Linear topologies induced by bilinear forms
by Vinnie Hicks Miller

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Mathematics
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Abstract:
The purpose of this paper is to identify some linear topologies associated in a natural way with a continuous, bilinear form 0 on a vector space E over an arbitrary discrete field k. The finest topology on E for which the canonical maps (Formula not captured by OCR) is continuous is denoted by(Formula not captured by OCR) denotes the extension of these topologies by sums to the tensor algebra and by quotients to the Clifford algebra.

For V a fixed totally isotropic subspace of E, (Formula not captured by OCR)the topology is defined by taking a neighborhood basis at zero of subspaces (Formula not captured by OCR) a finite dimensional subspace of E. It is shown that if(Formula not captured by OCR) is a linearly topologized space with,nontrivial topology T then the following are equivalent: (Formula not captured by OCR) for some totally isotropic V, (ii) (Formula not captured by OCR) is continuous, (iii) T has a zero neighborhood basis of sets (Formula not captured by OCR) with (Formula not captured by OCR)(iv)(Formula not captured by OCR) and (Formula not captured by OCR)is Hausdorff. If the case where dim E = and V is orthogonally closed, it is proved that(Formula not captured by OCR) is a topological algebra.

An investigation of the completion(Formula not captured by OCR) of a space E with(Formula not captured by OCR) topology and bilinear form (Formula not captured by OCR) , shows that E can be decomposed as follows: (Formula not captured by OCR), where (Formula not captured by OCR) is the algebraic dual of (Formula not captured by OCR) and H2 are totally isotropic for (Formula not captured by OCR) (Formula not captured by OCR) for (Formula not captured by OCR) and(Formula not captured by OCR) is nondegenerate iff V is closed (Formula not captured by OCR) These completions coincide with the locally linearly(Formula not captured by OCR)compact spaces on which the form (Formula not captured by OCR) is continuous.

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ABSTRACT

The purpose of this paper is to identify some linear topologies associated in a natural way with a continuous, bilinear form ϕ on a vector space E over an arbitrary discrete field k. The finest topology on E for which the canonical maps $\omega_p : \otimes^p E \to \otimes^p E$ is continuous is denoted by $\otimes^p \tau$; $\otimes \tau$ denotes the extension of these topologies by sums to the tensor algebra and by quotients to the Clifford algebra. For V a fixed totally isotropic subspace of E, the $\tau_\phi V$ topology is defined by taking a neighborhood basis at zero of subspaces $V \cap F^\perp$, F a finite dimensional subspace of E. It is shown that if (E, τ, ϕ) is a linearly topologized space with nontrivial topology τ then the following are equivalent: (i) $\tau \geqslant \tau_\phi V$ for some totally isotropic V, (ii) $\phi : E \times E \to k$ is continuous, (iii) τ has a zero neighborhood basis of sets U_α with $E = \bigcup_\alpha U_\alpha^\perp$ (iv) $\otimes \tau |_E = \tau$ and (v) $(C(E), \otimes \tau)$ is Hausdorff. If the case where $\dim E = \aleph_0$ and V is orthogonally closed, it is proved that $(C(E), \otimes \tau_\phi V)$ is a topological algebra.

An investigation of the completion $(\tilde{E}, \tilde{\tau}, \tilde{\phi})$ of a space E with $\tau_\phi V$ topology and bilinear form ϕ , shows that E can be decomposed as follows: $\tilde{E} = H_2^* \oplus H_2 \oplus H_1$, where $\tilde{V} = H_2^*$ is the algebraic dual of H_2, H_2^* and H_2 are totally isotropic for $\tilde{\phi}$, $\tilde{\phi}(h_2^*, h_2) = h_2^*(h_2)$ for $h_2^* \in H_2^*$ and $h_2 \in H_2$, $H_1 \perp (H_2^* + H_2)$; $\tilde{\phi}$ is nondegenerate iff V is closed; $\tilde{\tilde{\tau}} = \tau_{\tilde{\phi}} \tilde{V} = \tau_{\tilde{\phi}} H_2^*$. These completions coincide with the locally linearly $\tau_{\tilde{\phi}} \tilde{V}$ -compact spaces on which the form $\tilde{\phi}$ is continuous.

CHAPTER I

INTRODUCTION

The present paper is part of a larger program concerned with an algebraic theory of quadratic forms on infinite dimensional vector spaces over arbitrary fields, [5], [6] and [7], and in particular of the orthogonal group of such forms. For the finite dimensional case a substantial literature exists, (see Dieudonné: La géométrie des groupes classiques, [4], which also contains a comprehensive bibliography). The techniques used there do not extend however to the infinite dimensional case. In the hope that topological techniques may replace the finite methods, a natural topology, compatible in a certain sense with the quadratic form, is introduced. More specifically, by means of canonical topologies $\mathcal{T}$, certain subgroups of the full orthogonal group would be singled out for investigation, namely the subgroups of $\mathcal{T}$-continuous orthogonal automorphisms. These topologies may also be of service in the classification of spaces into classes unique up to orthogonal isomorphism.

In the finite dimensional case, consideration of the Clifford algebra has been very fruitful. For both finite and infinite dimensional spaces there is an intimate relationship between the orthogonal group and the group of Clifford algebra isomorphisms. So it seems reasonable to require that the topology on the vector space extend first to a suitable topology on the tensor product, and then, by the usual sum and quotient operations, to a Hausdorff topology on the Clifford algebra; finally, the Clifford algebra topology must induce the initial topology on the underlying vector space. This leads to consideration of the $\mathcal{T}_\phi V$ topologies, and their completions which are of the same type.

CHAPTER II

DEFINITIONS AND GENERAL REMARKS

In this chapter the definitions and notational conventions to be used in the remainder of the work are collected. Known results which will be needed later are stated here without proof. When no reference is cited for the proof the reader should consult the excellent summary in Köthe [9] for further details.

Sections are numbered for later reference.

1. The discussion in Chapters III and IV will be concerned with an infinite dimensional vector space E over a (commutative) field k with characteristic unequal to 2. For V a subspace of E the notation $V=k(v_\alpha)_{\alpha\in I}$ is intended to convey that V has as a basis the v_α indexed by α in I. When the index set is denumerably infinite this expression becomes $V=k(v_i)_{i\geq 1}$ and for V of finite dimension $V=k(v_i)_{1\leq i\leq m}$ and similar expressions will be employed. If the v_α are generators of V but not necessarily linearly independent we write $V=[v_\alpha ; \_\,]$.

2. It is assumed that E is equipped with a bilinear form $\phi: E\times E \longrightarrow k$ (that is $\phi(\alpha x_1+\beta x_2, y)=\alpha\phi(x_1,y)+\beta\phi(x_2,y)$ and $\phi(x, \alpha y_1+\beta y_2)=\alpha\phi(x, y_1)+\beta\phi(x, y_2)$).
Further ϕ will be assumed to be symmetric $(\phi(x,y)=\phi(y,x))$ and nondegenerate (for every $x\neq 0$ there is a y with $\phi(x,y)\neq 0$). A subspace F is called semisimple when the form ϕ is nondegenerate on $F\times F$. If the length of x is zero, $(\phi(x,x)=0)$, x is called an

isotropic vector. A subspace with no isotropic vectors except the zero vector is called anisotropic, while a subspace in which every vector is isotropic is totally isotropic. A subspace F is totally isotropic iff ϕ is identically zero on $F \times F$ since $\phi(x, y) = \frac{1}{2}[\phi(x+y, x+y) - \phi(x, x) - \phi(y, y)]$. By Zorn's lemma each totally isotropic subspace is contained in a maximal totally isotropic subspace.

3. Two vectors x and y are orthogonal, and we write $x \perp y$, iff $\phi(x, y) = 0$. If x is orthogonal to every vector in a subspace F we write $x \perp F$. By definition $F^{\perp} = \{x \in E\,;\ x \perp F\}$. When F is finite dimensional, $\dim F = \operatorname{codim} F^{\perp}$, [2]. For F and G subspaces of E, $(F+G)^{\perp} = F^{\perp} \cap G^{\perp}$. It is always the case that $F \subset F^{\perp\perp}$. If $F = F^{\perp\perp}$ then F is said to be $\perp$-closed. Every finite dimensional subspace of E is $\perp$-closed, [2].

4. For a direct sum $F \oplus G$ with $x \perp y$ for every $x \in F$ and $y \in G$ we write $F \overset{\perp}{\oplus} G$.

If E has denumerably infinite dimension and $V = k(v_i)_{i \geq 1}$ is a totally isotropic $\perp$-closed subspace of E then there is a totally isotropic closed subspace $V' = k(v_i')_{i \geq 1}$ such that $E = (V \oplus V') \overset{\perp}{\oplus} G$ with $\phi(v_i, v_j') = \delta_{ij}$ (i.e., zero if $i \neq j$ and one if $i = j$). For a proof see [8]; the proof given there carries over to our case.

5. If $f : (E, \phi) \to (E', \phi')$ preserves the form, that is if

$\phi'(f(x), f(y)) = \phi(x, y)$ for all x and y in E, then f is called a metric (orthogonal) map. A metric automorphism of a vector space (E, ϕ) is called an isometry and the group of all isometries is called the orthogonal group.

