Active RC networks utilizing the voltage follower
by James Llafet Hogin

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY in Electrical Engineering
Montana State University
© Copyright by James Llafet Hogin (1966)

Abstract:
The (RCVF) is defined as an active network whose passive portion consists of a three-port grounded
RC structure and whose active portion consists of the three-terminal grounded voltage follower. The
necessary and sufficient conditions on a voltage transfer function for its realization by the (RCVF) are
developed. Synthesis methods for the passive portion of the (RCVF) are reviewed, developed, and
discussed. A digital computer program is presented which demonstrates the feasibility of optimizing
(RCVF) networks by the method of steepest descent. A discussion and analysis of practical voltage
followers and the practical considerations involved with their use is presented. In addition, a useful
three stage voltage follower is developed which has very high input impedance and a gain very close to
unity.

Illustrative examples and experimental verification are given. It is concluded that the (RCVF) is a
highly useful and practical network.
ACTIVE RC NETWORKS UTILIZING THE VOLTAGE FOLLOWER

by

JAMES LLAFET HOGIN

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY in

Electrical Engineering

Approved:

[Signatures]

Dean, Graduate Division

MONTANA STATE UNIVERSITY
Bozeman, Montana

March, 1966
ACKNOWLEDGEMENTS

The author particularly wishes to thank Professor B. J. Bennett for his encouragement and helpful suggestions. He is also grateful for the financial support provided by a Douglas Aircraft Company scholarship.
TABLE OF CONTENTS

Vita
Acknowledgments
Table of Contents
List of Tables
List of Figures
List of Symbols
Abstract
I. Introduction
   A. The Active Network
   B. The Voltage Follower
   C. Comparison of Active and Passive Transfer Function Synthesis
II. Statement of Problem
III. Necessary and Sufficient Conditions for the Realization of the Voltage Transfer Function $T_V$
   A. Surplus Factor Utilization
IV. (RCVF) Realization Techniques
   A. A Pictorial Representation of the Passive Synthesis Problem
   B. Review of Previously Proposed Configurations

Page

ii
iii
iv
vi
vii
x
xii
1
2
4
5
11
13
21
26
26
30
LIST OF TABLES

6.1 Frequency-attenuation tabulation of experimental and theoretical values for network of Figure 6.8.
6.2 Frequency-attenuation tabulation of experimental and theoretical values for network of Figure 6.9.
6.3 Frequency-attenuation tabulation of experimental and theoretical values for network of Figure 6.10.
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>The active RC network utilizing the voltage follower.</td>
</tr>
<tr>
<td>1.2</td>
<td>Realization of $\frac{E_{out}}{I_{in}} = \frac{s^2 + 2.828s + 4}{s^2 + 1.414s + 1}$ with a lossless network terminated by resistors.</td>
</tr>
<tr>
<td>1.3</td>
<td>Realization of $\frac{E_{out}}{E_{in}} = 0.2 \frac{s^2 + 2.828s + 4}{s^2 + 1.414s + 1}$ by Pantell's method.</td>
</tr>
<tr>
<td>1.4</td>
<td>Simple realization of $\frac{E_{out}}{I_{in}} = \frac{s^2 + 2.828s + 4}{s^2 + 1.414s + 1}$</td>
</tr>
<tr>
<td>1.5</td>
<td>Realization of $\frac{E_{out}}{E_{in}} = \frac{s^2 + 2.828s + 4}{s^2 + 1.414s + 1}$ with the (RCVY) network to realize $T = \frac{a_2 s^2 + a_1 s + a_0}{p_2 s^2 + p_1 s + p_0}$.</td>
</tr>
<tr>
<td>3.1</td>
<td>The &quot;A&quot; and &quot;B&quot; two-port unbalanced networks.</td>
</tr>
<tr>
<td>3.2</td>
<td>Interconnection of &quot;A&quot; and &quot;B&quot; networks.</td>
</tr>
<tr>
<td>3.3</td>
<td>The &quot;A&quot; and &quot;B&quot; networks connected to the voltage follower.</td>
</tr>
<tr>
<td>4.1</td>
<td>A possible current distribution.</td>
</tr>
<tr>
<td>4.2</td>
<td>Current distribution of passive portion of (RCVY) network to realize $T = \frac{a_2 s^2 + a_1 s + a_0}{p_2 s^2 + p_1 s + p_0}$.</td>
</tr>
<tr>
<td>4.3</td>
<td>Network configuration with input current distribution of Figure 4.2.</td>
</tr>
<tr>
<td>4.4</td>
<td>Current distribution for case of $a_1/a_0 \leq r_1/r_0$.</td>
</tr>
<tr>
<td>4.5</td>
<td>Current distribution for case of $r_1/r_0 \leq a_1/a_0$.</td>
</tr>
<tr>
<td>4.6</td>
<td>The basic ladder network.</td>
</tr>
</tbody>
</table>
4.7 The basic decomposed ladder network.

4.8 Decomposed ladder network with shunt arm capacitors and series arm resistors.

4.9 Current distribution for basic ladder network realization of \( y_{11} \).

4.10 Pole-zero distributions for \( y_{11} \) (0) to \( y_{11} \) (6).

4.11 Three basic building blocks for the fourth-order band-pass transfer function realization.

4.12 Current distribution for network corresponding to transfer function of Eq. (4.23).

4.13 Network realization for \( T_v = \frac{s^2+2.828s+4}{s^2+1.414s+1} \).

4.14 Network realization for \( T_v = \frac{1}{s^4+2.6132s^3+3.4143s^2+2.6132s+1} \).

4.15 Network realization for \( T_v = \frac{2.172s^2}{s^4+1.414s^3+3s^2+1.414s+1} \).

5.1 Flow chart for network optimization computer program.

5.2 Configuration of example network.

5.3 Optimized network for realization of \( T_v = \frac{1}{s^4+2.6132s^3+3.4143s^2+2.6132s+1} \).

6.1 Current distribution diagram for nonunity voltage follower gain \( k \).

6.2 Voltage follower of Reference 32.

6.3 Altered network.
6.4 System diagram for networks of Figures 6.2 and 6.3.

6.5 Three stage voltage follower network.

6.6 Current distribution for twin-T network.

6.7 Network for realization of $T_v = 0.1s/s^2 + 0.1s + 1$.

6.8 (RCVF) for realization of second-order band-pass transfer function denormalized to 300,000 ohms and 150 cps.

6.9 (RCVF) for realization of fourth-order low-pass transfer function denormalized to 1 megohm and 85 cps.

6.10 (RCVF) for realization of fourth-order band-pass transfer function denormalized to 500,000 ohms and 39.8 cps.
**LIST OF SYMBOLS**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>RC</td>
<td>Abbreviation for the term &quot;resistance-capacitance&quot;.</td>
</tr>
<tr>
<td>RCVF</td>
<td>Abbreviation for the active network composed of resistors, capacitors, and a single grounded voltage follower.</td>
</tr>
<tr>
<td>$E_i$</td>
<td>Voltage parameter.</td>
</tr>
<tr>
<td>$i_i$</td>
<td>Current parameter.</td>
</tr>
<tr>
<td>$y_{ij}$</td>
<td>Short-circuit admittance parameters.</td>
</tr>
<tr>
<td>$T_v$</td>
<td>Voltage transfer function.</td>
</tr>
<tr>
<td>$s$</td>
<td>Complex frequency variable.</td>
</tr>
<tr>
<td>$N_{ij}(s)$</td>
<td>Numerator polynomial of $y_{ij}$ for $i=j$. Negative of numerator polynomial for $i \neq j$.</td>
</tr>
<tr>
<td>$D(s)$</td>
<td>Denominator polynomial of $y_{ij}$.</td>
</tr>
<tr>
<td>$A(s)$</td>
<td>Numerator polynomial of $T_v$.</td>
</tr>
<tr>
<td>$P(s)$</td>
<td>Denominator polynomial of $T_v$.</td>
</tr>
<tr>
<td>$R(s)$</td>
<td>Polynomial equal to $P(s)-A(s)$.</td>
</tr>
<tr>
<td>$E(s)$</td>
<td>Augmenting polynomial such that $P(s)+E(s)$ has negative real zeros.</td>
</tr>
<tr>
<td>$a_i$</td>
<td>Coefficients of $A(s)$.</td>
</tr>
<tr>
<td>$p_i$</td>
<td>Coefficients of $P(s)$.</td>
</tr>
<tr>
<td>$r_i$</td>
<td>Coefficients of $R(s)$.</td>
</tr>
<tr>
<td>$e_i$</td>
<td>Coefficients of $E(s)$.</td>
</tr>
<tr>
<td>$d_i$</td>
<td>Coefficients of $D(s)$.</td>
</tr>
<tr>
<td>$Y_i$</td>
<td>Admittance parameter.</td>
</tr>
</tbody>
</table>
Impedance parameter.
Resistance parameter.
Capacitance parameter.
Zeros of admittance.
Poles of admittance.
Optimization parameter.
Sensitivity of T with respect to k.
Polynomial root sensitivity (p = any polynomial root).
Coefficients of T
Capacitance distribution parameter.
Resistance distribution parameter.
Capacitance summation parameter.
Capacitance ratio parameter.
Resistance ratio parameter.
Optimization parameter weighting functions.
Second-order polynomial with zeros lying on unit circle in the RHP.
Angle zeros of F(s) make with the positive real axis.
Augmenting polynomial for F(s).
Zeros of B(s).
Coefficients of B(s).
Function of the coefficients of B(s).
ABSTRACT

The (RCVF) is defined as an active network whose passive portion consists of a three-port grounded RC structure and whose active portion consists of the three-terminal grounded voltage follower. The necessary and sufficient conditions on a voltage transfer function for its realization by the (RCVF) are developed. Synthesis methods for the passive portion of the (RCVF) are reviewed, developed, and discussed. A digital computer program is presented which demonstrates the feasibility of optimizing (RCVF) networks by the method of steepest descent. A discussion and analysis of practical voltage followers and the practical considerations involved with their use is presented. In addition, a useful three stage voltage follower is developed which has very high input impedance and a gain very close to unity.

Illustrative examples and experimental verification are given. It is concluded that the (RCVF) is a highly useful and practical network.
I. INTRODUCTION

Active RC networks are networks containing only resistors, capacitors and active elements. These networks are attractive at the lower frequencies where magnetic elements are large, expensive and difficult or impossible to realize. In addition, the advent of microcircuit technology, where a stable inductance parameter is difficult to attain, has resulted in an increased interest in this type of network. The need for a power supply for the active elements is often unimportant if, as frequently happens, the network is a part of a system for which power is required regardless of the network.

The purpose of this study is to develop synthesis procedures concerned with a particular type of active RC network: an active RC network whose active element consists of a single ideal voltage follower. The ideal voltage follower is defined as a 2-port network whose input admittance and output impedance are zero and whose ratio of output to input voltage is unity. The cathode, emitter, or source follower circuit may be considered a practical approximation to the ideal voltage follower.

In addition to developing synthesis procedures, this study is concerned with the optimization of the synthesized
networks by means of the method of steepest descent as applied by the digital computer. Finally, the study is concerned with the practical demonstration of the synthesis methods and with the problem of obtaining a voltage follower which closely approximates the ideal.

A. THE ACTIVE NETWORK

The active RC network configuration of concern in this research is shown in Figure 1.1. The passive portion of the network consists of a 3-port grounded RC structure. The active portion consists of the 3-terminal grounded voltage follower defined above. For convenience, this type of active network will be referred to as the (RCVF).

An important consideration in the design of active networks is the sensitivity of the desired network function to variations of the active element portion of the active network. This consideration is in addition to that of the sensitivity associated with variations of the passive network elements. It should be pointed out, however, that from a practical point of view, the gain of the voltage follower proposed as the active element in this study can be made more stable than alternate types of active devices which require a gain other than unity. This observation is based
Figure 1.1. The active RC network utilizing the voltage follower (RCVF).
on the fact that the voltage follower is concerned with the direct comparison of input voltage and output voltage rather than with their indirect comparison. Thus, a direct comparison of active networks on the basis of the magnitude of the sensitivity function associated with the active element could be misleading. It is more informative to specify network function variations caused by expected variations in the active device. In conclusion, the active element proposed for this study may be made highly stable and at least partially compensate for the fact that the networks are quite sensitive to changes in the active element.

An additional reason for considering the voltage follower as an active device is that, in its elementary forms, it is an extremely simple and widely used device.

B. THE VOLTAGE TRANSFER FUNCTION

The development of the voltage transfer function for the (RCVF) proceeds as follows. Referring to the passive portion of the network shown in Figure 1.1 and observing the conventions shown, the first of the short-circuit admittance equations is

\[ I_1 = Y_{11} E_1 + Y_{12} E_2 + Y_{13} E_3. \]  

(1.1)

For the particular configuration shown in Figure 1.1, \( I_1 = 0 \)
and $E_3 = E_1$. Equation (1.1) then becomes

$$y_{11}E_1 + y_{12}E_2 + y_{13}E_1 = 0$$

(1.2)

and the voltage transfer function $T_v$ may be solved as

$$T_v = \frac{E_{\text{out}}}{E_{\text{in}}} = \frac{-y_{12}}{y_{11} + y_{13}}$$

(1.3)

Since the poles of the short circuit admittance parameters are identical (provided common factors have not been cancelled), we may conveniently define

$$y_{ij} = \begin{cases} \frac{N_{ij}}{D} & \text{for } i=j \\ -\frac{N_{ij}}{D} & \text{for } i \neq j \end{cases}$$

(1.4)

where the $N$'s and $D$ are polynomials, and write the voltage transfer function as

$$T_v = \frac{E_{\text{out}}}{E_{\text{in}}} = \frac{N_{12}}{N_{11} - N_{13}} \equiv \frac{A(s)}{P(s)}$$

(1.5)

This manner of defining the $y_{ij}$ allows us to consider all of the coefficients of the $N_{ij}$ as having the same sign (which we arbitrarily choose positive).

C. COMPARISON OF ACTIVE AND PASSIVE TRANSFER FUNCTION SYNTHESIS

The main difference between active and passive transfer function synthesis is that passive synthesis is generally
concerned with the direct realization of a given transfer quantity by means of one of the transfer immittances. Active synthesis is generally concerned with the indirect realization of the same transfer quantity by means of the sum or difference of two transfer immittances. The active elements of the active network serve as sum or difference taking devices.

As an example, let us consider a given transfer function $T_v$ and examine some of the ways this function may be realized with passive circuitry. Then it will be shown how this same function may be realized with the (RCVF) active network.

Consider the transfer function $T_v = K \frac{s^2 + 2.828s + 4}{s^2 + 1.4l4s + 1}$ where $K$ is an arbitrary positive constant. This function may be realized by a passive network since its poles lie in the left-half of the complex frequency plane (LHP). The function is a minimum phase function since its zeros also lie in the LHP. Guillemin (25) shows that a minimum phase transfer function cannot be realized by a lossless network terminated in a single resistor. He shows, however, a realization consisting of a lossless network terminated by more than one resistor. Applying this method to the present case results in the network of Figure 1.2. (See Appendix III-A
Figure 1.2. Realization of $E_{out}/E_{in} = \frac{1}{s^2 + 2.828s + 4/s^2 + 1.414s + 1}$ with a lossless network terminated by resistors.

Figure 1.3. Realization of $E_{out}/E_{in} = 0.2(s^2 + 2.828s + 4/s^2 + 1.414s + 1)$ by Pantell's method.
for numerical computations). The chief disadvantage of this realization, besides the large number of elements, is that the input and output do not share a common ground.

Other possibilities for the realization of $T_V$ include its realization as a driving point impedance or admittance ($T_V$ is positive real), its realization in the form of a constant resistance ladder (29), or by Pantell's method (41). $T_V$ has been realized with the resultant network shown in Figure 1.3 (see Appendix III-B for numerical computations). Note that Pantell's method has the advantage of a driving and terminating resistor. Finally, if one does not care about terminating resistors, etc. and a simple network is sought which realizes $T_V$, the configuration of Figure 1.4 is among the simplest available. This network was not synthesized by any rigorous method, but by the application of heuristic means.

To realize $T_V$ by means of the (RCVF), we follow methods to be presented later in this thesis to develop the network shown in Figure 1.5. Note that the number of passive elements required is less than the number required in the majority of the passive networks presented above. Also note that the same number of reactive and only one additional resistive element is required by the (RCVF) realization than
by the simplest of the passive networks.
Figure 1.4. Simple realization of \[
\frac{E_{\text{out}}}{E_{\text{in}}} = \frac{\frac{s^2 + 2.828s + 4}{s^2 + 1.414s + 1}}{s^2 + 2.828s + 4}
\]

Figure 1.5. Realization of \[
\frac{E_{\text{out}}}{E_{\text{in}}} = \frac{s^2 + 2.828s + 4}{s^2 + 1.414s + 1}
\] with the (RCVF).
II. STATEMENT OF PROBLEM

Attention will now be given to a statement of the problem for this research. The basic problem of developing synthesis procedures for the (RCVF) is broken into the following five parts:

1. Determine the necessary and sufficient conditions on a voltage transfer function $T_{v} = \frac{A(s)}{P(s)}$ such that it may be realized in the configuration of the (RCVF) as shown in Figure 1.1.

2. Given the desired voltage transfer function $T_{v} = \frac{A(s)}{P(s)}$ whose numerator and denominator polynomials meet the necessary and sufficient conditions specified above, determine the corresponding network functions of the passive RC portion of the (RCVF).

3. Realize the determined network functions in the form of a four-terminal passive RC network such that when the network is used in conjunction with the voltage follower as in Figure 1.1, the overall network possesses the voltage transfer function $T_{v} = \frac{A(s)}{P(s)}$.

4. Take advantage of the digital computer topological methods of circuit analysis presently available and the method of steepest descent in order to improve and optimize the realized networks.

