Simultaneous convergence in two metrics
by Eric Drake

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Mathematics
Montana State University


Abstract:
On a metric space certain concepts of convergence for sequences are defined. Using these concepts, several relations between metrics on a set X are introduced and their interdependence studied. Included are the relations "equivalent" (identical topologies), "comparable" (one topology includes another), And "uniconvergent" (the identity map from X bearing one metric, to X bearing a second metric has closed graph). For X a commutative group, convergence in a translation-invariant metric d is more conveniently studied by introducing the associated metron (the function p: X → R such that □x □ X[p(x) = d(x,o)]). The same is true of relations.

Five classes of metrons on a linear space are considered; metrons, scalar-continuous metrons (the product of scalar and vector is a continuous function of the scalar component), quasinorms (the product of scalar and vector is a jointly continuous function of the scalar and vector components), norms, and inner product norms. A typical question studied is whether, on a given linear space, all metrons of a given class bear a given relation to each other. For norms, only four of the relations studied remain distinct, while for complete norms all coincide.

It is proved that non-uniconvergent metrons on a one-dimensional space, and non-uniconvergent inner product norms on a space of countably-infinite Hamel dimension exist.

The scalar-continuous metron is considered in detail. On finite dimensional spaces it has simple continuity properties, yet allows surprising convergence behavior. Counterexamples illustrating non-comparable and incomplete scalar-continuous metrons on a one-dimensional space are constructed. Some questions relating to completion remain unsolved.

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CONTENTS

# ABSTRACT

On a metric space certain concepts of convergence for sequences are defined. Using these concepts, several relations between metrics on a set X are introduced and their interdependence studied. Included are the relations "equivalent" (identical topologies), "comparable" (one topology includes another), and "uniconvergent" (the identity map from X bearing one metric, to X bearing a second metric has closed graph). For X a commutative group, convergence in a translation-invariant metric d is more conveniently studied by introducing the associated metron (the function $p: X \to R$ such that $\forall x \in X[p(x) = d(x,o)]$). The same is true of relations.

Five classes of metrons on a linear space are considered: metrons, scalar-continuous metrons (the product of scalar and vector is a continuous function of the scalar component), quasi-norms (the product of scalar and vector is a jointly continuous function of the scalar and vector components), norms, and inner product norms. A typical question studied is whether, on a given linear space, all metrons of a given class bear a given relation to each other. For norms, only four of the relations studied remain distinct, while for complete norms all coincide.

It is proved that non-uniconvergent metrons on a one-dimensional space, and non-uniconvergent inner product norms on a space of countably-infinite Hamel dimension exist.

The scalar-continuous metron is considered in detail. On finite dimensional spaces it has simple continuity properties, yet allows surprising convergence behavior. Counterexamples illustrating non-comparable and incomplete scalar-continuous metrons on a one-dimensional space are constructed. Some questions relating to completion remain unsolved.

## INTRODUCTION

In the course of reading Mikhlin "The problem of the minimum of a quadratic functional" I encountered the following situation:

"A linear space D is embedded in each of two normed linear spaces H and H'. A sequence of elements in D is supposed convergent to a point u in the norm of H and to a point u' in the norm of H'. It is then proved that for the particular spaces and norms under consideration there is a one-one correspondence between such limit points u and u'." (Mikhlin p.14).

It struck me that I was not familiar with general conditions that would guarantee such a result in even the simple case in which only one linear space is involved:

"A sequence of elements in a linear space is supposed convergent to a point u of the space in one metric, and to a point u' of the space in another metric. When are u and u' necessarily the same point?"

This problem formed the motivation for the present work; however, it forms but a small part of a wider ranging investigation.

In general terms, this thesis is concerned with how the properties of a metric control the convergence behavior of a sequence in a linear space.

Different types of metric and a variety of relations between metrics are considered. The typical question studied is whether, on a given linear space, all metrics of a given type bear a given relation to each other. An example of a statement of this nature is the well known result that, on a given finite-dimensional linear space, all norms are equivalent.

