Time series analysis of irrigation return flow

by Michael Earl Nicklin

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Civil Engineering
Montana State University

Abstract:
The potential of time series analysis as a tool for quantifying irrigation return flows along a reach of the Beaverhead River near Dillon, Montana is investigated. Univariate time series analysis is used to determine the stochastic character of both the irrigation diversion series and the irrigation return flow series. Box and Jenkins' transfer function identification procedures are used to determine the nature of the diversion-return flow relationship. Analysis of the 1974 irrigation season yielded a transfer function relationship in which return flows are most strongly dependent upon irrigation diversions that were made 54 days earlier. The 1974 relationship also indicates that 53.5 percent of the irrigation season diversion volume returns to the Beaverhead reach during the same irrigation season.

Although the study demonstrates the potential of the transfer function methodology as a tool for return flow quantification, it also shows how that potential is severely limited by data inadequacies. Proper application of time series analysis to irrigation return flow systems requires reliable and sufficient records of daily stream flows and daily diversions.

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This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

8/10/83 — Date

Richard L Brustkern — Chairperson, Graduate Committee

Approved for the Major Department

8/8/83 — Date

Theodore J Williams — Head, Major Department

Approved for the College of Graduate Studies

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Michael Malone — Graduate Dean

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Date 8/17/83

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TABLE OF CONTENTS

TABLE OF CONTENTS--Continued

LIST OF TABLES

LIST OF FIGURES

Figures Page

ABSTRACT

The potential of time series analysis as a tool for quantifying irrigation return flows along a reach of the Beaverhead River near Dillon, Montana is investigated. Univariate time series analysis is used to determine the stochastic character of both the irrigation diversion series and the irrigation return flow series. Box and Jenkins' transfer function identification procedures are used to determine the nature of the diversion-return flow relationship. Analysis of the 1974 irrigation season yielded a transfer function relationship in which return flows are most strongly dependent upon irrigation diversions that were made 54 days earlier. The 1974 relationship also indicates that 53.5 percent of the irrigation season diversion volume returns to the Beaverhead reach during the same irrigation season.

Although the study demonstrates the potential of the transfer function methodology as a tool for return flow quantification, it also shows how that potential is severely limited by data inadequacies. Proper application of time series analysis to irrigation return flow systems requires reliable and sufficient records of daily streamflows and daily diversions.

INTRODUCTION

In the semiarid West the major water use is agricultural irrigation. A good water management plan must therefore properly quantify those volumes of water that are diverted, utilized, and returned in connection with the irrigation operation. Quantification of these various volume components is complicated by the fact that although diversion data may be available, the diverted discharge often greatly exceeds crop consumptive requirements. Consequently, an unknown portion of the water eventually percolates into the groundwater system and finally back into the stream adjacent to the irrigated area.

A state-of-the-art study by Nicklin and Brustkern (1981) determined that most current mathematical methodologies dealing with irrigation return flow analysis center around research-oriented computer models. Typically these models employ either a finite difference or finite element numerical solution procedure. The major data needs for such return flow models are groundwater levels, hydraulic conductivities, storage coefficients, aquifer boundary information, and surface water measurements.

A mathematical technique known as time series analysis offers a potential analysis approach for return flow quantification. Time series analysis involves the development of mathematical descriptions for data sets (such as streamflow hydrographs) that arise sequentially over time. Although no applications of time series analysis to irrigation return flows could be identified in the literature, numerous applications to other hydrologic time series have been documented. The approach has been particularly useful in forecasting stochastic events such as streamflow. Salas et al. (1980) provide a good review of traditional hydrologic uses of time series. Time series analysis is also widely used in other areas such as control theory for machines and electronic circuits.

The time series characterization of irrigation return flows as herein presented requires only surface water input data, some of which is often readily available. Streamflow records, in particular, are routinely gathered at numerous locations and constitute a generally reliable network of time series observations. Successful application of time series analysis procedures to this data base provides information on the timing and quantity of return flows. Such information is critical for planners who must make decisions relative to additional irrigation diversions and the impacts of changing irrigation practices upon downstream users.