5. Demonstrate the practicality of the above items by
building and testing some example networks.

Item 2 above is complicated by the fact that if there is one way then in general there are an infinite number of ways that $T_v$ may be broken up into the network functions of a passive RC four-terminal network. Thus, we are faced with the possibility of selecting the one set of network functions which optimizes some attribute or combination of attributes of the network. Such attributes include minimizing the spread of element values, minimizing the sensitivity of $T_v$ to variations in circuit values, etc. A significant portion of this research is devoted to this optimization problem.
III. NECESSARY AND SUFFICIENT CONDITIONS FOR THE REALIZATION OF THE VOLTAGE TRANSFER FUNCTION $T_v$.

Considering $T_v$ as the ratio of polynomials $A(s)$ to $P(s)$, this section will be concerned with the establishment of the necessary and sufficient conditions on $A(s)$ and $P(s)$ such that a desired stable voltage transfer function $T_v$ may be realized with the (RCVF) configuration. Since we are only interested in stable transfer functions, it follows immediately that the zeros of $P(s)$ are confined to the left-half of the complex frequency plane with the possible exception of a single zero at the origin.

Next we call on the Fialkow condition (29) which states: For an unbalanced two-port with no mutual inductance the numerator coefficients of $-y_{12}$, $y_{11}$, and $y_{22}$ are all nonnegative, the coefficients of $-y_{12}$ being no greater than the corresponding ones in $y_{11}$ or $y_{22}$.

This condition can obviously be extended to the three-port unbalanced passive RC network of Figure 1.1 with the resulting statement that:

For an unbalanced RC three-port with no mutual inductance the numerator coefficients of $-y_{13}$, $-y_{12}$, and $y_{11}$ are all nonnegative, the coefficients of $-y_{12}$ being no greater than the corresponding coefficients in $(y_{11}+y_{13})$. 
This latter condition follows from the fact that if we form a two-port network from the three-port passive network of Figure 1.1 by joining terminals 2 and 3 together, then the short-circuit transfer admittance of the two-port network is the sum of the short-circuit admittance parameters \( y_{12} + y_{13} \) of the three-port network. The second part of the Fialkow condition for unbalanced two-ports may then be restated as "the coefficients \(- (y_{12} + y_{13})\) being no greater than the corresponding ones in \( y_{11} \)." If \( y_{13} \) is added to each side of the inequality, the previously stated condition is verified. Another way of expressing this condition is that the coefficients of \( (N_{11} - N_{12}) \) of equation (1.5) must be greater than or equal to the corresponding coefficients of \( N_{12} \).

Since the numerator \( A(s) \) of the transfer function \( T_v \) is the numerator of the transfer admittance of a passive RC unbalanced two-port, it follows that the zeros of the transfer function may appear anywhere except on the positive real axis.

The following necessary condition for the voltage transfer function of the (RCVF) may now be stated:

The voltage transfer function of an unbalanced RC three-port network used in conjunction with a voltage
follower network as shown in Figure 1.1 must be a ratio of nonnegative coefficient polynomials in s with the coefficients of the numerator polynomial being no greater than the corresponding coefficients in the denominator polynomial. In addition, for stable transfer functions, the zeros of the denominator polynomial must be located in the LHP with the possible exception of a single zero at the origin. The zeros of the numerator polynomial may appear anywhere except on the positive real axis.

A synthesis procedure will now be given which shows that a voltage transfer function fulfilling the above necessary condition may always be realized. Thus, the stated condition on a transfer function will be shown to be sufficient as well as necessary for its realization.

Consider the two-port unbalanced RC networks, "A" and "B", shown in Figure 3.1. Assume that the short-circuit admittance parameters of both networks have the identical denominator polynomial \( D(s) \) and that their numerator polynomials are defined as in equation (1.4) with the addition of the subscript A or B to denote network "A" or "B" respectively. If D volts are placed on the input terminals of each network and the output terminals grounded, then from the
Figure 3.1. The "A" and "B" two-port unbalanced RC networks.

Figure 3.2. Interconnection of "A" and "B" networks.
short-circuit admittance equations we know that currents are established as shown in Figure 3.1. Now if the "A" and "B" networks are interconnected as shown in Figure 3.2, the same currents will flow since the conditions previously imposed still apply. The interconnection forms a new three-port (terminal (1) to ground, terminal (2) to ground, and terminal (3) to ground) unbalanced RC network whose short-circuit admittance parameters are seen to be

\[
\begin{align*}
Y_{11} &= \frac{N_{11A} + N_{11B}}{D} = Y_{11A} + Y_{11B} \\
Y_{12} &= \frac{-N_{12A}}{D} = Y_{12A} \quad (3.1) \\
Y_{13} &= \frac{N_{12A} + N_{12B} - N_{11A} - N_{11B}}{D} 
\end{align*}
\]

If this network is used for the passive portion of the (RCVF), the overall voltage transfer function will be

\[
T_v = \frac{-Y_{12}}{Y_{11} + Y_{13}} = \frac{N_{12A}}{N_{11A} + N_{11B} + N_{12A} + N_{12B} - N_{11A} - N_{11B}} \\
T_v = \frac{N_{12A}}{N_{12A} + N_{12B}} = \frac{A(s)}{P(s)} \quad (3.2)
\]

This equation is interesting in that its element polynomials are polynomials which are the numerators of RC transfer admittances and not numerators of RC driving point admittances. The equation suggests the following synthesis procedure for a transfer function which meets the previously
stated necessary conditions.

1. Choose a polynomial D whose zeros are restricted to the negative real axis.

2. Synthesize an RC "A" network whose transfer admittance is

\[ \gamma_{12A} = \frac{-A(s)}{D(s)} \]

This may always be done provided the coefficients of A(s) are positive (the necessary condition guarantees that they are) and that the order of D and the magnitudes of its coefficients are chosen sufficiently large (see Section 15-3 of reference 25).

3. Synthesize an RC "B" network whose transfer admittance is:

\[ \gamma_{12B} = \frac{-(P(s)-A(s))}{D(s)} \]

Again the coefficients of (P(s)-A(s)) are guaranteed to be positive by the necessary condition, and the "B" network may always be realized provided the order of D and the magnitudes of its coefficients are chosen sufficiently large.

4. After being synthesized, the "A" and "B" networks are connected as in Figure 3.2 and joined to a voltage follower as shown in Figure 1.1. This new overall connection is shown in Figure 3.3.
Figure 3.3. "A" and "B" networks connected to the voltage follower.
The voltage transfer function of the active configuration will then be \( T_v = \frac{A(s)}{P(s)} \). Thus, the previously quoted condition is sufficient as well as necessary.

The synthesis procedure outlined above will, in general, produce configurations with an excessive number of elements. For example, consider the band pass voltage transfer function

\[
T_v = \frac{0.1s}{s^2 + 1s + 1} \tag{3.3}
\]

If we applied the suggested synthesis procedure, we could separate the function as:

\[
\gamma_{12A} = \frac{-0.1s}{D} \quad \text{and} \quad \gamma_{12B} = \frac{-(s^2 + 1)}{D} \tag{3.4}
\]

\( \gamma_{12B} \) may be realized in the form of a twin-T RC network which requires 3 capacitors and 3 resistors for a total of 6 elements. \( \gamma_{12A} \) could then be realized with 2 elements. It is known, however, that it is possible to realize the transfer function with fewer elements (reference 14) than 8. Thus the suggested synthesis procedure, although powerful, in general requires an excessive number of elements. For this reason, it is desirable to examine other methods of synthesis. The next chapter will be concerned with this problem.
A. SURPLUS FACTOR UTILIZATION

Before going on to other synthesis methods, it is appropriate to examine the possibility of utilizing surplus factors to convert a transfer function $T_v$ which apparently does not meet the necessary and sufficient conditions of this chapter into a transfer function which does meet these conditions.

It is well known in circuit theory that there are cases when it is possible to multiply a polynomial with negative coefficients by a polynomial with LHP zeros such that the resultant polynomial has nonnegative coefficients. Thus, if we have a $T_v$ whose poles are restricted to the LHP and its numerator polynomial contains some negative coefficients, there still exists a possibility that surplus factors might be utilized so that $T_v$ meets the prescribed necessary and sufficient conditions.

In the literature, the overall problem is broken down into the fundamental problem of finding a polynomial $B(s)$ with LHP zeros such that $B(s) \cdot F(s)$ has nonnegative coefficients where $F(s) = s^2 - (2 \cos \phi)s + 1$ and $0 < \phi < \pi/2$. This is a valid simplification of the problem since factors of the form of $F(s)$ may be separated out of any polynomial with
negative coefficients (provided an appropriate frequency transformation is made). These factors may then be individually operated upon to produce nonnegative coefficient polynomials.

One approach has been to show that a polynomial of the type \( B(s) = (s+1)^n \) could produce the desired coefficients and to reason then that for sufficiently small and distinct \( \epsilon_i \), the surplus factor \( B(s) = \prod_{i=1}^{n} (s+1 + \epsilon_i) \) will approach the same result \((28,40)\).

The smaller the value of \( \phi \), the larger the value of \( n \) required to produce nonnegative coefficients. As \( \phi \) approaches zero, the required \( n \) approaches \( \infty \). The optimum choice of \( \beta \) for \( B(s) = (s+\beta)^n \) will be considered so that for a given \( n \), \( \phi \) may take on the smallest value possible consistent with the nonnegative coefficient restriction on \( F(s)B(s) \). It is shown that the optimum choice of \( \beta \) for \( n \) odd is \( \beta_{\text{opt}} = 1 \), but that for \( n \) even \( \beta_{\text{opt}} = \sqrt{\frac{n}{n+4}} \) or \( \sqrt{\frac{n+4}{n}} \). The corresponding values for \( \phi_{\text{min}} \) are \( \cos^{-1} (\frac{n+1}{n+3}) \) and \( \cos^{-1} \sqrt{\frac{n}{n+4}} \).

Consider
\[
B(s) = (s+\phi)^n = b_0 s^n + b_1 s^{n-1} + \ldots + b_{n-1} s + b_n \quad (3.5)
\]
where
\[ b_j = \frac{n!}{(n-j)!j!} \beta^j \quad (j=0,1,2,\ldots,n) \quad (3.6) \]

The product of \( B(s) \) and \( F(s) \) may then be written as:
\[ B(s)F(s) = \sum_{j=-1}^{n+1} (b_{j-1}+b_{j+1}-2 \cos\beta b_j)s^{n+1-j} \quad (3.7) \]

where \( b_{-2}=b_{-1}=b_{n+1}=b_{n+2}=0 \)

Thus, it is seen that the following set of inequalities must be satisfied in order that \( B(s)F(s) \) have nonnegative coefficients:
\[ G_j = \frac{b_{j-1}+b_{j+1}}{b_j} \cdot 2 \cos \quad (j=0,1,\ldots,n) \quad (3.8) \]

Substituting the values from Eq. (3.6) into (3.8)
\[ G_j = \frac{\beta(n-j)}{\beta(n-j+1)} + \frac{\beta(n-j+1)}{(j+1)} \quad (3.9) \]

The basic problem may now be equated to that of choosing a nonnegative value of \( \beta \) such that the minimum of the set \( (G_0, G_1, \ldots, G_n) \) is maximized. Call this value of \( G \), \( G_{\min \max^*} \).

As seen from Eq. (3.9), for arbitrary \( n \), the \( G_j \) are continuous functions of \( \beta \) which possess no more than one minimum and no maxima in the range \( 0 \leq \beta < \infty \). In addition, if \( g(\beta) \) is defined as \( \min (G_0, G_1, \ldots, G_n) \), then \( g=0 \) for \( \beta=0 \) and \( \infty \). Thus, the optimum value of \( \beta \) corresponds to an intersection of two of the \( G \) curves. The values of \( \beta \) for which \( G_1 \) and \( G_j \) intersect may be computed by means of
equation (3.9) as
\[ \beta = \sqrt{\frac{(i+1)(j+1)}{(n-1)(n-j+1)}} \quad (i, j=0, 1, \ldots, n; i \neq j) \] (3.10)

An examination of this equation shows that for a given \( j \), the value of \( i > j \) which results in a minimum value for \( \beta \) is \( i = j + 1 \). Since \( g = G_j \) for \( \beta = 0 \), it follows that as \( \beta \) is increased from 0, \( g \) corresponds successively to \( G_0, G_1, \ldots, G_n \).

Substituting \((j+1)\) for \( i \) in equation (3.10), the optimum value of \( \beta \) is that element of the set
\[ \beta = \sqrt{\frac{(j+2)(j+1)}{(n-j)(n-j+1)}} \quad (j=0, \ldots, n-1) \] (3.11)

which produces the largest value of \( G_j \) from Eq. (3.9).

That is, what permissible value of \( j \) maximizes
\[ G_j = 2\sqrt{\frac{(n-j)(j+1)}{(n-j+1)(j+2)}} \] (3.12)

Differentiating with respect to \( j \):
\[ \frac{\partial G_j}{\partial j} = \frac{(n-2j+1)(n+2)}{(n-j)^{3/2}(j+1)^{3/2}(n-j+1)^{3/2}(j+2)^{3/2}} \] (3.13)

which is seen to be positive, negative, and zero for \( j < \frac{n-1}{2} \), \( j = \frac{n-2}{2} \), and \( j = \frac{n-1}{2} \) respectively. It follows that the optimum value of \( \beta(\beta_{opt}) \) and the corresponding value of minimum \( \phi = \cos^{-1} \frac{G_{\min \max}}{2} \) is found by substituting \( j = \frac{n-1}{2} \) for \( n \) odd and \( j = \frac{n-1}{2} \pm \frac{1}{2} \) for \( n \) even into Eqs. (3.11) and (3.12).

For the case of \( n \) even, it is found that the two possible values of \( j \) result in identical values of \( G_{\min \max} \). Thus,
The above shows the optimum surplus factor of the form $(s+\beta^n)$. In the present case, we are not limited to surplus factors which lie on the negative real axis. LHP complex conjugate zeros are permissible and, in general, they are superior to negative real axis zeros. The problem of finding their optimum location appears to be much more difficult and no solution is offered to this problem. The question, however, is left as a problem challenge.

\[
\begin{align*}
\beta_{\text{opt}} &= 1 \\
\phi_{\text{min}} &= \cos^{-1}\left(\frac{n+1}{n+3}\right) \\
& \quad \text{for } n \text{ odd} \\
\beta_{\text{opt}} &= \sqrt{\frac{n}{n+4}} \text{ or } \sqrt{\frac{n+4}{n}} \\
\phi_{\text{min}} &= \cos^{-1}\left(\sqrt{\frac{n}{n+4}}\right) \\
& \quad \text{for } n \text{ even}
\end{align*}
\]
This section is concerned with the synthesis of special network configurations to be used as the passive portion of the (RCVF). These configurations are of interest since, in general, they require fewer elements than the corresponding networks realized by the general synthesis procedure of Chapter III. Section A of this chapter gives a pictorial representation of the passive synthesis problem which simplifies the viewing of the problem and offers insight into the sensitivity problem. Section B reviews network configurations presented by other authors. This material is solely concerned with second-order transfer functions. Section C presents a newly developed synthesis procedure for the bi-quadratic transfer function. Section D presents a ladder network configuration for low-pass and high-pass transfer functions. Section E presents a special network configuration for fourth-order band pass transfer functions. Finally, Section F presents several numerical examples of the synthesis procedures presented in this chapter.

A. A PICTORIAL REPRESENTATION OF THE PASSIVE SYNTHESIS PROBLEM

There are a number of ways to state the requirements on the passive portion of the (RCVF). A simple way of viewing
the problem and one which offers some insight into the sensitivity problem is offered below.

Consider a (RCVF) voltage transfer function $T_V$ meeting the necessary and sufficient conditions of Chapter III.
From equation (1.5)
$$T_V = \frac{A(s)}{P(s)} = \frac{A(s)}{R(s)+A(s)} = \frac{N_{12}}{N_{11}-N_{13}} \quad (4.1)$$
where $R(s) = P(s)-A(s) = r_n s^n + r_{n-1} s^{n-1} + \ldots + r_0$
has nonnegative coefficients since the coefficients of $A(s)$ are required to be less than or equal to the corresponding coefficients of $P(s)$.

Since the polynomials $N_{11}$, $N_{12}$, and $N_{13}$ are the numerators of short-circuit RC admittance parameters, it is possible to associate the polynomials $A(s)$, $P(s)$, and $R(s)$ with the currents flowing in a four-terminal RC network whose short-circuit admittance parameters possess these numerator polynomials. This possible current distribution is shown in Figure 4.1 and is only valid when terminal (l) is driven by a voltage source equal to the characteristic polynomial $B(s)$ of the network, and the other terminals are grounded.

Since $\frac{P(s)+E(s)}{D(s)}$ is a driving point admittance, the zeros of its numerator and denominator must be interlaced
Figure 4.1. A possible current distribution.

Figure 4.2. Current distribution of passive portion of (RCVF) network to realize $T_v = (a_2s^2 + a_1s + a_0) / (p_2s^2 + p_1s + p_0)$. 

\[ P(s) + E(s) = \frac{a_2s^2 + a_1s + a_0}{p_2s^2 + (p_1 + e_1)s + p_0} \]

\[ R(s) = r_2s^2 + r_1s + r_0 \]
along the negative real axis. Thus $E(s)$ may be thought of as a polynomial whose coefficients serve the sole purpose of augmenting the corresponding coefficients of $P(s)$ in such a manner that the zeros of the resulting polynomial are shifted onto the negative real axis. Since $E(s)$ is also the numerator polynomial of the sensitivity function associated with the gain of the voltage follower (as will be shown later), it is desirable to choose $E(s)$ so that its effect on this sensitivity is minimized. This problem is discussed fully in Chapter V.

In conclusion, Figure 4.1 is a pictorial representation of the following synthesis problem.

Synthesize a four-terminal RC network whose characteristic polynomial $D$ possesses arbitrary negative real axis zeros, such that when terminal (1) is driven by $D$ volts and the other terminals are shorted to ground, $A(s)$ amps flow through terminal (2), $E(s)$ amps flow through terminal (3), and $R(s)$ amps flow through terminal (4). Once this network is realized, it is connected as shown in Figure 1.1 with terminal (4) becoming the ground terminal. The resulting voltage transfer function is then, of course, $T_v = A(s)/P(s)$.

