A vector-metric theory of gravity
by Ronald Ward Hellings

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Physics
Montana State University

Abstract:
A new theory of gravity is presented in which gravity is produced by a massless vector field in addition to the usual metric field. It-is found that the theory is compatible with present solar system experiments and cosmological observations. The theory also predicts the existence of black holes and of the most general types of weak plane waves.

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ABSTRACT

A new theory of gravity is presented in which gravity is produced by a massless vector field in addition to the usual metric field. It is found that the theory is compatible with present solar system experiments and cosmological observations. The theory also predicts the existence of black holes and of the most general types of weak plane waves.

CHAPTER I

INTRODUCTION

Whenever a new theory is proposed, it should be made clear why the theory is needed. Usually this involves showing how existing theories fail to adequately explain certain experimental or observational results. Yet this need not be the only justification for a new theory, and it is not the justification for the new theory of gravity to be presented here.

As a matter of fact, there is presently nothing observationally wrong with Einstein's general relativity,[1] the most widely accepted theory of gravity today. As time goes on and the evidence mounts in favor of general relativity, it is strange to remember that a few years ago, as recently as 1960, only two of general relativity's predictions which differed from Newton's theory had actually been observed,[2] and based on only these two the theory was almost universally accepted. One thing which probably gave impetus to the growth of experimental gravity during the sixties and certainly was the motivation behind the recent light deflection and time-delay experiments, was the appearance in 1961 of a new respectable, covariant theory of gravity--the Brans-Dicke theory.

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[1]See a summary of experimental results in C.M. Will, Lectures in B. Bertotti, ed., Proceedings of Course 56 of the International School of Physics "Enrico Fermi", Academic Press (1973).

[2]These were the deflection of light and the perihelion precession of mercury. The third "classical test" of relativity, the gravitational red shift, was originally confirmed, but subsequent observations showed the results to be inconclusive. It was not until the Pound-Rebka experiment in 1960 that the red shift was verified. See E. Finlay-Freundlich, Philos. Mag. 45, 303 (1954); and R.V. Pound and G.A. Rebka, Phys. Rev. Letters 4, 337 (1960).

In fact, what had been largely missing during the period 1916-1960 was the usual interplay between theory and experiment. Most theorists were satisfied with general relativity and were reluctant to consider any other possibility. Most experimenters were reluctant to tackle any of the great difficulties involved in measuring the extremely weak post-Newtonian effects of gravity, especially when the expected outcome of the experiment would merely be a further confirmation of what everyone already "knew".

There is an additional difficulty for the experimenter that arises when there is only one theory present, and that is that it is not clear which experiments are significant. This is especially true for general relativity which predicts a value of zero for the result of many experiments. A null experimental value is really only meaningful against a background of other theories predicting non-null results for the same experiment. For example, general relativity predicts that all neutral test bodies fall at the same rate in a gravitational field (weak equivalence principle), and that the result of any local experiment is independent of the velocity of the apparatus (part of the strong equivalence principle). If the second prediction must follow from the first in any reasonable theory of gravity, then an experimental result verifying the second prediction does not increase our confidence in general relativity. However, if a theory appears which predicts the first but not the second (as the Vector-Metric Theory does), then a verification of the second prediction serves to eliminate the interloper and increase our confidence in general relativity. The reason that the present theory is needed, as are other theories, is to point the way to future significant experiments, and to make "non-null" some of the latent null predictions of general relativity.

A. Metric Theories. An analysis by R. H. Dicke[3] has shown that the high precision null experiments--Eötvös experiments, Hughes-Drevor experiments, and ether drift experiments--rule out the existence of vector or additional tensor fields (besides the metric field) which couple directly to matter. As pointed out by Will and Nordtvedt,[4] however, vector and tensor fields may exist along with the metric field as long as the equation of motion for matter does not include them explicitly. These fields may couple to the metric field and be involved in determining the functional form of the metric, but once the metric functions are found the matter equations of motion will depend only on them and on the matter variables (position, velocity, etc.). Theories which have this property are called "metric theories".[5]

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[3] R. H. Dicke, The Theoretical Significance of Experimental Relativity, Gorden and Breach (1965).

[4] C. Will and K. Nordtvedt, Jr., Ap. J. 177, 757 (1972).

[5] K. Thorne and C. Will, Ap. J. 163, 595 (1970).

Gravitational experimental results depend on the metric fields and can be expected to indirectly detect the existence of the vector or tensor fields which go into determining the metric. In non-gravitational experiments, however, the metric only serves as a background. Coordinates are always chosen so that the metric is locally Lorentz, and the effect of whatever went into determining the metric is washed out. The Eötvös, Hughes-Drevor, and ether experiments are all non-gravitational in this sense. They are based on the fundamental idea of a metric and measure non-metric perturbations to geodesic behavior.

B. Machian Effects. Will and Nordtvedt[6] have studied extensively the effects of gravity which have been variously called "Machian",[7] "preferred-frame", and "ether". Of these labels the last two are far inferior, implying the existence of absolute space in logical contradiction to the spirit of all relativity. In fact, one need not postulate absolute space in order to find a frame which is in some sense preferred. Since gravity is a long-range universal interaction, one might expect the global distribution of matter to affect local gravitational physics in a Machian way and to establish a preferred frame as the mean rest frame of the universe.

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[6]Will and Nordvedt, op cit.

[7]The term "Machian" has been loosely applied to refer to the determination of the properties of space by the global distribution of matter. See Ernst Mach, The Science of Mechanics, Open Court Publishing (1902), Chapter II.

As Will and Nordvedt point out the mystery is not, "How can we have a preferred rest frame in space?", but "How can we ever avoid having one, related to the universe rest-frame". Theories which have only a metric field, or only a metric and a scalar field, avoid these effects in the following way.

1. A theory which contains only a metric field yields local gravitational physics which is identical in all frames which are asymptotically Minkowskiian. This follows from the invariance under boosts of $\eta_{\mu\nu}$ (the asymptotic form of $g_{\mu\nu}$), the only field coupling the local system asymptotically to the universe; and from general covariance, which allows us to find a coordinate system in which the metric takes this Minkowski form at the boundary between the universe and the local system.