This study investigates the potential of time series analysis as a tool for quantifying irrigation flows.

TIME SERIES ANALYSIS

Definition of Time Series

A time series is a collection of numerical observations arranged according to their order of occurrence. Although the data may in general arise sequentially over time or space, this study is concerned only with data that arises sequentially over time. Time series data can be characterized as having (1) deterministic properties, (2) random properties or (3) both deterministic and random properties. A time series has strictly deterministic properties if it can be exactly resynthesized. On the other hand, a time series with any random properties cannot be exactly resynthesized. Owing to the various possible outcomes, it can only be described by utilizing probability theory. Most hydrologic time series exhibit both deterministic and random characteristics.

The deterministic component of a hydrologic time series may consist of either or both of the following elements:

1. Trends—such as linear or quadratic changes in the level of a hydrologic series over time (for example, Figure 1 shows a deterministic trend and a random component), and,
2. Periodicities—such as cyclical variations in the level of a hydrologic series due primarily to seasonal meteorological changes (for example, Figure 2 shows a deterministic periodicity and a random component).

Random (or stochastic) properties in hydrologic data are due primarily to the host of uncertainties related to the meteorological cycle. Annual hydrologic measurements such as total precipitation often approximate strictly random processes.

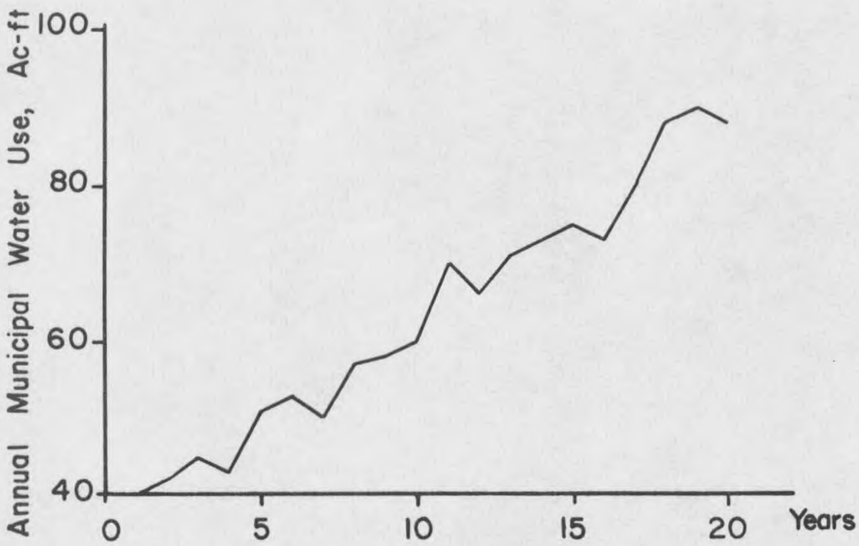

Figure 1. A conceptual municipal water use series provides an example of a time series exhibiting a trend plus a random component.

