ (see the examples in section 3).

Let (E,ϕ) be a space over such a field. The set E_* of all vectors $x \in E$ of length $\|x\| = 0$ is a subspace of E (this is precisely the subspace of vectors whose length $\|x\|$ is a "trace" if ϕ is considered a sesquilinear form. For this concept see, for example, La geometrie des groups classiques by J. Dieudonne, Berlin 1955, page 19.) Clearly E_* is an invariant subspace of E (invariant under orthogonal automorphisms of E). An algebraic complement of E_* in E will always be called L in the sequel:

$$E = E_* \oplus L$$

Since $\dim L = \dim E/E_*$, the dimension of L is invariant also. L contains no isotropic vectors, hence it possesses an orthogonal basis, and by the same token, $\dim L \leq [k:k^2] \leq 2$.

Let, for a fixed L, K be an algebraic complement of $E_* \cap L^\perp$ in E_*,

$$E = (E_* \cap L^\perp) \oplus K \oplus L \ , \quad E_* = (E_* \cap L^\perp) \oplus K \qquad (1)$$

The following investigations will be based throughout on such a decomposition (1) of E. We shall now convince ourselves, that $\dim K$ does not depend on the particular L chosen, more precisely we prove:

Lemma 4.1: Let (E,ϕ) be an arbitrary semi-simple k-vector space. If K and L are as described above then

$$\dim K + \dim (E_* + E_*^{\perp}/E_*^{\perp}) = \dim L$$

Proof: Let G be an algebraic complement of $E_* + E_*^{\perp}$ in E, and H an algebraic complement of E_* in $E_* + E_*^{\perp}$. Thus $E = E_* \oplus H \oplus G$ and $\dim H = \dim (E_* + E_*^{\perp}/E_*)$. It is sufficient to show $\dim K = \dim G$ for this implies $\dim K + \dim H = \dim G + \dim H = \dim (G \oplus H)$. Since $L = G \oplus H$ and $E_* \subset H^{\perp}$, we have $E_* \cap L^{\perp} = E_* \cap (G \oplus H)^{\perp} = E_* \cap H^{\perp} \cap G^{\perp} = E_* \cap G^{\perp}$. Now using the fact $E_* \cap L^{\perp} = E_* \cap G^{\perp}$, we obtain the chain of equalities:

$$\dim K = \dim (E_*/E_* \cap L^{\perp}) = \dim (E_*/E_* \cap G^{\perp}) = \dim (E_* + G^{\perp}/G^{\perp}).$$

$E_* + G^{\perp}$ is closed (see section 2.1) and $E_*^{\perp} \cap G = (0)$ by construction, hence one obtains $E_* + G^{\perp} = (E_* + G^{\perp})^{\perp\perp} = (E_*^{\perp} \cap G^{\perp\perp})^{\perp} = (E_*^{\perp} \cap G)^{\perp} = (0)^{\perp} = E$.
Thus

$$\dim K = \dim (E_* + G^{\perp}/G^{\perp}) = \dim (E/G^{\perp}) = \dim G.$$

From (1) we see that $(E_* \cap L^{\perp}) \cap \operatorname{rad} E_*$ is orthogonal to E, hence, as E is semi-simple $(E_* \cap L^{\perp}) \cap \operatorname{rad} E_* = (0)$. In other words, we can always choose K in such a way that $\operatorname{rad} E_* \subset K$. In the following discussion V will always be the space $E_* \cap L^{\perp}$ (for some choice of L) and K will be the algebraic complement of V in E_*.

In the following classification, we use the fact that E_*, $E_*^{\perp}$, and $\operatorname{rad} E_*$ are invariant subspaces of E. The scheme for dividing our investigation into cases is as follows: The first division is according to the $\dim L$; i.e., three cases: $\dim L = 0, 1, \text{or } 2$. Then in each of these cases we apply Lemma 4.1 and further subdivide by the allowable values of $\dim E_*^{\perp}$ and $\dim (\operatorname{rad} E_*)$. This yields 10 different cases; for a summary, see page 21.

We now proceed with the classification:

Case 1: $\dim E/E_* = 0$. We have $E = E_*$ and ϕ is alternate. Since E is of countable dimension, E possesses a sympletic basis, thus $E = E_{(0)}$.

Case 2.1: $\dim E/E_* = 1$, $\dim E_*^{\perp} = 0$, $\dim (\operatorname{rad} E_*) = 0$.

Let L be any algebraic complement of E_* in E, $V = E_* \cap L^{\perp}$ and k some algebraic complement of V in E_*, then $E_* = V \oplus k$ and $E = V \oplus k \oplus L$. By Lemma 4.1 $k = k(\omega)$ and $\omega \not\perp V$ since $rad\ E_* = (0)$. Let L be spanned by the vector ℓ.

We need the fact that the orthogonal complement $V^{\perp E_*}$ of V in E_* is (0). Suppose there is a non-zero $y \in E_*$ such that $y \perp V$. Let α and $\beta \neq 0$ be such that $(\alpha \ell + \beta y, \omega) = 0$. Thus $(\alpha \ell + \beta y) \perp E_*$, a contradiction to $dim\ E_*^{\perp} = (0)$.