This pictorial representation as discussed above and shown in Figure 4.1 will be used extensively in this chapter.
B. REVIEW OF PREVIOUSLY PROPOSED CONFIGURATIONS

Before presenting new synthesis techniques, a review of existing techniques will be given. Sallen and Key (14) present a catalog of 18 circuits which, when used in conjunction with an amplifier of gain \( K_0 \), produce second order transfer functions of the general form

\[
T_v = \frac{a_2s^2+a_1s+a_0}{p_2s^2+p_1s+p_0}
\]  

(4.2)

Although the amplifier gain \( K_0 \) of these circuits is neither restricted to nor required to be unity in all cases, many of the cataloged circuits are useful at this value of gain. Sallen and Key, in fact, point out the advantage of using this value. When unity gain is permissible, the networks of Sallen and Key reduce to the (RCVF) case. Those networks which are permissible are 10 in number. A study of them shows that there is a wide choice of network element values for the realization of a specific network function belonging to a given network function form. It is interesting to note that none of the networks of Sallen and Key are valid for the realization of the network function of the form \( T_v = \frac{A(s)}{P(s)} = \frac{a_2s^2+a_1s+a_0}{p_2s^2+p_1s+p_0} \) in the (RCVF) configuration. It is also interesting to note that the work of Balabanian and Patel (15) which is an extension of that of Sallen and Key is completely inapplicable to the (RCVF)
case in that the gain of the active element is in the vicinity of unity but that a gain of exactly unity is inadmissible. For the synthesis of the networks of Sallen and Key, the reader is referred to their report (14).

C. A REALIZATION TECHNIQUE FOR THE BIQUADRATIC TRANSFER FUNCTION

Consider the biquadratic transfer function

\[ T_v = K \frac{a_2 s^2 + a_1 s + a_0}{p_2 s^2 + p_1 s + p_0} \]  

(4.3)

whose coefficients satisfy the necessary and sufficient conditions of the \((RCVF)\). Assume all coefficients are greater than zero and \(K\) is made sufficiently small such that the resulting numerator coefficients are less than the corresponding denominator coefficients. In order to realize \(T_v\) as the voltage transfer function of the \((RCVF)\), it is sufficient to realize a four-terminal network whose current distribution as defined in Section IV A is shown in Figure 4.2.

Since the complex left-half plane zeros of a quadratic polynomial may be shifted onto the negative real axis by increasing the coefficient of the first order term, it is permissible that \(E(s)\) be no more complicated than \(E(s) = e_1 s\).
It will now be shown that the biquadratic $T_v$ with the previously mentioned restrictions may be realized with no more than three capacitors and four resistors. The effect of removing the restrictions on the coefficients will then be examined and it will be shown that the required number of elements may often be reduced.

Assume the characteristic polynomial, $D(s)$, of the four-terminal passive network to be realized is $D(s) = d_1 s + 1$ where $d_1 = \min(a_1/a_0, r_1/r_0)$ and is greater than zero since $a_0$, $a_1$, $r_0$, $r_1$ are all greater than zero.

Now consider the network configuration shown in Figure 4.3. It is easily verified that this configuration has an input current distribution the same as that of Figure 4.2 and that the characteristic polynomial is of the form $D(s) = d_1 s + 1$. Thus, the driving point current may be considered as $p_2 s^2 + (p_1 + e_1) s + p_0$ as shown in Figure 4.3.

To calculate the current distribution for the remainder of the network, it is possible to perform a continued fraction expansion of the driving point admittance to obtain the values of $R_1$ and $C_1$ as:

$$R_1 = \frac{1}{p_0} \quad C_1 = p_1 + e_1 - d_1 p_0$$
Figure 4.3. Network configuration with input current distribution of Figure 4.2.

\[ I_a = p_2 s^2 + (p_1 + e_1)s + p_0 \]
\[ I_b = p_2 s^2 + (p_1 + e_1 - d_1p_0)s \]

Figure 4.4. Current distribution for case of \((a_1/a_0) \leq (r_1/r_0)\).
The current flowing through \( R_1 \) is then seen to be \( (d_1 p_0 s + p_0) \) amperes and the current through \( C_1 \) is \( p_2 s^2 + c_1 s \). The continued fraction expansion is continued to obtain

\[
C_2 = \frac{p_2}{d_1 - \frac{p_2}{c_1}} \quad \text{and} \quad R_2 = \frac{d_1 - \frac{p_2}{c_1}}{c_1}
\]

The voltage \( V_R \) and the currents through \( C_2 \) and \( R_2 \) are shown in Figure 4.3. It is easily seen that \( C_1, C_2, \) and \( R_2 \) are positive provided \( e_1 \) is chosen sufficiently large.

The elements \( R_1, C_2 \) and \( R_2 \) are now divided in such a manner that paths for the currents \( \left( \frac{r_0 a_1}{a_0} s + r_0 \right), (a_1 s + a_0), (r_2 s^2), (a_2 s^2), (r_1 - \frac{r_0 a_1}{a_0}) s, \) and \( e_1 s \) are provided for the case of \( (a_1/a_0)^2 = (r_1/r_0) \). For the case of \( (r_1/r_0)^2 = (a_1/a_0) \), the elements are divided in such a manner that paths for the currents \( (r_1 s + r_0), \left( \frac{a_0 r_1}{r_0} s + a_0 \right), (r_2 s^2), (a_2 s^2), (a_1 - \frac{a_0 r_1}{r_0}) s, \) and \( e_1 s \) are provided. The current distributions for these two cases are shown in Figures 4.4 and 4.5, respectively.

It is readily seen that the networks of Figures 4.4 and 4.5 may be put into the form of Figure 4.2. Thus, these two networks will realize in the (RCVF) configuration any biquadratic voltage transfer function \( T_v = \frac{a_2 s^2 + a_1 s + a_0}{p_2 s^2 + p_1 s + p_0} \) to within a constant multiplier \( K \) provided all coefficients are greater than zero.
Figure 4.5. Current distribution for case of \( \frac{r_1}{r_0} \equiv \frac{a_1}{a_0} \)
The preceding was based on the fact that $K$ be chosen sufficiently small that the numerator coefficients of $T_v$ be less than the corresponding denominator coefficients. Since maximum gain is usually desired, it is desirable to increase $K$ until at least one of the numerator coefficients is equal to its corresponding denominator coefficient. In other words, $K$ is increased until at least one of the coefficients of the polynomial $R(s)$ is zero.

Assuming that only one of the coefficients of $R(s)$ is reduced to zero, three possibilities will be examined. If $r_2$ is zero, it is noted that the number of required capacitors is reduced by one. If $r_0$ is zero, the number of required resistors is reduced by one. For the case of $r_1 = 0$, the proposed network configuration no longer holds, and one must either back off on the value of $K$ or attempt a different network configuration. The four remaining cases, when two or three of the coefficients of $R(s)$ simultaneously vanish as $K$ is increased, are similarly analyzed.

D. LADDER NETWORK REALIZATION FOR LOW-PASS AND HIGH-PASS TRANSFER FUNCTIONS

A review of a synthesis procedure proposed by the present author (16) for the synthesis of arbitrary order low-
pass active filters in the (RCVF) configuration will be given. As an extension of this work, it will be proven that the 4th order low-pass transfer function may always be realized by this method. In addition, the method will be extended to include high-pass filters.

Consider the basic ladder network shown in Figure 4.6 and then consider each $Y_j (j=1,2,...,n)$ decomposed into two subnetworks whose admittances are $Y'_j$ and $Y''_j$ such that $Y'_j + Y''_j = Y_j$. Each of the $Y_j$ ground terminals may be disconnected from ground and soldered together to form a new terminal as shown in Figure 4.7. This altered network may be thought of as a decomposed ladder network. The notation has been changed in Figure 4.7 to agree with the current distribution notation used in previous sections. It is seen that the purpose of decomposing the admittances is to shunt the currents $R(s)$ and $E(s)$ to separate terminals. The success of the synthesis procedure presented in this section depends upon the ability to perform this separation. Note that an impedance added in series with terminal (1) or an admittance placed from terminal (2) to the reference node has no effect on the voltage transfer function of the corresponding (RCVF).

A synthesis procedure for low-pass transfer functions in the form of the decomposed ladder will now be given with
Figure 4.6. The basic ladder network.

Figure 4.7. The basic decomposed ladder network.
no guarantee that the resulting passive elements have non-negative values. Experience has shown that all of the network functions so far encountered of the low-pass form

\[ T_v = \frac{1}{p_n s^n + p_{n-1} s^{n-1} + \cdots + p_1 s + 1} = \frac{1}{P(s)} \]  

may be realized. A proof is offered that all functions of this form for the case \( n=4 \) may be realized by the decomposed ladder network. The proposed synthesis procedure now follows with the shunt arms of Figures 4.6 and 4.7 assumed to be single capacitors and the series arms assumed to be single resistors as shown in Figure 4.8.

1. Choose a polynomial \( E(s) \) of degree less than or equal to \( n \) and a polynomial \( D(s) \) of degree \( (n-1) \) such that \( y_{11} = \frac{P(s) + E(s)}{D(s)} \) is a driving point admittance conforming to the configuration of Figure 4.6. Magnitude scale \( y_{11} \) so that \( P(0) + E(0) = D(0) = 1 \). Since \( P(0) = 1 \), then \( E(0) = 0 \). Note that the zeros of \( [P(s) + E(s)] \) and \( D(s) \) must be interlaced along the negative real axis of the complex frequency plane.

2. Perform a continued fraction expansion on \( \frac{P(s) + E(s)}{D(s)} \) to determine the element values for the configuration shown in Figure 4.6. At each step of the expansion, label the denominator polynomial of the ratio of polynomials used to determine
Figure 4.8. Decomposed ladder network with shunt arm capacitors and series arm resistors.

\[ D(s) = K(s + \delta_1)(s + \delta_4)(s + \delta_6) \]
\[ V_a = K_2(s + \delta_2)(s + \delta_5) \]
\[ V_b = K_3(s + \delta_3) \]
\[ V_c = K_4 \]

\[ I_a = k_1 s(s + \delta_1)(s + \delta_4)(s + \delta_6) \]
\[ I_b = k_2 s(s + \delta_2)(s + \delta_5) \]
\[ I_c = k_3 s(s + \delta_3) \]
\[ I_d = k_4 s \]

\[ k_j = 0.5K_j \quad (j=1,2,3,4) \]

Figure 4.9. Current distribution for basic ladder network realization of \( y_{11} \).
The appendix of reference 16 contains a proof that the synthesis procedure outlined above is valid. The reference does not contain any guarantee that the element values so realized are nonnegative. It is shown below that the 4th order low-pass transfer functions may always be realized with nonnegative elements by this method.

Figure 4.9 shows the basic three-terminal passive RC network which must be decomposed in order to realize a 4th order low-pass transfer function by the method outlined in this section. As explained previously, this network has been synthesized from an RC driving point admittance.
where the trailing coefficient of the numerator and denominator polynomials have been normalized to unity. If the current and voltage distribution of Figure 4.9 is calculated, it is seen to have the form shown. Since the object of a continued fraction expansion of $y_{11}$ is to reduce the order of the polynomials associated with the current and voltage distributions by one at each step of the expansion, the distributions of Figure 4.9 are valid.

The pole-zero pattern of the continued fraction expansion of $y_{11}$ will now be reviewed to show that $\delta_1 = \delta_2 = \cdots = \delta_6$. It is well known that the poles and zeros of $y_{11}$ must be interlaced along the negative real axis. Thus, $y_{11}$ is of the form

$$y_{11} = K \frac{(s+\eta_1)(s+\eta_3)(s+\eta_5)(s+\eta_7)}{(s+\delta_1)(s+\delta_4)(s+\delta_6)}$$

where $\eta_1 \leq \delta_1 \leq \eta_3 \leq \delta_4 \leq \eta_5 \leq \delta_6 \leq \eta_7$ are all positive real numbers. The pole-zero distribution of $y_{11}$ is shown in Figure 4.10(a). The first step of the continued fraction expansion is to remove the pole of $y_{11}$ at infinity. The admittance remaining after this removal is $y_{11}^{(1)} = y_{11}^{(0)} - C_1 s$ where $y_{11}^{(0)}$ is the original $y_{11}$. Since $C_1$ is guaranteed positive,
Figure 4.10. Pole-zero distributions for $y_{11}^{(0)}$ to $y_{11}^{(6)}$. 
it is easily verified that the effect of the above pole removal is to shift all of the zeros to the left except for the most negative zero which disappears. The poles are unchanged. The resulting pole-zero pattern for $y_{11}^{(1)}$ is shown in Figure 4.10 (b). The next step in the expansion is to shift the most negative pole of $y_{11}^{(1)}$ to infinity. This is accomplished by subtracting a positive constant $R_1$ from $\frac{1}{y_{11}^{(1)}}$. The resulting admittance is $y_{11}^{(2)} = \frac{1}{\frac{1}{y_{11}^{(1)}}} - R_1$.

The effect of this extraction is to leave the zeros untouched and to shift all poles to the left. $R_1$ is chosen so as to shift the most negative pole to infinity. The resulting pole-zero pattern for $y_{11}^{(2)}$ is shown in Figure 4.10 (c). The preceding two steps are repeated until the order of the admittance is reduced to zero. The pole-zero patterns for each of these subsequent steps is shown in Figure 4.10. Notice that the poles of $y_{11}^{(k)}$ for $k$ even become the zeros of the currents flowing through the capacitors in the current distribution diagram of Figure 4.9. A study of the regions to which these poles are restricted leads to a subscript notation for the $\delta$'s such that $\delta_1 < \delta_2 < \ldots < \delta_6$. The one exception to this inequality is that $\delta_3$ may be larger than $\delta_4$. This exception is of no consequence in the following development.
If the proposed synthesis procedure is to be successful, 
\([P(s)+E(s)]\) and \(D(s)\) must be chosen to have produced currents 
through the capacitors of Figure 4.9 such that a set of \(h\)'s 
may be chosen to satisfy the equality

\[
\begin{align*}
h_1 s(s+\delta_1)(s+\delta_4)(s+\delta_6) + h_2 s(s+\delta_2)(s+\delta_5) + h_3 s(s+\delta_3) + h_4 s &= \\
p_4 s^4 + p_3 s^3 + p_2 s^2 + p_1 s
\end{align*}
\]

(4.7)

where \(C=h_j \geq k_j\) \((j=1,2,3,4)\). Equating coefficients leads to 
the following set of equalities

\[
\begin{align*}
h_1 &= p_4 \quad \text{(4.8)} \\
h_1 (\delta_1 + \delta_4 + \delta_6) + h_2 &= p_3 \quad \text{(4.9)} \\
h_1 (\delta_1 \delta_4 + \delta_1 \delta_6 + \delta_4 \delta_6) + h_2 (\delta_2 + \delta_5) + h_3 &= p_2 \quad \text{(4.10)} \\
h_1 (\delta_1 \delta_4 \delta_6) + h_2 (\delta_2 \delta_5) + h_3 \delta_3 + h_4 &= p_1 \quad \text{(4.11)}
\end{align*}
\]

From the polynomial \(D(s)=d_3 s^3 + d_2 s^2 + d_1 s + 1=K_1 (s+\delta_1)(s+\delta_4)(s+\delta_6)\),
we have

\[
\begin{align*}
(\delta_1 + \delta_4 + \delta_6) &= d_2 / d_3 \quad \text{(4.12)} \\
\delta_1 \delta_4 + \delta_1 \delta_6 + \delta_4 \delta_6 &= d_1 / d_3 \quad \text{(4.13)} \\
\delta_1 \delta_4 \delta_6 &= 1 / d_3 \quad \text{(4.14)}
\end{align*}
\]

If \((\delta_1 + \delta_4 + \delta_6)\) is chosen equal to \(p_3/p_4\), then a solution of 
equations (4.8) through (4.14) acquires the \(h\)'s in terms of 
the coefficients of \(P(s)\) and \(D(s)\) and the quantity \(\delta_3\).

\[
\begin{align*}
h_1 &= p_4 \quad \text{(4.15)} \\
h_2 &= 0 \quad \text{(4.16)} \\
h_3 &= p_2 - p_4 \frac{d_1}{d_3} \quad \text{(4.17)} \\
h_4 &= p_1 - \frac{d_1}{d_3} - \delta_3 (p_2 - p_4 \frac{d_1}{d_3}) \quad \text{(4.18)}
\end{align*}
\]
Since the \( k \)'s in the necessary conditions \( 0 \leq h_j \leq k_j \) 
\((j=1,2,3,4)\) may be made arbitrarily large by choosing the \( K \) 
of Equation (4.6) arbitrarily large, the conditions actually reduce to 
\( 0 \leq h_j \) \((j=1,2,3,4)\). These conditions are reflected 
in Equations (4.17) and (4.18) as
\[
\frac{d_1}{d_3} = 0 \quad (4.19)
\]
\[
\frac{p_2-p_4}{d_3} - \delta_3 (p_2-p_4 \frac{d_1}{d_3}) = 0 \quad (4.20)
\]

Since \( p_2 \) and \( p_4 \) are greater than zero and the quantity 
\((\delta_1 \delta_4 + \delta_1 \delta_6 + \delta_4 \delta_6) = \frac{d_1}{d_3} \), may be made as small as desired by 
making \( \delta_1 \) and \( \delta_4 \) (for example) arbitrarily small while holding 
\((\delta_1 + \delta_4 + \delta_6) = \frac{d_2}{d_3} \) constant, the condition of Equation (4.19) 
may always be fulfilled. Likewise \((\delta_1 \delta_4 \delta_6) = \frac{1}{d_3} \) may also be 
made as small as desired by making \( \delta_1 \) and \( \delta_4 \) arbitrarily small. The quantity \((\delta_1 - \delta_3) \) may be made as small as desired by 
choosing the zero of \([P(s) + E(s)] \) nearest the origin \( \eta_1 \) 
arbitrarily close to the \( \delta_1 \) term. Since \( \delta_1 \) in turn may be 
chosen as small as desired, it follows that \( \delta_3 \) may be chosen 
as small as desired. Thus, the condition of Equation (4.20) 
may always be fulfilled, and it follows that \([P(s) + E(s)] \) 
and \( D(s) \) may always be chosen such that the transfer function 
\( T_v \) of Equation (4.4) with \( n=4 \) may always be realized.