In chapter one a large part of the background for the thesis is presented, and a structure built up which enables the key problems to be identified. Some of this material will be familiar to mathematicians and is presented with as little elaboration as possible. Where no detailed proof or reference for a result is given, the demonstration should be found to require only a few, straightforward steps.

By contrast, the main results of the thesis follow from proofs involving rather detailed, though quite elementary, real analysis, and requiring some ingenuity in their construction. They are given, largely, in chapters two and three.

In chapter four, the more noteworthy results obtained are listed, and a set of further research problems related to this work is offered.

Appendix A identifies some of the notation used throughout.

The pre-requisites for reading the thesis are modest. The usual background of intermediate algebra and analysis is assumed. Though the thesis is in the field of functional analysis, the presentation is almost self-contained and little use is made of high-powered results from that discipline. However, some familiarity with the elements of axiomatic set theory and of point set topology is desirable; appendix B gives references for set theory, and appendix C outlines relevent results from topology.

References to the bibliography are given in the form: (Wilansky (1) p.52).

References to numbered items in the thesis are given in the forms:

2,3 ---- Item 3 of section 2 of the same chapter;

1,2,3 ---- Item 3 of section 2 of chapter 1.

## CHAPTER ONE: METRIC AND LINEAR SPACES.

We present in this introductory chapter certain basic concepts in the theories of metric and linear spaces. In the last section of this chapter we summarize the principal results obtained in this thesis.

### S1 METRIC SPACES.

Let X denote any set, and R the real number system.

1.1 Definition. A function $d: X \times X \to R$ is a metric d on X iff $\forall x,y,z \in X$:

(a) $d(x,y) \geq 0$ (non-negative)

$d(x,y) = 0 \iff x = y$ (total)

(b) $d(x,y) = d(y,x)$ (symmetric)

(c) $d(x,y) \leq d(x,z) + d(z,y)$ (triangle inequality).

Note (1) the non-negativity follows from the remainder of the definition (put $y = x$ in (c));

(2) the triangle inequality and symmetry imply

$$|d(x,z) - d(z,y)| \leq d(x,y) \leq d(x,z) + d(z,y).$$

A set X is a metric space iff a metric is defined in X.

We shall, for brevity, refer to a sequence of elements of X (precisely $\langle x^n \rangle_{n=1}^{\infty} \in X^{\omega}$) by writing $\langle x^n \rangle \in X^{\omega}$. The unordered elements of $\langle x^n \rangle$ are denoted by $\{x^n\} \subset X$.

1.2 Defnintion. Given a metric d on X we say that a sequence $\langle x^n \rangle$ of elements of X converges in metric d to an element a of X iff $d(x^n,a) \to 0$

i.e. $\forall \varepsilon > 0 \ \exists N \ \forall n > N \ [d(x^n,a) < \varepsilon]$.

1.3 Theorem. A sequence in X can converge in a fixed metric to at most one element of X.

Proof. Suppose $d(x^n,a) \to 0$ and $d(x^n,b) \to 0$.

Then $d(a,b) \leq [d(a,x^n) + d(x^n,b)] \to 0$.

Thus $d(a,b) = 0$ and $a = b$. //

1.4 Definition. A sequence $\langle x^n \rangle$ of elements of X is Cauchy in metric d iff $d(x^m,x^n) \to 0$,

i.e. $\forall \varepsilon > 0 \ \exists N \ \forall m,n > N \ [d(x^m,x^n) < \varepsilon]$.

1.5 Definition. Given a set X we say that metric d is complete in X (and X is complete in metric d) iff every sequence in X, Cauchy in d, converges in d to an element of X, i.e. iff

$\forall \langle x^n \rangle \in X^{\omega}\colon d(x^m,x^n) \to 0 \Longrightarrow \exists a \in X \ [d(x^n,a) \to 0]$.

1.6 Definition. Two sequences $\langle x^n \rangle$, $\langle y^n \rangle$ of elements of X are

1) <u>relatively convergent in metric d</u> iff $d(x^n,y^n) \to 0$;

2) <u>relatively Cauchy in metric d</u> iff $d(x^m,y^n) \to 0$.