Since $dim\ L = 1$ the range of the norm form is a single class in g_k. We now show that any two spaces falling in our case are isomorphic if and only if the range of the respective norm forms coincide. One half of the assertion is trivial. Let us then assume that E and $\bar{E}$ are spaces falling in our case, and that the ranges of the norm forms coincide. Thus if $\bar{\omega}$ and $\bar{\ell}$ are analogous objects as above, we may assume that $\|\ell\| = \|\bar{\ell}\|$ in the decomposition:

$$\begin{aligned} E &= V \oplus k(\omega) \oplus k(\ell), \quad \ell \perp V, \text{ and} \\ \bar{E} &= \bar{V} \oplus k(\bar{\omega}) \oplus k(\bar{\ell}), \quad \bar{\ell} \perp \bar{V}. \end{aligned} \tag{1}$$

In general we shall have $(\ell, \omega) \neq (\bar{\ell}, \bar{\omega})$. We therefore replace $\bar{\ell}$ by $\bar{\ell} + \gamma t$ where t is some vector in $\bar{V}$ with $(t, \bar{\omega}) \neq 0$ ($\bar{V}^{\perp \bar{E}_*} = (0)$ as we have seen). Thus we can achieve that $(\bar{\ell}, \bar{\omega}) = (\ell, \omega)$ with $\|\ell\| = \|\bar{\ell}\|$ because $\|t\| = 0$. Let $\bar{\bar{V}} = \bar{E}_* \cap k(\bar{\ell})^{\perp}$. Since $(\ell, \omega) \neq 0$ we have $\bar{\omega} \notin \bar{\bar{V}}$; i.e., $\bar{\omega}$ still spans a complement of $\bar{\bar{V}}$ in $\bar{E}_*$. We now have the following decomposition:

$$\begin{aligned} \bar{E} &= \bar{\bar{V}} \oplus k(\bar{\omega}) \oplus k(\bar{\ell}), \text{ and} \\ \|\ell\| &= \|\bar{\ell}\|, \quad \bar{\ell} \perp \bar{\bar{V}}, \quad (\bar{\ell}, \bar{\omega}) = (\ell, \omega). \end{aligned} \tag{2}$$

E_* and $\bar{E}_*$ are trivially semi-simple in our case. Furthermore $V^{\perp E_*} = \bar{\bar{V}}^{\perp \bar{E}_*} = (0)$ as we have seen. Hence by 2.4 there exists an isomorphism $T_*: E_* \longrightarrow \bar{E}_*$ with $T_*(V) = \bar{\bar{V}}$, $T(\omega) = \bar{\omega}$. In view of (2) we can extend T_* to an isomorphism $T: E \rightarrow \bar{E}$ simply by sending ℓ into $\bar{\ell}$. This proves the assertion.

If $\|\ell\| = \alpha$, then the space $E(\alpha)$ falls into our case as is readily verified. We have therefore found all non-isomorphic spaces which fall into the present case:

If $[k:k^2]=1$ there is only one type and $E_{(1)}$ is a representative.

If $[k:k^2]>1$ the spaces $E_{(\alpha)}$ where α runs through a set of representatives of g_k are non-isomorphic and they represent all spaces up to isomorphism.

Case 2.2: $\dim E/E_* = 1$, $\dim E_*^{\perp} = 1$, $\dim(\operatorname{rad} E_*) = 0$. In this case we simply have $E = E_* \oplus E_*^{\perp}$. E_* is therefore semi-simple and possesses a sympletic basis, thus $E_* = E_{(0)}$. If $\alpha = \|\ell\|$ where ℓ spans $E_*^{\perp}$, then E is completely determined by the orthogonal sum $E = E_{(0)} \oplus k(\ell)$. Two spaces E and $\bar{E}$ in this case are isomorphic if and only if $E_*^{\perp}$ and $\bar{E}_*^{\perp}$ are isomorphic. Thus:

If $[k:k^2] = 1$ there is only one type, namely $E_{(0)} \overset{\perp}{\oplus} \Delta_{(1)}$.

If $[k:k^2] > 1$ the spaces $E_{(0)} \overset{\perp}{\oplus} \Delta_{(\alpha)}$ where α runs through a set of representatives of g_k are non-isomorphic and they represent all spaces up to isomorphism.

Case 2.3: $\dim E/E_* = 1$, $\dim E_*^{\perp} = 1$, $\dim(\operatorname{rad} E_*) = 1$. Upon decomposition we have $E = V \oplus K \oplus L$ where by Lemma 4.1 $K = k(\omega)$, $L = k(\ell)$, and $\omega \perp E_*$. V is semi-simple, for if not, there exists $x(\neq 0) \in V$ such that $x \perp V$, but then $x \perp K \oplus L$ which implies $x \perp E$. This contradicts the assumption that E is semi-simple. V has a sympletic basis, therefore $V \cong E_{(0)}$. Now $(\omega, \ell) \neq 0$, however we can find $\lambda \neq 0$ such that $(\ell, \ell + \lambda\omega) = 0$, thus $K \oplus L = \Delta_{(\alpha\alpha)}$ where $\alpha = \|\ell\|$. E now has the form $E = E_{(0)} \overset{\perp}{\oplus} \Delta_{(\alpha\alpha)}$.