A means for realizing high-pass functions of the form
based on the methods of this section will now be given. The method is based on the following steps.

1. Perform a high-pass to low-pass transformation on the $T_v$ of Equation (4.21) to obtain a $T_v$ of the form of Equation (4.4).

2. Realize this transformed $T_v$ by the methods of this section.

3. Divide the admittance of each of the elements of the realized network by $s$. This is equivalent to multiplying the characteristic polynomial by $s$. Thus, the current distribution of the network is unchanged.

4. Now perform a low-pass to high-pass transformation on the network to obtain the high-pass realization of the original $T_v$.

It is easily verified that the proposed high-pass synthesis method is valid.

E. A NETWORK CONFIGURATION FOR FOURTH-ORDER BAND-PASS TRANSFER FUNCTIONS.

A passive network configuration is now presented which may be used in conjunction with the (RCVF) to realize a
fourth-order band-pass transfer function of the form

\[ T_v = \frac{a_2 s^2}{p_4 s^4 + p_3 s^3 + p_2 s^2 + p_1 s + 1} \]  

(4.22)

where the coefficients are assumed to meet the necessary conditions of Chapter III. With respect to active device sensitivity and spread of element values, this network represents the best of a considerable number of possible configurations investigated.

The three basic building blocks of the proposed configuration are shown in Figure 4.11. These basic blocks are placed in parallel, and elements are disconnected from ground and joined together to form terminals (2) and (3) of the sought after three-port (RCVF) passive network. These terminals are properly marked in Figure 4.11. Since the three networks are to be paralleled, they may be assumed to be driven by the same voltage source \((D(s))\) for the purpose of current distribution calculation. Thus, it is seen that the present configuration was obtained by an examination of the current distribution for various subnetworks.

A discussion of the choice of network parameters to realize a transfer function of the form of Equation (4.22) will now be given. First choose a third-order denominator
Figure 4.11. Three basic building blocks for the fourth-order band-pass transfer function realization.
polynomial

\[ D(s) = (s+\delta_1)(s+\delta_2)(s+\delta_3) \]  

(4.23)

where \( \delta_1, \delta_2, \delta_3 \), are unequal positive real quantities. Referring to Figure 4.11, \( D_1 \) is seen to be \((s+\delta_2)(s+\delta_3)\) and \( D_2 \) is \((s+\delta_1)(s+\delta_3)\). Notice that the current flowing through the grounded capacitor of the "a" network is \( s^4+(\delta_2+\delta_3)s^3+\delta_2\delta_3s^2 \). If \((\delta_2+\delta_3)\) is chosen equal to \( p_3/p_4 \) and the admittance of the "a" network is magnitude scaled by the factor \( p_4 \), this current becomes \( p_4s^4+p_3s^3+\delta_2\delta_3p_4s^2 \). The "a" network is seen to take care of the realization of the two leading coefficients of the polynomial \( P(s) \). Also notice that the current flowing through the grounded resistor of the "b" network is \( \eta_2s^2+\eta_2(\delta_1+\delta_3)s+\eta_2\delta_1\delta_3 \). If \((\delta_1+\delta_3)/\delta_1\delta_3 \) is chosen equal to \( p_1 \) and the admittance level of the "b" network is scaled by \( 1/\eta_2\delta_1\delta_3 \), this current becomes \((1/\delta_1\delta_3)s^2+p_1s+1\) and may be considered to realize the two trailing coefficients of \( P(s) \).

The "c" network accomplishes the realization of the \( T_y \) numerator and denominator quadratic term. \( \eta_3 \) and \( \eta_4 \) of Figure 4.11 (c) are chosen to interlace with the \( \delta \)'s to produce an RC admittance function. The admittance \( Y = s(s+\eta_3)(s+\eta_4)/(s+\delta_1)(s+\delta_2)(s+\delta_3) \) is then realized in the configuration shown. A current distribution computation for the "c" network shows that the current through the final resistor is
The admittance level of the network may then be scaled. Since a quadratic term current \( \left( \frac{1}{\delta_1 \delta_3} + \delta_2 \delta_3 p_4 \right) \) has already been realized, the scaling factor is chosen such that the resulting current flowing through the final resistor is \( (p_2 - \frac{1}{\delta_1 \delta_3} - \delta_2 \delta_3 p_4) s^2 \). It is seen that this current is the \( A(s) \) portion of the realization and is equal to \( a_2 \). Thus, the proposed network realized the desired band-pass transfer function to within a constant multiplier. It is apparent that the success of the method depends on the ability to choose the \( \delta \)'s such that the quantity \( (p_2 - \frac{1}{\delta_1 \delta_3} - \delta_2 \delta_3 p_4) s^2 \) is positive. Notice that no restriction has been placed on the relative magnitudes of the \( \delta \)'s. The element values of the "c" network are not shown as they become quite complicated in terms of the \( \delta \)'s and \( \eta \)'s. It is far easier to work with numerical values. The synthesis method as presented is only valid when \( (\delta_2 + \delta_3) \) may be chosen equal to \( p_3 / p_4 \) and \( (\delta_1 + \delta_3) / \delta_1 \delta_3 \) simultaneously chosen to equal \( p_1 \). It is obvious that there are cases where this may not be done. For those cases it is suggested that a current distribution diagram applied to the subnetworks might suggest alternative realizations.

F. EXAMPLES

This section will present numerical examples of the syn-
thesis procedures presented in the previous sections of Chapter IV. Each example problem will be presented in the current distribution form of Figure 4.1.

For the first example, consider the biquadratic transfer function of Section I.C.

\[ T_v = K \frac{s^2+2.828s+4}{s^2+1.414s+1} \]  \hspace{1cm} (4.23)

This transfer function will be realized by the methods of Section IV.C. The current distribution for this \( T_v \) is shown in Figure 4.12 for the case of \( K=\% \) (maximum value allowable). Since \( \frac{a_1}{a_0} < \frac{r_1}{r_0} \), we take the network of Figure 4.4 and easily calculate the element values. The resultant network is shown in Figure 4.13 for the case of \( e_1 \) having been chosen equal to 3.293. Note that \( e_1 \) could have been chosen considerably less since the only requirement on \( e_1 \) is that the polynomial \( s^2 + (1.414 + e_1)s + 1 \) have negative real zeros. This network connected to the voltage follower is shown in Figure 1.5.

The next example will demonstrate the low-pass ladder network realization of Section IV.D. Consider the Butterworth low-pass transfer function

\[ T_v = \frac{1}{s^4+2.613s^3+3.414s^2+2.613s+1} \]  \hspace{1cm} (4.24)

The first step of the procedure consists of choosing a poly-
\[ P(s) + E(s) = s^2 + (1.414+e_1)s + 1 \]

\[ A(s) = \frac{\%s^2 + 0.707s + 1}{s^2 + 4.707s + 1} \]

\[ E(s) = e_1s \]

\[ R(s) = \frac{\%s^2 + 0.707s}{\%s^2 + 0.707s} \]

Figure 4.12. Current distribution for network corresponding to transfer function of Eq. (4.23).

\[ P(s) + E(s) = \frac{s^2 + 2.828s + 4}{s^2 + 1.414s + 1} \]

\[ A(s) = \frac{\%s^2 + 0.707s + 1}{s^2 + 4.707s + 1} \]

\[ E(s) = 3.293s \]

\[ R(s) = \frac{\%s^2 + 0.707s}{\%s^2 + 0.707s} \]

Figure 4.13. Network realization for \( T_v = \frac{s^2 + 2.828s + 4}{s^2 + 1.414s + 1} \)
nomial $E(s)$ of degree less than or equal to four and a polynomial $D(s)$ of degree three such that $y_{11} = \left[ P(s) + E(s) \right] / D(s)$ is a driving point admittance conforming to the configuration of Figure 4.6. The polynomials

$$E(s) = 12.9710s^3 + 25.1919s^2 + 11.1407s$$

and

$$D(s) = 2.655s^3 + 6.939s^2 + 5.2260s + 1$$

fulfill these requirements.

A continued fraction expansion of $\left[ P(s) + E(s) \right] / D(s)$ is performed to determine the element values for the configuration shown in Figure 4.6. The $D_j$ polynomials are identified during this procedure as shown below.

$$C_1s = 0.3765s$$

$$D_1 = 2.655s^3 + 6.939s^2 + 5.2260s + 1$$

$$12.9710s^3 + 26.6386s^2 + 13.3774s + 1$$

$$R_1 = 0.2047$$

$$C_2s = 8.728s$$

$$D_2 = 1.4862s^2 + 2.4876s + 0.7953$$

$$\sqrt{12.9710s^3 + 26.6386s^2 + 13.3774s + 1}$$

$$12.9710s^3 + 21.7126s^2 + 6.9405s$$

$$R_2 = 0.3017$$

$$4.9260s^2 + 6.4369s + 1$$

$$\sqrt{1.4862s^2 + 2.4876s + 0.7953}$$

$$1.4862s^2 + 1.9420s + 0.3017$$
\[-55-\]

\[C_{ss} = 9.027s\]

\[D_2 = 0.5456s + 0.4936 \pm \sqrt{\frac{4.9260s^2 + 6.4369s + 1}{4.9260s^2 + 4.4548s}}\]

\[R_2 = 0.2753\]

\[1.9821s + 1 \pm \sqrt{0.5456s + 0.4936 \pm 0.5456s + 0.2753}\]

\[C_{ss} = 9.0843s\]

\[D_4 = 0.2183 \pm \sqrt{1.9821s + 1 \pm 1.9821s}\]

\[R_4 = 0.2183\]

\[1 - \frac{\sqrt{0.2183}}{0.2183}\]

The \(C_j\) may now be determined as

\[C_{ss} = 8.728s\]

\[D_2 = 1.4862s^2 + 2.4876s + 0.7953 \pm \sqrt{12.9710s^3 + 25.1919s^2 + 11.1470s \pm 12.9710s^3 + 21.7126s^2 + 6.9405s}\]

\[C_{ss} = 6.377s\]

\[D_2 = 0.5456s + 0.4936 \pm \sqrt{3.4793s^2 + 4.2002s \pm 3.4793s^2 + 3.1477s}\]

\[C_{ss} = 4.828s\]

\[D_4 = 0.2183 \pm \sqrt{1.0525s \pm 1.0525s}\]

and \(C_j\) determined as \(C_j = C_j - C_j^n\). The above calculations
result in the network shown in Figure 4.14.

The final example is concerned with a demonstration of the fourth-order band-pass transfer function synthesis of Section IV.E. This example function is

\[ T_v = K \frac{s^2}{s^4 + 1.414s^3 + 3s^2 + 1.414s + 1} \]  

(4.27)

The first step is to choose the zeros of \( D(s) \). These are chosen as \( \delta_1 = 2.42, \delta_2 = 0.414, \delta_3 = 1.00 \). Note that the quantity \( p_2 - \frac{1}{\delta_1 \delta_3 \delta_4} \) is equal to 2.172. Thus, the above choice is permissible and leads to a value of \( K = 2.172 \) and

\[ D(s) = (s + 2.42)(s + 0.414)(s + 1.00) \]  

(4.28)

The \( \eta \)'s of the network are chosen as \( \eta_1 = 2.667, \eta_2 = 0.343, \eta_3 = 0.600, \) and \( \eta_4 = 1.667 \). These values lead to the network realization of Figure 4.15 with the terminals for the configuration of Figure 4.1 properly marked.
Figure 4.14. Network realization for \( T_v = \frac{1}{s^4 + 2.6132s^3 + 3.4143s^2 + 2.6132s + 1} \).
Figure 4.15. Network realization for $T_v = \frac{2.172s^2}{s^4 + 1.414s^3 + 3s^2 + 1.414s + 1}$. 
V. NETWORK OPTIMIZATION

This chapter investigates the possibility of utilizing the digital computer to optimize a given network configuration. A particular class of networks is selected for use as an example of the optimization process and a computer program is developed which optimizes any low-pass or, by a suitable transformation, any high-pass transfer function realized by the decomposed ladder method of Section IV.D. Section A of the present chapter is devoted to a discussion of the network parameters to be optimized. Section B is concerned with a description of the optimization computer program and the results of the program on an example network. The prime purpose of Section A is to define the optimization parameter Q while the latter section will be concerned with the mechanics involved in minimizing this parameter.

A. OPTIMIZATION PARAMETERS

The first parameter to be discussed is the sensitivity parameter. Bode first introduced the term "sensitivity" in connection with feedback analysis (30). Since then the term has become commonly defined as the logarithmic derivative of $T(s,k)$ with respect to the logarithm of $k$ where $T(s,k)$ is a network function of interest (in the present case the voltage
transfer function $T_v$) and $k$ is a variable element. Thus, the sensitivity parameter is

$$S_k = \frac{\partial (\ln T)}{\partial (\ln k)} = \frac{k}{T} \frac{\partial T}{\partial k} \quad (5.1)$$

The above definition of sensitivity may be looked upon as representing the ratio of a percentage change in a given network function to a percentage change in a variable element of the network function. Thus, ignoring phase angles, if the network function varies by 0.01 db for a 1 db variation in $k$, we say the sensitivity is 0.01. As an example of the use of the sensitivity parameter in the present investigation, consider that the gain of the active device departs from unity and is represented by the variable $k$. This represents the actual situation since in any practical case, the active device will depart from the ideal by some amount. Expression (1.5) which is the expression for $T_v$ in terms of the applicable admittance polynomials must then be altered to

$$T_v = \frac{N_{12}}{N_{11} - kN_{13}} = \frac{A(s)}{P(s)} \quad (5.2)$$

Applying the definition of (5.1) to this equation results in

$$S_k^T_v = \frac{kN_{13}}{N_{11} - kN_{13}} \quad (5.3)$$

Since $k$ is approximately unity and $(N_{11} - kN_{13})$ is $P(s)$,

Equation (5.3) may be written as

$$S_k^T_v \approx \frac{N_{13}}{P(s)} \quad (5.4)$$
Since the network realization has no control of \( P(s) \), this equation shows that the only possibility of sensitivity minimization lies in the choice of \( N_{13} \) which is the \( E(s) \) of the pictorial representation of Figure 4.1.

An alternate approach to sensitivity is to utilize the pole-zero sensitivity which follows when the sensitivity of \( T(s,k) \) is obtained in terms of its poles and zeros (31). Let

\[
T(s,k) = g \prod_{i=1}^{m} \frac{1}{T_i(s+Z)}
\]

where the \( Z_i, P_i, \) and \( g \) are, in general, functions of some parameter \( k \). Taking the logarithm of each side, differentiating with respect to \( k \), and multiplying both sides by \( k \)

\[
S^T_k = S^g_k + \sum_{i=1}^{m} \tilde{S}^Z_k \cdot \frac{1}{s+Z} - \sum_{i=1}^{n} \tilde{S}^P_k \cdot \frac{1}{s+P} \tag{5.6}
\]

where \( \tilde{S}^Z_k = k \frac{\partial Z}{\partial k}; \tilde{S}^P_k = k \frac{\partial P}{\partial k} \)

are defined respectively as the zero and pole sensitivity of the network function \( T \).

Still another type of sensitivity is "coefficient sensitivity" (18). If \( T(s,k) = \sum_{i=0}^{n} t_i(k)s^i \), then coefficient sensitivity is defined as
which is seen to be the ratio of a percentage change in a given polynomial coefficient to a percentage change in the variable element $k$.

In the present investigation, we are interested in the variable element $k$ representing the values of the components in the passive portion of the network as well as the gain of the active portion of the (RCVF).

Other network parameters which will be of interest are element distribution (33), element sum, and element ratio. Capacitance and resistance distribution are defined respectively as

$$C_D = \sum_{i=1}^{n_c} (C_i - \bar{C})^2$$  \hspace{1cm} (5.7)

and

$$R_D = \sum_{i=1}^{n_r} (R_i - \bar{R})^2$$  \hspace{1cm} (5.8)

where $\bar{C}$ and $\bar{R}$ are the average values of network capacitance and resistance respectively. Capacitance sum $C_s$ is the total capacity of the network

$$C_s = \sum_{i=1}^{n_c} C_i$$  \hspace{1cm} (5.9)

This quantity is of interest since it is proportional to the
volume and cost of the resultant network.

Element ratio is defined as the ratio of maximum element value to minimum element value. Thus, capacitance and resistance ratio are defined respectively as

\[ C_R = \frac{C_{\text{max}}}{C_{\text{min}}} \quad \text{and} \quad R_R = \frac{R_{\text{max}}}{R_{\text{min}}} \]  

(5.10)

It is desirable, in practice, to minimize each of the parameters listed above. In general, however, it is impossible to minimize them simultaneously. In order to reduce a multitude of optimization parameters to a single optimization parameter to be minimized, an overall optimization parameter will now be defined together with its associated optimization problem. The optimization parameter \( Q \) will be defined as

\[
Q = \sum_{i} w_{k,a_i} |S_{a_i}^k| + \sum_{i} w_{k,P_i} |S_{P_i}^k| + \sum_{j} w_{y,P_i} |S_{y,P_i}^j| + \sum_{i} y_{y,a_i} |S_{y}^a_i| + \sum_{i} \sum_{j} w_{C,j,P_i} |S_{C,j}^P_i| \\
+ \sum_{i} \sum_{j} w_{C,j,R} a_i |S_{C,j}^R_i| + \sum_{i} \sum_{j} w_{R,j,P_i} |S_{R,j}^P_i| + \sum_{i} \sum_{j} w_{R,j,R} |S_{R,j}^R| \\
+ w_{C,D} C_{D} + w_{R,D} R_{D} + w_{C,S} C_{S} + w_{C,R} C_{R} + w_{R,R} R_{R}
\]  

(5.11)

where all quantities have previously been defined (k is the gain of the voltage follower) except for the w's which are arbitrarily assigned weighting factors. These may be used
to upgrade or downgrade the relative importance of the component network parameters. The corresponding optimization problem is to minimize $Q$ for a given set of weighting factors $w$ subject to the usual network constraints of nonnegative element values.