6. A quadratic form is a function $Q: E \to k$ of the vectorspace E into the groundfield k with the following properties: (i) $Q(\lambda x) = \lambda^2 Q(x)$ and (ii) $\bar{\phi}$ defined by $\bar{\phi}(x, y) = \frac{1}{2}[Q(x+y) - Q(x) - Q(y)]$ is a bilinear form. If ϕ is a bilinear form on E and Q is defined by $Q(x) = \phi(x, x)$ then Q is a quadratic form called the quadratic form associated with ϕ, and $\phi = \bar{\phi}$. Throughout this paper when quadratic forms are introduced it is understood that they are associated with the bilinear form under discussion.

7. With the tensor algebra T(E) defined as usual over the vector space (E,ϕ), (for an English reference see [3]), let Q be the quadratic form associated with ϕ and let I be the two-sided ideal generated by the elements $x \otimes x - Q(x)$ in T(E). Then the Clifford algebra C(E) is by definition T(E)/I. The equivalence class of $x_1 \otimes \dots \otimes x_m$ will be denoted by $x_1 \circ \dots \circ x_m$. If $E = k(e_\alpha)_{\alpha \in J}$ with J asymmetrically ordered by $<$, then for $S = \{\alpha_1, \dots, \alpha_m\}$, $\alpha_1 < \alpha_2 < \dots < \alpha_m$ let $e_S = e_{\alpha_1} \circ \dots \circ e_{\alpha_m}$. The e_S together with the scalar 1 are a basis for C(E) (for a proof see [2]). In particular if $x_1, \dots, x_m$ are linearly independent elements of E, then $x_1 \circ \dots \circ x_m \neq 0$.

If f:E⟶A is a linear map of a k-vector space (E,ϕ) into a k-algebra A, then f extends to a unique algebra homomorphism g mapping $T(E) \longrightarrow A$ which is the identity on k. If in addition $f(x)^2 = Q(x)$ then f extends uniquely to an algebra homomorphism $h: C(E) \rightarrow A$ which is the identity on k.

A Clifford algebra over a finite dimensional vector space is simple, that is there are no proper nontrivial two-sided ideals, [2]. For E of infinite dimension, C(E) is the union of the Clifford algebras over the finite dimensional subspaces of E so C(E) is again simple.

8. A <u>linear form</u> is a linear function (i.e., vector space homomorphism) into the ground field, $u: E \rightarrow k$. The set of all linear forms on E is denoted by E*. E* becomes a vector space over k, called the <u>dual space</u> of E, under the usual definition of addition and scalar multiplication of functions. If F and G are subspaces of (E, ϕ) then each x in F induces a map $\phi_x : G \rightarrow k$ with $\phi_x(y) = \phi(x, y)$. These maps are elements of G* because of the bilinearity of ϕ. If $F = k(x_\alpha)_{\alpha \in I}$ then the maps ϕ_{x_i} are linearly independent iff $F \cap G^\perp = (0)$. For $u_1, \ldots, u_m$ linearly independent functions from G*, there exist elements $y_1, \ldots, y_m$ of G with $u_i(y_j) = \delta_{ij}$ ([9], page 74). In particular if $k(x_i)_{1 \leq i \leq m} \cap G^\perp = (0)$ then there exist $y_1, \ldots, y_m$ in G with $\phi(x_i, y_j) = \delta_{ij}$.

9. A <u>linear topology</u> on a k-vector space E is a topology with a neighborhood basis at zero of linear subspaces U_α of E and a neighborhood basis at x composed of the linear manifolds $x + U_\alpha$. A linear

topology is always a uniform topology. It is Hausdorff if and only if $\bigcap_\alpha U_\alpha = (0)$. A vector space together with a Hausdorff linear topology is called a <u>linearly topologized space</u>. In the sequel we shall always assume that the field k carries the discrete topology. If $\mathcal{T}$ is a linear topology on a vector space E then the operations of addition and scalar multiplication are continuous (the product topology being taken on $E \times E$ and $k \times E$). If f is a linear function $E_1 \to E_2$ with linear topologies on E_1 and E_2 then f is continuous provided it is continuous at zero.

If E_1 and E_2 are linearly topologized spaces and ψ is a bilinear function $E_1 \times E_1 \to E_2$ then ψ is continuous iff it is continuous at (0,0) and the partial functions $\psi_x : y \to \psi(x,y)$ and $\psi_y : x \to \psi(x,y)$ are continuous at zero. This follows immediately from the identity

$$\psi(x,y) - \psi(x_0,y_0) = \psi(x-x_0, y-y_0) + \psi(x_0, y-y_0) + \psi(x-x_0, y_0).$$

In particular a multiplication defined on a linearly topologized space is continuous iff it is continuous at (0,0) are separately continuous at 0.

10. If the $(E_\alpha, \mathcal{T}_\alpha)$ are linearly topologized spaces then the <u>direct sum topology</u> $\oplus \mathcal{T}_\alpha$ on $\oplus E_\alpha$ is defined by taking as a zero neighborhood basis the spaces $\oplus U_\alpha$ with U_α running through the spaces of a zero neighborhood basis for $\mathcal{T}_\alpha$. The topological direct sum $(\oplus E_\alpha, \oplus \mathcal{T}_\alpha)$ is once more a linearly topologized space. The direct sum $E = F \oplus G$

is a topological direct sum iff the projection $p: F \oplus G \longrightarrow F$, with $p(x+y)=x$, is continuous ([9], page 95).

In the case of a linearly topologized space the definition of the quotient topology τ_q can be given more easily than in the general case. Let σ be the canonical homomorphism $E \longrightarrow E/F$, then the quotient topology is defined by taking for open sets in E/F the images under σ of the open sets in E. Under this definition σ is a continuous function. τ_q is Hausdorff iff F is topologically closed (equal to its closure).

11. Two vector spaces E_1, and E_2 form a dual pair with respect to a bilinear form $\psi : E_1 \times E_2 \longrightarrow k$ if for each x in E_1 there is a y in E_2 with $\psi(x, y) \neq 0$, and for each y in E_2 there is an x in E_1 with $\psi(x, y) \neq 0$. Subspaces F and G of (E, ϕ) are a dual pair with respect to the form induced by ϕ iff $F \cap G^{\perp} \neq (0)$ and $G \cap F^{\perp} \neq (0)$. The vector spaces E and E* are a dual pair with respect to the natural bilinear form $\psi(x, \mu) = \mu(x)$ for $x \in E$ and $\mu \in E^*$. For (E, τ) a linearly topologized space we denote by E' the subspace of E* consisting of τ-continuous linear forms. E' is called the topological dual space to E. $\langle E, E' \rangle$ is a dual pair for the natural bilinear form mentioned above, [9].

A particularly important linear topology is defined in terms of dual pairs. If $\langle E_1, E_2 \rangle$ is a dual pair with respect to ψ then the weak topology on E_1 with respect to E_2 has a neighborhood basis at zero of sets $F^0 = \{ x \in E_1 \; ; \; \phi(x, y) = 0$ for every $y \in E_2 \}$, F running through all finite dimensional subspaces of E_2. We follow Bourbaki and denote this topology by $\sigma(E_1, E_2)$; (for reference the notation used in Köthe is $\mathcal{T}_{ls}(E_2)$). If $\langle E_1, E_2\rangle$ is a dual pair with respect to ψ then $E_1' = E_2$ where E_1' consists of the $\sigma(E_1, E_2)$- continuous linear forms on E_1, ([9], page 90).

CHAPTER III

CANONICAL TOPOLOGIES ON THE CLIFFORD ALGEBRA

Let E be an infinite dimensional vector space over a commutative field k, characteristic k unequal 2, and ϕ a nondegenerate, symmetric bilinear form $E \times E \longrightarrow k$. Our eventual aim is to study continuous orthogonal groups, that is the groups of isometries of E which are continuous for some natural linear topology on E. In this chapter the meaning of "natural" linear topology is delimited and topologies with the specified properties are investigated.

If f is an isometry of E then f induces an algebra homomorphism $h: C(E) \to C(E)$ with $h(x_1 \circ \ldots \circ x_m) = f(x_1) \circ \ldots \circ f(x_m)$ and h the identity on k. This follows from section 7 in Chapter II, taking A = T(E) then A = C(E), since $f(x) \circ f(x) = Q(f(x)) = Q(x)$. Further, let E have basis $(e_\alpha)_{\alpha \in I}$ then $(f(e_\alpha))_{\alpha \in I}$ is also a basis for E. Basis elements $e_{i_1} \circ \ldots \circ e_{i_m}$ of C(E) are mapped by h onto a complete set of basis elements $f(e_{i_1}) \circ \ldots \circ f(e_{i_m})$ of C(E), so h is bijective. Thus every isometry f of E induces an algebra isomorphism h of C(E). Conversely, if h is an algebra isomorphism of C(E) which maps E onto E and is the identity on k then the restriction of h to E is an isometry, for $\phi(h(x), h(x)) = h(x) \circ h(x) = h(x \circ x) = h(\phi(x, x)) = \phi(x, x)$. Because of this canonical relation between the isometries of E and algebra isomorphisms of C(E), we shift our attention to the problem of topologizing the Clifford Algebra.