Again in this case the range of the norm form is a single class in g_k. We show that any two spaces E and $\bar{E}$ in this case are isomorphic if and only if the ranges of their respective norm forms coincide. If the ranges of the norm forms agree we simply map basis vectors. If E and $\bar{E}$ are isomorphic with decompositions $E = E_{(0)} \overset{\perp}{\oplus} \Delta_{(\alpha\alpha)}$ and $\bar{E} = \bar{E}_{(0)} \overset{\perp}{\oplus} \Delta_{(\beta\beta)}$, then clearly $\alpha \sim \beta$. Thus:

If $[k:k^2] = 1$ there is only one type, namely $E_{(0)} \overset{\perp}{\oplus} \Delta_{(11)}$.

If $[k:k^2] > 1$ the spaces $E_{(0)} \overset{\perp}{\oplus} \Delta_{(\alpha\alpha)}$ where α runs through a set of representatives of $\mathcal{G}_k$ are non-isomorphic and they represent all spaces up to isomorphism.

Case 3.1: $\dim E/E_* = 2$; $\dim E_*^{\perp} = 0$, $\dim(\mathrm{rad}\, E_*) = 0$. Decomposing E and applying Lemma 4.1 we obtain $E = V \oplus K \oplus L$ with $K = k(\omega_1, \omega_2)$ and $L = k(\ell_1, \ell_2)$. In this case $V^{\perp} \cap K = (0)$. For suppose there exists $x\ (\neq 0) \in K$ such that $x \perp V$. Completing x to a basis of K we have $K = k(x, y)$. If $(x, y) = 0$, then $x \perp E_*$ which contradicts $\dim E_*^{\perp} = (0)$. If $(x, y) \neq 0$, let $\bar{\ell}$ be a vector in L such that $(\bar{\ell}, x) = 0$ and choose λ for which $(\bar{\ell} + \lambda x, y) = 0$. Clearly $(\bar{\ell} + \lambda x) \perp E_*$ contradicting $\dim E_*^{\perp} = (0)$.

We now proceed to give a normal form for E . Let ℓ_1' and ℓ_2' be any orthogonal basis of L ; i.e., $L = k(\ell_1', \ell_2')$. Since $\mathrm{codim}\, k(\ell_1')^{\perp} = 1$ there exists $\omega_1' \in K$ such that $(\omega_1', \ell_1') = 0$. Let $k(\omega)$ be an algebraic complement of $k(\omega_1')$ in K . If $(\omega_1', \omega) \neq 0$, then select $x_0 \in V$ with $(\omega_1', x_0) \neq 0$. For suitable λ we form $\bar{\omega} = \omega + \lambda x_0$ for which $(\bar{\omega}, \omega_1') = 0$. If $(\ell_2, \bar{\omega}) = 0$, then let $\bar{\omega} = \omega_2'$, if not select μ such that $\omega_2' = \omega_1' + \mu \bar{\omega}$ and $(\omega_2', \ell_2) = 0$. Now multiplying ω_1' and ω_2' by suitable scalars γ and δ we obtain $(\ell_1', \omega_2'') = (\ell_2', \omega_1'') = 1$ with $\omega_2'' = \delta \omega_2'$ and $\omega_1'' = \gamma \omega_1'$.

Dropping the primes and summarizing we have:

$$E = V \oplus k(\omega_1, \omega_2) \oplus k(\ell_1, \ell_2),\ \|\omega_1\| = \|\omega_2\| = 0,\ \|\ell_1\| = \alpha\ (\neq 0)$$
$$\|\ell_2\| = \beta\ (\neq 0),\ \alpha \not\sim \beta,\ (\omega_1, \omega_2) = (\ell_1, \ell_2) = (\omega_1, \ell_1) = (\omega_2, \ell_2) = 0,\ (\omega_1, \ell_2) = (\omega_2, \ell_1) = 1 \qquad (3)$$

In this case we have that any two spaces are isomorphic. To see this we proceed as follows: Let E and $\bar{E}$ be any two spaces of this case in normal form (3); i.e., E is as in (3) and $\bar{E}$ has the decompositions $\bar{E} = \bar{V} \oplus \bar{K} \oplus \bar{L}$, $\bar{K} = k(\bar{\omega}_1, \bar{\omega}_2)$, $L = k(\bar{\ell}_1, \bar{\ell}_2)$, $\|\bar{\ell}_1\| = \delta\ (\neq 0)$, $\|\ell_2\| = \gamma\ (\neq 0)$, $\delta \not\sim \gamma$ and the basis elements satisfying the analogues of conditions (3). Next we show that E can be written in normal form as $E = V' \oplus K' \oplus \Delta(\delta\gamma)$. Since $[k : k^2] = 2$, there exists a_1, b_1, a_2 and b_2 such that $a_1^2 \alpha + b_1^2 \beta = \delta$ and $a_2^2 \alpha + b_2^2 \beta = \gamma$. Let $\ell_1' = a_1 \ell_1 + b_1 \ell_2$ and $\ell_2' = a_2 \ell_1 + b_2 \ell_2 + \mu \omega$, where μ is chosen so that $(\ell_1', \ell_2') = 0$.

Call $L' = k(\ell_1', \ell_2')$ and $V' = L'^{\perp} \cap E_*$, then $E = V' \oplus K' \oplus L'$. By the preceding, we may put this in normal form without changing ℓ_1' and ℓ_2'.