1.7 Theorem. For sequences $\langle x^n \rangle$, $\langle y^n \rangle$ of elements of X, and in terms of any one metric on X:

1) $\langle x^n \rangle$ convergent $\Rightarrow$ $\langle x^n \rangle$ Cauchy;

2) $\langle x^n \rangle$, $\langle y^n \rangle$ relatively Cauchy $\Leftrightarrow$ $\langle x^n \rangle$, $\langle y^n \rangle$ Cauchy and relatively convergent.

Proof.

1) Suppose $d(x^n,a) \to 0$.

Then $d(x^m,x^n) \leq [d(x^m,a) + d(a,x^m)] \to 0$

2) Suppose $d(x^m,y^n) \to 0$.

Then clearly $d(x^n,y^n) \to 0$,

$d(x^m,x^n) \leq [d(x^m,y^n) + d(y^n,x^n)] \to 0$, and similarly

$d(y^m,y^n) \to 0$.

Suppose $d(x^m,x^n) \to 0$ and $d(x^n,y^n) \to 0$.

Then $d(x^m,y^n) \leq [d(x^m,x^n) + d(x^n,y^n)] \to 0$. //

## S2 RELATIONS BETWEEN METRICS ON A GIVEN SET.

Recall that a relation is a class of ordered pairs of sets. If A is a relation and $\langle u,v\rangle \in A$, we shall write u A v.

2.1 Definition. Given any relation A we define:

1) $A^{-1}$, the inverse of A, by $A^{-1} = \{\langle u,v\rangle \mid \langle v,u\rangle \in A\}$

i.e. $u\, A^{-1}\, v$ iff v A u.

2) $\overline{A}$, the symmetric restriction of A, by $\overline{A} = A \cap A^{-1}$

i.e. $u\, \overline{A}\, v$ iff [u A v and v A u].

3) A', the symmetric extension of A, by $A' = A \cup A^{-1}$

i.e. u A' v iff [u A v or v A u].

We may read u A' v as "u,v are A-comparable", and $u\, \overline{A}\, v$ as "u,v are A-symmetric". In case A is an equivalence relation we may read u A v as "u,v are A-equivalent".

2.2 Theorem. For any relations A,B, and for any class S:

1) $\overline{A} \subset A \subset A'$.

2) $\overline{A}$ reflexive on S <=> A reflexive on S <=> A' reflexive on S

3) $\overline{A}$, A' are symmetric.

4) A transitive => $\overline{A}$ transitive.

5) $A \subset B \iff A^{-1} \subset B^{-1}$.

6) $A \subset B \Rightarrow [\overline{A} \subset \overline{B}$ and $A' \subset B']$.

Proof. 4) Suppose A transitive.

$$\text{Then } [u\ \bar{A}\ v \text{ and } v\ \bar{A}\ w] \iff \begin{bmatrix} u\ A\ v \text{ and } v\ A\ u \\ \text{and } v\ A\ w \text{ and } w\ A\ v \end{bmatrix}$$

$$\Rightarrow \quad [u\ A\ w \text{ and } w\ A\ u]$$

$$\iff \quad u\ \bar{A}\ w. \ //$$

We now introduce certain relations defined on a collection M of metrics on a given set X.

2.3 Definition. For metrics $d_1$, $d_2$ $\varepsilon$ M:

1) $\underline{d_1 B\ d_2}$ iff $\exists k > 0\ \forall x,y \in X\ [k\ d_1(x,y) \geq d_2(x,y)]$.

2) $\underline{d_1 C\ d_2}$ iff there exists a function $f: R \to R$, continuous at 0 with $f(0) = 0$, such that $\forall x,y \in X\ [f[d_1(x,y)] \geq d_2(x,y)]$.

3) $\underline{d_1 D\ d_2}$ iff $\forall \langle x^n \rangle, \langle y^n \rangle \in X^\omega [d_1(x^n,y^n) \to 0 \Rightarrow d_2(x^n,y^n) \to 0]$.