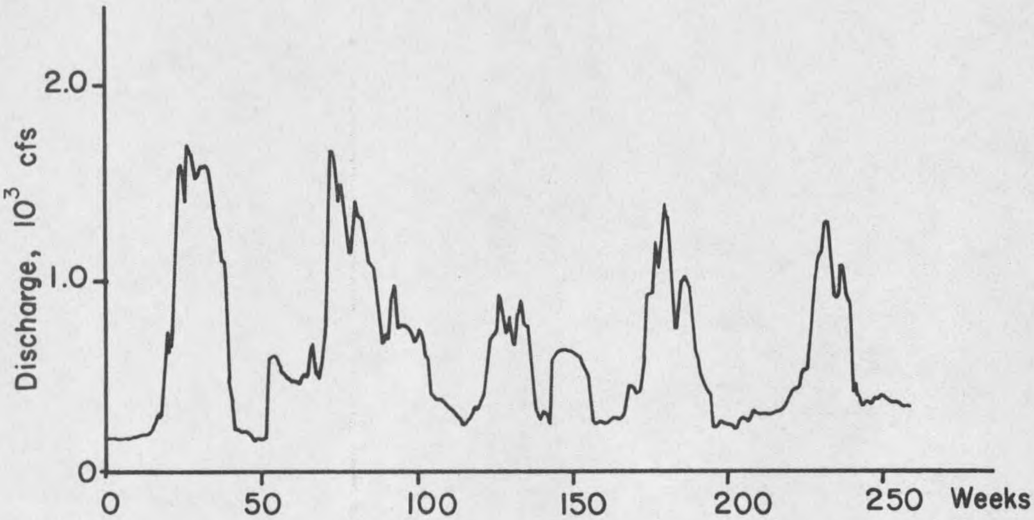

Figure 2. The stream discharge of the Beaverhead River at the Barretts gaging station provides an example of a time series exhibiting both periodic and random components.

Time series analysis generally requires that a mathematical description of the time series data be developed. Although time series processes may be continuous or discrete, a discrete time series description is generally employed (out of either necessity or convenience) to describe the hydrologic process. A discrete time series may evolve from a continuous time series process in two ways:

1. A discrete time series results from sampling a continuous time series at discrete intervals. For example, measurements of water levels in an observation water well at monthly increments result in a discrete time series from a continuous process of water level fluctuation.
2. A discrete time series also results from accumulating or averaging a continuous variable over discrete time intervals. A relevant example is provided by a record of streamflows. The raw data set is a continuous record of water level stages when viewed on a recorded strip chart. Using a stage-discharge relationship, the raw data set is converted to a discrete time series of mean daily flows.

Methodologies for Time Series Analysis

Two mathematical analyses have been particularly useful in the interpretation of time series data. One assesses the behavior of the data set in the time domain, time domain analysis, and the other assesses the behavior in the frequency domain, frequency domain (or spectral) analysis. This investigation concentrates on time domain analysis procedures.

In time domain analysis, a model of the time series is generally developed. Although the model identification and development processes, which will be discussed later, are mechanical and methodical, a physical understanding of the time series process is an invaluable aid in model identification.

Assessment of a single time series record is often termed univariate time series analysis. Assessment of the relationship between two dependent time series records is frequently termed transfer function analysis of a bivariate process. Statistical descriptions of univariate time series will be discussed in the section that follows. Later, transfer function methods pertinent to bivariate processes will be presented.

Statistical Descriptions of a Univariate Time Series

A time series sample record over a given interval is often termed a realization of the underlying process. The collection of all possible records over the given interval of time is usually designated as the ensemble (analogous to the sample space in probability theory). The analysis of a univariate time series sample (realization) provides information regarding interdependencies within the series.

The three most important parameters generally used to characterize a stationary (stationarity is defined later) stochastic time series are the mean, variance, and autocovariance. The expected value of a time series, X(t), is its mean, μ

$$E[X(t)] = \mu \tag{1}$$

The expected value of the second moment about the mean is the time series variance, σ^2

$$E[\left\{X(t) - \mu\right\}^2] = \sigma^2 \tag{2}$$

Furthermore, the expected value of the autocovariance between a time series, X(t), and the same time series displaced τ time units, $X(t+\tau)$, is defined as

$$E[\left\{X(t) - \mu\right\}\left\{X(t+\tau) - \mu\right\}] = \gamma_\tau \tag{3}$$

Inspection of Equation (3) reveals that $\gamma_0 = \sigma^2$ when $\tau = 0$. The time series sample statistics for the mean, variance, and autocovariance are respectively

$$\hat{\mu} = \frac{1}{N} \sum_{t=1}^{N} x(t) \tag{4}$$

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{t=1}^{N} \left\{ x(t) - \hat{\mu} \right\}^2 \tag{5}$$

$$\hat{\gamma}_\tau = \frac{1}{N} \sum_{t=1}^{N-\tau} \left\{ x(t) - \hat{\mu} \right\} \left\{ x(t+\tau) - \hat{\mu} \right\} \tag{6}$$

The use of upper case and lower case letters will designate population and sample variables respectively for the remaining discussion.