An important consideration in optimization problems is that of the constraints. It may be reasoned in the present problem that the constraints are, in effect, absent. This is true since the network element values are continuous functions of the independent parameters. The element values of the initial network are all positive. If an element value becomes negative, it must have passed through a value of zero. This means that either the value of $C_R$ or $R_R$ passed through infinity. As long as $w_{C_R}$ and $w_{R_R}$ have finite values, the condition of negative element value will either never occur or else signify that too large a step size has been taken in the optimization process. This latter condition is handled by taking the step over again with reduced step size. Thus, in the present problem, the constraints are effectively absent.

An examination of the optimization parameter $Q$ shows that, once a network has been synthesized, all of the components of $Q$ may easily be evaluated with the possible ex-
ception of the component sensitivity parameters. The evaluation of the component sensitivity parameters involves increasing each element in turn by one-percent, evaluating the appropriate transfer admittances, and determining the change in the coefficients of the transfer function $T_v$. Fortunately, there are digital computer routines available which evaluate the transfer admittances of a network from a description of its topology.

There are a number of ways a solution to the optimization problem may be attempted. It should be pointed out at the outset that the optimization problem is extremely complicated and that no exact solution is attempted. One possibility for an approximate solution is to perform a purely random choice of the independent parameters of the subject network a large number of times with $Q$ being evaluated for each of the resultant networks. That network which corresponds to the lowest value of $Q$ is then selected as the final design.

Another possibility, and the one used in this investigation, is the method of steepest descent (35). This method is often compared in the literature to the predicament of being lost in the woods. Since civilization is usually concentrated at the lower elevations, one possibility for
reaching shelter is to travel at all times in that direction which maximizes one's descent. In optimization problems this corresponds to computing gradients at each step in the development of the solution and taking a new step in the direction of steepest descent. This process is continued until no further significant improvement is possible. The method consists of seeking a local minimum and it is seen that other minima are ignored. Since the computation of gradients can require considerable computational effort, a number of steps may be taken in a given direction in order to fully exploit a given gradient computation. This possibility is taken advantage of in the present investigation.

A preliminary step in the present optimization problem is the determination of a reasonable network topology and initial element values for a given transfer function $T_v$. This topology may either be selected from those presented in this report or devised by other means. The experience and ability of the network designer are utilized in this step. A "reasonable" network is defined here as one which appears to be amenable to optimization. The network synthesized in this step is called the initial network.

The computer program to be described in the next section accepts the initial network and optimization parameter
weighting factors and, by applying the method of steepest
descent, determines improved networks.

B. COMPUTER PROGRAM DESCRIPTION.

The computer program for network optimization has been
written in FORTRAN for the IBM 1620 (60K) computer with disk
storage capability. As explained previously, the computer
program is not written for a general network but for a par­
ticular class of networks. This class of networks is the
low-pass decomposed ladder network of Section IV.D. Since
many of the subroutines of the computer program may be used
if similar computer programs are written for other classes
of networks, and since the present computer program is valid
for a widely used class of network functions, the FORTRAN
statements for the computer program are printed in Appendix
I. In addition, the method and purpose of each subroutine
are explained in this section. This should enable the read­
er to recognize and adapt those routines which are of inter­
est to him.

The network optimization computer program consists of
16 subroutines and one main program. The main program has
been broken into five linked subprograms because of the
limited core storage of the 1620 computer. These five sub­
programs, which have been named CHOKE, CHTWO, CHTRI, CHFOUR and CHFIVE, are together considered as the main program. The flow chart for this main program is shown in Figure 5.1. The circuit analysis portion of the program is accomplished in terms of topological formulas based on linear graph theory. No attempt is made here to present the basic theory or to explain the terminology. The interested reader is referred to references 36 and 37 for an exposition of the theory and familiarization with the terminology. Before discussing the flow chart, each of the 16 subroutines will be explained.

SUBROUTINE RDIN. This subroutine is concerned with reading in input data by means of the card reader and computing some constants which will be used later in the program. The input data consists of the following quantities.

- **NOD** Number of branches in a complete tree of the network.
- **NELP** Number of network elements exclusive of the auxiliary elements.
- **NO** Order of the denominator of $T_\nu$.
- **NTYP** Array for identifying the type of each element of the network. +1 for a capacitor, 0 for a resistor and -1 for an inductor (inductors are not used in the present case).
Figure 5.1. Flow chart for network optimization computer program.
Figure 5.1. (Continued).
STP = STBES
CALL TKSTP

ALBAR(I) = ALPH(I), I = 1, NO
BTBAR(I) = BETA(I), I = 1, NMT

VOBAR = VOBJ

IF SENSE SWITCH 3
  DECS = ((VORJ - VOBJ) / VORJ) * 10000.
  DECS = ABSF(DECS)
ENDIF

IF DECS - 5.0 >= 0
  PRINT OUTPUT HEADING FOR FINAL NETWORK
ENDIF

CALL DMPOT
CALL EXIT

Figure 5.1. (Concluded).
IB : Identifying number of an element connected between the input and reference node of the subject network.

IJ : Identifying number of an element connected between the input and output node of the network.

KB : Identifying number of an element connected between the output and reference node of the network.

IFR : Array for identifying the node "from" which each element is connected.

ITO : Array for identifying the node "to" which each element is connected.

SP : Array for the coefficients of the denominator (P(s)) of T_v. SP(I) is the coefficient of the s term raised to the (I-1) power.

ALBAR : Array for zeros of the initial (P(s)+E(s)) polynomial.

BTBAR : Array for zeros of the initial D(s) polynomial.

WK : Array for the weighting factors \( w_{xpi} \).

WY : Array for the weighting factors \( w_{xpi} \).

WELM : Array for the weighting factors \( w_{C,jp_i} \) and \( w_{R,jp_i} \).

WCD : Set equal to the weighting factor \( w_{CD} \).
WRD Set equal to the weighting factor $w_{RD}$.
WCS Set equal to the weighting factor $w_{CS}$.
WCR Set equal to the weighting factor $w_{CR}$.
WRR Set equal to the weighting factor $w_{RR}$.

The subroutine computes the following quantities.

- **NSP** Number of coefficients in $P(s)$ (NO+1).
- **NTT** Number of branches in a 2-tree (NOD-1).
- **NVB** Number of independent variables associated with the network (2xNO-2).
- **NOMI** (NO-1).
- **NOMT** (NO-2).
- **NFAR** Number of core storage locations required for the storage of one tree (5 elements may be stored in one core storage location).

**SUBROUTINE TRCOM.** Determines and stores those 2-trees of the network which are utilized in the determination of the numerator polynomial of $y_{12} - y_{13}$. Upon completion of this subroutine, the applicable 2-trees are stored in the ITR array with each member of the array containing as many as five network element identification numbers. The quantity NTS is set equal to the number of 2-trees stored in the ITR array.

**SUBROUTINE TRDET.** This subroutine is called by TRCOM for
the purpose of determining if a given set of network branches forms a network tree. Given a set of NOD integers (LD array) identifying a set of branches of a network consisting of NOD independent node pairs, it is determined if the set forms a tree of the network. The subroutine returns the integer variable NEL as 0 if the set forms a tree. Otherwise NEL is set equal to K to denote that member of the LD array which represents the branch which first forms a circuit with those branches previously examined (the branches are examined beginning with branch number LD(NOD)). The method of determining whether or not a given set of branches constitutes a tree is the following.

1. Take each branch beginning with the branch identified by LD(NOD) and proceed through LD(NOD-1), etc., to LD(1). As each branch is selected, determine if the nodes of this branch when considered with the nodes of those branches already selected:
   a. Adds no new nodes.
   b. Adds one new node.
   c. Adds two new nodes.

2. If two new nodes are added, the branch selected forms a subgraph which is not connected to any which may have already been considered.

3. If one new node is added, the branch being considered forms an extension to one of the subgraphs
previously generated. No circuits are generated in this case.

4. If no new nodes are generated, there are two possibilities.
   a. Two unconnected subgraphs have been connected to form a single subgraph. No circuits are generated in this case.
   b. A circuit has been generated by the addition of this node in which case the quantity $NEL$ is set equal to the subscript of the member of the LD array which is currently involved, and the examination of branches is terminated since the set of branches being examined cannot possibly form a tree.

5. If all branches are examined without encountering the condition of 4b, the set of branches constitutes a tree and the quantity $NEL$ is set equal to zero.

SUBROUTINE TRCL. This subroutine classifies a tree into one of six categories. Only one of these categories is of interest in the present case, but since the routine is of use for other configurations, it is discussed here. The subroutine will accept a set of NOD integers (LD array) identi-
fying the branches of a tree of a network and assign the tree to one of six classifications by setting the integer variable ICLS equal to the classification number of the tree. Before defining the classification numbers, it is necessary to define several terms.

**INPUT TREE**  A tree which contains the branch IB.

**OUTPUT TREE**  A tree which contains the branch KB.

**PLUS TREE**  An input tree which forms a tree when branch KB is substituted for IB but does not form a tree when branch IJ is substituted for IB.

**MINUS TREE**  An input tree which forms a tree both when branch KB is substituted for IB and again when IJ is substituted for IB.

The ICLS integers may now be defined.

ICLS = 1, the tree does not fall into any of the categories given below.

= 2, the tree is an input tree.

= 3, the tree is an output tree.

= 4, the tree is both an input and output tree.

= 5, the tree is both an input and minus tree.

= 6, the tree is both an input and plus tree.

**SUBROUTINE SEQGEN.** This subroutine is an aid in the generation of all possible combinations of N things taken M at a
time. In the present case, the objective is to generate all possible trees from a given set of elements and nodes. Specifically, the subroutine takes a set of integers,
\[ LD(M), LD(M-1), \ldots, LD(2), LD(1) \]
where \( LD(1) > LD(2) > \ldots > LD(M-1) > LD(M) \) and \( LD(1) \leq N \), and generates a new set of integers:
\[ LD'(M), LD'(M-1), \ldots, LD'(2), LD'(1) \]
subject to the same constraints given above and the additional constraint that the quantity
\[
\sum_{j=1}^{n} (LD'(j) - LD(j))N^j
\]
is positive and the minimum value possible. In other words, the next highest in rank of \( N \) integers taken \( M \) at a time has now been generated.

An additional feature of this subroutine is the ability to skip over combinations. Suppose we have the sequence
\[ LD(M), LD(M-1), \ldots, LD(K+1), LD(K), LD(K-1), \ldots, LD(1) \]
and, instead of generating the next ranking sequence, we desire to skip to the next ranking sequence after
\[ LD(M), LD(M-1), \ldots, LD(K+1), LD(K), (N+2-K), \ldots, (N-1), N. \]
This is accomplished by the inclusion of the integer variable \( NEL \) in the input array of the subroutine where \( NEL \) is set equal to \( K \). If \( NEL \) is set less than 2, normal sequencing occurs. Upon returning from the subroutine, if \( NEL \) is zero,
the sequence presented to the subroutine was the highest ranking possible and the generation of all possible combinations was previously completed. For any other condition, K will be unchanged if its initial value was greater than 1 or set to 2 if its initial value was less than 2.

SUBROUTINE SYNTHZ. This subroutine performs the low-pass synthesis procedure of Section IV.D. Given the coefficients of P(s) (SP array), the NO zeros of the polynomial (P(s) +E(s)), and the (NO-2) zeros of the polynomial D(s), this subroutine computes the (3xNO-2) element values (ELEM array) of the resultant low-pass filter. The first NO element values correspond to R₁ through Rₙ respectively (see Figure 4.8). The next two element values correspond to C_1' and C_2 respectively. The remaining element values alternately correspond to C_j' and C_j'' (j=3,4,...,NO). Subroutine SYNTHZ calls on subroutines POAD and POMUL. These two subroutines are explained below.

SUBROUTINE POAD. The input array of this subroutine is (POC, NOC, POD, NOD, RPO, NPO). The subroutine adds the NOD coefficients (POD array) of a (NOD-1) order polynomial (POD(1) is the trailing coefficient) to the NOC coefficients (POC array) of a (NOC-1) order polynomial. The coefficients of the resulting polynomial are stored in the
RPO array and the quantity NPO is set equal to the number of coefficients of the resulting polynomial.

SUBROUTINE POMUL. The input array of this subroutine is (AN, IAN, POD, NOD, POC, NOC, KERR). The subroutine multiplies an (IAN-1) order polynomial, whose IAN coefficients are stored in the AN array, by a (NOD-1) order polynomial whose coefficients are stored in the POD array. The coefficients of the resulting polynomial are stored in the POC array and the quantity NOC is set equal to the number of coefficients of the resulting polynomial.

SUBROUTINE EVLVTR. This subroutine evaluates the 2-tree admittance products of the NTS 2-trees stored in the ITR array. The admittance product for 2-tree number I is stored at location VTR(I).

SUBROUTINE SENSIT. This subroutine evaluates the sensitivities of the coefficients of P(s) to changes in the element values of the network synthesized by subroutine SYNTHZ. These sensitivities are stored in the SELM array. SELM(M, J) gives the percentage change in coefficient $p_{M-1}$ of the polynomial P(s) due to a 1% increase in the element value of the low-pass network whose value is stored at location ELEM(J).
SUBROUTINE EVOB. The sole purpose of this subroutine is to evaluate the optimization parameter $Q$ of Equation (5.11). This quantity is stored at location $VOBJ$.

SUBROUTINE DMPOT. This is an output subroutine which handles the data associated with the initial network and the final network as explained in the previous section. The subroutine tests sense switch 1. If sense switch 1 is off, the output is printed on the on line printer only. If on, output is also punched on cards.

SUBROUTINE EVGRAD. The independent variables of the present optimization problem are the $NO$ zeros of $(P(s)+E(s))$ and $(NO-2)$ zeros of $D(s)$. Thus, the number of independent variables is $NVB=2\times NO-2$. The purpose of the present subroutine is to evaluate the $NVB$ gradients associated with the optimization problem. The computed gradients are stored in the $GRAD$ array and are weighted in direct proportion to the magnitude of the associated polynomial zero.

SUBROUTINE STCOMP. This subroutine determines the size of step to be taken in the method of steepest descent. The step size is stored at location $STP$.

SUBROUTINE TKSTP. This subroutine takes the step of value
STP and computes the resultant optimization parameter VOBJ.

SUBROUTINE CONSTR. This subroutine checks to see that a given set of zeros for \((P(s)+E(s))\) (ALPH array) and a given set of zeros for \(D(s)\) (BETA array) are properly interlaced. Upon returning from the subroutine, the integer variable NCON is zero if the zeros are properly interlaced and set to one if they are not properly interlaced.

This completes the explanation of the 16 subroutines. The main program of Figure 5.1 will now be discussed. First, subroutine RDIN is called to read the input data from cards. Since the elements of the IFR and ITO arrays are to be used later as subscripts and since these elements contain the integer 0, each of the elements are incremented by 1. Next, subroutine TRCOM is called to determine and store the applicable 2-trees of the input network. The ALPH and BETA arrays are set equal to the ALBAR and BTBAR arrays respectively. Subroutines SYNTHZ, EVLVTR, SENSIT and EVOB are then called in turn and it is seen that at this point the value of the objective function of the network initially input has been computed along with the element values of the network. VOBAR is set equal to the current value of the objective function and an output heading for the initial network is printed. Subroutine DMFOT is next called to print
out the computed parameters of the initial network. The remaining portion of the main program is concerned with the improvement of the initial network by the method of steepest descent. As each improvement to the network is accomplished, the value of the resulting objective function is typed out if sense switch 1 is turned on. The program is terminated automatically if the objective function improvement obtainable for a given set of gradients is less than 0.05% of its previous value. If it is desired to terminate the program before this condition is fulfilled, sense switch 3 is turned on. Upon termination, subroutine DMPOT is called to print out the parameters of the final improved network and the program is terminated.

The method of preparing the input data will now be explained. Refer to the explanation of subroutine RDIN for a definition of the variables to be discussed below. The first input card is used to input the 7 integer variables NO, NEIP, NO, IB, IJ, KB and NELM. These are input as 7-two digit numbers which occupy the first 14 columns of the first input card. Card number 2 is used to input the NTYP array. A maximum of 30-two digit numbers may occupy the first 60 columns of this card. The same format is used to input the IFR array on card number 3 and the ITO array on card number 4. The first 50 columns of the next card are
used to input five F10.2 numbers corresponding to the values of WCD, WRD, WCS, WCR and WRR respectively. Next, the WEIL array is input by placing as many as 8 F10.2 values on a card until the array is input. This same format is used to input the WK, WY, SP, ALBAR and BTBAR arrays in turn.

C. EXAMPLE NETWORK.

An example is now presented to clarify the preparation of the input data and to demonstrate the accomplishments which are possible with the program. The transfer function selected is the basic low-pass Butterworth fourth-order transfer function

$$T_v = \frac{1}{s^4 + 2.6132s^3 + 3.4143s^2 + 2.6132s + 1}$$  \hspace{1cm} (5.11)

It is known that the application of the synthesis procedure of Section IV.D. will result in the 10 element network configuration of Figure 5.2. The elements must be placed in the positions shown in order that the computer program operate properly. Node and element identifying numbers are properly assigned and, since there are no elements connected across the input and output terminals of the network, dummy (auxiliary) elements are added to the network and assigned the identification numbers 11 and 12. This is done so that the quantities IB and KB will have values other than zero. An
Figure 5.2. Configuration of example network.

Figure 5.3. Optimized network for realization of

$$T_v = \frac{1}{s^4 + 2.6132s^3 + 3.4143s^2 + 2.6132s + 1}$$
examination of Figure 5.2 enables the determination of the following input variables.