4) $\underline{d_1 E\ d_2}$ iff $\forall \langle x^n \rangle, \langle y^n \rangle \in X^\omega [d_1(x^m,y^n) \to 0 \Rightarrow d_2(x^m,y^n) \to 0]$.

5) $\underline{d_1 F\ d_2}$ iff $\forall \langle x^n \rangle \in X^\omega\ \forall a \in X\ [d_1(x^n,a) \to 0 \Rightarrow d_2(x^n,a) \to 0]$.

6) $\underline{d_1 G\ d_2}$ iff $\forall \langle x^n \rangle \in X^\omega\ \forall a \in X\ \exists b \in X$

$$[d_1(x^n,a) \to 0 \Rightarrow d_2(x^n,b) \to 0].$$

7) $\underline{d_1 H\ d_2}$ iff $\forall \langle x^n \rangle \in X^\omega\ [d_1(x^m,x^n) \to 0 \Rightarrow d_2(x^m,x^n) \to 0]$.

8) $\underline{d_1 I\ d_2}$ iff $\forall \langle x^n \rangle \in X^\omega\ \forall a \in X\ [d_1(x^n,a) \to 0 \Rightarrow d_2(x^m,x^n) \to 0]$.

9) $\underline{d_1 U\ d_2}$ iff $\forall \langle x^n \rangle \in X^\omega\ \forall a,b \in X$

$$[(d_1(x^n,a) \to 0 \text{ and } d_2(x^n,b) \to 0) \Rightarrow a = b].$$

We introduce also the symmetric restriction and the symmetric extension (2.1) of each of the relations in 1) - 8); relation 9) is already symmetric, so $\overline{U} = U = U'$.

2.4 Theorem. The relations introduced in 2.3, defined on a collection M of metrics on a given set, have the following properties:

1) All are reflexive on M.

2) U and all relations of types $\overline{A}$ and A' are symmetric.

3) For A = B, C, D, E, F, G, H, relations of types A and $\overline{A}$ are transitive.

4) Thus $\overline{B}$, $\overline{C}$, $\overline{D}$, $\overline{E}$, $\overline{F}$, $\overline{G}$, $\overline{H}$ are equivalence relations on M.

Proof. Immediate from 2.1-2.3. //

2.5 Theorem. The implications portrayed in Figure 1 hold between the relations introduced in 2.3, all defined on the same collection of metrics on a given set (where, for example, for brevity the symbolism $\overline{B} \Rightarrow B$ is used to denote $d_1 \overline{B}\, d_2 \Rightarrow d_1 B\, d_2$ i.e. $\overline{B} \subset B$):