Starting with a linearly topologized space (E, τ), there are many ways of constructing linear topologies on the tensor products $\overset{p}{\otimes} E$, Here we shall consider two tensor product topologies, the τ_e topology, corresponding to the ϵ-product of Schwartz and the "projective" topological tensor product topology corresponding to that of Grothendieck. These topologies have been studied in [7].

A linear topology on the tensor products extends canonically by taking the direct sum topology on the tensor algebra and then the quotient topology to a linear topology on the Clifford algebra. If this extension is to be useful it must induce the initial topology when restricted to E. We now investigate whether this is the case for either the ϵ-product or the projective tensor product extensions.

Since it will quickly become apparent that the ϵ-product topology is not suitable in the sense just mentioned, we shall describe it only briefly. For further detail the reader is referred to [7]. The τ_e topology is the finest linear topology on $\overset{p}{\underset{1}{\otimes}} E$ for which the canonical multilinear map $\overset{p}{\underset{1}{\pi}} E_i \longrightarrow \overset{p}{\underset{1}{\otimes}} E_i$ is uniformly continuous. For each p, τ_e has a neighborhood basis at zero of sets

$$\hat{U}_p = U_p \otimes E \otimes E \otimes \ldots \otimes E + E \otimes U_p \otimes E \otimes \ldots \otimes E + \ldots + E \otimes E \otimes E \otimes \ldots \otimes U_p$$

each summand containing p factors and the U_p running through a zero neighborhood basis for the topology τ on E. A zero neighborhood basis for the tensor algebra consists of sets $\hat{U} = \overset{\infty}{\underset{p=1}{\oplus}} \hat{U}_p$ and a zero neighborhood basis for the Clifford algebra of the sets $\sigma\hat{U}$, where σ is the canonical map $T(E) \longrightarrow C(E)$. These extensions as well as the ϵ-product topologies on the tensor products will be denoted by τ_e.

Theorem 1: If $(E,\mathcal{T})$ is discrete then $(C(E),\mathcal{T}_e)$ is discrete. If $(E,\mathcal{T})$ is not discrete then $(C(E),\mathcal{T}_e)$ is trivial.

Proof: If $(E,\mathcal{T})$ is discrete then (0) is in the zero neighborhood basis for $\mathcal{T}$. In the expression for $\hat{U}_p$ in the preceding paragraph taking $U_p=(0)$ for every p gives $\hat{U}=(0)$ and $\sigma(\hat{U})=(0)$. Since (0) is thus in the zero neighborhood basis for $(C(E),\mathcal{T}_e)$, the latter is discrete in this case.

On the other hand if $(E,\mathcal{T})$ is not discrete and $\sigma(\hat{U}) = \sigma(U_1+U_2\otimes E+E\otimes U_2+U_3\otimes E\otimes E+\ldots)$ is an arbitrary set in the zero neighborhood basis of $(C(E),\mathcal{T}_e)$ then every element of the form $x_1\circ x_2\circ\ldots\circ x_m$ is in $\sigma\hat{U}$. For since $\mathcal{T}$ is not discrete, there is an element $y\neq 0$ in U_{m+2}, and since E is semisimple, there is a $z\in E$ with $\phi(y,z)=1/2$, $y\circ z+z\circ y=1$, and so $x_1\circ\ldots\circ x_m = y\circ z\circ x_1\circ\ldots\circ x_m + z\circ y\circ x_1\circ\ldots\circ x_m \in \sigma\hat{U}$. $\sigma(\hat{U})$ is thus seen to be a subspace of C(E) containing a set of generators of C(E), hence $\sigma(\hat{U}) = C(E)$. In this case $\mathcal{T}_e$ is the trivial topology on C(E).

So requiring that the $\mathcal{T}_e$ topology on C(E) induce the initial topology $\mathcal{T}$ on E would leave for consideration only the uninteresting cases where $\mathcal{T}$ is discrete or trivial. For this reason the $\mathcal{T}_e$ topology will not be discussed further.

We now turn our attention to the projective tensor product topology $\overset{p}{\underset{1}{\otimes}}\mathcal{T}$ on the tensor product $\overset{p}{\underset{1}{\otimes}}E$. In [7] it is shown that there is a unique linear topology on $E \otimes E$ with the following properties: (1) the canonical bilinear map $\omega_2 : E \times E \longrightarrow E \otimes E$ is continuous and (2) if f is a bilinear continuous map of $E \times E$ into a linearly topologized k-vector space G then the induced linear map $E \otimes E \longrightarrow G$ is continuous. The proof extends to $\overset{p}{\underset{1}{\otimes}} E$. $\overset{p}{\underset{1}{\otimes}} \mathcal{T}$ is by definition this unique topology. Clearly $\overset{p}{\underset{1}{\otimes}} \mathcal{T}$ is the finest linear topology on $\overset{p}{\underset{1}{\otimes}} E$ for which $\omega_p : \overset{p}{\underset{1}{\prod}} E \longrightarrow \overset{p}{\underset{1}{\otimes}} E$ is continuous. If $\mathcal{T}$ is Hausdorff so is $\overset{p}{\underset{1}{\otimes}} \mathcal{T}$ (for details see [7]).

A neighborhood basis at zero for the $\overset{2}{\underset{1}{\otimes}} \mathcal{T}$ topology is given by the subspaces $\hat{U}_2 = U_2 \otimes U_2 + \sum'_{x \in E} [x] \otimes U_{2x} + \sum'_{x \in E} U_{2x} \otimes [x]$

with U_{2x} and U_2 running through a zero neighborhood basis of $\mathcal{T}$. This is so since ω_2 is continuous if and only if it is continuous at (0,0) and is separately continuous at $(x,0)$ and $(0,x)$ for every $x \in E$. If ω_2 is to be continuous for a topology $\overline{\mathcal{T}}$ on $E \otimes E$ then every set in the $\overline{\mathcal{T}}$ zero neighborhood basis must contain a space of the form $\hat{U}_2$. Conversely, the spaces $\hat{U}_2$ define a linear topology on E for which ω_2 is continuous. The same reasoning applies for any p, so a zero neighborhood basis for $\overset{p}{\underset{1}{\otimes}} \mathcal{T}$ consists of the sets

$$\hat{U}_p = U_p \otimes \dots \otimes U_p + \sum'_{x \in E} \sum'_{perms} [x] \otimes U_{px} \otimes \dots \otimes U_{px} + \sum'_{x, y \in E} \sum'_{perms} [x] \otimes [y] \otimes U_{pxy} \otimes \dots \otimes U_{pxy}$$

$$+ \dots + \sum'_{x_1, \dots, x_{p-1} \in E} \sum'_{perms} [x_1] \otimes [x_2] \otimes \dots \otimes [x_{p-1}] \otimes U_{p x_1 x_2 \dots x_{p-1}},$$ with the subscripted U's running through a zero neighborhood basis for $\mathcal{T}$ and

$\sum'_{perms} [\alpha] \otimes U_{p\alpha} \otimes \ldots \otimes U_{p\alpha} = [\alpha] \otimes U_{p\alpha} \otimes \ldots \otimes U_{p\alpha} +$
$U_{p\alpha} \otimes [\alpha] \otimes U_{p\alpha} \otimes \ldots \otimes U_{p\alpha} + \ldots + U_{p\alpha} \otimes U_{p\alpha} \otimes \ldots \otimes [\alpha]$ and with similar meanings for the other $\sum'_{perms}$ symbols. Henceforth in this chapter

$\sum_{\alpha_1, \ldots, \alpha_q \in E} \sum'_{perms}$ will be abbreviated by $\sum'$ and $\hat{U}_p$ will be written simply $U_p \otimes \ldots \otimes U_p + \sum' [\alpha] \otimes U_{p\alpha} \otimes \ldots \otimes U_{p\alpha} +$
$\sum' [\alpha] \otimes [y] \otimes U_{p\alpha y} \otimes \ldots \otimes U_{p\alpha y} + \ldots$. Taking sums and quotients, the projective tensor product topologies induce linear topologies on T(E) and C(E), both denoted by $\otimes\mathcal{T}$.

We now determine for which topologies $\mathcal{T}$ on E the induced topology $\otimes\mathcal{T}/_E$ is equal to $\mathcal{T}$.

Theorem 2: If E has a zero neighborhood basis of subspaces no one of which is totally isotropic then $\otimes\mathcal{T}/_E$ is trivial.

Proof: Let $\sigma(\hat{U}) = \sigma(\overset{\infty}{\underset{1}{\oplus}} \hat{U}_p) = \sigma(U_1 + U_2 \otimes U_2 + \sum' [\alpha] \otimes U_{2\alpha}$
$+ U_3 \otimes U_3 \otimes U_3 + \ldots)$ be an arbitrary space from the zero neighborhood basis for $\otimes\mathcal{T}$ on C(E). We claim that an arbitrary element α of E is an element of $\sigma(\hat{U})$. For U_α is not totally isotropic so there exists a $y \in U_\alpha$ with $Q(y) \neq 0$. $\alpha = \frac{1}{Q(y)} y \circ y \circ \alpha =$
$\sigma((\frac{1}{Q(y)}) y \otimes y \otimes \alpha) \in \hat{U}_3 \subset \sigma(\hat{U})$. But this implies $\sigma(\hat{U}) \cap E = E$ for arbitrary $\sigma(\hat{U})$, hence the assertion of the theorem.