2. A theory which contains a metric field and a scalar field yields physics which is identical in all asymptotic Minkowski frames. This follows from invariance of $\eta_{\mu\nu}$ and the scalar field under boosts (the scalar field, of course, is generally invariant).

In the present theory, we include with the metric a vector field, K_μ, whose components depend on the choice of Lorentz frame. In some Lorentz frame, the vector field at a point will have a zeroth component only. This is the only frame in which space is locally isotropic[8] and

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[8] Any non-zero space part of the background vector field must point in some preferred direction, destroying the local isotropy of space.

we define this frame to be the preferred frame in which, for simplicity, we will do much of the work to follow. The apparent isotropy of the matter distribution seen through the telescope leads us to believe that this preferred frame is also the mean rest frame of the universe, and that the preferred frame is determined in a "Machian" way.

C. Summary of Results. In the chapters to follow, we compute the predictions of the vector-metric theory for: 1) solar system experiments, 2) the existence of black holes or event horizons, 3) gravitational radiation, and 4) cosmology. The results are as follows:

1. Solar system experiments. It is found that with restrictions on the strength of some coupling constants the theory is compatible with all existing solar system experiments, and that for a particular choice of the coupling constants the theory makes the same predictions as does general relativity. This last point is of particular importance since it means that no solar system experiment presently envisioned can differentiate between this theory and general relativity. It is evident that experiments which are to choose between these theories must involve higher order effects such as occur in a) gravitational radiation, b) cosmology, or c) extremely precise solar system experiments. It is not clear which type of experiment offers the best possibility, but theories such as these should stand as a challenge to the gravitational experimenter to devise new and better ways to measure the extremely small effects of post-Newtonian and post-post-Newtonian gravity.

2. Black holes. It is found that an event horizon exists in a static spherically symmetric configuration. Moreover, this solution has the same metric behavior close to the horizon as the Swarzschild solution in general relativity.

3. Gravitational radiation. We have found that there are two classes of radiation in the theory, depending on the type of coupling between the vector and metric fields. If the coupling is "scalar-type" (that is, K_μ appears only as $K_\mu K^\mu$ in the interaction Lagrangian), then we find waves of class N_3,[9] the class typical of scalar-metric theories. In the most general coupling, the class is II_6, reflecting the vector nature of the interaction. Also, these last waves are seen to propagate at speeds either greater or less than the speed of light, depending on the values of the coupling constants. This is discussed in Chapter V.

4. Cosmology. The vector-metric theory is found to possess an acceptable closed cosmology, the precise behavior depending on the coupling constants as in Brans-Dicke theory. It is found that the local constant of gravity is dependent on the cosmological strength of the vector field. The observed gravitational constant is obtained if one postulates an energy density in the universe which is 1,000 times the observed stellar matter density. This requirement sounds less outrageous

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[9] For a description of this classification, see D. Eardley, *et al.*, Phys. Rev. Letters 30, 884 (1973). There is also a summary of the scheme in Chapter V of this work.

when we realize that general relativity requires a factor of 100 more energy in the universe than the stellar matter presently observed in order to produce a closed universe.

CHAPTER II

THE VECTOR-METRIC THEORY OF GRAVITY

A. Notation. The notation used in the remainder of this work follows that of Adler, Bazin, and Schiffer's Introduction to General Relativity.[1]

1. Greek indices take on values from 0 to 3. Latin indices run from 1 to 3. Repeated indices are summed over their range of values.

2. The metric tensor is denoted $g_{\mu\nu}$ and has signature (+ - - -). $\eta_{\mu\nu}$ is the Minkowski metric tensor with the same signature. Occasionally g_{ss} will be used to mean g_{11}, g_{22}, or g_{33}, and there will be no sum over s.

3. A comma (,) indicates ordinary partial differentiation. A vertical bar (|) indicates covariant differentiation:

$$A_{\mu|\nu} = A_{\mu,\nu} - \Gamma_{\mu\nu}^{\alpha} A_{\alpha} .$$

4. Sign conventions in the differential geometry are as follows:

$$\Gamma_{\mu\nu}^{\alpha} = \frac{1}{2} g^{\alpha\beta} (g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}),$$

$$R^{\alpha}_{\mu\beta\nu} = \Gamma^{\alpha}_{\beta\mu,\nu} - \Gamma^{\alpha}_{\mu\nu,\beta} + \Gamma^{\alpha}_{\lambda\nu}\Gamma^{\lambda}_{\beta\mu} - \Gamma^{\alpha}_{\lambda\beta}\Gamma^{\lambda}_{\mu\nu},$$

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$$

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[1] R. Adler, M. Bazin, and M. Schiffer, Introduction to General Relativity, McGraw-Hill (1965).

5. The box operator is defined $\Box^2(\ ) \equiv \eta^{\alpha\beta}(\ )_{,\alpha\beta}$ and we define a covariant box operator $\Box^2(\ ) \equiv g^{\alpha\beta}(\ )_{|\alpha\beta}$.

6. We work in units where $c = 1$.

B. The Lagrangian. Einstein's theory of gravity can be derived from variation of the action integral:

$$A = \int \sqrt{-g}[16\pi G L_m + R] d^4x$$

where g is the determinant of $g_{\mu\nu}$,

G is Newton's gravitational constant,

$L_m = L_m(g_{\mu\nu}$, matter variables) is the matter Lagrangian which is a function of $g_{\mu\nu}$ and its derivatives and of whatever variables are used to describe the state of matter in the system (position, velocity, etc.).

The requirement that the action be invariant under an infinitesimal variation of $g_{\mu\nu}$ produces the field equations of general relativity. Variation of the matter variables produces the equations of motion of matter.