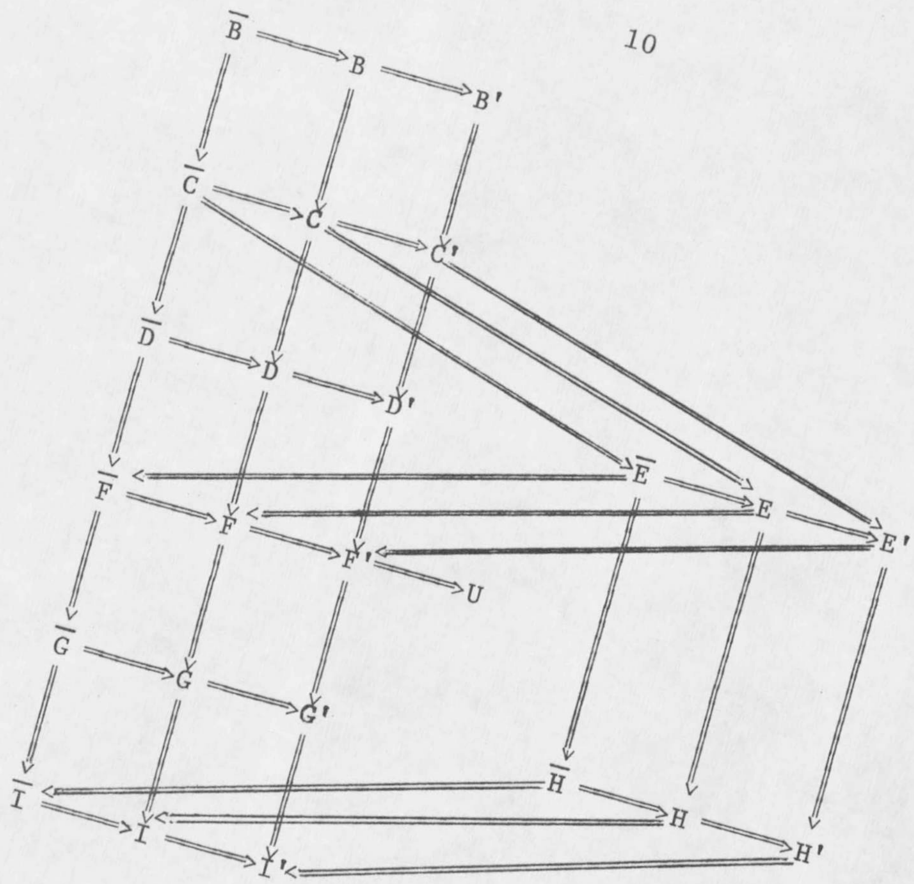


Figure 1.

Proof. The implications follow easily with some use of 1.3, 1.7 and 2.2. //

2.6 Theorem. Continuing 2.5, the further implications hold:

1) $F \Leftrightarrow [G$ and $U]$.

2) $[D$ and $H] \Rightarrow E$.

3) If $d_2$ is complete: $d_1 I\ d_2 \Leftrightarrow d_1 G\ d_2$;

if $d_1$ is complete: $d_1 I\ d_2 \Leftrightarrow d_1 H\ d_2$.

4) $[d_1 F\ d_3$ and $d_2 F\ d_3] \Rightarrow d_1 U\ d_2$.

Proof. The statements follow easily with some use of 1.3 and 1.7. //

Certain of the relations introduced in 2.3 are important enough to be given names. In the literature, for the particular case of relation F (only):

$d_1 F\ d_2$ is read "$d_1$ is stronger than $d_2$" or "$d_2$ is weaker than $d_1$";

$d_1 F'\ d_2$ is read "$d_1$, $d_2$ are comparable" (rather than F-comparable);

$d_1 \overline{F}\ d_2$ is read "$d_1$, $d_2$ are equivalent" (rather than $\overline{F}$-equivalent).

We shall read $d_1 U\ d_2$ as "$d_1$, $d_2$ are uniconvergent".

These relations are involved in the following characterizations.

2.7 Theorem. On a given set, metric $d_1$ is stronger than metric $d_2$ ($d_1 F\, d_2$) iff the $d_1$ topology is stronger than (includes) the $d_2$ topology. Thus $d_1$ and $d_2$ are equivalent ($d_1 \overline{F}\, d_2$) iff the $d_1$ and $d_2$ topologies are identical. (Wilansky (2) p.27.)

2.8 Theorem. On a given set X, $d_1$ and $d_2$ are uniconvergent ($d_1 U\, d_2$) iff the identity map from X bearing the $d_1$ topology to X bearing the $d_2$ topology has closed graph. (Wilansky (1) p.195.)

## S3 LINEAR SPACES.

3.1 Definition. A set X is a linear space over K, (X,K), iff there exist operations of "addition" of the elements ("vectors") of X, denoted for vectors x and y by $x \dot{+} y$, and of "multiplication" of the elements of X by the elements ("scalars") of a commutative field K, denoted for a vector x and a scalar k by k.x, such that:

A) X is a commutative group with respect to addition;

B) $\forall h,k \in K \quad \forall x,y \in X$:

(a) $k.x \in X$

(b) $(h + k).x = h.x \dot{+} k.x$

(c) $k.(x \dot{+} y) = k.x \dot{+} k.y$

(d) $h.(k.x) = (hk).x$

(e) $1.x = x$ (where 1 denotes the scalar multiplicative identity).

For simplicity we shall omit the dots and hereafter denote the addition of vectors x and y by $x + y$, and the multiplication of a vector x by a scalar k by kx.