Autocorrelation Function

Dividing Equation (3) by Equation (2) yields the autocorrelation coefficient, ρ_τ, between pairs of time series values that are separated by τ time units,

$$\rho_\tau = \frac{\gamma_\tau}{\sigma^2} \tag{7}$$

and as previously noted $\sigma^2 = \gamma_0$ for $\tau = 0$. Therefore

$$\rho_\tau = \frac{\gamma_\tau}{\gamma_0} \tag{8}$$

The sample estimate of the autocorrelation coefficient for time displacement τ is

$$\hat{\rho}_\tau = \frac{\hat{\gamma}_\tau}{\hat{\gamma}_0} \tag{9}$$

For a given time series sample, the collection of autocorrelation coefficients for all possible time displacements is referred to as the autocorrelation function. The graphical representation of the autocorrelation function is sometimes referred to as the correlogram. The autocovariance and therefore the autocorrelation function is a second order property of a time series. The mean is a first order property and the variance is of course a second order property. As can be seen from Equations (6) and (8), the autocorrelation function may range from 1 to −1 (negative values resulting from negative correlations) with a maximum of one always occurring at $\tau = 0$. The autocorrelation function is symmetric about zero.

The shape of the graphed autocorrelation function serves as a guide in choosing an appropriate statistical model for characterizing a time series data set. That is it identifies the similarities between the realization, x(t), and the realization shifted τ units. For example, a realization with the strongest positive dependencies between the most closely spaced values and progressively weaker dependencies between more widely separated values will, for small lags, exhibit large autocorrelation coefficients, and, for large lags, exhibit small autocorrelation coefficients. Thus, the autocorrelation function will decay in this case as τ is increased. A typical autocorrelation function for such a dependence is shown in Figure 3. Because of the function's previously mentioned symmetry, it is customary to represent the autocorrelation function to the right of the origin only. For a realization with no significant time dependencies between neighboring values, the autocorrelation coefficient will be insignificant even at small lags.

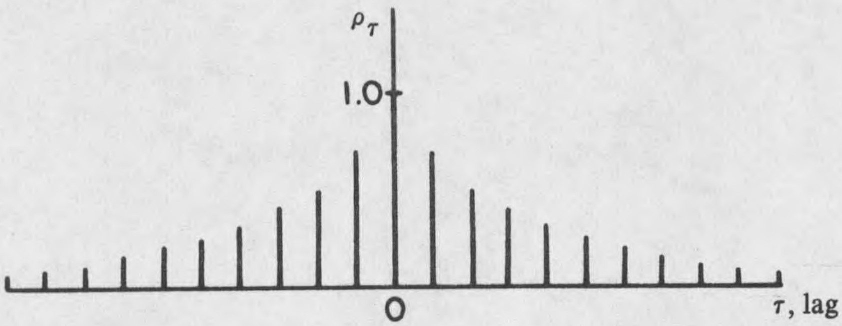

Figure 3. An example of an autocorrelation function.

Partial Autocorrelation Function

Another parameter useful in model identification is the partial autocorrelation coefficient. It is often important to measure the autocorrelation between the dependent variable, X(t), and a particular independent variable, X(t+τ), with the effects of intervening variables removed. For example, if X(t) is significantly autocorrelated to X(t+1) and

X(t+1) is significantly autocorrelated to X(t+2), the result may well be a significant autocorrelation between X(t) and X(t+2) that is reflected in the autocorrelation coefficient ρ_2. The partial autocorrelation coefficient removes the effect of the intervening variable X(t+1) on the autocorrelation between X(t) and X(t+2), and thus provides an indication of the direct dependency between X(t) and X(t+2). More generally, the partial autocorrelation coefficient between X(t) and X(t+τ) removes the effects of the intervening variables X(t+1), X(t+2), . . ., X(t+τ−1).