In 1961, Brans and Dicke[2] proposed a theory in which gravitation was produced by two fields--a metric tensor field and an auxiliary scalar field. Their action integral is written

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[2] C. Brans and R. H. Dicke, Phys. Rev. 124, 925 (1961).

$$A = \int \sqrt{-g}[16\pi L_m + \phi R + \frac{\omega}{\phi} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu}] d^4x,$$

where ϕ is the scalar field and ω is a dimensionless parameter. Once again, L_m is a function of $g_{\mu\nu}$ and matter variables only; ϕ does not enter. The Brans-Dicke field equations come from independent variation of $g_{\mu\nu}$ and ϕ. The equations of motion of matter come from variation of the matter variables in L_m. Since ϕ does not appear in L_m, the equation of motion of matter will involve only the metric and will produce a metric theory of gravity.

Our reason for reviewing these theories is that the new theory to be presented here adds onto general relativity in a way similar to that of the Brans-Dicke theory. We propose a theory of gravity in which a massless vector field appears in addition to the metric field. Committed to the spirit as well as to the law of general covariance in physics, we introduce no a priori fields or reference frames into the theory. We require a Lagrangian subject to the following conditions:

1. The Lagrangian density is a four-scalar density.
2. It generates positive definite free field energies for both the metric and the vector fields.
3. It produces a "metric theory".
4. It generates field equations containing no higher than second derivatives of the fields.

Such a Lagrangian is

$$A = \int \sqrt{-g}[16\pi G_0 L_m + R - F_{\mu\nu}F^{\mu\nu} + \omega K_\mu K^\mu R + \eta K^\mu K^\nu R_{\mu\nu}]d^4x \qquad \text{(II.1)}$$

where $L_m = l_m(g_{\mu\nu}$, matter variables) as before,

$F_{\mu\nu} = K_{\mu|\nu} - K_{\nu|\mu}$ in analogy with electrodynamics,

K_μ is the vector field,

ω and η are dimensionless parameters, and

G_0 is an a priori or "bare" gravitational constant.

C. The Field Equations. The field equations are calculated by requiring that the action, equation II.1, be stationary under independent variation of the fields. Details of the derivation are given in Appendix E. Variation of $g_{\mu\nu}$ gives the equation

$$\begin{aligned} R_{\mu\nu} &- \tfrac{1}{2}g_{\mu\nu}R + \omega(K_\mu K_\nu R + KR_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}KR + K_{|\mu\nu} - g_{\mu\nu}\Box^2 K) \\ &+ \eta K^\alpha K^\beta(g_{\mu\alpha}R_{\nu\beta} + g_{\nu\beta}R_{\mu\alpha} - \tfrac{1}{2}g_{\mu\nu}R_{\alpha\beta}) - \tfrac{1}{2}\eta[g_{\mu\nu}(K^\alpha K^\beta)_{|\alpha\beta} \\ &+ \Box^2(K_\mu K_\nu) - (K_\mu K^\alpha)_{|\nu\alpha} - (K_\nu K^\alpha)_{|\mu\alpha}] \\ &+ 2F_{\mu\alpha}F^\alpha{}_\nu + \tfrac{1}{2}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} = -8\pi G_0 T_{\mu\nu} \end{aligned} \qquad \text{(II.2)}$$

where $K \equiv K_\alpha K^\alpha$ and $T_{\mu\nu} \equiv \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}(\sqrt{-g}16\pi G_0 L_m)$. The contraction of equation II.2 is

$$R + (3\omega + \tfrac{1}{2}\eta)\Box^2 K + \eta(K^\alpha K^\beta)_{|\alpha\beta} = 8\pi G_0 g^{\mu\nu}T_{\mu\nu} \equiv 8\pi G_0 T. \qquad \text{(II.3)}$$

Variation of K_μ in equation II.1 gives

$$\omega RK^\mu + \eta R^{\mu\nu}K_\nu + 2F^{\mu\nu}{}_{|\nu} = 0. \qquad \text{(II.4)}$$

D. The Solution. In the chapters to follow, the above equations will be solved in four different contexts. To each of the cases, there corresponds some approximation or symmetrization of the metric which can be used to simplify the field equations. The four cases are as follows:

1. Weak Gravity. It is assumed that there exists a region of space where gravity is weak and the motion of sources is slow. By weak gravity it is meant that, for each massive source of the metric, $\frac{GM}{r} << 1$ (where r is the distance from the source of mass M). Slow motion of the sources naturally means $v^2 << 1$. These conditions are satisfied in the solar system and in most local regions of the universe. When the conditions are satisfied, one can make a general expansion of the metric in powers of both $\frac{GM}{r}$ and v^2. Keeping terms second order in the combination (that is, terms like G^2M^2/r^2 and GMv^2/r) produces the Parameterized Post-Newtonian (PPN) metric of Nordtvedt and Will.[3] In Chapter III, we obtain the PPN metric for the vector-metric theory.

2. Event Horizon. Vishveshwara[4] has shown that, for a static metric, a surface of infinite redshift (i.e., a surface on which $g_{00} = 0$)

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[3] C. M. Will and K. Nordtvedt, Jr., Ap. J. 177, 757 (1972).

[4] C. V. Vishveshwara, J., Math. Phys. 9, 1319 (1968).

is also an event horizon or a one-way membrane. Therefore, the question of whether or not the vector-metric theory predicts the existence of black holes as does general relativity can be answered by a search for a surface of infinite redshift in a static configuration, which for simplicity we also take to be spherically symmetric. Ideally one would like to have an exact solution for this configuration, but this turns out to be an extremely difficult problem whose solution has not yet been found. Rather, we have expanded the fields in a power series about the event horizon, keeping the leading terms only.

3. Radiation. Weak plane gravitational waves are described by writing the metric and vector fields as the sum of a constant background part and a wave disturbance perturbation:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$K^{\mu} = \phi^{\mu} + A^{\mu}$$

and requiring that $h_{\mu\nu} << \eta_{\mu\nu}$ and that $A^{\mu} << \phi^{\mu}$. Here

$\eta_{\mu\nu}$ is the Minkowski metric tensor,

ϕ^{μ} is the cosmological vector background,

$h_{\mu\nu}$ is the metric wave,

A^{μ} is the vector wave;

$h_{\mu\nu}$ and A^{μ} are assumed proportional to $\exp(ik_{\mu}x^{\mu})$, with k_{μ} as some constant propagation vector. Solutions giving relationships between

ϕ_μ, $h_{\mu\nu}$, A_μ and k_μ for weak radiation are found in Chapter V.