We let: $-x$ denote the additive inverse of x,

$x - y$ denote $x + (-y)$,

0, o denote the zero elements of K, X respectively,

Q, R, C denote the fields of rational, real, and complex numbers respectively.

3.2 Definition. A set L is a <u>linear manifold of (X,K)</u> iff $L \subset X$ and L is a linear space over K with the same operations of addition and multiplication as (X,K).

For any (X,K), {o} is a linear manifold of (X,K), the <u>zero linear space</u>.

3.3 Definition. Let (X,K) be given and let S be a subset of X. A <u>linear combination from S</u> is an element $\sum_{i=1}^{n} k_i s^i$ of X where the coefficients $k_1,\ldots k_n$ are scalars, $s^1,\ldots s^n$ are distinct elements of S, and n is finite.

A linear combination is <u>non-trivial</u> iff at least one scalar coefficient is non-zero.

3.4 Definition. Let (X,K) be given. A subset S of X is _linearly dependent_ iff o is a non-trivial linear combination from S; otherwise _linearly independent_.

3.5 Definition. A _Hamel basis for (X,K)_ is a subset S of X such that every element of X can be represented as a unique linear combination from S.

(By "unique" we mean that the scalar coefficient of any element s of S is the same in every linear combination from S containing s that represents a given element of X.)

3.6 Theorem. A subset S of X is a Hamel basis for (X,K) iff every element of X can be represented as a linear combination from S, and S is linearly independent.

Proof. Immediate (3.5). //

3.7 Theorem. 1) Every non-zero linear space has a Hamel basis. (Wilansky (1) p.16).

2) Any two Hamel bases for a linear space are in one-one correspondence. (Wilansky (1) p.17).

3.8 Definition. The _Hamel dimension_ of a linear space is the cardinality of any Hamel basis for the space.

3.9 Definition. Two linear spaces $(X,K)$ and $(X',K)$ are <u>linearly isomorphic</u> iff there exists a function $f: X \to X'$ which is

(a) bijective (one-one and onto),

(b) linear i.e. $\forall x,y \in X \; \forall k \in K$

$$[f(x + ky) = f(x) + k\, f(y)].$$

3.10 Theorem. Two linear spaces over the same scalar field are linearly isomorphic iff they have the same Hamel dimension. (Wilansky (1) p.20).

3.11 Example. 1) Let $(R^n,R)$ denote the space of real n-tuples. A Hamel basis may be taken as $\{\delta^k\}_{k=1}^{n}$ where $\delta^k = \langle 0,0,\ldots 0,1,0,\ldots 0\rangle$, the 1 occuring as the $k^{th}$ coordinate.

Then every n-dimensional linear space is linearly isomorphic to $(R^n,R)$.

2) Let $(R^\infty,R)$ denote the space of real sequences with only a finite number of non-zero terms. A Hamel basis may be taken as $\{\delta^k\}_{k=1}^{\infty}$ where now $\delta^k = \langle 0,0,\ldots 0,1,0,\ldots\rangle$, the 1 occuring as the $k^{th}$ coordinate.

Then every linear space of countably-infinite Hamel dimension is linearly isomorphic to $(R^\infty,R)$.

## S4 TRANSLATION-INVARIANT METRICS AND METRONS ON A COMMUTATIVE GROUP.

On a set which is a commutative group with respect to an operation of "addition" we may introduce the following two concepts.

4.1 Definition. On a commutative group X a metric d is a translation-invariant metric (t.i.m.) iff

$\forall x,y,z \in X\ [d(x + z, y + z) = d(x,y)]$.

4.2 Definition. On a commutative group X a function $p: X \to R$ is a metron (m.) iff $\forall x,y \in X$:

(a) $p(x) \geq 0$; $p(x) = 0 \iff x = 0$ (positive definite)

(b) $p(x) = p(-x)$ (symmetric)

(c) $p(x + y) \leq p(x) + p(y)$ (sub-additive).

Note (1) $p(x) \geq 0$ follows from the remainder of the definition (put $y = -x$ in (c));

(2) sub-additivity and symmetry imply

$$|p(x) - p(y)| \leq p(x - y) \leq p(x) + p(y).$$

We show the two concepts are closely related.