On the other hand, if the conditions of Theorem 2 are not met then some linear neighborhood V of zero is totally isotropic. Intersecting

V with the spaces of the zero neighborhood basis gives a zero neighborhood basis of totally isotropic subspaces. In this case we have

Theorem 3: Let (E,τ) have a zero neighborhood basis $\{U_\alpha\}$ of totally isotropic subspaces. Then $\otimes\tau/_E = \tau$ if and only if $E = \bigcup_\alpha U_\alpha^\perp$.

Proof of necessity: Let $x \in E$. We shall show $x \in \bigcup_\alpha U_\alpha^\perp$. For $x = 0$ this conclusion is immediate so suppose $x \neq 0$. Since τ is Hausdorff there is a U_α with $x \notin U_\alpha$; $\otimes\tau/_E = \tau$ so there is a $\hat{U}$ with $\sigma(\hat{U}) \cap E \subset U_\alpha$; $\hat{U} = U_1 + U_2 \otimes U_2 + \sum_1 [x] \otimes U_{2x} + U_3 \otimes U_3 \otimes U_3 + \sum_1 [x] \otimes U_{3x} \otimes U_{3x} + \sum_1 [x] \otimes [y] \otimes U_{3xy} + \ldots$. Suppose by way of contradiction that $x \notin U_{3xx}^\perp$. Then there is a $y \in U_{3xx}$ with $\phi(x,y) = 1$; therefore $x = \phi(x,y)x = x \circ y \circ x + y \circ x \circ x \in \sigma(\hat{U})$ which, since $x \in E$, implies $x \in U_\alpha$ a contradiction. We conclude $x \in U_{3xx}^\perp$.

Proof of sufficiency: Since each $\sigma(\hat{U}) \cap E = \sigma(U_1 + \ldots) \cap E \supset U_1$, $\otimes\tau/_E \leq \tau$. Now suppose U_1 is an arbitrary space in the τ zero neighborhood basis. We shall construct $\hat{U}$ such that $\sigma(\hat{U}) \cap E \subset U_1$. First take $U_m \subset U_1$ for all m. By hypothesis for every x there is a U_x such that $x \perp U_x$. Take $U_{mx_1 \ldots x_m} = U_m \cap \bigcap_{i=1}^{m} U_{x_i}$ and $\hat{U} = U_1 + U_2 \otimes U_2 + \sum_1 [x] \otimes U_{2x} + \ldots$. We first note that $\sigma(\sum_{perms} [x_1] \otimes \ldots \otimes [x_m] \otimes U_{mx_1 \ldots x_m} \otimes \ldots \otimes U_{mx_1 \ldots x_m}) =$

$\sigma([x_1]\otimes \ldots [x_m]\otimes U_{m x_1 \ldots x_m}\otimes \ldots \otimes U_{n x_1 \ldots x_m})$ because if $u \in U_{m x_1 \ldots x_m}$ then $u \circ x_i = -x_i \circ u, \ 1 \leq i \leq m$ So every element of $\sigma(\hat{U})$ is of the form $t = \sum_{finite} t_i \circ u_i,\ u_i \in U_1$. Suppose by way of contradiction $x \in \sigma(\hat{U}) \cap E$ but $x \notin U_1$. Let $(e_\alpha)_{\alpha \in A}$ be a basis for the vectorspace U_1. The set $(x, e_\alpha)_{\alpha \in A}$ being linearly independent can be extended to a basis for E. Since $x \in \sigma(\hat{U})$, by the remarks above x is of the form

$$\sum_{i=1}^{n} t_i \circ u_i = \sum_{i=1}^{n} t_i \circ \left(\sum_{j=1}^{m} \lambda_{ij} e_{\alpha_j}\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_{ij}\, t_i \circ e_{\alpha_j}.$$

Multiplying through by $e_{\alpha_1} \circ e_{\alpha_2} \circ \ldots \circ e_{\alpha_m}$ gives $x \circ e_{\alpha_1} \circ \ldots \circ e_{\alpha_m} = 0$ since U_1 is totally isotropic. But this is not possible since $x \circ e_{\alpha_1} \circ \ldots \circ e_{\alpha_m}$ is an element of a basis of the Clifford Algebra. Hence $\sigma(\hat{U}) \cap E \subset U_1$. $\therefore \otimes\tau|_E = \tau$.

As an immediate consequence of the construction in the proof of Theorem 3 we have for future reference the

Corollary: If (E,τ) has a zero neighborhood basis of totally isotropic subspaces $\{U_\alpha\}$ then $(T(E), \otimes\tau)$ has a zero neighborhood basis of sets $\hat{U}$ such that t in $\sigma(\hat{U})$ implies $t = \sum_{finite} t_j \circ e_j$ with the e_j linearly independent elements from a single totally isotropic U_α.

Next we shall describe the topologies for which the conditions of Theorem 3 are realized.

Definition: Let U be a fixed totally isotropic subspace of E. The $\mathcal{T}_\phi U$ topology is defined by taking as a neighborhood basis at zero the spaces $U \cap F^\perp$, F a finite dimensional subspace of E, [5]. This gives a linear topology which is Hausdorff since ϕ is assumed to be nondegenerate.

Theorem 4: If (E,$\mathcal{T}$) is a linearly topologized space with a zero neighborhood basis of totally isotropic subspaces U_α and $E = \bigcup_\alpha U_\alpha^\perp$ then $\mathcal{T} \geqslant \mathcal{T}_\phi U_\alpha$ for every α. Conversely, if (E,$\mathcal{T}$) is a linearly topologized space with $\mathcal{T} \geqslant \mathcal{T}_\phi V$ for some totally isotropic subspace V of E then $\mathcal{T}$ has a zero neighborhood basis of totally isotropic subspaces U_α and $E = \bigcup_\alpha U_\alpha^\perp$.

Proof: To show $\mathcal{T} \geqslant \mathcal{T}_\phi U_{\alpha_0}$, let $U_{\alpha_0} \cap F^\perp$ be a set in the $\mathcal{T}_\phi U_{\alpha_0}$ zero neighborhood basis, $F = k(x_i)_{1 \leqslant i \leqslant m}$. Since $E = \bigcup_\alpha U_\alpha^\perp$ there exist zero neighborhoods U_{α_i} with $x_i \perp U_{\alpha_i}$. $U_{\alpha_0} \cap \bigcap_{i=1}^{m} U_{\alpha_i} \subset U_{\alpha_0} \cap F^\perp$ proving the contention.

Conversely suppose $\mathcal{T} \geqslant \mathcal{T}_\phi V$. The spaces $V \cap F_\beta^\perp$ with F_β a finite dimensional subspace of E are by hypothesis part of a zero neighborhood basis $\{U_\alpha\}$ for $\mathcal{T}$. Since $V \cap F_\beta^\perp$ is totally isotropic the U_α may be chosen totally isotropic. But the $(V \cap F_\beta^\perp)^\perp$ already cover E since $E = \bigcup_\beta F_\beta$ and $F_\beta \subset F_\beta^{\perp\perp} \subset (V \cap F_\beta^\perp)^\perp$. Therefore $E = \bigcup_\alpha U_\alpha^\perp$.

It is interesting to note that when $\mathcal{T}=\mathcal{T}_\phi V$, V of infinite dimension and codimension, the $\mathcal{T}\otimes\mathcal{T}$ topology on $E\otimes E$ is strictly finer than the $\mathcal{T}_e$ topology. This will be proved at the end of this chapter at which time certain lemmas and theorems will be available to make the proof easy.

The $\mathcal{T}_\phi V$ topologies are related in the following natural way to the form ϕ .

Theorem 5: For $(E,\mathcal{T})$ a linearly topologized space with symmetric, nondegenerate, bilinear form ϕ , the following are equivalent:
(i) Q , the associated quadratic form, is $\mathcal{T}$ -continuous.
(ii) $\mathcal{T}\geqslant\mathcal{T}_\phi V$ for some totally isotropic subspace V of E.
(iii) $\phi : E\times E\longrightarrow k$ is continuous (for the product topology on $E\times E$).

Proof: If Q is $\mathcal{T}$ -continuous then the conditions of Theorem 4 are satisfied. For the continuity of Q at zero implies $\mathcal{T}$ has a totally isotropic zero neighborhood V , hence a neighborhood basis at zero of totally isotropic subspaces U_α . And if $x\in E$, by continuity at x there exists a U_α with $Q(x+U_\alpha)=Q(x)$. Since U_α is totally isotropic this implies $x\perp U_\alpha$. Therefore $E=\bigcup_\alpha U_\alpha^\perp$, and applying Theorem 4 we have (i) implies (ii).

If $\mathcal{T}\geqslant\mathcal{T}_\phi V$, V totally isotropic, Theorem 4 can be used to show $\phi : E\times E\longrightarrow k$ is continuous at (x,y). For $E=\bigcup_\alpha U_\alpha^\perp$, so there exists a totally isotropic U_α with $x\perp U_\alpha$ and $y\perp U_\alpha$.
$\phi(x+U_\alpha\ ,\ y+U_\alpha)=\phi(x,y)$. So (ii) implies (iii).

It is apparent that the continuity of ϕ implies that of Q, completing the proof.

The topology on C(E) can only be considered admissible if continuous orthogonal automorphisms of E induce continuous algebra isomorphisms of C(E) and conversely. The projective tensor product topology has this essential property as the following theorem shows.