4. Cosmology. The Robertson-Walker cosmological metric with a neutral dust model for matter is used to establish a cosmology for the vector-metric theory. The metric is assumed to be of the form

$$g_{00} = 1$$

$$g_{ij} = - \frac{S(t)^2}{(1 + \frac{\kappa}{4}r^2)^2} \delta_{ij}.$$

In Chapter VI, the field equations are solved giving S, the "size" of the universe, as a function of time.

CHAPTER III

THE PPN METRIC

Will and Nordvedt[1] have arrived at a general form for the first post-Newtonian metric valid in any inertial coordinate frame.

$$g_{00} = 1 - 2\sum_i \frac{GM_i}{r_i} + 2\beta\left(\sum_i \frac{GM_i}{r_i}\right)^2 - (2\gamma + 1 + \alpha_3 + \zeta_1)\sum_i \frac{GM_i v_i^2}{r_i}$$

$$- 2(1 - 2\beta + \zeta_2)\sum_i \frac{GM_i}{r_i} \sum_{j\neq i} \frac{GM_j}{r_{ij}} + \zeta_1\sum_i \frac{GM_i}{r_i^3}(\vec{v}_i\cdot\vec{r}_i)^2$$

$$+ \alpha_2\sum_i \frac{GM_i}{r_i^3}(\vec{w}\cdot\vec{r}_i)^2 + (\alpha_1 - \alpha_2 - \alpha_3)\sum_i \frac{GM_i}{r_i}w^2$$

$$+ (\alpha_1 - 2\alpha_3)\sum_i \frac{GM_i}{r_i}(\vec{w}\cdot\vec{v}_i)$$

$$g_{0k} = \frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1)\sum_i \frac{GM_i}{r_i}v_i^k + \frac{1}{2}(1 + \alpha_2 - \zeta_1)\sum_i \frac{GM_i}{r_i^3}$$

$$\times(\vec{v}_i\cdot\vec{r}_i)x_i^k + (\tfrac{1}{2}\alpha_1 - \alpha_2)\sum_i \frac{GM_i}{r_i}w^k + \alpha_2\sum_i \frac{GM_i}{r_i^3}(\vec{w}\cdot\vec{r}_i)x_i^k$$

$$g_{\ell m} = -\delta_{\ell m}\left(1 + 2\gamma\sum_i \frac{GM_i}{r_i}\right)$$

where x_i^k are cartesian components of the i^{th} source-to-field-point vector,

v_i^k are cartesian velocity components of the i^{th} source $(v^k = \frac{dx^k}{dt})$,

w^k are cartesian components of the velocity of the inertial coordinate system relative to the universe rest frame.

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[1] C. M. Will and K. Nordtvedt, Jr., Ap. J. 177, 757 (1972).

$$r_i = \left[\sum_{k=1}^{3} (x_i^{\,k})^2\right]^{1/2},$$

$$r_{ij} = \left[\sum_{k=1}^{3} (x_i^{\,k} - x_j^{\,k})^2\right]^{1/2},$$

M_i is the gravitational mass of the i^{th} body, and

G is the effective gravitational constant.

It should be appreciated that this form is based on very few assumptions (see Nordtvedt[2]). The parameters β, γ, α_1, α_2, α_3, ζ_1, ζ_2 in the metric are theory-dependent and may depend on cosmological factors through the influence of cosmological fields. In general relativity the PPN parameters have the value

$$\gamma = \beta = 1$$

$$\alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = 0,$$

and in the Brans-Dicke scalar-metric theory

$$\gamma = \frac{1 + \omega}{2 + \omega}$$

$$\beta = 1$$

$$\alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = 0.$$

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[2] K. Nordtvedt, Jr., Phys. Rev. 169, 1017 (1968).

Inspection of the PPN metric shows that one can obtain all the PPN parameters by calculating in the universe rest frame, and that one can obtain all but ζ_2 by considering a single source. For most of our work we therefore use

$$g_{oo} = 1 - 2\frac{GM}{r} + 2\beta\frac{G^2M^2}{r^2} - (2\gamma + 1 + \alpha_3 + \zeta_1)\frac{GM}{r}v^2 + \zeta_1\frac{GM}{r^3}(\vec{r} \cdot \vec{v})^2$$

$$g_{ok} = \frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1)\frac{GM}{r}v^k + \frac{1}{2}(1 + \alpha_2 - \zeta_1)\frac{GM}{r^3}(\vec{r} \cdot \vec{v})x^k \qquad \text{(III.1)}$$

$$g_{\ell m} = -\delta_{\ell m}(1 + 2\gamma\frac{GM}{r} + 2\delta\frac{G^2M^2}{r^2})$$

where it is noted that a post-post-Newtonian term is added to $g_{\ell m}$.