The partial autocorrelation coefficient for a realization and the same realization displaced τ time units, $\hat{\rho}_{\tau\tau}$, may be estimated from

$$\hat{\rho}_{\tau\tau} = \begin{cases} \hat{\rho}_1 & \text{if } \tau = 1 \quad \text{(no intervening variables)} \\ \dfrac{\hat{\rho}_\tau - \sum_{j=1}^{\tau-1} \hat{\rho}_{\tau-1,j}\,\hat{\rho}_{\tau-j}}{1 - \sum_{j=1}^{\tau-1} \hat{\rho}_{\tau-1,j}\,\hat{\rho}_j} & \text{if } \tau = 2,3,\ldots \end{cases} \qquad (10)$$

where $\hat{\rho}_{\tau j} = \hat{\rho}_{\tau-1,j} - \hat{\rho}_{\tau\tau}\,\hat{\rho}_{\tau-1,\tau-j}$ for $j = 1,2,\ldots,\tau-1$

Note that $\hat{\rho}_\tau$ is the sample autocorrelation between the variable x(t) and the variables displaced τ units, x(t+τ).

The collection of partial autocorrelation coefficients for all possible time displacements is referred to as the partial autocorrelation function. The shape of the graphed partial autocorrelation function is a guide in choosing a statistical model for a given time series. For example, if X(t) is significantly autocorrelated with X(t+1) but not with X(t+2), then the partial autocorrelation function will indicate a significant partial autocorrelation only at a displacement of one as shown in Figure 4.

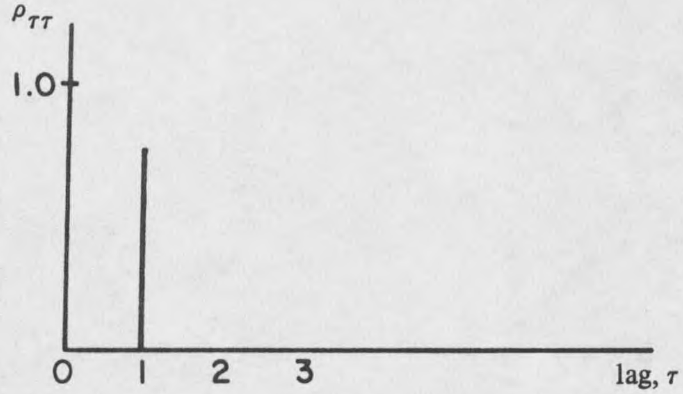

Figure 4. A typical partial autocorrelation function for a time series with a time dependency at lag 1 only.

Stationarity

A stochastic time series is stationary if its statistical properties do not change with time. Strict stationarity of a stochastic time series requires that statistical properties of all orders do not change over time. For example, strict stationarity for a normally distributed time series would require that the mean, variance, and autocovariance not change with time. For time series that are not normally distributed, and thus cannot be completely characterized by the mean, variance and autocovariance, strict stationarity requires constancy of the remaining characterization parameters.

Fortunately in most circumstances, strict stationarity is not a requirement for developing an adequate model. Stationarity of order two is usually considered sufficient. The normally distributed time series of the previous paragraph is completely characterized by the first and second order parameters of the mean, variance and autocovariance and thus is not only stationary to order two but also strictly stationary. For other distributions, constancy of the mean, variance and autocovariance would indicate the required second order stationarity but not strict stationarity.

The mean, variance, and autocovariance are statistics utilized in the determination of the autocorrelation function. Consequently, the autocorrelation function is useful in assessing the stationarity of a time series data set. If the autocorrelation function fails to dampen or cut off quickly to zero, nonstationarity in the first and/or second order statistics is the usual cause. Another set of conditions implied by the concept of stationarity is described in a later section.