In order to establish the isomorphism we need that E_* is semi-simple and $V^{\perp E_*} = (o)$. ($\overline{E}_*$ semi-simple and $\overline{V}^{\perp \overline{E}_*} = (o)$). That E_* is semi-simple is seen by noting that $\mathrm{rad}\, E_* = (o)$ in this case. Also since $\mathrm{rad}\, E_* = (o)$ we have $V^{\perp E_*} \subset V$. Suppose $V^{\perp E_*} \neq 0$, then there exists $x_0 \in V$ such that $x_0 \perp V$ and $x_0 \perp K$ since E_* is semi-simple. Further suppose $x_0 \not\perp \omega_1$, then for a suitable λ we will have $\overline{\ell}_2 = \ell_2 + \lambda x_0$ with $(\overline{\ell}_2, \omega_1) = 0$. Let $L' = k(\ell_1, \overline{\ell}_2)$, then $L'^{\perp} \cap E_* = V \oplus k(\omega_2)$ which implies $\dim K \leq 1$. This is a contradiction for this case.

Since E_* and $\overline{E}_*$ are semi-simple and have sympletic bases, we conclude $E_* = E_{(0)}$ and $\overline{E}_* = \overline{E}_{(0)}$ which implies $E_* \cong \overline{E}_*$. By (2.4) there exists an isomorphism $T_* : E_* \longrightarrow \overline{E}_*$ such that $T(V) = \overline{V}$, $T(\omega_1) = \overline{\omega}_1$ and $T(\omega_2) = \overline{\omega}_2$. In view of the normal form (3) we can extend T_* to an isomorphism $T : E \longrightarrow \overline{E}$ by $T/_{E_*} = T_*$ and $T(\ell_1) = \overline{\ell}_1$, $T(\ell_2) = \overline{\ell}_2$. Therefore $E \cong \overline{E}$.

One can readily verify that the space $E = E_{(\alpha)} \overset{\perp}{\oplus} E_{(\beta)}$, $\alpha \not\sim \beta$ is of this case and thus can be considered as a representative.

<u>Case 3.2:</u> $\dim E/E_* = 2$; $\dim E_*^{\perp} = 1$, $\dim(\mathrm{rad}\, E_*) = 0$. Decomposition of E yields $E = V \oplus K \oplus L$ with $K = k(\omega)$ and $L = k(\ell_1, \ell_2)$. By application of Lemma 4.1 we have $\ell_1 \perp E_*$ for suitable ℓ_1. Since L admits an orthogonal basis, ℓ_2 can be chosen perpendicular to ℓ_1. Moreover, multiplication of ω by a suitable scalar will yield $(\omega', \ell_2) = 1$ with $\omega' = \alpha\omega$. In the following we will delete the prime.

In this case we will show that two spaces $E = V \oplus K \oplus L$ and $\overline{E} = \overline{V} \oplus \overline{K} \oplus \overline{L}$ are isomorphic if and only if $L \cong \overline{L}$.

Let T be an isomorphism $T : E \longrightarrow \overline{E}$ of two spaces in this case with decompositions (normalized as above) $E = V \oplus k(\omega) \oplus k(\ell_1, \ell_2)$, $\ell_1 \perp E_*$, $\overline{E} = \overline{V} \oplus k(\overline{\omega}) \oplus k(\overline{\ell}_1, \overline{\ell}_2)$, and $\overline{\ell}_1 \perp \overline{E}_*$. Since

$T(E_*) = \bar{E}_*$ we have $T(E_*^{\perp}) = \bar{E}_*^{\perp}$, thus $T(\ell_1) = \bar{\ell}_1$ because $E_*^{\perp} = k(\ell_1)$ and $\bar{E}_*^{\perp} = k(\bar{\ell}_1)$. It follows that $\|\ell_1\| \sim \|\bar{\ell}_2\|$. Also $T(E_*^{\perp\perp}) = \bar{E}_*^{\perp\perp}$ and we find $E_*^{\perp\perp} = E_* \oplus k(\ell_1)$ and $\bar{E}_*^{\perp\perp} = \bar{E}_* \oplus k(\bar{\ell}_1)$. Again the range of the norm form must be equal on $E_*^{\perp\perp}$ and $\bar{E}_*^{\perp\perp}$ respectively hence $\|\ell_2\| \sim \|\bar{\ell}_2\|$. Since ℓ_1, ℓ_2, and $\bar{\ell}_1$, $\bar{\ell}_2$ are orthogonal bases, we have therefore $L \cong \bar{L}$.

Before proceeding with the second half of the proof, let us show that for any space E which falls in our case, E_* is semi-simple and $V^{\perp E_*} = (0)$. $\dim(\operatorname{rad} E_*) = 0$ thus E_* is semi-simple. Now, no algebraic complement $k(\omega)$ of V in E_* can be perpendicular to V (otherwise $\omega \in \operatorname{rad} E_*$) hence $V^{\perp E_*} \subset V$. Suppose $V^{\perp E_*} \neq (0)$, then there exists $x_0 \in V$ such that $x_0 \perp V$. Since $(x_0, \omega) \neq 0$ (otherwise $x_0 \perp E_*$ and $x_0 \perp L$ would imply $x_0 \perp E$) we can find $d \neq 0$ such that $\bar{\ell}_2 = \ell_2 + d x_0$ and $(\bar{\ell}_2, \omega) = 0$. Clearly $\bar{\ell}_2 \perp E_*$ and since $\ell_1 \perp E_*$ we would have $\dim E_*^{\perp} = 2$ which is a contradiction to $\dim E_*^{\perp} = 1$. Thus $V^{\perp E_*} = (0)$.