In addition to the PPN metric expansion, we will need a similar expansion for K_μ,

$$K_o = \sqrt{\phi}[1 + a_1\frac{GM}{r} + b\frac{G^2M^2}{r^2} + a_2\frac{GM}{r}v^2 + a_3\frac{GM}{r^3}(\vec{r} \cdot \vec{v})^2 + f\frac{GM}{r}(\vec{r} \cdot \vec{a})] \qquad \text{(III.2)}$$

$$K_\ell = \sqrt{\phi}[d\frac{GM}{r}v^\ell + d'\frac{GM}{r^3}(\vec{r} \cdot \vec{v})x^\ell]$$

where ϕ is a constant, as are the parameters a_i, b, f, d, d'. These expansions are substituted into the field equations (II.2, II.3, and II.4), the details of the solution being given in Appendix A. The linearized equations are first solved for a static source, giving for γ

$$\gamma = 1 + \omega\phi\frac{2\omega - n - 2}{1 - \omega\phi(4\omega - 1)}$$

Light deflection and retardation experiments[3] show that $\gamma \approx 1$, so we specialize to that case. There are three ways this condition can be realized. The first is to require that

$$\phi << 1.$$

This weak cosmological field condition reduces all results arbitrarily close to general relativity. For that reason, it is the least interesting. There are two other conditions which give $\gamma \approx 1$. These are

$$\omega = \frac{1}{2}\eta + 1 \qquad \text{(Case I)}$$

and (III.3)

$$\omega = 0 \qquad \text{(Case II)}$$

Proceeding with the solutions for each case, it is found that

Case I	Case II
$\gamma = 1$	$\gamma = 1$
$a_1 = -1$	$a_1 = \frac{1}{2}\eta$

and the gravitational constant in each case is renormalized

Case I	Case II
$G = \dfrac{G_0}{1 + \frac{1}{2}\eta\phi}$	$G = \dfrac{G_0}{1 + \eta\phi + \frac{1}{4}\eta^2\phi}$

(III.4)

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3 John D. Anderson, et al., Proceedings of the Conference on Experimental Tests of Gravitational Theories, (NASA-JPL Technical Memorandum 33-499, 1970).

This means that G, the effective gravitational constant which enters the metric and determines the strength of gravity in the solar system, depends on background strength of K_μ far from the solar system. In Chapter VI it is seen from cosmological considerations that this produces a time development of G coupled to the evolution of the universe; and that the extreme weakness of G in the solar system can be viewed as being due to a renormalization of $G_0(\sim 1)$ by a large cosmological $\eta\phi$.

The parameters β and δ come from the static solution of the field equations to second order. For the two cases we get

(Case I)

$$\beta = 1$$

which agrees with the value in general relativity and

$$\delta = 1 + \frac{1 - 4\omega}{2\phi^{-1} + 2\omega(1 - 4\omega)} \qquad \text{(III.5)}$$

which does not, general relativity having $\delta = 1$. Unfortunately, this parameter does not affect existing experiments to a measurable degree. In the other case, β and δ are found to be

(Case II)

$$\beta = 1 + \frac{\frac{1}{4}\eta(\eta + 2)(\eta + 4)\phi}{4 + \eta\phi(\eta + 4)}$$

$$\delta = 1 - \frac{1}{2}\eta\phi(\eta + 4)\frac{(\eta + 3) - \frac{1}{4}\eta\phi(\eta + 4)}{4 + \eta\phi(\eta + 4)} \qquad \text{(III.6)}$$

The experimental result $\beta \approx 1$ requires $\eta \approx 0, -2,$ or -4 (none of which provides the small α_1 required below). The unobservability of δ effects in present solar system experiments limits $\phi \sim 1$ or less.

Solving the total dynamic linearized field equations gives the additional PPN parameters:

(Case I)
$$\zeta_1 = \alpha_3 = 0$$
$$\frac{1}{2}\alpha_1 = \alpha_2 = \frac{4\eta}{4\phi^{-1} + 4 + 6\eta + \eta^2} \quad \text{(III.7)}$$

(Case II)
$$\zeta_1 = \alpha_3 = 0$$
$$\alpha_1 = \frac{2\eta\phi(3 + \eta)}{1 + \eta\phi + \frac{1}{4}\eta^2\phi} \quad \text{(III.8)}$$
$$\alpha_2 = \frac{1}{2}\alpha_1 - \frac{1}{2}\,\frac{3\eta\phi(\eta + 2)}{2 + 4\eta\phi + \eta^2\phi}$$

An examination of the configuration of a point source inside a massive spherical shell yields the second order two-mass PPN parameter;

$$\zeta_2 = 0$$

for both cases.

The ζ parameters measure 4-momentum non-conservation and are expected to be zero in theories derived from Lagrangian action principles.[4] The α parameters measure the existence of "Machian" effects

[4] Will and Nordtvedt, op cit.

in gravitation. Except for the case $\eta = 0$, both cases predict such preferred frame effects. Will and Nordtvedt[5] have analyzed various geophysical and planetary orbital effects to arrive at upper limits on the α parameters. The restrictions are[6]

$$\alpha_1 < .1$$

$$\alpha_2 < .1$$

In Case I, there are two ways these restrictions can be met:

$$\eta \gtrsim 34$$
$$\eta \lesssim .1 \qquad \text{(III.9)}$$

Of special interest is the case where $\eta = 0$ ($\omega = 1$) in equation III.7; then α_1 and α_2 are strictly zero. In this case, renormalization of G_0 is lost and the total set of PPN parameters is identical to those of general relativity. δ still differs, however, as can be seen by setting $\omega = 1$ in equation III.5.

$$\delta = 1 + \frac{1}{2}\frac{3\phi}{3\phi - 1}$$

In Case II, the α restrictions require

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[5]C. M. Will and K. Nordtvedt, Jr., Ap. J. 177, 775 (1972).

[6]The original value of $\alpha_2 < .03$ given by Nordtvedt and Will probably represents an overly optimistic evaluation of gravimeter results. See K. Nordtvedt, Jr., Science 178, 1157 (1972).

$$\eta \leq \frac{1}{60}$$

Case II already had $\omega \simeq 0$, so this additional restriction on the magnitude of η shows this case to represent weak coupling to the metric.

To sum up, experimental results limit the theory to a form fulfilling one of several conditions. The first condition is that of weak cosmological vector field,

$$\phi << 1,$$

which is not interesting because one can always postulate any kind of field one wants as long as it is too weak to affect experimental results. A second condition is Case II where

$$\omega << 1$$

$$\eta << 1$$

which is also not interesting, involving as it does an extremely weak coupling of the vector field to the metric. The last possible way of satisfying experimental restrictions is that of Case I, which is

$$\omega = \frac{1}{2}\eta + 1$$

$$\eta \gtrsim 34 \quad \text{or} \quad \eta \lesssim .1.$$

In the interest of generality, subsequent sections will not require these conditions a priori but will refer to them from time-to-time.