Theorem 6: Let f be an orthogonal automorphism of (E,ϕ), g the corresponding algebra isomorphism of T(E) (with $g(x_1 \otimes \dots \otimes x_m) = f(x_1) \otimes \dots \otimes f(x_m)$), and let h be the corresponding algebra isomorphism of C(E) (with $h(x_1 \circ \dots \circ x_m) = f(x_1) \circ \dots \circ f(x_m)$). If $\mathcal{T} \geq \mathcal{T}_\phi V$, V totally isotropic, then f is $\mathcal{T}$-continuous if and only if h is $\otimes\mathcal{T}$-continuous.

Proof: Suppose f is $\mathcal{T}$-continuous. Let $\hat{U} = U_1 + U_2 \otimes U_2 + \sum [x] \otimes U_{2x} + \dots$ be a set in the T(E) zero neighborhood basis. For every U_m (resp. $U_{mf(x)f(y)\dots}$) there is a V_m (resp. $V_{mxy\dots}$) such that $f(V_m) \subset U_m$ (resp. $f(V_{mxy\dots}) \subset U_{mf(x)f(y)\dots}$).

$$g(\hat{V}) = g(V_1 + V_2 \otimes V_2 + \textstyle\sum_1 [x] \otimes V_{2x} + \dots \subset$$

$$U_1 + U_2 \otimes U_2 + \textstyle\sum_1 [f(x)] \otimes U_{2f(x)} + \dots = \hat{U}$$ establishing the continuity of g.

Clearly $h\sigma = \sigma g$ so $h\sigma(\hat{V}) = \sigma g(\hat{V}) \subset \sigma(\hat{U})$, and h is likewise $\otimes\mathcal{T}$-continuous.

Conversely if h is an algebra isomorphism of C(E) with $h/_E = f$ and $h/_k = 1/_k$ then we already know $h/_E = f$ is an orthogonal automorphism of E. Since $\tau \geqslant \tau_\phi V$, $\tau = \otimes\tau/_E$, so the continuity of h implies the continuity of $h/_E$.

Applying the theorem to f^{-1} and h^{-1} gives the

Corollary: With the hypotheses of Theorem 6, f is open if and only if h is.

Although not essential, it would be desirable to have a Hausdorff topology on C(E). We first consider separate continuity of multiplication in T(E) since this result will be used in the proof of Hausdorff. Later in the chapter the subject of continuity of multiplication will be discussed in more detail.

Theorem 7: Multiplication is separately continuous in (T(E), $\otimes\tau$).

Proof: First consider multiplication on the left by

$s = x_1 \otimes \dots \otimes x_p \in \overset{p}{\underset{1}{\otimes}} E$. For every q the map $\overset{q}{\underset{1}{\Pi}} E \to \overset{p+q}{\underset{1}{\Pi}} E \to \overset{p+q}{\underset{1}{\otimes}} E$ with $(y_1, \dots, y_q) \to (x_1, \dots, x_p, y_1, \dots, y_q) \to x_1 \otimes \dots \otimes x_p \otimes y_1 \otimes \dots \otimes y_q$ is continuous and so induces a continuous map $\overset{q}{\underset{1}{\otimes}} E \to \overset{p+q}{\underset{1}{\otimes}} E$ with $y_1 \otimes \dots \otimes y_q \to x_1 \otimes \dots \otimes x_p \otimes y_1 \otimes \dots \otimes y_q$. (by the definition of $\overset{p+q}{\underset{1}{\otimes}} E$). Addition gives a $\otimes\tau$ continuous map T(E)$\to$T(E) with $t \to s \otimes t$.

Now let $s' = \sum_{i=1}^{m} s_i$ be an arbitrary element of T(E),

$s_i = s_{i_1} \otimes \dots \otimes s_{i_{p_i}}$. Left multiplication by s_i is continuous at zero so for every s_i there is a $\hat{V}_i$ such that $[s_i] \otimes \hat{V}_i = \hat{U}$. Therefore $[s'] \otimes \bigcap_{i=1}^{m} \hat{V}_i \subset \hat{U}$. Left multiplication by s' is continuous at zero and therefore continuous, since $\otimes\mathcal{T}$ is a linear topology. The symmetric argument proves right multiplication is also continuous.

Using Theorem 7 we can prove the

Corollary: If A is a two sided ideal in $(T(E), \otimes\mathcal{T})$ then $\bar{A}$, the topological closure of A, is also a two sided ideal.

Proof: A is a linear subspace of the linearly topologized space T(E), so $\bar{A}$ is also; in particular $\bar{A}$ is closed under differences. Now let $t \in$ T(E) and $s \in \bar{A}$, and let $t \otimes s + \hat{U}$ be an arbitrary basic neighborhood of $t \otimes s$. Since multiplication is separately continuous at $(t, 0)$ there is a $\hat{V}$ such that $[t] \otimes \hat{V} \subset \hat{U}$. And $s \in \bar{A}$ implies $s + \hat{V}$ meets A at some point $s + v$. Then $t \otimes (s + v) = t \otimes s + t \otimes v$, is in both A and $t \otimes s + \hat{U}$. So $t \otimes s \in \bar{A}$ from which we conclude $\bar{A}$ is a left ideal. The proof of right ideal is similar.

With the corollary above we are in a position to prove $(C(E), \otimes\mathcal{T})$ is Hausdorff for $\mathcal{T} \geqslant \mathcal{T}_\phi V$:

Theorem 8: $\mathcal{T} \geqslant \mathcal{T}_\phi V$ for V some totally isotropic subspace of E if and only if $(C(E), \otimes\mathcal{T})$ is Hausdorff.

Proof: Using Theorem 4 it suffices to show that $(C(E), \otimes\mathcal{T})$ is Hausdorff iff $\mathcal{T}$ has a zero neighborhood basis of totally isotropic subspaces U_α and $E=\bigcup_\alpha U_\alpha^\perp$ The topology $\otimes\mathcal{T}$ on C(E) was obtained by quotients from the $\otimes\mathcal{T}$ topology on T(E). Under these circumstances it is well known (see for example [9]) that (C(E), $\otimes\mathcal{T}$) is Hausdorff iff $I=\bar{I}$ (I the two sided ideal in T(E) generated by the elements $x\otimes x - Q(x)$ or equally well by the elements $x\otimes y + y\otimes x - 2\phi(x,y)$.

Suppose $\otimes\mathcal{T}$ is Hausdorff. $I=\bar{I}$ is a proper ideal in T(E) so in particular $-1\notin\bar{I}$. Therefore there exists $\hat{U}=U_1+U_2\otimes U_2+\sum[x]\otimes U_{2x}+\ldots$ in the usual zero neighborhood basis for T(E) with $-1+\hat{U}$ disjoint from I. We claim U_2 is totally isotropic. For if this were not so there would be an $x\in U_2$ with $\|x\|\neq 0$. Put $y=x/2\|x\|$, then y is also in U_2 and $\phi(x,y)=\frac{1}{2}$. So $x\otimes y+y\otimes x-1\in(-1+\hat{U})\cap I$, contradiction. We may therefore assume that all the U_α are totally isotropic.

We claim in addition that for each $x\in$ E, $x\perp U_{2x}$. If not there would be a $y\in U_{2x}$ with $\phi(x,y)=\frac{1}{2}$, and then $x\otimes y+y\otimes x-1\in(-1+\hat{U})\cap I$ as before; contradiction. $\therefore E=\bigcup_\alpha U_\alpha^\perp$.

To prove the converse we assume $E=\bigcup_\alpha U_\alpha^\perp$ for some totally isotropic zero neighborhood basis $\{U_\alpha\}$. By the corollary to Theorem 3 proved earlier, T(E) has a zero neighborhood basis of sets $\hat{U}$ such that if $t\in\sigma(\hat{U})$ then $t=\sum_{j=1}^{m} t_j \circ e_j$ with the e_j linearly independent elements from a single totally isotropic zero neighborhood

U_{α_0}. We claim this implies $1 \notin \bar{I}$. For if $1 \in \bar{I}$ then $1 + \hat{U}$ meets I and so $1 + \sigma(\hat{U})$ meets (0) say in $1 + t$. We have $0 = 1 + t = 1 + \sum_{j=1}^{m} t_j \circ e_j$. Multiplying by $e_1 \circ \ldots \circ e_m$ gives $0 = e_1 \circ \ldots \circ e_m$ which is impossible since the e_j are linearly independent.

Thus it is clear that $1 \notin \bar{I}$; in particular $I \neq T(E)$. But C(E) = T(E)/I is a simple algebra and $I \subset \bar{I}$, so $I = \bar{I}$. Thus $\otimes\mathcal{T}$ is Hausdorff.

The discrete topology on E would give all the essential properties thus far ascribed to the $\mathcal{T}_\emptyset V$ topologies. The next theorem guarantees that we are not dealing with just the discrete case.

Theorem 9: A $\mathcal{T}_\emptyset V$ topology is discrete iff V is finite dimensional.

Proof: As noted in Chapter II, for a finite dimensional subspace F of E, $\dim F = \operatorname{codim} F^\perp$.