NOD=5  NELP=10  NO=4  IB=11
IJ=5  KB=12  NELM=12  NTYP(1)=0
NTYP(2)=0  NTYP(3)=0  NTYP(4)=0  NTYP(5)=1
NTYP(6)=1  NTYP(7)=1  NTYP(8)=1  NTYP(9)=1
NTYP(10)=1  NTYP(11)=0  NTYP(12)=0  IFR(1)=1
IFR(2)=2  IFR(3)=3  IFR(4)=4  IFR(5)=1
IFR(6)=2  IFR(7)=3  IFR(8)=3  IFR(9)=4
IFR(10)=4  IFR(11)=1  IFR(12)=5  ITO(1)=2
ITO(2)=3  ITO(3)=4  ITO(4)=0  ITO(5)=5
ITO(6)=0  ITO(7)=5  ITO(8)=0  ITO(9)=5
ITO(10)=0  ITO(11)=0  ITO(12)=0

The weighting functions were initially set to 1 with the thought that these quantities could be altered after running through the computer program once in order to obtain a feel for the magnitudes of the various elements of the optimization parameter. It was found desirable to increase WEB and WEE to 100 and 10 respectively because of the characteristically low value of ED and EE in comparison with the other optimization parameter elements. Likewise, the WK's and WY's were increased to 10 and 50 respectively.

The remaining quantities which are to be input are set as follows:
SP(1)=1.0
SP(4)=2.6132
ALBAR(2)=0.2500
BTBAR(1)=0.1000

SP(2)=2.6132
SP(5)=1.0
ALBAR(3)=0.3500
BTBAR(2)=0.3000

ALBAR(1)=0.0500
ALBAR(4)=4.0000

These quantities are punched at their proper input card locations and are shown on lines 3 through 19 on page 122 of Appendix II. The results are interpreted and discussed below. See pages 123 and 124 for the printout.

1. The results are printed out in two steps. First, the parameters of the initial network are printed, and then the parameters of the final network.

2. The first item considered is the pole and zero locations of the input admittance $y_{ll}$. It is shown that the initial zero locations of 0.05, 0.25, 0.35, and 4.00 are shifted to the final zero locations of 0.0644, 0.4432, 1.0438, and 1.5467. The poles are shifted from 0.1000, 0.3000, and 2.2132 to 0.2823, 0.8960, and 1.4348. Note that these quantities are the distances from the origin to the location of the poles or zeros on the negative real axis and not their actual location.

3. The next item considered is the coefficients of the input admittance $y_{ll}$. The results show that the input admittance is shifted from $57.143s^4 + 265.714s^3$...
Next, the sensitivities of the coefficients of $P(s)$ with respect to variations in the network element values are printed out. There are too many of these quantities to be listed individually here. As an example, note that the optimization process changes the sensitivity of coefficient 3 with respect to element 10 from 0.534 to 0.309.

Note that the capacitance distribution parameter is reduced from 1160 to 33.87, the resistance distribution parameter from 0.0861 to 0.00622, and the total capacity of the network is reduced from 89.35 to 27.98. The capacitance and resistance ratios are reduced from 508.74 to 20.18 and 4.337 to 1.561 respectively.

The sensitivities of the coefficients of $P(s)$ with respect to the voltage follower gain are next considered. Notice that the sensitivity of coefficient 2 to a one percent variation in the voltage follower gain is decreased from 0.207% for the initial network to 0.0926% for the final network. Finally, the sensitivity of the coefficients of $P(s)$ with respect to the voltage follower input
conductance are considered. Notice that the sensitivity of coefficient \( 3 \) to a voltage follower with an input conductance equal to one percent of the reciprocal of the total resistance of the passive portion of the network is decreased from 0.393% to 0.0719%.

7. At this point, the sensitivities and other elements of the optimization parameter have been printed out. The next item to be considered is the element values of the associated networks. Notice that the 33.778 farad capacitor of the initial network has been reduced to 6.312 farads in the final network.

8. The final item considered is the value of the optimization parameter. Notice that it has been decreased from an initial value of 1884.5 to 126.58. This, of course, may be considered a worthwhile improvement.

The final optimized network is shown in Figure 5.3. Several observations and recommendations with regard to the optimization computer program will now be made. First of all, it was observed that the optimization parameter converged to the same value for several starting points. This suggests, although it certainly does not prove, that the
minimum obtained is a global minimum and not a local minimum.

Although the optimization parameter converged to the same value for each starting point, the rate of convergence is greatly influenced by the starting point. It is felt that it would be advantageous to carry out an initial search technique which would randomly generate a number of starting points. That network which results in the lowest value of optimization parameter would then be selected as the initial network of the optimization program.

A factor which was found to influence greatly the rate of convergence is the weighting factors applied to the gradients. In the present case, it was found most advantageous to weight the gradients by the magnitude squared of their associated independent parameters. For other configurations, it may be desirable to change this to a weighting of the gradients by the magnitude or some other function of its associated independent parameter. Any desired change is carried out by altering statements 3 and 4 of subroutine EVGRAD.
VI. PRACTICAL CONSIDERATIONS

Section A of this chapter will consider the effect of the departure of the voltage follower from the ideally assumed conditions of zero input admittance and unity voltage gain. Section B will be concerned with the design of voltage follower networks and Section C will present the experimental results of some example networks.

A. THE NON-IDEAL VOLTAGE FOLLOWER

As seen in Figure 1.1, if the voltage follower possesses a finite input admittance \( Y_a \), this admittance may be considered to appear across the input terminals of the passive RC network. If \( Y_a \) is assumed to consist of a conductance \( G_a \) in parallel with a susceptance \( C_a \), then \( Y_a = G_a + C_a s \) and the characteristic equation \( D(s) \) of the RC network is unchanged. Thus, the quantity \( N_{11} - N_{13} \) is increased by the quantity \( G_a D + C_a D_s \). One way in which the effect of \( Y_a \) may be taken into account is the following:

1. Synthesize a passive network for the transfer function \( T_v = A(s)/P(s) \) as though \( Y_a \) does not exist. One of the results of this step is the determination of the admittance scaling factor \( K \) and the characteristic polynomial \( D(s) \) of the network.

2. Now repeat the synthesis procedure for the trans-
The transfer function \( T_v = A(s)/P'(s) \) where \( P'(s) = P(s) - G_a D(s) - C_a s D(s) \). If the \( K \) and the \( D(s) \) of the new network are the same as those of the previously synthesized network, then the transfer function \( T_v = A(s)/P(s) \) is obtained when the second network is used in conjunction with a voltage follower whose input admittance is \( Y_a = G_a + C_a s \).

To take account of a voltage follower gain \( k \) other than unity recall that the voltage transfer function \( T_v \) is from Equation (1.5)

\[
T_v = \frac{N_{12}}{N_{11} - kN_{13}} \quad (6.1)
\]

This equation may be interpreted in the terminology of the current distribution diagram of Figure 4.1 as though the quantity \( kE(s) \) is subtracted from the current of terminal 1 to determine \( P(s) \) instead of the quantity \( E(s) \). An alternate current diagram representation is shown in Figure 6.1. If the quantity \( P'(s) \) is set equal to \( (P(s) - (1 - kE(s)) \) and the usual synthesis procedures are applied, then the resulting passive network, when used in conjunction with a voltage follower whose gain is \( k \), will produce a transfer function \( T_v = A(s)/P(s) \).

B. VOLTAGE FOLLOWER CIRCUITS
Figure 6.1. Current distribution diagram for non-unity voltage follower gain.

Figure 6.2. Voltage follower of Reference 32.
This section is concerned with the problem of designing practical voltage follower circuits. As previously mentioned, the cathode, emitter or source follower circuit offers a simple approximation to a voltage follower. Except for the less critical applications, however, these circuits have been found to depart too far from the ideal to perform satisfactorily. It will be shown that voltage followers which very closely approximate the ideal may be designed.

The first thought that occurs is that a two stage amplifier using the field effect transistor as an input stage would make an ideal network. Analysis of such a network reveals, however, that while extremely high input impedances are attainable, a gain of the order of 0.98 is all that is possible. A review of the literature does not immediately reveal a simple satisfactory two stage voltage follower. The two stage voltage follower to be presented here, however, is one which is adopted from the literature. The original circuit (32) is shown in Figure 6.1. This circuit has an input impedance of approximately 50,000 ohms and a gain of approximately 0.998. The altered network shown in Figure 6.2 has a much higher input impedance and a gain much closer to unity. This improvement is accomplished by substituting a field effect transistor for a conventional transistor.
Notice also that the circuitry of the altered network is greatly simplified and that the low frequency response has been extended down to d-c. To gain insight into the operation of the two networks, consider the system diagram of Figure 6.3 which is valid for both circuits. $A_1$ and $A_2$ represent the voltage gains of the first and second stages respectively. Note that the first stage of each amplifier is operated in the common base configuration but that the load impedance for the stage in Figure 6.1 is low in comparison with that for the corresponding stage in Figure 6.2. Since the voltage gain of the common base configuration is proportional to load impedance, $A_1$ for the first network is much lower than $A_1$ for the second network. Straightforward analysis of Figure 6.3 show that

$$A = \frac{e_{\text{out}}}{e_{\text{in}}} = \frac{A_1 A_2}{A_1 A_2 + 1 - A_2} \quad (6.2)$$

$$Z_{\text{in}} = \frac{e_{\text{in}}}{I_{\text{in}}} = \frac{Z_i}{1 - A} \quad (6.3)$$

where $Z_i$ is the emitter-base impedance of the input stage. Note that $A_2$ is essentially the voltage gain of an emitter follower and may be considered approximately 0.95. Typical values of $A_1$ are 35 and 500 for the circuits of Figures 6.1 and 6.2 respectively. Substituting typical values into equations (6.2) and (6.3) results in $A=0.9985$ and $A_{\text{in}}=23,300$
Figure 6.3. Altered network.

Figure 6.4. System diagram for networks of Figures 6.2 & 6.3.
ohms for Figure 6.1. The corresponding results for the network of Figure 6.2 are $A \approx 0.9999$ and $Z_{in} \approx 350,000$ ohms. Thus, improvement has been accomplished by allowing the voltage gain $A_1$ to approach the maximum attainable from a conventional transistor.

A three stage voltage follower which follows naturally from the network of Figure 6.2 is shown in Figure 6.5. Actually, the network contains four transistors. Since one of the transistors only behaves as a current source to properly bias the first stage, the network is classified as a three stage network. The result accomplished by this network is to make $A_2$ of Figure 6.3 approach much closer to unity. Notice that the low frequency response of this network also extends to d-c. This network will be used in the experimental verification of Section C.

C. EXPERIMENTAL RESULTS.

This section will present the experimental results of three transfer functions realized by the methods of this report. The three transfer functions include a second-order band-pass transfer function, a fourth-order low-pass transfer function and a fourth-order band-pass transfer function. These transfer functions in normalized form are
Figure 6.5. Three stage voltage follower network.
respectively

\[ T_v = \frac{0.1s}{s^2 + 0.1s + 1} \quad (6.4) \]

\[ T_v = \frac{1}{s^4 + 2.6132s^3 + 3.4143s^2 + 2.6132s + 1} \quad (6.5) \]

\[ T_v = \frac{s^2}{s^4 + 1.414s^3 + 3s^2 + 1.414s + 1} \quad (6.6) \]

The transfer function (6.4) corresponds to a band-pass function whose center frequency is 1 radian per second and whose pass band is 0.1 radian per second. Equation (6.5) is the low-pass fourth-order maximally flat (Butterworth) characteristic whose cutoff frequency is one radian per second. Equation (6.6) is the fourth-order maximally flat band-pass transfer function whose center frequency is one radian per second and whose pass-band is one radian per second. This means that the upper cutoff frequency is 1.619 radians per second and the lower cutoff frequency is 0.619 radians per second.

An explicit synthesis procedure for the second-order band-pass transfer function was not presented in Chapter IV. One of the networks of Sallen and Key may be utilized or, more simply, the current distribution diagram for the famil-
A twin-T RC network may be computed as shown in Figure 6.6. If $R$, $k_R$, and $k_C$ are chosen 0.8, 1.67, and 1 respectively, a network with positive elements results which can be converted to the form of Figure 6.7 for use in conjunction with the (RCVF) to realize the transfer function of (6.4). This network is shown in Figure 6.8.

The fourth-order low-pass function of Equation (6.5) is the same as that used in the computer optimization example of Chapter V. The network used in the present case is one that was obtained from the computer program before the computer program was completely debugged. The network is not an optimum one although it turns out to be very nearly optimum.

The procedure of IV E may be applied in the case of the transfer function of Equation (6.6). Application of this procedure results in the network of Figure 6.6. This network realizes (6.6) within a constant multiplier.

The realized networks will be impedance and frequency scaled to typical values. The circuit for the second-order band-pass transfer function was impedance scaled to 300,000 ohms and frequency scaled to 150 cycles per second. The resultant (RCVF) configuration is shown in Figure 6.7.
Figure 6.6. Current distribution for twin-T network.

Figure 6.7. Network for realization of $T_v = \frac{0.1s}{s^2 + 0.1s + 1}$. 
Figure 6.8. (RCVF) for realization of second-order band-pass transfer function denormalized to 300,000 ohms and 150 cps.

Figure 6.9. (RCVF) for realization of fourth-order low-pass transfer function denormalized to 1 megohm and 85 cps.
The circuit for the fourth-order low-pass transfer function is impedance scaled to 1 megohm and frequency scaled to 85 cps. This denormalized network is shown in Figure 6.8.

The circuit for the fourth-order band-pass transfer function is impedance scaled to 500,000 ohms and frequency scaled to 39.8 cps. The denormalized network is shown in Figure 6.9.

The networks of Figures 6.7, 6.8, and 6.9 were built and tested using the voltage follower circuit of Figure 6.4. A comparison of theoretical and experimental results for these networks are shown in Tables 6.1 through 6.3.
Figure 6.10. (RCVF) for realization of fourth-order band-pass transfer function denormalized to 500,000 ohms and 39.8 cps.
### Table 6.1

Frequency-Attenuation Tabulation of Experimental and Theoretical Values for Network of Figure 6.8

<table>
<thead>
<tr>
<th>Frequency (CPS)</th>
<th>DB Attenuation (Experimental)</th>
<th>DB Attenuation (Theoretical)</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>17.0</td>
<td>47.00</td>
<td>38.80</td>
<td>1.198</td>
</tr>
<tr>
<td>25.0</td>
<td>36.00</td>
<td>35.31</td>
<td>0.690</td>
</tr>
<tr>
<td>38.0</td>
<td>32.00</td>
<td>31.35</td>
<td>0.646</td>
</tr>
<tr>
<td>55.0</td>
<td>28.00</td>
<td>27.46</td>
<td>0.531</td>
</tr>
<tr>
<td>75.0</td>
<td>24.00</td>
<td>23.54</td>
<td>0.458</td>
</tr>
<tr>
<td>109.0</td>
<td>17.00</td>
<td>16.35</td>
<td>0.646</td>
</tr>
<tr>
<td>119.1</td>
<td>14.00</td>
<td>13.55</td>
<td>0.446</td>
</tr>
<tr>
<td>125.1</td>
<td>12.00</td>
<td>11.56</td>
<td>0.438</td>
</tr>
<tr>
<td>130.5</td>
<td>10.00</td>
<td>9.64</td>
<td>0.551</td>
</tr>
<tr>
<td>134.8</td>
<td>8.00</td>
<td>7.46</td>
<td>0.530</td>
</tr>
<tr>
<td>138.5</td>
<td>6.00</td>
<td>5.50</td>
<td>0.497</td>
</tr>
<tr>
<td>140.1</td>
<td>5.00</td>
<td>4.57</td>
<td>0.424</td>
</tr>
<tr>
<td>141.7</td>
<td>4.00</td>
<td>3.61</td>
<td>0.387</td>
</tr>
<tr>
<td>142.6</td>
<td>3.50</td>
<td>3.06</td>
<td>0.336</td>
</tr>
<tr>
<td>143.3</td>
<td>3.00</td>
<td>2.63</td>
<td>0.398</td>
</tr>
<tr>
<td>144.2</td>
<td>2.50</td>
<td>2.10</td>
<td>0.300</td>
</tr>
<tr>
<td>144.9</td>
<td>2.00</td>
<td>1.69</td>
<td>0.285</td>
</tr>
<tr>
<td>145.8</td>
<td>1.50</td>
<td>1.21</td>
<td>0.259</td>
</tr>
<tr>
<td>146.8</td>
<td>1.00</td>
<td>0.74</td>
<td>0.167</td>
</tr>
<tr>
<td>147.9</td>
<td>0.50</td>
<td>0.33</td>
<td>0.103</td>
</tr>
<tr>
<td>150.2</td>
<td>0.00</td>
<td>0.00</td>
<td>-0.003</td>
</tr>
<tr>
<td>152.6</td>
<td>.50</td>
<td>.48</td>
<td>.015</td>
</tr>
<tr>
<td>154.0</td>
<td>1.00</td>
<td>1.06</td>
<td>-0.062</td>
</tr>
<tr>
<td>154.9</td>
<td>1.50</td>
<td>1.50</td>
<td>-0.002</td>
</tr>
<tr>
<td>155.8</td>
<td>2.00</td>
<td>1.97</td>
<td>-0.024</td>
</tr>
<tr>
<td>156.6</td>
<td>2.50</td>
<td>2.41</td>
<td>-0.089</td>
</tr>
<tr>
<td>157.3</td>
<td>3.00</td>
<td>2.90</td>
<td>-0.093</td>
</tr>
<tr>
<td>158.3</td>
<td>3.50</td>
<td>3.34</td>
<td>-0.152</td>
</tr>
<tr>
<td>159.3</td>
<td>4.00</td>
<td>3.89</td>
<td>-0.109</td>
</tr>
<tr>
<td>161.1</td>
<td>5.00</td>
<td>4.83</td>
<td>-0.168</td>
</tr>
<tr>
<td>163.0</td>
<td>6.00</td>
<td>5.76</td>
<td>-0.237</td>
</tr>
<tr>
<td>167.3</td>
<td>8.00</td>
<td>7.62</td>
<td>-0.377</td>
</tr>
<tr>
<td>172.8</td>
<td>10.00</td>
<td>7.57</td>
<td>-0.427</td>
</tr>
<tr>
<td>180.4</td>
<td>12.00</td>
<td>11.69</td>
<td>-0.303</td>
</tr>
<tr>
<td>189.8</td>
<td>14.00</td>
<td>13.72</td>
<td>-0.277</td>
</tr>
<tr>
<td>207.9</td>
<td>17.00</td>
<td>16.54</td>
<td>-0.452</td>
</tr>
<tr>
<td>236.1</td>
<td>20.00</td>
<td>19.49</td>
<td>-0.500</td>
</tr>
<tr>
<td>303.1</td>
<td>24.00</td>
<td>23.63</td>
<td>-0.311</td>
</tr>
<tr>
<td>459.2</td>
<td>28.00</td>
<td>27.47</td>
<td>-0.528</td>
</tr>
<tr>
<td>618.2</td>
<td>32.00</td>
<td>31.77</td>
<td>-0.223</td>
</tr>
<tr>
<td>939.0</td>
<td>36.00</td>
<td>35.70</td>
<td>-0.291</td>
</tr>
<tr>
<td>1476.0</td>
<td>40.00</td>
<td>39.77</td>
<td>-0.229</td>
</tr>
</tbody>
</table>
TABLE 6.2
FREQUENCY-ATTENUATION TABULATION OF EXPERIMENTAL AND THEORETICAL VALUES FOR NETWORK OF FIGURE 6.9