CHAPTER IV

THE EVENT HORIZON

We first produce the exact field equations for a source which is spherically symmetric, static, and at rest in the universe. Then an approximate solution is found which is valid near the event horizon. Choosing Swarzschild-type variables, the fields are written

$$g_{00} = e^{\nu}$$

$$g_{rr} = -e^{\lambda}$$

$$g_{\theta\theta} = -r^2$$

$$g_{\phi\phi} = -r^2 \sin^2\theta$$

$$K_0 = \sqrt{\phi} e^{\mu}$$

$$K_\ell = 0$$

where μ, ν, and λ are functions of r only. The spherical symmetry and time independence are manifest in the form of the metric. The absence of motion through the universe is reflected in the vanishing of K_ℓ. If a spherically symmetric black hole were moving with respect to the rest frame of the universe, an asymmetry could be communicated to the metric via a non-zero space part of the vector field, producing an asymmetric event horizon. This possibility is worth investigating for its astrophysical implications, though that will not be done here.

Continuing with the solution, we write the action integral in terms of the Swarzschild variables. First,

$$R = \frac{e^{-\lambda}}{r^2}(2e^{\lambda} - 2 + 2r\lambda' + \frac{1}{2}r^2\lambda'\nu' - r^2\nu'' - 2r\nu' - \frac{1}{2}r^2\nu'^2)$$

$$R_{00} = e^{\nu-\lambda}(\frac{1}{4}\lambda'\nu' - \frac{1}{2}\nu'' - \frac{1}{4}\nu' - \frac{1}{4}\nu'^2)$$

$$K_{0|r} = (\mu' - \frac{1}{2}\nu')\sqrt{\phi}e^{\mu}$$

$$K_{r|0} = -\frac{1}{2}\nu'\sqrt{\phi}e^{\mu}$$

with all other covariant derivatives of K_μ being zero. (Prime denotes differentiation with respect to r.) Equation II.1 can then be written

$$A = \int d^4xr^2e^{\frac{\nu+\lambda}{2}}[16\pi G_oL_m + R + \omega\phi Re^{2\mu-\nu} + \eta\phi R_{00}e^{2\mu-2\nu} + 2\phi\mu'^2e^{2\mu-\nu-\lambda}]$$

Invariance of the action under variation of μ gives the equation

$$4\frac{dP}{dr} + (2\omega + \eta)\frac{dQ}{dr} - 4F - 4\omega\phi e^{\zeta}(e^{\lambda} - 1 + r\lambda') = 0. \qquad \text{(IV.1)}$$

Variation of ν gives

$$(2\omega + \eta)\frac{dP}{dr} - (2\omega + \eta)\frac{dQ}{dr} - e^{\frac{\nu-\lambda}{2}}(e^{\lambda} - 1 + r\lambda') + \omega\phi e^{\zeta}(e^{\lambda} - 1 + r\lambda')$$

$$+ F = -8\pi G_oT_o{}^o. \qquad \text{(IV.2)}$$

Variation of λ gives

$$e^{\frac{\nu-\lambda}{2}}(e^{\lambda} - 1 - r\nu') + \omega\phi e^{\zeta}(e^{\lambda} - 1 + r\nu' - 4r\mu') - F = 0. \qquad \text{(IV.3)}$$

In all of the above

$$\zeta = 2\mu - \frac{1}{2}\nu - \frac{1}{2}\lambda$$

$$P = \phi r^2 e^{\zeta} \mu'$$

$$Q = \phi r^2 e^{\zeta} \nu'$$

$$F = (2\omega + \eta)\phi r^2 e^{\zeta} \frac{1}{2}\nu'(\mu' - \frac{1}{2}\nu') + \phi r^2 e^{\zeta} \mu'^2$$

$$T_0^{\ 0} \equiv - \frac{\delta L_m}{\delta \nu}.$$

These three field equations can be made simpler both for exact solution and for the present approximate solution, by some recombination. Adding together IV.1 and IV.2 gives

$$(2\omega + \eta)\frac{d}{dr}(P - Q) - e^{\frac{\nu-\lambda}{2}}(r\nu' + r\lambda') + 2\omega\phi e^{\zeta}(e^{\lambda} - 1 - r\zeta') = 0 \tag{IV.4}$$

where the solution is to be taken external to the source ($T_0^{\ 0} = 0$). One-fourth of equation IV.1 subtracted from equation IV.3 gives

$$\frac{dP}{dr} + \frac{1}{4}(2\omega + \eta)\frac{dQ}{dr} - 2\omega\phi e^{\zeta}(e^{\lambda} - 1 - r\nu') - e^{\frac{\nu-\lambda}{2}}(e^{\lambda} - 1 - r\zeta') = 0. \tag{IV.5}$$

The field equations for a static source (IV.1 to IV.5) have been written in a form which, it is hoped, should be amenable to exact solution, though we have not made progress in this direction. Here we will describe an approximate solution, valid near an event horizon, obtained by a method which should be useful in other complicated theories of gravity as well.