Stationarity of a time series data set is a requirement for developing a stochastic model of that data set. However, as will be shown later, there are effective ways to transform nonstationary data sets into stationary forms.

Models for Univariate Time Series

A group of models which often provides parsimonious representations of univariate time series are discussed in this section. The models described herein apply to either (1) stationary time series or (2) nonstationary time series transformable to stationary time series (the mechanics of this transformation are described later in this section). Model selection is based on the modeler's knowledge of the physical process and on his interpretation of the sample autocorrelation and partial autocorrelation functions of the process time series.

Moving Average Models

Based on an idea by Yule (1927), Box and Jenkins (1976) have developed a mathematical model that is useful in characterizing certain stochastic time series. The time series, Z(t), is regarded as a set of successive highly dependent values which are generated from a series of independent and randomly distributed variables, A(t), often termed white noise. By definition, the mean of a white noise process is zero and the variance is constant, or

$$E[A(t)] = \mu_A = 0 \qquad (11)$$

$$E[\{A(t) - \mu_A\}^2] = \sigma_A^2 = C \tag{12}$$

Furthermore, the autocovariance of a white noise series is zero,

$$E[A(t)\,A(t+\tau)] = 0 \tag{13}$$

for lags τ not equal to zero. The time series (and model) may be represented as

$$Z(t) = \mu + A(t) - \theta_1 A(t-1) - \theta_2 A(t-2) - \ldots - \theta_q A(t-q) \tag{14}$$

The time series is thus dependent upon a "moving average" involving current and previous values from the white noise series. The previous values are weighted by the θ coefficients. The order of the moving average is indicated by q. A moving average model of order q is often abbreviated as MA(q).

Equation (14) can be more concisely written as

$$Z(t) = \mu + \theta^q(B)\,A(t) \tag{15}$$

where

$$\theta^q(B) = -\theta_0 B^0 - \theta_1 B^1 - \theta_2 B^2 - \ldots - \theta_q B^q$$

and by definition $\theta_0 = -1$. The symbol B is a backstep operator. It serves to transform A(t) as follows

$$B^1 A(t) = A(t-1)$$

$$B^2 A(t) = A(t-2)$$

$$\vdots$$

$$B^q A(t) = A(t-q)$$

Another form of Equation (15) is

$$\tilde{Z}(t) = \theta^q(B)\,A(t) \tag{16}$$

where $\tilde{Z}(t) = Z(t) - \mu$. $\tilde{Z}(t)$ then represents the deviation of the time series from its mean, μ.

A time series that results from a moving average model of order q is always stationary. First order stationarity is easily demonstrated. The expected value of a white noise process equals zero or

$$E[A(t)] = \mu_A = 0$$

Thus, since $\theta^q(B)$ is a constant operator,

$$E[\theta^q(B)\,A(t)] = \theta^q(B)\,E[A(t)]$$

and consequently

$$E[\theta^q(B)\,A(t)] = 0.$$

Second order stationarity may be shown by forming the product of $\tilde{Z}(t)\tilde{Z}(t-\tau)$ and taking expectations term by term,

$$E[\tilde{Z}(t)\tilde{Z}(t-\tau)]$$

$$= E[\left\{1 - \theta_1 B^1 - \theta_2 B^2 - \theta_3 B^3 - \ldots - \theta_q B^q\right\} A(t) \cdot \left\{1 - \theta_1 B^1 - \theta_2 B^2 - \theta_3 B^3 - \ldots - \theta_q B^q\right\} A(t-\tau)]$$

$$= E[\left\{1 - \theta_1 B^1 - \theta_2 B^2 - \theta_3 B^3 - \ldots - \theta_q B^q\right\} A(t) \cdot \left\{B^{\tau} - \theta_1 B^{\tau+1} - \theta_2 B^{\tau+2} - \theta_3 B^{\tau+3} - \ldots - \theta_q B^{\tau+q}\right\} A(t)] \quad (17)$$