Let E and $\bar{E}$ be two spaces of this type with normalized decompositions: $E = V \oplus k(\omega) \oplus k(\ell_1, \ell_2)$, $\bar{E} = \bar{V} \oplus k(\bar{\omega}) \oplus k(\bar{\ell}_1, \bar{\ell}_2)$, and $\ell_1 \perp E_*$, $\bar{\ell}_1 \perp \bar{E}_*$, $(\ell_1, \omega) = (\bar{\ell}_1, \bar{\omega}) = 1$, $(\ell_1, \ell_2) = (\bar{\ell}_1, \bar{\ell}_2) = 0$, (4)
$\|\ell_1\| = \|\bar{\ell}_1\|$, $\|\ell_2\| = \|\bar{\ell}_2\|$.

E_* and $\bar{E}_*$ are semi-simple and both have sympletic bases thus $E_* \cong E_{(0)}$ and $\bar{E}_* \cong \bar{E}_{(0)}$ whence $E_* \cong \bar{E}_*$. By (2.4) there exists an isomorphism $T_*: E_* \longrightarrow \bar{E}_*$ with $T_*(V) = \bar{V}$ and $T_*(\omega) = \bar{\omega}$. From (4) we can extend T_* to an isomorphism $T: E \longrightarrow \bar{E}$ by $T/E_* = T_*$, $T(\ell_1) = \bar{\ell}_1$ and $T(\ell_2) = \bar{\ell}_2$.

An example of a space of this type is $E = E(\alpha) \overset{\perp}{\oplus} E(\beta)$, $\alpha \not\sim \beta$. Also we can generate all non-isomorphic types be letting α and β run through a set of representatives of $\mathcal{G}_k$ such that $\alpha \not\sim \beta$.

<u>Case 3.3</u>: $\dim E/E_* = 2$, $\dim E_*^{\perp} = 1$, $\dim(\operatorname{rad} E_*) = 1$. By (4.1) we have $E = V \oplus K \oplus L$ with $K = k(\omega_1, \omega_2)$ and $L = k(\ell_1, \ell_2)$, $\|\ell_1\| \not\sim \|\ell_2\|$. In this case we have $\dim(V^{\perp} \cap K) = 1$. Since $\dim(\operatorname{rad} E_*) = 1$ there exists $\omega_1' \in K$ such that $\omega_1' \perp E_*$. Let ω_1'

and ω_2' be a basis of K ; i.e., $K = k(\omega_1', \omega_2')$. Suppose $\omega_2' \perp V$, if $\omega_2' \perp \omega_1'$ then $\omega_2' \perp E_*$ which contradicts $\dim(\mathrm{rad}\, E_*) = 1$. Assume $(\omega_2', \omega_1') \neq 0$ then let $\ell_3 \in L$ be such that $(\ell_3, \omega_1') = 0$. For a suitable λ we have $\overline{\ell_3} = \ell_3 + \lambda\omega_1'$ such that $(\overline{\ell_3}, \omega_2') = 0$. Thus $\overline{\ell_3} \perp E_*$ which contradicts $\dim E_*^{\perp} = 1$. Hence by dropping the primes we have $\omega_1 \perp E_*$ and $\omega_2 \not\perp V$. Moreover as in case 3.1 we can assume that the decomposition is normalized, i.e., $(\omega_1, \omega_2) = (\ell_1, \ell_2) = 0$, $(\ell_1, \omega_1) = (\ell_2, \omega_2) = 0$ and $(\ell_1, \omega_2) = (\ell_2, \omega_1) = 1$.

Let E and $\overline{E}$ be two spaces of this type in normal form, i.e., $E = V \oplus k(\omega_1, \omega_2) \oplus k(\ell_1, \ell_2)$ and $\overline{E} = \overline{V} \oplus k(\overline{\omega}_1, \overline{\omega}_2) \oplus k(\overline{\ell_1}, \overline{\ell_2})$. We will show that $E \cong \overline{E}$ if and only if $\|\ell_1\| \sim \|\overline{\ell_1}\|$.

First suppose T is an isomorphism such that $T: E \longrightarrow \overline{E}$, then $T(E_*) = \overline{E}_*$, $T(\mathrm{rad}\, E_*) = \mathrm{rad}\, \overline{E}_*$, and $T((\mathrm{rad}\, E_*)^{\perp}) = (\mathrm{rad}\, \overline{E}_*)^{\perp}$. From our normal form we read off $(\mathrm{rad}\, E_*)^{\perp} = k(\omega_1)^{\perp} = E_* \oplus k(\ell_1)$ and $(\mathrm{rad}\, \overline{E}_*)^{\perp} = k(\overline{\omega}_1)^{\perp} = \overline{E}_* \oplus k(\overline{\ell}_1)$. The ranges of the norm form must be equal, hence $\|\ell_1\| \sim \|\overline{\ell}_1\|$.

Secondly, suppose E and $\overline{E}$ are as above with $\|\ell_1\| = \|\overline{\ell}_1\|$. In general $\|\ell_2\| \not\sim \|\overline{\ell_2}\|$, however we can rewrite E such that $E = V' \oplus k(\omega_1', \omega_2') \oplus k(\ell_1', \ell_2')$, E is in normal form and $\|\ell_2'\| = \|\overline{\ell_2}\|$. Since $\|\ell_1\| \not\sim \|\ell_2\|$ there exists a and b such that $a^2\|\ell_1\| + b^2\|\ell_2\| = \|\overline{\ell_2}\|$. Now $(\omega_2, \ell_1) \neq 0$ so we can find a λ for which $\ell_2' = a\ell_1 + b\ell_2 + \lambda\omega_2$ and $(\ell_2', \ell_1) = 0$. Let $L' = k(\ell_1, \ell_2')$ and decompose $E = V' \oplus K' \oplus L'$ and then normalize.