4.3 Theorem. Given a commutative group X, there is a one-one correspondence between the members of the set D of all translation-invariant metrics on X and the members of the set P of all metrons on X given by defining:

the <u>metron p associated with a translation-invariant metric d</u> by

(1) $\forall x \in X\ [p(x) = d(x,o)]$;

the <u>translation-invariant metric d associated with a metron p</u> by

(2) $\forall x,y \in X\ [d(x,y) = p(x - y)]$.

Proof. (a) For any $d \in D$ it is readily verified that the function p defined by (1) satisfies the requirements for a metron (4.2) i.e. $p \in P$.

(b) For any $p \in P$ is is readily verified that the function d defined by (2) satisfies the requirements for a metric (1.1). Furthermore $d(x + z,y + z) = p(x + z - y - z) = p(x - y) = d(x,y)$. Thus d is a translation-invariant metric i.e. $d \in D$.

(c) In addition to defining p(x), (1) also defines a function $f: D \to P$; likewise a function $g: P \to D$ is defined by (2). We shall show that $f = g^{-1}$.

For any $d_1 \in D$, $f(d_1) = p \in P$ and $g(p) = d_2 \in D$. But $\forall x,y \in X\ [d_2(x,y) = p(x - y) = d_1(x - y,o) = d_1(x,y)]$. Hence $g \circ f$ is the identity map on D.

For any $p_2 \in P$, $g(p_2) = d \in D$ and $f(d) = p_1 \in P$. But $\forall x \in X\ [p_1(x) = d(x,o) = p_2(x - o) = p_2(x)]$. Hence $f \circ g$ is the identity map on P. Thus $f = g^{-1}$. //

When metrics are translation-invariant we shall find it simpler to work with the associated metrons than with the metrics themselves.

4.4 Example. On any set X the <u>discrete metric</u> d is defined by:

$$\forall x,y \in X: \ d(x,y) = \begin{cases} 0 & (x = y) \\ 1 & (x \neq y). \end{cases}$$

When X is a commutative group the discrete metric is translation-invariant, and the associated <u>discrete metron</u> p is defined by:

$$\forall x \in X: \ p(x) = \begin{cases} 0 & (x = o) \\ 1 & (x \neq o). \end{cases}$$

On a commutative group X bearing a metron p, definitions 1.2, 1.4, 1.5, 1.6, become: A sequence $\langle x^n \rangle \in X^\omega$ <u>converges in p</u> to $a \in X$ iff $p(x^n - a) \to 0$ i.e. $\forall \varepsilon > 0\ \exists N\ \forall n > N\ [p(x^n - a) < \varepsilon]$; A sequence $\langle x^n \rangle \in X^\omega$ is <u>Cauchy in p</u> iff $p(x^m - x^n) \to 0$ i.e. $\forall \varepsilon > 0\ \exists N\ \forall m,n > N\ [p(x^m - x^n) < \varepsilon]$;

A metron p is complete in X (and X is complete in p) iff every sequence in X, Cauchy in p, converges in p to an element of X i.e. iff

$$\forall \langle x^n \rangle \in X^\omega: p(x^m - x^n) \to 0 \Rightarrow \exists a \in X [p(x^n - a) \to 0];$$

Sequences $\langle x^n \rangle$, $\langle y^n \rangle \in X^\omega$ are

1) relatively convergent in p iff $p(x^n - y^n) \to 0$;

2) relatively Cauchy in p iff $p(x^m - y^n) \to 0$.

Theorem 1.3, 1.7 also have direct analogs.

Whenever a set X bears a metric d (metron p) it will be understood to bear the d topology (p topology i.e. the associated t.i.m. topology).

4.5 Definition. We denote:

a set X bearing a metric d by (X,d);

a commutative group X bearing a metron p by (X,p);

a linear space (X,K) bearing a metric d, a metric and linear space, by (X,K,d);

a linear space (X,K) bearing a metron p, a metron and linear space, by (X,K,p);

the set of real numbers bearing the Euclidean topology by R.