The expectation of the cross products involving the independently distributed white noise variables is zero at displacements $\tau \neq 0$ (recall Equation (13)). Thus for $\tau = 1$, for example,

$$E[\tilde{Z}(t)\tilde{Z}(t-1)] = \sigma_A^2(-\theta_1 + \theta_1\theta_2 + \theta_2\theta_3 + \ldots + \theta_{q-1}\theta_q)$$

Generalizing, the autocovariance is

$$\gamma_\tau = \sigma_A^2 \sum_{j=0}^{q-\tau} \theta_j \theta_{j+\tau} \quad (18)$$

Recall that $\theta_0 = -1$. When $\tau = 0$, Equation (17) yields

$$\gamma_0 = \sigma_A^2(1 + \theta_1{}^2 + \theta_2{}^2 + \theta_3{}^2 + \ldots + \theta_q{}^2)$$

and so the variance may be written

$$\sigma^2 = \gamma_0 = \sigma_A^2 \sum_{j=0}^{q} \theta_j^2 \qquad (19)$$

Equations (18) and (19) demonstrate that the autocovariance and variance are independent of time and thus stationarity to order two of moving average processes is justified.

An important requirement for moving average processes is the requirement of invertibility. Although invertibility applies only to moving average processes, its importance is most easily demonstrated in the section that follows.

<u>Autoregressive Models</u>

A moving average process $\tilde{Z}(t)$ may also be characterized in terms of a current white noise disturbance, A(t), and all past time series observations of $\tilde{Z}(t)$. For example, a moving average process of order one,

$$\tilde{Z}(t) = (1 - \theta_1 B^1)\, A(t) \qquad (20)$$

may be rewritten as

$$\frac{1}{1 - \theta_1 B^1}\, \tilde{Z}(t) = A(t)$$

Dividing, yields

$$(1 + \theta_1 B^1 + \theta_1^2 B^2 + \theta_1^3 B^3 + \ldots)\, \tilde{Z}(t) = A(t) \qquad (21)$$

Operating on $\tilde{Z}(t)$ and rearranging gives

$$\tilde{Z}(t) = - \theta_1 \tilde{Z}(t-1) - \theta_1^2 \tilde{Z}(t-2) - \ldots + A(t) \qquad (22)$$

This relationship is termed an autoregressive process as the variable $\tilde{Z}(t)$ is regressed on past observations of itself. Thus a moving average process of order one may be transformed into an autoregressive process of infinite order. The moving average process is said to be invertible if the infinite series converges. As can be seen from the equation, convergence will occur only if $|\theta_1| \not> 1$. Consequently not all moving average processes of order one are invertible. If $|\theta_1| > 1$, the current deviation $\tilde{Z}(t)$ depends upon events $\tilde{Z}(t-1)$, $\tilde{Z}(t-2)$, . . .

with event weights increasing as events become more far removed in time. This is physically unreasonable and so invertibility is required.

Using Box and Jenkins' notation, Equation (21) may be written

$$\phi(B)\,\tilde{Z}(t) = A(t) \tag{23}$$

where

$$\phi(B) = (1 + \theta_1 B^1 + \theta_1{}^2 B^2 + \ldots).$$

Autoregressive processes may be more generally expressed as

$$\phi^p(B)\,\tilde{Z}(t) = A(t) \tag{24}$$

where

$$\phi^p(B) = -\phi_0 B^0 - \phi_1 B^1 - \phi_2 B^2 - \ldots - \phi_p B^p$$

and

$$\phi_0 = -1$$

Equation (24) may also be written as

$$\tilde{Z}(t) = \phi_1 \tilde{Z}(t-1) + \phi_2 \tilde{Z}(t-2) + \ldots + \phi_p \tilde{Z}(t-p) + A(t) \tag{25}$$

The time series is thus dependent upon previous values of the time series weighted by the ϕ coefficients and the current white noise value A(t). The order of the autoregressive process is indicated by p. An autoregressive process of order p is often abbreviated AR(p).