From the above we may assume that in E and $\overline{E}$, $\|\ell_1\| = \|\overline{\ell_1}\|$ and $\|\ell_2\| = \|\overline{\ell_2}\|$.

In order to continue we need that $F = V \oplus k(\omega_2)$ is semi-simple and $V^{\perp_F} = (0)$. First we claim $(\mathrm{rad}\, F) \cap V = (0)$. For suppose $x \in (\mathrm{rad}\, F) \cap V$ and $x \neq 0$, then $x \perp \omega$, $x \perp L$ which implies $x \perp E$. This is a contradiction since E is semi-simple. Therefore we may assume $\mathrm{rad}\, F \subset k(\omega_2)$, but $\omega_2 \not\perp V$, which implies $\mathrm{rad}\, F = (0)$. Secondly suppose $x \in V^{\perp_F}$, $x \neq 0$, then $x = v + \lambda\omega_2$ where $v \in V$. Now $(x + \lambda\omega_2, v) = 0$ implies $\lambda(\omega_2, v) = 0$ and $(v + \lambda\omega_2, \omega_2) \neq 0$ (if $(v + \lambda\omega_2, \omega_2)$ were 0 then $v + \lambda\omega_2 \in \mathrm{rad}\, F = 0$) implies

$(v, \omega_2) \neq 0$, thus $\lambda = 0$. Therefore it follows that $V^{\perp_F} \subset V$. Let $x_0\ (\neq 0) \in V^{\perp_F}$, then since $(x_0, \omega_2) \neq 0$, a suitable λ can be found such that $\ell_1' = \ell_1 + \lambda x_0$ and $(\ell_1', \omega_2) = 0$. Now consider $E = E_* \oplus L'$ where $L' = k(\ell_1', \ell_2)$. Since $\ell_1' \perp E_*$ and $\omega_1 \perp E_*$ we have $\dim E_*^{\perp} \geq 2$ which is a contradiction.

Therefore F and $\overline{F}$ have sympletic bases, in other words $F \cong \overline{F}$. Moreover since $V^{\perp_F} = \overline{V}^{\perp_{\overline{F}}} = (0)$, by (2.4) there exists an isomorphism $T_1 : F \to \overline{F}$ and $T_1(v) = \overline{v}$, $T_1(\omega_2) = \overline{\omega_2}$. We extend T_1 to an isomorphism $T : E \longrightarrow \overline{E}$ by $T/_F = T_1$, $T(\omega_1) = \overline{\omega_1}$, $T(\ell_1) = \overline{\ell_1}$, and $T(\ell_2) = \overline{\ell_2}$.

An example of this type space is $E = E(\alpha) \overset{\perp}{\oplus} \Delta(\beta\beta)$, $\alpha \not\sim \beta$. By letting α run through a set of representatives of $\mathcal{G}_k$ such that $\alpha \not\sim \beta$ we generate precisely all isomorphism types.

Case 3.4: $\dim E/_{E_*} = 2$, $\dim E_*^{\perp} = 2$, $\dim(\operatorname{rad} E_*) = 0$. By (4.1) E has the decomposition $E = V \overset{\perp}{\oplus} L$ with $V = E_*$ and $L = k(\ell_1, \ell_2)$. L is non-isotropic, hence $L = \Delta(\alpha\beta)$, $\|\ell_1\| = \alpha\ (\neq 0)$, $\|\ell_2\| = \beta\ (\neq 0)$ and $\alpha \not\sim \beta$. Furthermore since V is semi-simple E can be written as $E = E(0) \overset{\perp}{\oplus} \Delta(\alpha\beta)$.

In this case we find that two spaces $E = E(0) \overset{\perp}{\oplus} \Delta(\alpha\beta)$ and $\overline{E} = \overline{E}(0) \overset{\perp}{\oplus} \Delta(\delta\gamma)$ are isomorphic if and only if $\Delta(\alpha\beta) \cong \Delta(\delta\gamma)$.

Suppose T is an isomorphism such that $T : E \longrightarrow \overline{E}$, then $T(E_*) = \overline{E}_*$ which implies $T(E_{(0)}^{\perp}) = \overline{E}_{(0)}^{\perp}$. But $E_{(0)}^{\perp} = \Delta(\alpha\beta)$ and $\overline{E}_{(0)}^{\perp} = \Delta(\delta\gamma)$ thus $\Delta(\alpha\beta) \cong \Delta(\delta\gamma)$.

Conversely, let T_1 be an isomorphism $T_1 : \Delta(\alpha\beta) \longrightarrow \Delta(\delta\gamma)$. We can extend T_1 to an isomorphism $T : E \longrightarrow \overline{E}$ by $T/_{\Delta(\alpha\beta)} = T_1$ and $T(E_0) = \overline{E}_0$. $T/_{E_0}$ is defined by mapping basis vectors into basis vectors.