The event horizon around a static spherically symmetric source is a surface $r = r_0 = \text{const}$ on which $g_{00} = 0$. Close to the horizon $\frac{r - r_0}{r_0} < 1$ and we can expand the fields in a power series.

$$g_{00} = e^{\nu} = \left(\frac{r - r_0}{r_0}\right)^s \left[a + \sum_{n=1}^{\infty} a_n \left(\frac{r - r_0}{r_0}\right)^n\right]$$

$$g_{rr} = -e^{\lambda} = -\left(\frac{r - r_0}{r_0}\right)^t \left[b + \sum_{n=1}^{\infty} b_n \left(\frac{r - r_0}{r_0}\right)^n\right] \tag{IV.6}$$

$$K_0 = \sqrt{\phi}\, e^{\mu} = \left(\frac{r - r_0}{r_0}\right)^u \left[c + \sum_{n=1}^{\infty} c_n \left(\frac{r - r_0}{r_0}\right)^n\right].$$

Requiring that $s > 0$ guarantees that $r = r_0$ is indeed an event horizon. Sufficiently close to the horizon, the metric and vector fields may be approximated by their leading terms alone. When this approximation is substituted into equations IV.5, IV.4, and IV.3, they become, respectively

$$\lambda(u - s)[(w - 1)x^{w-2} + 2x^{w-1}] - \frac{a}{c^2}x^{p-1}(s + t) + 2\omega x^w\left(bx^t - 1 - \frac{w}{x}\right)$$
$$= 0 \tag{IV.7}$$

$$\left(u + \frac{1}{4}\lambda s\right)[(w - 1)x^{w-2} + 2x^{w-1}] - \frac{a}{c^2}x^p\left(bx^t - 1 - \frac{s}{x}\right)$$
$$- 2\omega x^w\left(bx^t - 1 - \frac{w}{x}\right) = 0 \tag{IV.8}$$

$$\lambda x^{w-2}\frac{1}{2}s(u - \frac{1}{2}s) + x^{w-2}u^2 - \frac{a}{c^2}x^p(bx^t - 1 - \frac{s}{x})$$

$$+ \omega x^w(bx^t - 1 - \frac{s}{x} - 4\frac{u}{x}) = 0 \qquad (V.9)$$

where we have defined

$$x = \frac{r - r_0}{r_0} ,$$

and

$$w = 2u - \frac{1}{2}s - \frac{1}{2}t$$

$$p = \frac{1}{2}s - \frac{1}{2}t$$

$$\lambda = 2\omega + \eta .$$

The details of the solution for s, t, and u are given in Appendix B. The result is

$$g_{00} = a\,\frac{r - r_0}{r_0}$$

$$g_{rr} = -[1 + \frac{1}{4}\frac{c^2}{a}(2\omega + \eta + 4)]\frac{r_0}{r - r_0}$$

$$K_0 = c\,\frac{r - r_0}{r_0}$$

with a, c, and r_0 still arbitrary. The arbitrariness of a stems from the freedom to redefine time anyway we choose. c just represents the freedom in the cosmological value of the vector field.

Since the metric has the same dependence on $r - r_0$ as does the Swarzschild metric, the behavior near the horizon is similar to that of general relativity. In particular, the singularity in g_{rr} at $r = r_0$ is only a coordinate singularity, the physical components of the Riemann tensor remaining regular across the boundary.

CHAPTER V

RADIATION

A. Linearized Gravitational Waves. We now consider the propagation of weak gravitational waves through a region of empty space. The fields are written

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$K^{\mu} = \phi^{\mu} + A^{\mu}$$

where $\eta_{\mu\nu}$ and ϕ^{μ} are the constant cosmological background values of the fields and $h_{\mu\nu}$ and A^{μ} are the local wave perturbations. By "weak" waves, it is meant that, over the region of space to be considered, $h_{\mu\nu} << \eta_{\mu\nu}$ and $A^{\mu} << \phi^{\mu}$. Coordinates are chosen so that $\eta_{\mu\nu}$ is the Minkowski metric. In all that follows we consider plane waves, written

$$h_{\mu\nu} = \varepsilon_{\mu\nu} e^{ik_{\mu}x^{\mu}}$$

$$A^{\mu} = a^{\mu} e^{ik_{\mu}x^{\mu}} .$$

k_{μ} is a constant propagation four-vector which will be taken to represent waves moving in the z-direction, i.e., $k_{\mu} = (\omega,0,0,-k)$. The speed of propagation of the radiation is

$$c = \frac{\omega}{k}$$

and may or may not be equal to the speed of light (1 in the units we have chosen).

Substituting these expressions for the fields into the field equations and discarding terms second order or higher (i.e., terms like A^2, Ah, h^2) produces the following linearized equations (details of the derivation are given in Appendix C).

The g (contracted) equation (equation II.3) is

$$R + \eta\phi^\alpha\phi^\beta R_{\alpha\beta} + 3\Box^2 P - 2\eta\phi^\alpha F_\alpha{}^\beta{}_{,\beta} = 0$$

where $P \equiv (2\omega + \eta)(\phi^\alpha A_\alpha - \frac{1}{2}\phi^\alpha\phi^\beta h_{\alpha\beta})$ has been defined for notational ease. The K_μ equation (equation II.4) is

$$\omega\phi_\mu R + \eta\phi^\alpha R_{\mu\alpha} + 2F_\mu{}^\alpha{}_{,\alpha} = 0$$

The $g_{\mu\nu}$ equation (II.2) is simplified by adding combinations of other equations to it;

$$R_{\mu\nu}(1 + \omega\phi^2) + \eta\phi^\alpha\phi^\beta R_{\mu\alpha\nu\beta} - \omega\phi_\mu\phi_\nu R + \tfrac{1}{2}g_{\mu\nu}\Box^2 P + P_{,\mu\nu}$$

$$+ g_{\mu\nu}\phi^\alpha F_\alpha{}^\beta{}_{,\beta} - \tfrac{1}{2}\eta\phi^\alpha(F_{\alpha\mu,\nu} + F_{\alpha\nu,\mu}) - (\tfrac{1}{2}\eta + 2)(\phi_\mu F_\nu{}^\alpha{}_{,\alpha} - \phi_\nu F_\mu{}^\alpha{}_{,\alpha})$$

$$= 0.$$

There are a total of fourteen functions which appear in these equations, ten components of $h_{\mu\nu}$ and four of K_μ. However, except for the scalar P, $h_{\mu\nu}$ appears only in the components of the Riemann tensor of which only six are independent, and A_μ appears only in $F_{\mu\nu,\alpha}$ which only has three independent components. Therefore, a careful check of any solution is necessary to be sure that it is consistent with all fourteen independent equations.