A comparison of Equation (20), an autoregressive form, and (23), a moving average form, indicates that it is parsimoniously unwise to select an autoregressive model to represent a process which could be sufficiently explained by a moving average model of low order (just as it will be shown parsimoniously unwise to use a moving average model to represent a process which is sufficiently modeled by an autoregressive model of low order).

The invertibility condition may be extended to a moving average process of any order. Consider a moving average of order q

$$\tilde{Z}(t) = A(t) - \theta_1 A(t-1) - \ldots - \theta_q A(t-q)$$

or

$$\tilde{Z}(t) = \theta(B)\, A(t)$$

where $\theta(B) = \theta^q(B)$. It has already been shown that a first order moving average process is invertible if $|\theta_1| < 1$, or equivalently,

$$\pi(B) = (1 - \theta_1 B)^{-1} = \sum_{j=0}^{\infty} \theta_1{}^j B^j$$

converges. That is equivalent to saying that the root $B = \theta^{-1}$ of the equation $1 - \theta B = 0$ (the characteristic equation) lies outside of the unit circle (if $|\theta_1| < 1$ for invertibility, then $|1/\theta_1| > 1$). By similar reasoning, the invertibility condition for a MA(q) process requires that the roots of the characteristic equation

$$1 - \theta_1 B^1 - \theta_2 B^2 - \theta_3 B^3 - \ldots - \theta_q B^q = 0$$

lie outside the unit circle for invertibility.

Autoregressive processes are always invertible because the series

$$\phi^p(B) = 1 - \phi_1 B^1 - \phi_2 B^2 - \ldots - \phi_p B^p$$

is finite. However, an autoregressive process may or may not be stationary. Stationarity of an autoregressive process requires certain conditions on the ϕ coefficients in Equation (25). To insure stationarity for an autoregressive process of order p,

$$\phi^p(B)\, \tilde{Z}(t) = A(t)$$

where

$$\phi^p(B) = 1 - \phi_1 B^1 - \phi_2 B^2 - \ldots - \phi_p B^p$$

the roots of the characteristic equation

$$1 - \phi_1 B^1 - \phi_2 B^2 - \ldots - \phi_p B^p = 0$$

must lie outside of the unit circle. For illustration of the importance of this requirement, an autoregressive process of order one, AR(1), may be written as

$$\tilde{Z}(t) = \phi_1 \tilde{Z}(t-1) + A(t) \qquad (26)$$

or

$$(1 - \phi_1 B)\tilde{Z}(t) = A(t)$$

Rearranging

$$\tilde{Z}(t) = \frac{1}{1-\phi_1 B} A(t)$$

and then dividing yields

$$\tilde{Z}(t) = (1 + \phi_1 B^1 + \phi_1{}^2 B^2 + \ldots) A(t) \qquad (27)$$

The stationarity requirement for an AR(1) process is that the absolute value of ϕ_1 be less than one (or alternatively that the root of the characteristic equation $1 - \phi_1 B = 0$ be outside the unit circle). Note that when $|\phi_1| > 1$, the series is infinite and not convergent and the process is nonstationary.

A comparison of Equation (26), an autoregressive form, and (27), a moving average form, indicates that it is parsimoniously unwise to select a moving average model to represent a process which could be sufficiently explained by an autoregressive model of low order.

Mixed Autoregressive and Moving Average Models

Sometimes it is parsimoniously advantageous to use mixed autoregressive moving average models of order p and q (or ARMA(p,q)) to model certain processes. Such models often require fewer parameters for adequate description of stochastic processes. ARMA models with order $p \leqslant 2$ and $q \leqslant 2$ are adequate for many processes. An ARMA process (and model) with orders p and q may be written as