Representatives of all non-isomorphic spaces of this type can be obtained by letting α and β $(\alpha \not\sim \beta)$ run through a set of representatives of $\mathcal{G}_k$.

Case 3.5: $\dim E/_{E_*} = 2$, $\dim E_*^{\perp} = 2$, $\dim(\operatorname{rad} E_*) = 1$.

By (4.1) E has the decomposition $E = V \oplus K \oplus L$ such that $K = k(\omega)$, $L = k(\ell_1, \ell_2)$, $\|\ell_1\| \not\sim \|\ell_2\|$, and $\ell_1 \perp E_*$. Since $\dim(\mathrm{rad}\, E_*) = 1$ it follows that $\omega \perp E_*$. We have $\|\ell_1\| = \alpha$, $\|\ell_2\| = \beta$, $\alpha \not\sim \beta$, $\|\omega\| = 0$, and $(\ell_1, \ell_2) = 0$. Since $k(\omega) \oplus k(\ell_1, \ell_2)$ is semi-simple $(\ell_2, \omega) \neq 0$ because $(\ell_1, \omega) = 0$. Let $\ell_3 = \ell_2 + \frac{\alpha}{(\ell_2, \omega)} \omega$ then $\|\ell_3\| = \beta$ and $\ell_3 \perp \ell_1$ and ℓ_2. Thus $k(\omega) \oplus k(\ell_1, \ell_2)$ is of the form $\Delta(\beta\beta) \overset{\perp}{\oplus} \Delta(\alpha)$. V is semi-simple and has a sympletic basis, ergo $V \cong E_{(0)}$. E thus has the form $E = E_{(0)} \overset{\perp}{\oplus} \Delta(\beta\beta) \overset{\perp}{\oplus} \Delta(\alpha)$.

Let E and $\bar{E}$ be two spaces of this type with decompositions $E = E_{(0)} \overset{\perp}{\oplus} \Delta(\alpha\alpha) \overset{\perp}{\oplus} \Delta(\beta)$, $\|\ell_1\| = \|\ell_2\| = \alpha$, $\|\ell_3\| = \beta$ and $\bar{E} = \bar{E}_{(0)} \overset{\perp}{\oplus} \Delta(\delta\delta) \overset{\perp}{\oplus} \Delta(\gamma)$, $\|\bar{\ell}_1\| = \|\bar{\ell}_2\| = \delta$, and $\|\bar{\ell}_3\| = \gamma$. In this case we find that $E \cong \bar{E}$ if and only if $\Delta(\alpha\alpha) \overset{\perp}{\oplus} \Delta(\beta) = \Delta(\delta\delta) \overset{\perp}{\oplus} \Delta(\gamma)$

One half of the assertion is trivial. Let us assume then that T is an isomorphism such that $T : E \longrightarrow \bar{E}$. Since $E_* = E_{(0)} \oplus k(\ell_1 + \ell_2)$, $\bar{E}_* = \bar{E}_{(0)} \oplus k(\bar{\ell}_1 + \bar{\ell}_2)$ and $T(E_*^{\perp}) = \bar{E}_*^{\perp}$ it follows that $k(\ell_1 + \ell_2) \oplus k(\ell_3) \cong k(\bar{\ell}_1 + \bar{\ell}_2) \oplus k(\bar{\ell}_3)$. As before this implies $\|\ell_3\| \sim \|\bar{\ell}_3\|$, i.e., $\beta \sim \gamma$. Let us assume that $\|\bar{\ell}_3\| = \beta$. Thus we only need to show that $\Delta(\alpha\alpha) \oplus \Delta(\beta) \cong \Delta(\delta\delta) \oplus \Delta(\beta)$. Since $[k : k^2] = 2$ there exists d_1, d_2 such that $\delta = d_1^2 \alpha + d_2^2 \beta$ and $d_1 \neq 0$ since $\delta \not\sim \beta$. The desired isomorphism is obtained by the following transformation:

$$\bar{\ell}_1 = \frac{\delta}{d_1 \alpha} \ell_1 + \left(d_1 + \frac{\delta}{d_1 \alpha}\right) \ell_2 + d_2 \ell_3 \; ; \quad \bar{\ell}_2 = d_1 \ell_2 + d_2 \ell_3 ,$$

$$\bar{\ell}_3 = \frac{d_2 \beta}{d_1 \alpha} (\ell_1 + \ell_2) + \ell_3 .$$

We remark that by the above isomorphism we can conclude that the two spaces E and $\bar{E}$ are isomorphic if and only if $\Delta(\beta) \cong \Delta(\gamma)$.

<u>Case 3.6</u>: $\dim E/E_* = 2$, $\dim E_*^{\perp} = 2$, $\dim(\mathrm{rad}\, E_*) = 2$. Applying (4.2) to E we obtain $E = V \oplus K \oplus L$ with $K = k(\omega_1, \omega_2)$, $K \perp V$, and $L = k(\ell_1, \ell_2)$. As in case 3.1 we may assume $K \oplus L$ is normalized, i.e., $(\ell_1, \omega_1) = (\ell_2, \omega_2) = (\ell_1, \ell_2) = (\omega_1, \omega_2) = 0$ and $(\ell_1, \omega_2) = (\ell_2, \omega_1) = 1$. Also by a previous construction we can write $k(\omega_1, \ell_2) = \Delta(\alpha\alpha)$

and $k(\omega_2, \ell_1) = \Delta(\beta\beta)$ where $\|\ell_1\| = \beta$ and $\|\ell_2\| = \alpha$, $\alpha \not\sim \beta$. Since V is semi-simple and has a sympletic basis, $V = E_{(0)}$. Thus every space of this type is of the form $E = E_{(0)} \overset{\perp}{\oplus} \Delta(\alpha\alpha) \overset{\perp}{\oplus} \Delta(\beta\beta)$, $\alpha \not\sim \beta$.