If $\mathcal{T}_\emptyset V$ is discrete then $V \cap F^\perp = (0)$ for some finite dimensional subspace F. This implies that the sum $V + F^\perp$ is direct, so $\dim F = \operatorname{codim} F^\perp \geqslant \dim V$. In particular $\dim V$ is finite.

On the other hand, if $\dim V$ is finite the $\operatorname{codim} V^\perp = \dim V$. That is $E = V^\perp \oplus F$ for some finite dimensional space F. Therefore $(0) = E^\perp = V^{\perp\perp} \cap F^\perp = V \cap F^\perp$, which implies that $\mathcal{T}_\emptyset V$ is discrete.

Continuity of Multiplication

We turn our attention to the question of continuity of multiplication in $(C(E), \otimes\tau_\phi V)$. For denumerable (E,ϕ) we shall establish the remarkable fact that $(C(E), \otimes\tau_\phi V)$ is a topological algebra for closed V, (Theorem 10). It is not clear whether a similar result holds in the nondenumerable case. For τ strictly finer than $\tau_\phi V$, we shall give an example of a denumerable (E,ϕ) for which multiplication fails to be continuous in $(C(E), \otimes\tau)$. First we prove

Lemma 1: If $E = k(e_\alpha)_{\alpha\in I}$ then the sets

$$\hat{U} = U_1 + U_2 \otimes U_2 + \sum_{e_\alpha}{}' \sum_{perms}{}' [e_\alpha] \otimes U_{2e_\alpha} + U_3 \otimes U_3 \otimes U_3$$
$$+ \sum_{e_\alpha}{}' \sum_{perms}{}' [e_\alpha] \otimes U_{3e_\alpha} \otimes U_{3e_\alpha} + \sum_{e_\alpha e_\beta}{}' \sum_{perms}{}' [e_\alpha] \otimes [e_\beta] \otimes U_{3e_\alpha e_\beta} + \ldots$$

form a zero neighborhood basis for $\otimes\tau$ on T(E) when the subscripted U's run through a zero neighborhood basis for τ.

We shall again write $\sum'$ to mean $\sum\limits_{e_\alpha e_\beta \ldots}{}' \sum\limits_{perms}{}'$.

Proof: Clearly each $\otimes\tau$ zero neighborhood contains such a $\hat{U}$. Conversely, put $V_m = U_m$ and for $\alpha_1, \alpha_2, \ldots, \alpha_m \in k(e_\alpha)_{\alpha\in A}$, A finite, put $V_{m\alpha_1\ldots\alpha_m} = \bigcap\limits_{\alpha_i\in A} U_{m e_{\alpha_1}\ldots e_{\alpha_m}}$. Then

$$\sum_{\alpha_i\in A}{}' [e_{\alpha_1}] \otimes \ldots \otimes [e_{\alpha_m}] \otimes U_{m e_{\alpha_1}\ldots e_{\alpha_m}} \otimes \ldots \otimes U_{m e_{\alpha_1}\ldots e_{\alpha_m}} \supset [\alpha_1] \otimes \ldots \otimes [\alpha_m] \otimes V_{m\alpha_1\ldots\alpha_m} \otimes \ldots \otimes V_{m\alpha_1\ldots\alpha_m}$$

so $\hat{U}$ contains a $\otimes\tau$ zero neighborhood.

Theorem 10: If dim $E = \aleph_0$ and V is a closed totally isotropic subspace of E then $(T(E), \otimes\tau_\phi V)$ and $(C(E), \otimes\tau_\phi V)$ are topological algebras.

Proof: Since $\dim E = \aleph_0$ and V is closed and totally isotropic there is a decomposition of E into $(V \oplus V') \overset{\perp}{\oplus} G$ with $V = k(v_i)_{i \geq 1}$ and $V' = k(v_i')_{i \geq 1}$ both totally isotropic and $\phi(v_i, v_j') = \delta_{ij}$. The $\mathcal{T}_\phi V$ topology has a zero neighborhood basis of sets $k(v_i)_{i > m}$ since for F finite dimensional, $F \subset V + k(v_i')_{i \leq m} + G$ so $V \cap F^\perp \supset V \cap V^\perp \cap k(v_i')^\perp_{i \leq m} \cap G^\perp = V \cap k(v_i')^\perp_{i \leq m} = k(v_i)_{i > m}$. We shall need an enumerated basis for E, so let $E = k(e_i)_{i \geq 1}$ with $v_i = e_{2i}$ for $i \geq 1$. Then the sets $U_m^* = k(e_i)_{\substack{i > m \\ i\ \text{even}}}$ are a $\mathcal{T}_\phi V$ zero neighborhood basis. (They are not distinct. In fact $U_1^* = k(v_i)_{i > 0}$, $U_2^* = U_3^* = k(v_i)_{i > 1}$, etc.). The advantage of this numbering is that it yields the following simple criterion: $e_i \in U_m^*$ iff $e_i \in V$ and $i > m$. The U_m^* will be referred to as $*$-sets in the rest of the proof.

To show multiplication in T(E) is continuous at (0,0) let $\hat{U}' = U_1' + U_2' \otimes U_2' + \sum [e_i] \otimes U'_{2e_i} + \ldots$ be a set in the (T(E), $\otimes\mathcal{T}$) zero neighborhood basis. We must find $\hat{V} \otimes \hat{V} \subset \hat{U}'$. Clearly it suffices to find $\hat{V} \otimes \hat{V} \subset \hat{U} \subset \hat{U}'$. With this in mind we shrink $\hat{U}'$ somewhat, in order to make it more manageable, as follows. Choose inductively sets U_m which are $*$-sets and such that $U_1 \subset U_1' \cap U_1^*$ and $U_m \subset U_1 \cap U_2 \cap \ldots \cap U_{m-1} \cap U_m' \cap U_m^*$. Denote by U''_{me_m} the set

$$\bigcap_{t \leq m-1} \bigcap_{j_1, \ldots, j_t \leq m} U'_{me_{j_1} \ldots e_{j_t} e_m}$$

(i.e., the intersection of all sets $U'_{me_{j_1} \ldots e_{j_t} e_m}$ for which m is the largest e-subscript). Define the sets U_{ne_m} by induction on m to be $\not\ast$-sets contained in

$U_n \cap U_{ne_1} \cap \ldots \cap U_{ne_{m-1}} \cap U''_{ne_m} \cap U^*_m$ with U_{ne_1} a $\not\ast$-set contained in $U_n \cap U''_{ne_1} \cap U^*_1$. Since the U_n and U_{ne_m} are $\not\ast$-sets there are functions g_n with $U_n = U^*_{g_n(0)}$ and $U_{ne_m} = U^*_{g_n(m)}$. As a consequence of the construction we have $U^*_{g_n(0)} = U_n \subset U^*_n$ so

$$0 < n \leqslant g_n(0)$$

and $U^*_{g_n(m)} = U_{ne_m} \subset U^*_m$ so $\quad m \leqslant g_n(m).$

Also if $0 < i \leqslant j$ then $U_{ne_j} \subset U_{ne_i} \subset U_n$ so $U^*_{g_n(j)} \subset U^*_{g_n(i)} \subset U^*_{g_n(0)}$ therefore

$$g_n(0) \leqslant g_n(i) \leqslant g_n(j) \quad \text{for} \quad 0 < i \leqslant j$$

Take

$$\hat{U} = U_1 + U_2 \otimes U_2 + \sum [e_i] \otimes U_{2e_i} + U_3 \otimes U_3 \otimes U_3 + \sum [e_i] \otimes U_{3e_i} \otimes U_{3e_i} + \sum [e_i] \otimes [e_j] \otimes U_{3e_ie_j} + \ldots$$

with $U_{ne_{i_1} \ldots e_{i_m}} = U_{ne_{i_k}}$ where $i_k = \max(i_1, \ldots, i_m)$.

To define $\hat{V}$ we shall make use of a function used in enumerating $N \times N$, N the nonnegative integers. For $n, m \in N$ put $f(n, m) = \frac{1}{2}(n+m)(n+m+1) + n + 1$. Then $f(n_1, m_1) \leqslant f(n_2, m_2)$ iff either $n_1 + m_1 < n_2 + m_2$ or $n_1 + m_1 = n_2 + m_2$ and $n_1 \leqslant n_2$. For our purposes it suffices that f have the property that for any two pairs (n_1, m_1) and (n_2, m_2), $f(n_1, m_1)$ and $f(n_2, m_2)$ are comparable, and for only finitely many (n_1, m_1) is $f(n_1, m_1) \leqslant f(n_2, m_2)$. We now define the $\hat{V}_q$ for our $\hat{V}$. For prescribed $q, e_{j_1'}, \ldots, e_{j_m'}$ let

$$V_{q e_{j_1'}, \ldots e_{j_m'}} = \bigcap_{\substack{\text{all } p, i_m \geqslant 0 \text{ with} \\ f(p, i_m) \leqslant f(q, j_m)}} \Big[U^*_{g_{p+q}(j_m)} \cap U^*_{g_{p+q}(i_m)} \cap \bigcap_{i=1}^{g^p_{p+q}(j_m)} U^*_{g_{p+q}(i)} \cap \bigcap_{i=1}^{g^p_{p+q}(i_m)} U^*_{g_{p+q}(i)} \Big] \cap U^*_{j_m}$$

where $j_m = \max(j_1', \ldots, j_m')$. Finally take the expression for V_q to be the same as that for $V_{q e_{j_1'} \ldots e_{j_m'}}$ with j_m replaced by 0 throughout. g^p_{p+q} is defined iteratively by $g'_{p+q}(j_m) = g_{p+q}(j_m)$, and $g^p_{p+q}(j_m) = g_{p+q}(g^{p-1}_{p+q}(j_m))$. Put $\hat{V} = \bigoplus \hat{V}_p = V_1 + V_2 \otimes V_2 + \sum [e_i] \otimes V_{2e_i} + \ldots$ as usual. The reason for the choice of each part of $V_{q e_{j_1'} \ldots e_{j_m'}}$ will become apparent in the cases we consider in showing that $\hat{V} \otimes \hat{V} \subset \hat{U}$.