<table>
<thead>
<tr>
<th>FREQUENCY (CPS)</th>
<th>DB ATTENUATION (EXPERIMENTAL)</th>
<th>DB ATTENUATION (THEORETICAL)</th>
<th>DB ERROR</th>
</tr>
</thead>
<tbody>
<tr>
<td>50.0</td>
<td>0.00</td>
<td>0.06</td>
<td>-0.062</td>
</tr>
<tr>
<td>65.7</td>
<td>0.50</td>
<td>0.52</td>
<td>-0.021</td>
</tr>
<tr>
<td>71.8</td>
<td>1.00</td>
<td>1.00</td>
<td>-0.001</td>
</tr>
<tr>
<td>75.9</td>
<td>1.50</td>
<td>1.47</td>
<td>-0.025</td>
</tr>
<tr>
<td>79.5</td>
<td>2.00</td>
<td>2.00</td>
<td>-0.002</td>
</tr>
<tr>
<td>82.4</td>
<td>2.50</td>
<td>2.50</td>
<td>-0.004</td>
</tr>
<tr>
<td>84.9</td>
<td>3.00</td>
<td>2.99</td>
<td>-0.009</td>
</tr>
<tr>
<td>87.2</td>
<td>3.50</td>
<td>3.47</td>
<td>-0.022</td>
</tr>
<tr>
<td>89.7</td>
<td>4.00</td>
<td>4.04</td>
<td>-0.045</td>
</tr>
<tr>
<td>93.8</td>
<td>5.00</td>
<td>5.05</td>
<td>-0.050</td>
</tr>
<tr>
<td>97.5</td>
<td>6.00</td>
<td>6.01</td>
<td>-0.017</td>
</tr>
<tr>
<td>105.0</td>
<td>8.00</td>
<td>8.07</td>
<td>-0.077</td>
</tr>
<tr>
<td>113.0</td>
<td>10.00</td>
<td>10.31</td>
<td>-0.316</td>
</tr>
<tr>
<td>120.0</td>
<td>12.00</td>
<td>12.24</td>
<td>-0.248</td>
</tr>
<tr>
<td>128.0</td>
<td>14.00</td>
<td>14.38</td>
<td>-0.384</td>
</tr>
<tr>
<td>140.0</td>
<td>17.00</td>
<td>17.41</td>
<td>-0.416</td>
</tr>
<tr>
<td>153.0</td>
<td>20.00</td>
<td>20.46</td>
<td>-0.461</td>
</tr>
<tr>
<td>172.0</td>
<td>24.00</td>
<td>24.50</td>
<td>-0.504</td>
</tr>
<tr>
<td>195.0</td>
<td>28.00</td>
<td>28.85</td>
<td>-0.855</td>
</tr>
<tr>
<td>222.0</td>
<td>32.00</td>
<td>33.35</td>
<td>-1.356</td>
</tr>
<tr>
<td>267.0</td>
<td>36.00</td>
<td>39.76</td>
<td>-3.767</td>
</tr>
</tbody>
</table>
## Table 6.3
Frequency-Attenuation Tabulation of Experimental and Theoretical Values for Network of Figure 6.10

<table>
<thead>
<tr>
<th>Frequency (CPS)</th>
<th>Attenuation (Experimental)</th>
<th>Attenuation (Theoretical)</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.9</td>
<td>35.50</td>
<td>36.12</td>
<td>-0.623</td>
</tr>
<tr>
<td>5.9</td>
<td>32.00</td>
<td>32.77</td>
<td>-0.777</td>
</tr>
<tr>
<td>7.6</td>
<td>28.00</td>
<td>28.12</td>
<td>-0.124</td>
</tr>
<tr>
<td>9.3</td>
<td>24.00</td>
<td>24.29</td>
<td>-0.296</td>
</tr>
<tr>
<td>11.3</td>
<td>20.00</td>
<td>20.45</td>
<td>-0.51</td>
</tr>
<tr>
<td>13.3</td>
<td>17.00</td>
<td>17.06</td>
<td>-0.69</td>
</tr>
<tr>
<td>15.1</td>
<td>14.00</td>
<td>14.30</td>
<td>-0.300</td>
</tr>
<tr>
<td>16.4</td>
<td>12.00</td>
<td>12.42</td>
<td>-0.424</td>
</tr>
<tr>
<td>18.1</td>
<td>10.00</td>
<td>10.10</td>
<td>-0.107</td>
</tr>
<tr>
<td>19.7</td>
<td>8.00</td>
<td>8.06</td>
<td>-0.069</td>
</tr>
<tr>
<td>21.3</td>
<td>6.00</td>
<td>6.19</td>
<td>-0.19</td>
</tr>
<tr>
<td>22.2</td>
<td>5.00</td>
<td>5.21</td>
<td>-0.218</td>
</tr>
<tr>
<td>23.1</td>
<td>4.00</td>
<td>4.31</td>
<td>-0.318</td>
</tr>
<tr>
<td>23.8</td>
<td>3.50</td>
<td>3.67</td>
<td>-0.175</td>
</tr>
<tr>
<td>24.3</td>
<td>3.00</td>
<td>3.24</td>
<td>-0.249</td>
</tr>
<tr>
<td>24.9</td>
<td>2.50</td>
<td>2.77</td>
<td>-0.275</td>
</tr>
<tr>
<td>25.6</td>
<td>2.00</td>
<td>2.27</td>
<td>-0.278</td>
</tr>
<tr>
<td>26.5</td>
<td>1.50</td>
<td>1.72</td>
<td>-0.226</td>
</tr>
<tr>
<td>27.5</td>
<td>1.00</td>
<td>1.22</td>
<td>-0.228</td>
</tr>
<tr>
<td>29.0</td>
<td>0.50</td>
<td>0.68</td>
<td>-0.187</td>
</tr>
<tr>
<td>39.8</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>50.8</td>
<td>0.50</td>
<td>0.24</td>
<td>0.251</td>
</tr>
<tr>
<td>54.6</td>
<td>1.00</td>
<td>1.08</td>
<td>0.414</td>
</tr>
<tr>
<td>56.9</td>
<td>1.50</td>
<td>1.68</td>
<td>0.316</td>
</tr>
<tr>
<td>59.6</td>
<td>2.00</td>
<td>2.17</td>
<td>0.325</td>
</tr>
<tr>
<td>61.5</td>
<td>2.50</td>
<td>2.59</td>
<td>0.404</td>
</tr>
<tr>
<td>63.0</td>
<td>3.00</td>
<td>3.19</td>
<td>0.308</td>
</tr>
<tr>
<td>65.0</td>
<td>3.50</td>
<td>3.68</td>
<td>0.310</td>
</tr>
<tr>
<td>66.6</td>
<td>4.00</td>
<td>4.71</td>
<td>0.284</td>
</tr>
<tr>
<td>69.8</td>
<td>5.00</td>
<td>5.65</td>
<td>0.345</td>
</tr>
<tr>
<td>72.7</td>
<td>6.00</td>
<td>7.67</td>
<td>0.326</td>
</tr>
<tr>
<td>79.1</td>
<td>8.00</td>
<td>9.71</td>
<td>0.282</td>
</tr>
<tr>
<td>86.1</td>
<td>10.00</td>
<td>11.81</td>
<td>0.180</td>
</tr>
<tr>
<td>94.1</td>
<td>12.00</td>
<td>13.77</td>
<td>0.220</td>
</tr>
<tr>
<td>102.5</td>
<td>14.00</td>
<td>16.87</td>
<td>0.128</td>
</tr>
<tr>
<td>118.0</td>
<td>17.00</td>
<td>19.83</td>
<td>0.164</td>
</tr>
<tr>
<td>136.0</td>
<td>20.00</td>
<td>23.79</td>
<td>0.201</td>
</tr>
<tr>
<td>166.0</td>
<td>24.00</td>
<td>27.90</td>
<td>0.094</td>
</tr>
<tr>
<td>206.0</td>
<td>28.00</td>
<td>31.84</td>
<td>0.159</td>
</tr>
<tr>
<td>255.0</td>
<td>32.00</td>
<td>35.88</td>
<td>0.115</td>
</tr>
<tr>
<td>319.0</td>
<td>36.00</td>
<td>39.95</td>
<td>0.041</td>
</tr>
<tr>
<td>401.0</td>
<td>40.00</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
VII. CONCLUSION AND RECOMMENDATIONS

A. CONCLUSION

This thesis has described and demonstrated the feasibility of utilizing the grounded voltage follower as the active portion of an active RC network. Chapter III established the necessary and sufficient conditions for the realization of the voltage transfer function of such a device (the RCVF). Chapter IV described some synthesis procedures for the (RCVF) and Chapter V demonstrated that the digital computer may be utilized to optimize the synthesized networks. Finally, Chapter VI gave a demonstration of several transfer functions realized by the (RCVF) active network which indicates the practical value of the network by showing excellent agreement between predicted and actual results. Indeed, the results of both the theoretical and experimental investigation lead one to conclude that the (RCVF) is a highly useful and practical network.

B. RECOMMENDATIONS

This study has resulted in a number of possibilities for future work becoming apparent. Recommendations for future work are listed below.

1. The first, and probably the most obvious, recom-
mendation that comes to mind is the further investigation of synthesis techniques. Although Chapter III showed that a transfer function which meets specified necessary conditions may always be realized, the synthesis of optimum networks is another matter. Thus, the further investigation of networks for specific classes of transfer functions is indicated.

2. An area which was not touched on in this investigation but appears interesting is that of the possibility of varying transfer functions by switching. The concept of the current distribution diagram offers a great deal of insight into the problem. A simple example of the possibilities for switching may be seen by examining the network of Figure 6.7. This network is used in conjunction with the voltage follower to realize the transfer function

\[ T_v = \frac{a_1 s}{p_2 s^2 + a_1 s + p_0} \]

where \( a_1 = 0.1 \) and \( p_2 = p_0 = 1 \).

Notice that \( a_1 \) may be halved merely by splitting the 2-ohm resistor carrying the current \( 0.1 s \) into two 4-ohm resistors and switching one of them over to the terminal associated with the current \( E(s) \). The result of this operation is to cut the bandwidth of the band-pass function in half. The above example represents only one possibility of a number
which readily come to mind.

3. The next recommendation concerns the optimization of networks by means of the digital computer. It is obvious that network optimization is an extremely complex problem and that it has only been touched upon in this investigation. Only one optimization method (the method of steepest descent) has been attempted on a single class of networks. The optimization attempt was successful from the standpoint that significant improvement was accomplished, but the problem is important enough to deserve a great deal of further study. It is specifically recommended that the approach of continuously equivalent networks (20,33) be investigated. This approach is not only concerned with the optimization of a given network configuration but with the generation of new configurations.

4. Another important area for further investigation is that concerned with practical considerations. Such considerations would include temperature effects and compensation, component selection, voltage follower circuitry, the maximum impedance level of a network, the minimum frequency allowable, etc. Another important item under this heading would be the effect of joining two or more (RCVF) networks
together. These networks could be connected in cascade or some other desirable configuration.

5. The final recommendation concerns the driving point impedance of the (RCVF). This thesis has done nothing with this possibly important subject. Preliminary investigation has indicated that driving point function synthesis utilizing the voltage follower may be more difficult than transfer function synthesis. It is felt that the problem deserves a separate paper.
PLEASE NOTE:
This is not original copy. Appendix contains pages with extremely fine print. Filmed as received.

University Microfilms, Inc.
APPENDIX I

OPTIMIZATION COMPUTER PROGRAM LISTING
301 PUNCH 302
303 FORMAT(********2JX,13HFINAL NETWORK,2B8,6H*****)
304 CONTINUE
CALL LINK(CHFIVE)
END

302 PUNCH 303
304 FORMAT(********2JX,13HFINAL NETWORK,2B8,6H*****)
305 CONTINUE
CALL LINK(CHFIVE)
END

12 JOB 5
Z2FOR 5
*LDISK
SUBROUTINE CONSTR
DIMENSION LD(31),NTRP(31),VTR(150),SELM(11,30),SP(11),WELH(11,30),
1 ELEM(30),DT(11),K(11),MY(11),ALBAR(10),DTBAR(10),GRAD120,I(30),
2 ITR(30),ITR(1200),FDCO(11),FDCO(111),ALPH(10),BETA(10),FCCO(11),
3 TT(31)
COMMON NOD,NTT,NEP,NTS,NO,NSP,LD,NTRP,SELM,SP,WELH,ELEM,K,MY,
5 ALBAR,DTBAR,GRAD120,I(30),ITR(1200),FDCO(11),FDCO(111),ALPH(10),BETA(10),
6 FCCO(11),TT(31)

1 CALL HNPST
CALL EXIT
END

12 JOB 5
Z2FOR 5
*LDISK
SUBROUTINE CONSTR
DIMENSION LD(31),NTRP(31),VTR(150),SELM(11,30),SP(11),WELH(11,30),
1 ELEM(30),DT(11),K(11),MY(11),ALBAR(10),DTBAR(10),GRAD120,I(30),
2 ITR(30),ITR(1200),FDCO(11),FDCO(111),ALPH(10),BETA(10),FCCO(11),
3 TT(31)
COMMON NOD,NTT,NEP,NTS,NO,NSP,LD,NTRP,SELM,SP,WELH,ELEM,K,MY,
5 ALBAR,DTBAR,GRAD120,I(30),ITR(1200),FDCO(11),FDCO(111),ALPH(10),BETA(10),
6 FCCO(11),TT(31)

1 CALL HNPST
CALL EXIT
END

Z2 JOB 5
Z2 FOR 5
*LDISK
SUBROUTINE DMOTP
DIMENSION LD(31),NTRP(31),VTR(150),SELM(11,30),SP(11),WELH(11,30),
1 ELEM(30),DT(11),K(11),MY(11),ALBAR(10),DTBAR(10),GRAD120,I(30)
2)

1 CALL DMOTP
CALL EXIT
END

Z2 JOB 5
Z2 FOR 5
*LDISK
SUBROUTINE HNPST
DIMENSION LD(31),NTRP(31),VTR(150),SELM(11,30),SP(11),WELH(11,30),
1 ELEM(30),DT(11),K(11),MY(11),ALBAR(10),DTBAR(10),GRAD120,I(30)
2)

1 CALL EXIT
END

Z2 JOB 5
Z2 FOR 5
*LDISK
SUBROUTINE HNPST
DIMENSION LD(31),NTRP(31),VTR(150),SELM(11,30),SP(11),WELH(11,30),
1 ELEM(30),DT(11),K(11),MY(11),ALBAR(10),DTBAR(10),GRAD120,I(30)
2)

1 CALL EXIT
END
208 PUNCH 36,(HRE(1),HRE(1),I=1,S)
209 CONTINUE
210 IF(I=NELP) 55,53,53
211 PRINT 38,CO,H,C1,CR,RR
212 FORMAT(36H CAPACITANCE DISTRIBUTION PARAMETER=E17.8,35H RESISTANCE, 
213 INCE DISTRIBUTION PARAMETER=E17.8,29H TOTAL CAPACITANCE PARAMETE 
214 R=E17.8,36H CAPACITANCE RATIO=EI7.8,36H RESISTANCE RATIO=R1 
215 37.8,70HM SENSITIVITY OF COEFFICIENT M WITH RESPECT TO VOLTAGE FO 
216 ALLOVER GAIN K+/+H )
217 M=I
218 IF (SENSE SWITCH I) 210,45
219 PUNCH 38,CD,RD,CSUM,CR,RR
220 FORMAT (36H NETWORK ELEMENT VALUES)
221 PUNCH 39,CD,CR,RR
222 END
223 JOB 5
224 END
225 *LDISK
226 SUBROUTINE EVGRAD
227 DIMENSION LU(30),HTYP(30),VTR(500),SSELM(11),301,SWELM(11),301, 
228 I=SELV(30),DT(11),WX(11),ALVAR(11),BTBAR(11),GRAD(20),1FR(30) 
229 2,1TT(31),1TR(1001),FDCO(11),BT(11),ALPH(11)+BET(11)+FCO 
230 311 COMMON NOC,H,T,HTYP,HS=0,LD,HTYP=VTR,SEL=SWELM,MW=KM,NO= 
231 15,3,1HC,HTYP,EL=DT,WX,HTVR=ALVAR,BT=BTBAR,GRAD=GRMAX,1N=STP, 
232 32X4,HTVR=WX,1TR=1TTR,1TR=TR,1TR=1TR,1TTR=1TTR,1TTR=1TTR, 
233 1TTR=1TTR,1TTR=1TTR,1TTR=1TTR,1TTR=1TTR,1TTR=1TTR,1TTR=1TTR, 
234 1TTR=1TTR,1TTR=1TTR,1TTR=1TTR,1TTR=1TTR,1TTR=1TTR,1TTR=1TTR, 
235 1TTR=1TTR,1TTR=1TTR,1TTR=1TTR,1TTR=1TTR,1TTR=1TTR,1TTR=1TTR, 
236 1TTR=1TTR,1TTR=1TTR,1TTR=1TTR,1TTR=1TTR,1TTR=1TTR,1TTR=1TTR, 
237 DO 1 I=1,NO
238 1 ALPH(I) = ALBAR(I)
239 DO 2 I=1,NO+1
240 2 CALL(EVGRAD)
241 FORMAT(I=NO) 3-4
242 3 GRAD(I) = (VBAR-VOBAR) (ALPH(I)-ALVAR(I)) ALVAR(I)**2 
243 ALPH(I) = ALVAR(I)
244 IF(I=NSP) 5,6
245 6 M=I-NO
246 5 BET(I) = BETBAR(I)
247 DO 7 M=I-NO+1
248 7 BETBAR(I) = BETBAR(I)+BETBAR(I)
249 IF(I=1V) 8,9
250 8 BET(I) = BETBAR(I)
251 9 RETURN
252 END
SUBROUTINE EVLVR
DIMENSION LD(30),N(30),VT(30),S(30),C(30)
WORK(LD)
10 DO 20 I=1,N
20 CONTINUE
RETURN
END
LDO = 15 IF(NEL = 13) 13,12,12
13 L=1
12 LD(IN) = J
11 J = J + 1
10 GO TO 19
9 NEL = 5
8 RETURN
END