As was mentioned, coordinates were chosen so that the background metric is locally Minkowskiian and the z-axis is along the direction of propagation of the wave. There still remains the freedom of inertial frame which we choose to be the rest frame of the universe (i.e., that frame in which $\phi_k = 0$). The vanishing or non-vanishing of the various components of $R_{\mu\alpha\nu\beta}$ and A_μ will of course depend on this choice of Lorentz frame, but, once the linearized solutions have been worked out in this frame, they may be found in any frame by Lorentz transformations since we are working in a flat background metric.

B. Results. The solutions are worked out in detail in Appendix C. Here we simply present the results within a classification scheme worked out by Douglas Eardley, et al.[1] In this scheme, the six independent components of the Riemann tensor are combined into four Newman-Penrose functions, two of which are complex. The reason for this recombination is that each of these functions affects a gravitational wave antenna in a characteristic way. The reader is referred to the work of

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[1] D. Eardley, et al., Phys. Rev. Letters 30, 884 (1973).

Eardley, et al., for details. The four functions are (where the 3-axis is taken along the direction of propagation)

$$\psi_2 = -\frac{1}{6}R_{0303}$$

$$\psi_3 = \frac{1}{2}(-R_{0103} + iR_{0203})$$

$$\psi_4 = R_{0101} - R_{0202} + 2iR_{0102}$$

$$\Phi_{22} = -(R_{0101} + R_{0202})$$

Besides affecting an antenna, the passage of radiation could also affect the gravitational constant in PPN-type experiments (see Chapter III). The effect is proportional to

$$K_\mu K_\nu g^{\mu\nu} = (\phi_\mu + A_\mu)(\phi_\nu + A_\nu)(\eta^{\mu\nu} + h^{\mu\nu})$$

In the universe rest frame this becomes

$$K_\mu K_\nu g^{\mu\nu} = {\phi_o}^2 + 2\phi_o A_o - {\phi_o}^2 h_{oo}$$

where the fact that $h^{oo} = -h_{oo}$ has been used.

We describe the separate modes in Tables I and II.

TABLE I. $\eta = 0$; five independent modes.

Independent Mode	Speed ($c^2 =$)	Other Fields	Correction to $K_\mu K^\mu$
Φ_{22}	1	None	$\frac{1 + \omega\phi^2}{\omega}(-k^2)\Phi_{22}$
$\mathrm{Re}\psi_4$	1	None	None
$\mathrm{Im}\psi_4$	1	None	None
A_1	1	None	None
A_2	1	None	None

$\mathrm{Re}\psi_3 = \mathrm{Im}\psi_3 = \psi_2 = 0$

At this point it is recalled that the solution has been found only in a particular Lorentz frame. Eardley, et al., have shown, however, that if ψ_3 and ψ_2 are null waves ($c = 1$) and if $\psi_3 = \psi_2 = 0$ in any frame, then they are zero in all frames.

In the most general case ($\eta \neq 0$) it is found that all four of the Newman-Penrose functions are present, though they travel at different speeds.

TABLE II. $\eta \neq 0$; five independent modes.

Independent Mode	Speed ($c^2 =$)	Other Fields	Correction to $K_\mu K^\mu$
$\mathrm{Re}\psi_4$	$1 - \frac{\eta\phi^2}{1 + \omega\phi^2 + \eta\phi^2}$	None	None
$\mathrm{Im}\psi_4$	$1 - \frac{\eta\phi^2}{1 + \omega\phi^2 + \eta\phi^2}$	None	None
$\mathrm{Re}\psi_3$	$1 + \frac{1}{4}\frac{\eta^2\phi^2}{1 + \omega\phi^2 + \eta\phi^2}$	$\Box^2 A_1 = \frac{\eta\phi}{2c}\mathrm{Re}\psi_3$	None
$\mathrm{Im}\psi_3$	$1 + \frac{1}{4}\frac{\eta^2\phi^2}{1 + \omega\phi^2 + \eta\phi^2}$	$\Box^2 A_2 = \frac{\eta\phi}{2c}\mathrm{Im}\psi_3$	None
Φ_{22}	$1 + \frac{1}{3}\frac{\eta\phi^2 + \eta^2\phi^2}{1 + \omega\phi^2 + \eta\phi^2}$	$\psi_2 = \frac{c^2}{1 - c^2}\Phi_{22}$	$2\frac{1 + \omega\phi^2 + \eta\phi^2}{2\omega + \eta} \times (-c^2k^2)\Phi_{22}$
		$F_{0\ ,\alpha}^{\ \alpha} = \frac{\eta\phi}{2c^2}\Phi_{22}$ *	

*$F_{0\ ,\alpha}^{\ \alpha}$ is used instead of A_0 or A_3 to avoid possible non-physical "coordinate ripple" waves. See Appendix D.

Eardley, et al., have classified theories of gravity according to which of the metric-wave functions vanish, as follows:

Class II_6. $\psi_2 \neq 0$. All observers in such Lorentz frames measure a non-zero amplitude in the ψ_2 mode and agree on the value of this amplitude. (But they will generally disagree about the presence or absence and amplitude of all other modes.)

Class III_5. $\psi_2 \equiv 0 \neq \psi_3$. All observers agree on the absence of ψ_2 and the presence of ψ_3. (But they generally disagree about the presence or absence of ψ_4 and Φ_{22}.)

Class N_3. $\psi_2 \equiv 0 \equiv \psi_3$; $\psi_4 \neq 0 \neq \Phi_{22}$. All observers agree about the presence or absence of all modes.

Class N_2. $\psi_2 \equiv 0 \equiv \psi_3$; $\psi_4 \neq 0 \equiv \Phi_{22}$. All observers agree.

Class 0_1. $\psi_2 \equiv 0 \equiv \psi_3$; $\psi_4 \equiv 0 \neq \Phi_{22}$. All observers agree.

The version of the theory with $\eta = 0$ is therefore of class N_3, while the general $\eta \neq 0$ version is of class II_6. In addition the $\eta \neq 0$ case has gravitational waves traveling at speeds greater or less than the speed of light, depending on the values of ω and η. The slower speed is interesting but not shocking. The faster speed requires some additional discussion.