Let $s' = e_{i_1'} \otimes \ldots \otimes e_{i_m'} \otimes e_{i_{m+1}'} \otimes \ldots \otimes e_{i_p'} \in \hat{V}_p$ with $e_{i_k'} \in V_{p e_{i_1'} \ldots e_{i_m'}}$, $m+1 \leqslant k \leqslant p$. Let $t = e_{j_1'} \otimes \ldots \otimes e_{j_q'}$ with $e_{j_k'} \in V_{q e_{j_1'} \ldots e_{j_m'}}$, $m+1 \leqslant k \leqslant q$. Let $i_1 \leqslant i_2 \leqslant \ldots \leqslant i_m$ be the subscripts $i_1', \ldots, i_m'$ in their natural order and $i_{m+1} \leqslant \ldots \leqslant i_p$ the subscripts $i_{m+1}', \ldots, i_p'$ in their natural order. Since $e_{i_{m+1}} \in V_{p e_{i_1'} \ldots e_{i_m'}} \subset U^*_{i_m}$, $i_{m+1} > i_m$ giving the combined ordering $i_1 \leqslant \ldots \leqslant i_m < i_{m+1} \leqslant \ldots \leqslant i_p$. Similarly let $j_1 \leqslant \ldots \leqslant j_m < j_{m+1} \leqslant \ldots \leqslant j_q$ be the natural order of the j_k'. Note that $e_{i_{m+1}}, \ldots, e_{i_p}, e_{j_{m+1}}, \ldots, e_{j_q}$

are all in V .

We now show $s' \otimes t' \in \hat{U}$. The general nature of the next steps in the proof is this. Let $l_1 \leq l_2 \leq \dots \leq l_s \leq l_{s+1} \leq \dots \leq l_{p+q}$ be the subscripts $i_1, \dots, i_p, j_1, \dots, j_q$ arranged in order. Let $l_{s+1} > i_m, j_m$. Then $e_{l_{s+1}}, e_{l_{s+2}}, \dots$ are all in V . We show that we can always choose s so that $e_{l_{s+1}} \in U^*_{g_{p+q}(l_s)} = U_{p+q, e_{l_1}, \dots, e_{l_s}}$. Then $e_{l_{s+2}}, e_{l_{s+3}}, \dots, e_{l_{p+q}}$ are also in $U_{p+q, e_{l_1}, \dots, e_{l_s}}$ so

$$s' \otimes t' \in \sum_{perms} [e_{l_1}] \otimes \dots \otimes [e_{l_s}] \otimes U_{p+q}\, e_{l_1} \dots e_{l_s} \otimes \dots \otimes U_{p+q}\, e_{l_1} \dots e_{l_s} \subset \hat{U}.$$

We assume without loss of generality that $f(p, i_m) \leq f(q, j_m)$ hence i_m and j_m will not play symmetric roles in the sequel. Since $j_m = max(j'_1, \dots, j'_m)$ by the definition of $V_q\, e_{j'_1} \dots e_{j'_m}$ we have $e_{j_{m+1}} \in U^*_{g_{p+q}(j_m)}$, $e_{j_{m+1}} \in U^*_{g_{p+q}(i_m)}$, $e_{j_{m+1}} \in U^*_{g_{p+q}(i)}$ for $i < g^p_{p+q}(j_m)$ and $e_{j_{m+1}} \in U^*_{g_{p+q}(i)}$ for $i \leq g^p_{p+q}(i_m)$. In particular from the second of these conditions we have $j_{m+1} > g_{p+q}(i_m) \geq i_m$.

Case A: i_m or j_m is the immediate predecessor of j_{m+1} in the ordered list of subscripts. Since as noted above $e_{j_{m+1}} \in U^*_{g_{p+q}(i_m)}$ and $U^*_{g_{p+q}(j_m)}$ in these cases $s' \otimes t' \in \hat{U}$.

Since $i_m < j_{m+1}$ the only other possibility is that j_{m+1} is the immediate successor of some i_s, $s > m$.

Case B: $i_m \leq \dots \leq j_m \leq i_{s-k} \leq \dots \leq i_s \leq j_{m+1} \leq \dots$
Note that only i -subscripts occur between i_{s-k} and i_s . If $i_{s-k} > g_{p+q}(j_m)$ then $e_{i_{s-k}} \in U^*_{g_{p+q}(j_m)}$ (by the basic definition of the $*$-sets), and we're done. Similarly if $i_t > g_{p+q}(i_{t-1})$ for any t

with $s-k<t\leq s$ then $e_{i_t} \in U^*_{g_{p+q}}(i_{t-1})$ as desired. If on the other hand none of these alternatives occurs then $i_s \leq g_{p+q}(i_{s-1})$ and $i_{s-1} \leq g_{p+q}(i_{s-2})$ etc. So $i_s \leq g_{p+q}(i_{s-1}) \leq$

$$g^2_{p+q}(i_{s-2}) \leq \cdots \leq g^k_{p+q}(i_{s-k}) \leq g^{k+1}_{p+q}(j_m) \leq g^p_{p+q}(j_m)\,;$$

these inequalities follow since g_{p+q} is nondecreasing. But then $i_s \leq g^p_{p+q}(j_m)$, so as noted earlier $e_{j_{m+1}} \in U^*_{g_{p+q}}(i_s)$.

Case C: $j_m \leq \cdots \leq i_m \leq i_{m+1} \leq \cdots \leq i_s \leq j_{m+1} \leq \cdots$. The proof is the same as for Case B but with i_m replacing j_m throughout.

In the case where $i_{m+1} = i_1$ (resp. $j_{m+1} = j_1$) take $i_m = 0$ (resp. $j_m = 0$), and the proof goes through as above. This would be the case when $e_{i_1'} \otimes \cdots \otimes e_{i_p'} \in V_p \otimes \cdots \otimes V_p$ (resp. $e_{j_1'} \otimes \cdots \otimes e_{j_q'} \in V_q \otimes \cdots \otimes V_q$).

In every instance $s' \otimes t' \in \hat{U}$. Now a product of two arbitrary elements of $\hat{V}$ is a sum of terms of the form $s' \otimes t'$ hence also in $\hat{U}$, completing the proof that multiplication in (T(E), $\otimes\mathcal{T}$) is continuous at (0,0).

In Theorem 8 it was shown that multiplication in (T(E), $\otimes\mathcal{T}$) is separately continuous. Thus (T(E), $\otimes\mathcal{T}$) is a topological vector space with continuous multiplication, hence a topological algebra.

We now prove that continuity of multiplication in (T(E), $\otimes\mathcal{T}$) implies continuity of multiplication in (C(E), $\otimes\mathcal{T}$). Let $m : (s,t) \to s \otimes t$ be the multiplication in T(E) and σ the canonical map: $T(E) \to C(E) = T(E)/I$. Then $\sigma \circ m : T(E) \times T(E) \to C(E)$ is continuous and constant on equivalence classes modulo I, so it induces a well defined map $\bar{m}: (\sigma(s), \sigma(t)) \longrightarrow \sigma(s)\circ\sigma(t)$ which is in fact multiplication in C(E). Given $\sigma(s)\circ\sigma(t)\in \mathcal{O}$, $\mathcal{O}$ open in C(E), there exist $\mathcal{O}(s)$ and $\mathcal{O}(t)$ containing s and t respectively with $\sigma\circ m(\mathcal{O}(s)\times\mathcal{O}(t))\subset\mathcal{O}$. Since $\bar{m}\circ(\sigma\times\sigma) = \sigma\circ m$, $\bar{m}(\sigma(\mathcal{O}(s))\times\sigma(\mathcal{O}(t)))\subset\mathcal{O}$ and so $\bar{m}$ is continuous.

Multiplication need not be continuous in (T(E), $\otimes\mathcal{T}$) for $\mathcal{T} > \mathcal{T}_{\phi}V$ even when dim $E = \aleph_0$ and V is a maximal (hence closed) totally isotropic subspace as the example below will show. The next lemma will be used in the example and in the next theorem.