We now give a simple result that will be needed later.

4.6 Theorem. Let a commutative group X bear a metron q. Then a metron p: $(X,q) \to R$ is continuous everywhere on X iff p is continuous at zero.

Proof. Clearly p continuous everywhere => p continuous at o.

Suppose p is continuous at o. Then $\forall a \in X \quad \forall \langle x^n \rangle \in X^\omega$:

$$q(a + x^n - a) = q(x^n - o) \to 0 \Rightarrow p(a + x^n - a) = p(x^n)$$
$$\to p(o) = 0$$
$$\Rightarrow |p(a + x^n) - p(a)| \to 0. \quad //$$

Consider the relations introduced in 2.3, but defined now on a collection M of translation-invariant metrics on a given commutative group X. Let $\underline{M}$ be the collection of metrons associated with the members of M by the bijection f: $M \to \underline{M}$ given by 4.3(1).

Corresponding to any relation A on M we introduce a relation $\underline{A}$ on $\underline{M}$ defined by:

$$\forall d_1, d_2 \in M \; [f(d_1) \; \underline{A} \; f(d_2) \Leftrightarrow d_1 \; A \; d_2].$$

The sets M and $\underline{M}$ are isomorphic with respect to the relations A and $\underline{A}$, and f is an isomorphism from M to $\underline{M}$.

The relation $\underline{A}$ between metrons corresponding to any given relation A between metrics introduced in 2.3 may be written directly in terms of metrons using $d(x,y) = p(x - y)$. Henceforth we shall employ this form only (4.7) of the relations, but for convenience relations between metrons will be identified by the same symbols that originally identified the corresponding relations between metrics.

4.7 Definition. Let $\underline{M}$ be a collection of metrons on a given commutative group X. For metrons $p_1, p_2 \in \underline{M}$:

1) $\underline{p_1 B\, p_2}$ iff $\exists k > 0\ \forall x \in X\ [k\, p_1(x) \geq p_2(x)]$.

2) $\underline{p_1 C\, p_2}$ iff there exists a function $f: R \to R$, continuous at 0 with $f(0) = 0$, such that $\forall x \in X\ [f[p_1(x)] \geq p_2(x)]$.

3) $\underline{p_1 D\, p_2}$ iff $\forall \langle x^n \rangle \in X^\omega [p_1(x^n) \to 0 \Rightarrow p_2(x^n) \to 0]$.

4) $\underline{p_1 E\, p_2}$ iff $\forall \langle x^n \rangle, \langle y^n \rangle \in X^\omega [p_1(x^m - y^n) \to 0 \Rightarrow p_2(x^m - y^n) \to 0]$.

5) $\underline{p_1 F\, p_2}$ iff $\forall \langle x^n \rangle \in X^\omega [p_1(x^n) \to 0 \Rightarrow p_2(x^n) \to 0]$.

6) $\underline{p_1 G\, p_2}$ iff $\forall \langle x^n \rangle \in X^\omega\ \exists c \in X$

$$[p_1(x^n) \to 0 \Rightarrow p_2(x^n - c) \to 0].$$

7) $\underline{p_1 H\, p_2}$ iff $\forall \langle x^n \rangle \in X^\omega [p_1(x^m - x^n) \to 0 \Rightarrow p_2(x^m - x^n) \to 0]$.

8) $\underline{p_1 I\, p_2}$ iff $\forall \langle x^n \rangle \in X^\omega [p_1(x^n) \to 0 \Rightarrow p_2(x^m - x^n) \to 0]$.

9) $\underline{p_1 U\, p_2}$ iff $\forall \langle x^n \rangle \in X^\omega\ \forall c \in X$

$$[(p_1(x^n) \to 0 \text{ and } p_2(x^n - c) \to 0) \Rightarrow c = o].$$

We introduce also the symmetric restriction and the symmetric extension (2.1) of each of the relations in 1) - 8); relation 9) is already symmetric, so $\overline{U} = U = U'$.

We are thus considering, apparently, twenty-five different relations on $\underline{M}$. However, the following theorem shows that, for