In this case every space is isomorphic to $E = E_{(0)} \overset{\perp}{\oplus} \Delta(\alpha\alpha) \overset{\perp}{\oplus} \Delta(\beta\beta)$. It is sufficient to show that any space $F = \Delta(ss) \overset{\perp}{\oplus} \Delta(\gamma\gamma)$ is isomorphic to $\Delta(\alpha\alpha) \overset{\perp}{\oplus} \Delta(\beta\beta)$. Let $\Delta(ss) = k(e_1, e_2)$ and $\Delta(\gamma\gamma) = k(e_3, e_4)$, then there exists field elements a, b, c and d such that $a^2\gamma + b^2 s = \alpha$ and $c^2\gamma + d^2 s = \beta$ (note that $[k : k^2] = 2$). We see that $k(a\ell_1 + b\ell_3, c\ell_2 + d\ell_4) = \Delta(\alpha\beta) \subset F$. F is finite and semi-simple, $\Delta(\alpha\beta)$ is semi-simple thus $\Delta(\alpha\beta) \oplus \Delta^{\perp}(\alpha\beta) = F$. $\Delta^{\perp}(\alpha\beta)$ is semi-simple and of dimension 2 therefore it must be isomorphic to one of P (hyperbolic plane), $\Delta(rr)$, or $\Delta(rs)$ $(r \not\sim s)$. Because $\Delta(\alpha\beta) \overset{\perp}{\oplus} \Delta(\alpha\beta)^{\perp} \cong \Delta(ss) \overset{\perp}{\oplus} \Delta(\gamma\gamma)$ their determinants must be equal up to a square factor. This implies $\Delta^{\perp}(\alpha\beta) = \Delta(rs)$, $r \not\sim s$. Again there exists x and y such that $\alpha = x^2 r + y^2 s$, call $h_1 = x g_1 + y g_2$ where $\|g_1\| = r$, $\|g_2\| = s$, and $\Delta(rs) = k(g_1, g_2)$. Thus $\Delta^{\perp}(\alpha\beta) \cong \Delta(\alpha\lambda)$, $\alpha \not\sim d$. At this point we have $\Delta(\alpha\beta) \overset{\perp}{\oplus} \Delta(\alpha d) \cong \Delta(ss) \overset{\perp}{\oplus} \Delta(\alpha\gamma)$ and by appealing to determinants again we obtain $d \sim \beta$. Thus $\Delta(\alpha\alpha) \overset{\perp}{\oplus} \Delta(\beta\beta) \cong \Delta(ss) \overset{\perp}{\oplus} \Delta(\gamma\gamma)$.

APPENDIX

TABLE I

MAJOR NON-ISOMORPHIC SPACES

case	dim E/E_*	dim $E_*^{\perp}$	dim $\mathrm{rad}\, E_*$	types of Spaces *
1	0	0	0	$E(0)$
2.1	1	0	0	$E(\nu)$
2.2	1	1	0	$E(0) \oplus \Delta(\nu)$
2.3	1	1	1	$E(0) \oplus \Delta(\nu\nu)$
3.1	2	0	0	$E(\alpha) \oplus E(\beta)$
3.2	2	1	0	$E(\nu) \oplus \Delta(\mu) \quad \nu \not\sim \mu$
3.3	2	1	1	$E(\alpha) \oplus \Delta(\beta\beta)$, $E(\nu) \oplus \Delta(\alpha\alpha) \quad \nu \not\sim \alpha$
3.4	2	2	0	$E(0) \oplus \Delta(\nu\mu) \quad \nu \not\sim \mu$
3.5	2	2	1	$E(0) \oplus \Delta(\alpha\alpha) \oplus \Delta(\beta)$, $E(0) \oplus \Delta(\beta\beta) \oplus \Delta(\nu)$, $\nu \not\sim \beta$
3.6	2	2	2	$E(0) \oplus \Delta(\alpha\alpha) \oplus \Delta(\beta\beta)$

* All the sums are orthogonal, $\{\alpha, \beta\}$ is some fixed basis of k over k^2; ν and μ run independently through a fixed set of representatives of g_k , subject only to conditions listed in the table. All the spaces thus obtained are mutually non-isomorphic and, in each case, they are up to orthogonal isomorphisms all k -spaces (E, ϕ) with $\dim E = \aleph_0$ and $\mathrm{char}\, k = 2$.

LITERATURE CITED

Gross, H., Ein Wittscher Satz in Falle von Vektorraumen abzahlbarer Dimension, J. Riene N. Angew, Math. (To appear shortly).

Gross, H., and Fischer, H. R., Quadratic Forms and Linear Topologies II, Math. Annalen. (To appear shortly).

Kaplansky, I., Forms on Infinite Dimensional Spaces, Anais, Acad. Brazil. Ciencias XXII (1950), 1-17.

O'Meara, O. T., Introduction to Quadratic Forms, Academic Press, New York, (1963).