SUBROUTINE STCOMP
DIMENSION LD(30) NTYP(30), VTR(500), SELM(11,30), SP(11), WEMP(11,30)
15, 16, 17, 18
DIMENSION LD(30) NTYP(30), VTR(500), SELM(11,30), SP(11), WEMP(11,30)
15, 16, 17, 18
1 STYP = 1
12 LD(IN) = J
11 J = J + 1
10 LD(IN) = J
9 RETURN
END

SUBROUTINE SCOGEN(NEL)
DIMENSION LD(30), NTYP(30), VTR(500), SELM(11,30), SP(11), WEMP(11,30)
15, 16, 17, 18
5, 6, 7
IF(NEL = 11) 3, 2, 1
3 LD(IN) = LD(11) + 1
2 RETURN
1 J = NEL + 1
10 IF (IFLD(NEL) = J) 6, 5, 5
5 IFEL(NEL) = 7, 8, 9
7 IFEL(NEL) = 7, 8, 9
6 J = NEL + 1
5 NEL = NEL + 1
4 RETURN
3 LD(IN) = LD(11) + 1
2 IFEL = 2
1 RETURN

SUBROUTINE SYNHZ
DIMENSION LD(30), NTYP(30), VTR(500), SELM(11,30), SP(11), WEMP(11,30)
15, 16, 17, 18
5, 6, 7
IF(NEL = 11) 3, 2, 1
3 LD(IN) = LD(11) + 1
2 RETURN
1 J = NEL + 1
10 IF (IFLD(NEL) = J) 6, 5, 5
5 IFEL(NEL) = 7, 8, 9
7 NEL = NEL + 1
6 J = NEL + 1
5 NEL = NEL + 1
4 RETURN
9 2 8:1
P O D (U = A L P H I + D)
C A L L P O M U L I A N i
I = L
P O D i N O D t P O C t N O C t K E R R)
L = I
A N (U = P O C l L)
I F (L - N O C) 4 t 5, 4
L = L + I
G O  T O  6
5 I A N = N O C
I F l I- N O M l I) 7 , 8 , 7
7 1 = 1 + 1
G O  T O  9
8 D O  10 I = 1 , I A N
10 D T l I )= A N  11 I
N D T = I A N
A N (Z) = I .
A Nl U = B E T A ( I )
I A N = Z
I = I
13 P O D ( I ) = B E T A l  I  + U
C A L L  P O M U L (A N ♦ I  A N , P O D , N O D ,P O C t N O C ,K E R R )
D O  il L = I t N O C
11 A N (L) = P O C (L)
I A N = N O C
I F ( I - N O M T ) 1 2 , 1 8 , IZ
IZ 1 = 1 + 1
G O  T O  13
18 D O  14 1 = 1 , IAN
14 B T l I )=AN I
N D T = I A N
D O  15 I = I , N O M T
15 A L P H ( I ) = A L B A R ( I ) - S T P S G R A D m
D O  2 - I = I , N O M T
M = I + N O
2 B E T A ( I ) = B T B A R ( I ) - S f P S G R A D ( M )
3 C A L L  C O N S T R
C A L L  S Y N T H Z
C A L L  E V L V T R
C A L L  S E N S I T
C A L L  E V O B
R E T U R N
E N D

118

SUBROUTINE T R C L I C L S )
D I M E N S I O N  L D (30 I  , N T Y P ( S O ) , V T R ( S O C )  , S E L M I 1 1 , 3 0 )  ,S P (11) , W E U 1 U  11 ,30 )  ,
I E L E M 1 I  30) , D T ( I l )  ,WKl 11) , W Y ( I l )  , A L B A R t 10) , U T B A R l 10 )  , G R A D  (20) ,I F R  ('30)
2 , 1 T O  (30) , I T R ( I O O O )  , F N C O I 11) ,  F D C O l  I U  ,BT 111)., A L P h  (10 ), B E T A  I  10) , F T C O
3 ( 11)
3 P+Beta+FTCO+CMAX+CMIN+RMAX+RMIN+CD+RD+C6M+V0BJ
D O  I = I 120
1 A L P H ( I ) = ALBAR(I)+STP+GRAD
2 D O  2 - I = I N O M T
M = I + N O
2 B E T A ( I ) = B T B A R ( I ) + STP+GRAD
3 C A L L  C O N S T R
S Y N T H Z
C A L L  E V L V T R
C A L L  S E N S I T
C A L L  E V O B
R E T U R N
E N D

Z Z J O B  5
Z Z F O R  5
*LDIS C
SUBROUTINE TESTP
D I M E N S I O N  L D (30 I  , V T R ( S O C )  , S E L M I 1 1 , 3 0 )  ,S P (11) , W E U 1 U  11 ,30 )  ,
I E L E M 1 I  30) , D T ( I l )  ,WKl 11) , W Y ( I l )  , A L B A R t 10) , U T B A R l 10 )  , G R A D  (20) ,I F R  ('30)
2 , 1 T O  (30) , I T R ( I O O O )  , F N C O I 11) ,  F D C O l  I U  ,BT 111)., A L P h  (10 ), B E T A  I  10) , F T C O
3 ( 11)
3 P+Beta+FTCO+CMAX+CMIN+RMAX+RMIN+CD+RD+C6M+V0BJ
D O  I = I 120
1 A L P H ( I ) = ALBAR(I)+STP+GRAD
2 D O  2 - I = I N O M T
M = I + N O
2 B E T A ( I ) = B T B A R ( I ) + STP+GRAD
3 C A L L  C O N S T R
S Y N T H Z
C A L L  E V L V T R
C A L L  S E N S I T
C A L L  E V O B
R E T U R N
E N D

Z Z J O B  5
Z Z F O R  5
*LDIS C
SUBROUTINE T R C L I C L S )
APPENDIX II

OPTIMIZATION COMPUTER PROGRAM INPUT AND OUTPUT DATA FOR EXAMPLE PROBLEM
ZZJOB 5
ZZXEOSCHONE
5100411051212
10101010101013009
1020304010203030404040105
20304000500050005000000

<table>
<thead>
<tr>
<th>1</th>
<th>100</th>
<th>1</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>10</th>
<th>10</th>
<th>10</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>50</td>
<td>50</td>
<td>50</td>
</tr>
<tr>
<td>1</td>
<td>26132</td>
<td>34143</td>
<td>26132</td>
</tr>
<tr>
<td>.05</td>
<td>.25</td>
<td>.35</td>
<td>4.</td>
</tr>
<tr>
<td>.1</td>
<td>.3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Sensitivity of Coefficient M with Respect to Voltage Follower Input Admittance

<table>
<thead>
<tr>
<th>Element</th>
<th>M</th>
<th>N</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>100.0000E-04</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>137.98167E-03</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>137.98167E-03</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>4</td>
<td>150.6114E-03</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>4</td>
<td>200.0000E-10</td>
</tr>
</tbody>
</table>

**Network Element Values**

- **Element 1**: 1.1294040E+00
- **Element 2**: 1.7644922E+00
- **Element 3**: 1.3779518E+00
- **Element 4**: 1.7435497E+00
- **Element 5**: 1.3198774E+01

**Objective Function Value**: 1.000000E+04

### Optimization Parameters

<table>
<thead>
<tr>
<th>N</th>
<th>J</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2.000000E-99</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>2.000000E-99</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2.000000E-99</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>2.000000E-99</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>2.000000E-99</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>2.000000E-99</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2.000000E-99</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2.000000E-99</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2.000000E-99</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>2.000000E-99</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>2.000000E-99</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>2.000000E-99</td>
</tr>
</tbody>
</table>

### Sensitivity of Coefficient M with Respect to Element Number J, M = N, J = N

<table>
<thead>
<tr>
<th>M</th>
<th>N</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>3.198774E+00</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4.557138E-04</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>9.544666E+01</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>9.544666E+01</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>9.544666E+01</td>
</tr>
</tbody>
</table>

### Objective Function Value

- **Value 1**: 393.57092E-03
- **Value 2**: 150.6114E-03
- **Value 3**: 200.0000E-10

### Capacitance Distribution Parameter

- **Value 1**: 1.1601307E+04
- **Value 2**: 8.6115873E-01

### Resistance Distribution Parameter

- **Value 1**: 1.1601307E+04
- **Value 2**: 8.6115873E-01

### Total Capacitance Parameter

- **Value 1**: 8.9351085E+02
- **Value 2**: 5.0874323E+03

### Resistance Ratio

- **Value 1**: 6.377117E+01
- **Value 2**: 6.377117E+01
### Final Network

<table>
<thead>
<tr>
<th>Zero Locations</th>
<th>Pole Locations</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04459769</td>
<td>-2.82347852</td>
</tr>
<tr>
<td>4.4338273</td>
<td>0.9630857</td>
</tr>
<tr>
<td>1.04377760</td>
<td>1.43481370</td>
</tr>
<tr>
<td>1.56730200</td>
<td></td>
</tr>
</tbody>
</table>

### Coefficients of Input Admittance Z(I+1)

<table>
<thead>
<tr>
<th>Numerator Coefficients</th>
<th>Denominator Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00000000</td>
<td>1.00000000</td>
</tr>
<tr>
<td>19.39866400</td>
<td>5.35470860</td>
</tr>
<tr>
<td>64.23079000</td>
<td>7.19828770</td>
</tr>
<tr>
<td>67.27739100</td>
<td>2.75482070</td>
</tr>
<tr>
<td>21.71604200</td>
<td></td>
</tr>
</tbody>
</table>

### Pole Locations

<p>| |</p>
<table>
<thead>
<tr>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.282347783</td>
</tr>
<tr>
<td>0.89603857</td>
</tr>
<tr>
<td>1.434801370</td>
</tr>
</tbody>
</table>

### Objective Function Value

- 1.2654031E+03

---

### Optimization Parameters

#### Sensitivity of Coefficient M with Respect to Element Number J:

<table>
<thead>
<tr>
<th>M = 1, J = 1</th>
<th>M = 1, J = 2</th>
<th>M = 1, J = 3</th>
<th>M = 1, J = 4</th>
<th>M = 1, J = 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000000E-99</td>
<td>0.000000E-99</td>
<td>0.000000E-99</td>
<td>0.000000E-99</td>
<td>0.000000E-99</td>
</tr>
</tbody>
</table>

---

### Sensitivity of Coefficient M with Respect to Voltage Follower Input Admittance

#### Network Element Values:

<table>
<thead>
<tr>
<th>Element</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2614052E+00</td>
</tr>
<tr>
<td>2</td>
<td>0.2952689E+00</td>
</tr>
<tr>
<td>3</td>
<td>0.2563107E+00</td>
</tr>
<tr>
<td>4</td>
<td>0.1867825E+00</td>
</tr>
<tr>
<td>5</td>
<td>0.3630010E+00</td>
</tr>
<tr>
<td>6</td>
<td>0.7324217E+00</td>
</tr>
<tr>
<td>7</td>
<td>0.7271281E+01</td>
</tr>
<tr>
<td>8</td>
<td>0.6312934E+01</td>
</tr>
<tr>
<td>9</td>
<td>0.5988513E+01</td>
</tr>
<tr>
<td>10</td>
<td>0.5677358E+01</td>
</tr>
</tbody>
</table>

---

### Capacitance Distribution Parameter

- 0.338674E+02

### Resistance Distribution Parameter

- 0.6222025E-02

### Total Capacitance Parameter

- 0.279754E+02

### Capacitance Ratio

- 0.2017692E+02

### Resistance Ratio

- 0.1500767E+01

### Sensitivity of Coefficient M with Respect to Voltage Follower Gain K:

<table>
<thead>
<tr>
<th>M = 1</th>
<th>M = 2</th>
<th>M = 3</th>
<th>M = 4</th>
<th>M = 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000000E-99</td>
<td>9.26594E+00</td>
<td>205.4949E+00</td>
<td>105.2902E+00</td>
<td>10.00000E+00</td>
</tr>
</tbody>
</table>

---

### Objective Function Value

- 1.2654031E+03
APPENDIX III

NUMERICAL COMPUTATIONS
This appendix presents the numerical computations for the networks of Figures 1.2 and 13.

A. NUMERICAL COMPUTATIONS FOR NETWORK OF FIGURE 1.2.

Guillemin presents a method for synthesizing the transfer function $Z_{12}(s) = \frac{E_{out}(s)}{I_{in}(s)}$, where $Z_{12}$ is minimum phase and stable, by means of a lossless network terminated by resistors (see pages 535-537 of Reference 25). In the present case, we desire to synthesize

$$Z_{12}(s) = K \frac{s^2 + 2.828s + 4}{s^2 + 1.414s + 1}. \quad (III.1)$$

Separating the numerator into its even and odd parts, we have

$$Z_{12} = Z_{12a} + Z_{12b} = \frac{K(s^2 + 4)}{s^2 + 1.414s + 1} + \frac{K(2.828s)}{s^2 + 1.414s + 1}. \quad (III.2)$$

Guillemin points out that since the numerator of $Z_{12}$ is a Hurwitz polynomial, its even and odd parts have simple zeros restricted to the j-axis. Chapter 11 of Reference 25 shows that transfer functions of the form of $Z_{12a}$ and $Z_{12b}$ may be realized by a lossless network terminated by a single resistor. Thus, the two transfer functions are realized separately and connected in series to form the total transfer function. Utilizing $K = \frac{1}{4}$ and applying Guillemin's procedure to $Z_{12a}$ and $Z_{12b}$ separately.
A network possessing the above $z$ parameters consists of a two section ladder network whose input shunt arm is the impedance $z_{12a}$ and whose output series arm is the impedance $z_{22a} - z_{12a}$. Thus,

$$z_{12a} = \frac{s^2 + 4}{1.414s} = 0.177s + \frac{1}{1.414s}$$  \hspace{1cm} (III.4)

which is a $0.177$ henry coil in series with a $1.414$ farad capacitor. The series arm is

$$z_{22a} - z_{12a} = \frac{s^2 + 1}{1.414s} - \frac{\frac{1}{2}s^2 + 1}{1.414s} = \frac{\frac{3}{2}s^2}{1.414s} = 0.530s$$  \hspace{1cm} (III.5)

which is a $0.530$ henry inductor.

Similarly $z_{12b}$ is realized as

$$z_{12b} = \frac{z_{12b}}{1 + z_{22b}} = \frac{\frac{1}{2}s^2 + 1}{s^2 + 1}$$  \hspace{1cm} (III.6)

and

$$z_{12b} = \frac{0.707s}{s^2 + 1} = \frac{1}{1.414s + \frac{1}{0.707s}}$$  \hspace{1cm} (III.7)

which is a $1.414$ farad capacitor in parallel with a $0.707$ henry coil. The series arm is
\[ z_{22b} - z_{12b} = \frac{1.414s}{s^2 + 1} - \frac{0.707s}{s^2 + 1} = \frac{0.707s}{s^2 + 1} = \frac{1}{1.414s + 0.707s} \]

which is a repeat of \( z_{12b} \).

\( Z_{12a} \) and \( Z_{12b} \) are now realized by connecting \( 1 \) ohm resistors across the output terminals of the above ladders. These two networks are then connected in series as shown in Figure 1.2 to realize \( Z_{12}(s) \).

B. NUMERICAL COMPUTATIONS FOR NETWORK OF FIGURE 1.3.

Pantell's method (41) is to be applied to the transfer function

\[ T_v(s) = \frac{E_{out}(s)}{E_{in}(s)} = k \frac{s^2 + 2.828s + 4}{s^2 + 1.414s + 1} \]

Pantell shows that a minimum phase and stable \( T_v \) may always be realized with resistive terminations at each end provided \( T_v(\omega) = 0 \) or \( T_v(\omega) = 0 \) and no poles are located on the \( j \) axis. Utilizing the surplus factor \( s^2 + 8.48s + 4 \) and letting \( K = k_1 k_2 \), \( T_v \) may be put into the form

\[ T_v(s) = k_1 \frac{s^2 + 8.48s + 4}{s^2 + 1.414s + 1} + k_2 \frac{s^2 + 2.828s + 4}{s^2 + 8.48s + 4} = f_1 f_2 \]

(III.10)
where both $f_1$ and $f_2$ are positive real. Setting $k_1=1$ and $k_2=0.2$ and following Pantell's procedure

$$Z_1 = \frac{\frac{1}{k_1 f_1} - k_2 f_2}{s^2 + 1.414s + 1} = \frac{0.2s^2 + 56.56s + 0.8}{s^2 + 8.48s + 4}$$

and

$$Y_1 = \frac{1}{k_2 f_2} = 5 \frac{s^2 + 8.48s + 4}{s^2 + 2.828s + 4}$$

Pantell's method relies on the fact that the surplus factor, $k_1$, and $k_2$ may always be chosen such that $f_1$, $f_2$, $Z_1$, and $Y_1$ are positive real. $Z_1$ becomes the input series arm of a ladder network and $Y_1$ its following shunt arm with $E_{out}$ taken across $Y_1$. $Z_1$ and $Y_1$ are realized with the results shown in Figure 1.3.


Hogin, J. L.
Active RC networks utilizing the voltage follower