Lemma 2. Let $E = V\oplus W$ with $V = k(e_\alpha)_{\alpha\in I}$ and $W = k(e_\alpha)_{\alpha\in J}$ have a topology for which there is a neighborhood basis at zero composed of sets of the form $U_L = k(N_\alpha)_{\alpha\in L}$, L running through some of the subsets of I. Let

$$\hat{U}_m = U_m\otimes\cdots\otimes U_m + \sum_{\alpha\in I\cup J}[e_\alpha]\otimes U_{me_\alpha}\otimes\cdots\otimes U_{me_\alpha} + \sum [e_\alpha]\otimes[e_\beta]\otimes U_{me_\alpha e_\beta}\otimes\cdots\otimes U_{me_\alpha e_\beta} + \cdots$$

with $U_{me_\alpha}\subset U_m$, $U_{me_\alpha e_\beta}\subset U_{me_\alpha}\cap U_{me_\beta}\cap\cdots$ and all subscripted U's from the zero neighborhood basis. If $e_{\alpha_0}\notin U_{me_{\alpha_1}}$ and $e_{\alpha_1}\notin U_m$ then $e_{\alpha_1}\otimes(\overset{m-1}{\underset{1}{\otimes}} e_{\alpha_0})\notin\hat{U}_m$, e_{α_0} and e_{α_1} elements of the basis $\{e_\alpha\}_{\alpha\in I\cup J}$.

Proof: The summands of $\hat{U}_m$ are of these types: either of the form $[e_{\alpha_1}]\otimes A$ with A containing a factor $U_{me_{\alpha_1}}\_$ or of the form $U_{m,}\_\otimes B$, or of the form $[e_\alpha]\otimes C$ with $\alpha\neq\alpha_1$. Since by hypothesis $[e_{\alpha_0}]\notin U_{me_{\alpha_1}}\_$, $[e_{\alpha_1}]\otimes A\subset F = [e_{\alpha_1}\otimes e_{\beta_1}\otimes\cdots\otimes e_{\beta_{m-1}};$ some $\beta_i\neq\alpha_0]$.

While $U_{m}\_\otimes B$ and $[e_\alpha]\otimes C\subset G = [e_{\gamma_1}\otimes e_{\gamma_2}\otimes\cdots\otimes e_{\gamma_m};\ \gamma_1\neq\alpha_1]$. $e_{\alpha_1}\otimes e_{\alpha_0}\otimes\cdots\otimes e_{\alpha_0}\notin F\oplus G$ and $\hat{U}_m\subset F\oplus G$ concluding the proof.

Example: Let $E = V \oplus W$ with $V = k(v_i)_{i \geq 1}$ and $W = k(w_i)_{i \geq 1}$ both totally isotropic and $\phi(v_i, w_j) = \delta_{ij}$. Take for τ the topology with neighborhood basis at zero of sets $U_m^{**} = k(v_{2^m i})_{i \geq 1}$. As proved in Theorem 10, the $\tau_\phi V$ topology has a zero neighborhood basis of sets $U_m^* = k(v_i)_{i > m}$. Each U_m^{**} contains some U_m^* (for example $U_m^{**} \supset U_m^*$) but not conversely so τ is strictly finer than $\tau_\phi V$.

In the zero neighborhood basis for $(T(E), \otimes\tau)$ consider any set $\hat{U} = U_1 + U_2 \otimes U_2 + \sum [v_i] \otimes U_{2v_i} + \sum [w_i] \otimes U_{2w_i} + \ldots$ of the general form given in lemma 2 and in particular with $U_m = U_m^{**}$ and $U_{mv_i} = U_m^{**} \cap U_i^{**}$. Let $\hat{V} = V_1 + V_2 \otimes V_2 + \ldots$ with the subscripted V's from the τ zero neighborhood basis and suppose by way of contradiction that $\hat{V} \otimes \hat{V} \subset \hat{U}$. $V_1 = U_q^{**} = k(v_{2^q i})_{i \geq 1}$ for some q. For i odd, $v_{2^q i} \in V_1$ but $v_{2^q i} \notin U_{q+1}$. $\bigcap_{i\, odd} U_{q+1, v_{2^q i}} \subset \bigcap_{i\, odd} U_{2^q i}^{**} = (0)$, so $V_q \not\subset \bigcap_{i\, odd} U_{q+1, v_{2^q i}}$. There is an odd i_1, and a $v_{j_0} \in V_q$ such that $v_{j_0} \notin U_{q+1, v_{2^q i_1}}$. And since i_1, is odd, $v_{2^q i_1} \notin U_{q+1}$. Therefore by the lemma $v_{2^q i_1} \otimes v_{j_0} \otimes \ldots \otimes v_{j_0} \notin \hat{U}_{q+1}$. On the other hand $v_{2^q i_1} \otimes v_{j_0} \otimes \ldots \otimes v_{j_0} \in V_1 \otimes V_q \otimes \ldots \otimes V_q \subset \hat{U}_{q+1}$, contradiction.

Examples can be given with V closed and totally isotropic, $\tau > \tau_\phi V$ and $\dim V > \aleph_0$ for which multiplication is not continuous. The state of affairs when $\tau = \tau_\phi V$ and $\dim V > \aleph_0$ is an open question.

In this chapter two topologies were considered on the tensor product $E \otimes E$. It is apparent from a comparison of the neighborhood basis at zero that $\mathcal{T}_e \leqslant \mathcal{T}\otimes\mathcal{T}$. In [7] it is shown that $\mathcal{T}_e = \mathcal{T}\otimes\mathcal{T}$ when $\mathcal{T}$ is the weak topology. On the other hand using several of our earlier results it is now easy to show that $\mathcal{T}_e$ is strictly coaser than $\mathcal{T}\otimes\mathcal{T}$ for $\mathcal{T}$ a $\mathcal{T}_\phi V$ topology, V of infinite dimension and codimension.

Theorem 11: Let $E = V \oplus H$ have topology $\mathcal{T}_\phi V$, V totally isotropic and of infinite dimension, $H = k(h_\alpha)_{\alpha\in I}$ also of infinite dimension. Then $\mathcal{T}_e < \mathcal{T}\otimes\mathcal{T}$.

Proof: Since card $I \geqslant \aleph_0$ there is a bijective function f mapping I onto its finite subsets. There is a neighborhood basis at zero for the $\mathcal{T}_\phi V$ topology of sets $U_\alpha = V \cap k(h_\beta)^\perp_{\beta\in f(\alpha)}$. For if F is finite dimensional then $V\cap F^\perp \supset V \cap (V + k(h_\beta)_{\beta\in f(\alpha)})^\perp$ $= V \cap k(h_\beta)^\perp_{\beta\in f(\alpha)}$. Let $V = k(v_\gamma)_{\gamma\in C}$. $\hat{U}_2 = V\otimes V + \sum [v_\gamma]\otimes V + \sum [h_\alpha]\otimes U_\alpha$ is a space in the $\mathcal{T}\otimes\mathcal{T}$ zero neighborhood basis (see Lemma 1). Suppose by way of contradiction that $\hat{U}_2 \supset E\otimes U + U\otimes E$, U in the $\mathcal{T}_\phi V$ neighborhood basis at 0. $\mathcal{T}_\phi V$ is Hausdorff but not discrete since dim $V \geqslant \aleph_0$ (Theorem 9), so there is a U_{α_0} with $U \not\subset U_{\alpha_0}$ i.e., some $v_{\alpha_1}\in U$, $v_{\alpha_1}\notin U_{\alpha_0}$. Then $h_{\alpha_0}\otimes v_{\alpha_1}$, h_{α_0} in the basis for H is clearly in $E\otimes U + U\otimes E$ but by Lemma 2 it is not in $\hat{U}_2$. Thus $\hat{U}_2$ contains no $\mathcal{T}_e$ zero neighborhood so $\mathcal{T}_e < \mathcal{T}\otimes\mathcal{T}$.

CHAPTER IV

THE COMPLETION OF $(E, \tau_\phi V)$

As in Chapter III assume that E is an infinite dimensional vector space over a field k and that ϕ is a nondegenerate, symmetric, bilinear form. Let V be a totally isotropic subspace of E and equip E with the topology $\tau = \tau_\phi V$. In several of the theorems which follow it will be convenient to consider the following decomposition: $E = V \oplus H_1 \oplus H_2$ with $V^\perp = H_1 \oplus V$, $V = k(v_\alpha)_{\alpha \in I}$, $H_1 = k(h_{1\alpha})_{\alpha \in J}$, $H_2 = k(h_{2\alpha})_{\alpha \in K}$, $H = H_1 \oplus H_2$. Such a decomposition is of course always possible.

The symbol $\sim$ will denote completion. The topology τ under consideration is always $\tau_\phi V$. Thus $\tilde{\tau}$ denotes the completion of the $\tau_\phi V$ topology.

The first theorem shows that the problem of completing E reduces to that of completing V.

Theorem 1: $\tilde{E} = \tilde{V} \oplus H$. $\tilde{\tau}/_H$ is the discrete topology.

Proof: If U is any set in the zero neighborhood basis for $\tau_\phi V$ then $U \cap H = (0)$. So (0) is a $\tau_\phi V/_H$ zero neighborhood, hence $\tau_\phi V/_H$ is discrete so H is already complete.

Let p_1 be the projection mapping $V \oplus H$ into V. Let $v + U$ be a neighborhood of $p_1(v+h) = v$. Since $p_1(v+h+U) = v+U$, p_1 is continuous. Therefore $V \oplus H$ is a topological direct sum and $\tilde{E} = \tilde{V} \oplus \tilde{H} = \tilde{V} \oplus H$.

The completion is only of interest if ϕ induces a continuous bilinear form on $\tilde{E}$. The next theorem guarantees that this will be the case.