Convergence structures on homeomorphism groups
by Wayne Richard Park

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Mathematics
Montana State University

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In Chapter IV, X is given a uniform convergence structure. Uniform convergence structures are then constructed on subgroups of H(X).

In each case, the question is asked whether these uniform convergence structures induce convergence group structures on the underlying ho-meomorphism subgroup.

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ABSTRACT

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INTRODUCTION

In 1959 H. Fischer formalized the concept of a convergence structure as a generalization of a topology. The generalization is a good one in that many of the fundamental ideas of topology carry over to this new category, and in addition, as the category is so much fuller, we can find convergence structures satisfying certain properties for which no such topologies exist.

The primary intent of this work is to show similar results with respect to homeomorphism groups. R. Arens has analyzed the properties of various topologies on the group of homeomorphisms, H(X), of a topological space X. Basically, meaningful results are obtained only after the assumption that the topological space X is at least locally compact and Hausdorff. In this work, by beginning with the assumption that X is given just an arbitrary convergence structure and considering homeomorphisms in this new category, we analyze convergence structures on the new H(X). In particular, we construct a convergence structure σ on H(X) which is the coarsest of the admissible convergence group structures on H(X). By an admissible convergence group structure we mean the convergence structure guarantees the continuity of the group operations and the evaluation mapping. Properties of this σ convergence structure are analyzed and compared with those of γ_c, the convergence structure of continuous convergence. A counterexample using the rational numbers shows that these two convergence structures are distinct.

The topological generalization of a homeomorphism group is a topological transformation group on a topological space X. We define the concept of a convergence transformation group on a convergence space X and obtain a characterization of these groups by a mapping property. Finally we define a convergence structure on a convergence transformation group in an analogous way to that of σ on H(X). It is shown that this convergence structure has properties similar to those of σ in this more general setting.

In the last chapter we let X be an arbitrary uniform convergence space. We construct uniform convergence structures on both the homeomorphism group H(X) and on U(X), the subgroup of uniformly continuous homeomorphisms of X. These uniform convergence structures are analyzed with respect to the question of whether they induce convergence group structures on the underlying homeomorphism group.

CHAPTER I

Let $(S,\leq)$ be a partially ordered set with the additional property that every subfamily $\{F_\alpha\}$ of S has an infimum $\bigwedge_\alpha F_\alpha$ in S. A <u>$\wedge$-ideal</u> is a subfamily T of S which satisfies:

1. $F, F' \in T \Rightarrow F \wedge F' \in T$, and

2. $F \in T$, $F' \geq F \Rightarrow F' \in T$.

A (proper) <u>filter</u> F on a non-empty set X is a $\wedge$-ideal in $(P(X), \subseteq)$ with the additional requirement that $\emptyset \notin F$. A <u>filter base</u> B is a family of subsets of X satisfying

1. $B, B' \in \mathcal{B} \Rightarrow$ there exists a $B'' \in \mathcal{B}$ such that $B'' \subseteq B \cap B'$,

and

2. $\emptyset \notin \mathcal{B}$.

We say that $\mathcal{B}$ generates a filter F if $\mathcal{B} \subseteq F$ and for every $F \in F$ there exists a $B \in \mathcal{B}$ such that $B \subseteq F$, and in this case we write $F = [\mathcal{B}]$. The family of filters F(X) on a non-empty set X forms a partially ordered set with infima with $\leq$ defined by $F \leq F' \Leftrightarrow F \subseteq F'$. If $\{F_\alpha\}$ is an arbitrary family of filters on X, then $\bigwedge_\alpha F_\alpha$ is just $\bigcap_\alpha F_\alpha$ and if for each α, $\mathcal{B}_\alpha$ is a filter base for F_α, then $\bigwedge_\alpha F_\alpha$ is generated by the filter base $\{\bigcup B_\alpha \mid B_\alpha \in \mathcal{B}_\alpha\}$.

A <u>convergence structure</u> τ on a non-empty set X is defined if for each $x \in X$ there is assigned a $\wedge$-ideal, τx, of filters of subsets of X with the added requirement that the filter of supersets of the point x, denoted by $\overset{o}{x}$, also is a member of τx. If X has a convergence structure τ, then (X,τ) is called a convergence space. If the filter $\mathcal{F} \in \tau x$, we say $\mathcal{F}$ converges to x.

If τ, τ' are two convergence structures on X, we say τ is coarser than τ', ($\tau \leq \tau'$), or τ' is finer than τ if for each $x \in X$, $\tau' x \subseteq \tau x$. With this, the family of convergence structures on X forms a complete lattice. [7, p.275].

If τ is a convergence structure on X, let $\psi\tau$ be the convergence structure on X defined by $\psi\tau x = \{\mathcal{F}' \in F(X) \mid \mathcal{F}' \geq \bigwedge_{\mathcal{F}\in\tau x} \mathcal{F}\}$. These convergence structures are called principal and the corresponding spaces are called principal convergence spaces. Every topology T on X defines a convergence structure τ_T on X, namely, $\tau_T\, x = \{\mathcal{F} \in F(X) \mid \mathcal{F} \geq \mathcal{V}(x)\}$ where $\mathcal{V}(x)$ is the neighborhood filter of x defined by the topology T. It should be noted if $T \leq T'$ for two topologies T and T', then $\tau_T \leq \tau_{T'}$. For every convergence structure τ on X there is a finest topology $\bar{\omega}\tau$ which is coarser than τ. Namely, consider those sets A in X such that for each $x \in A$, $A \in \mathcal{F}$ for each $\mathcal{F} \in \tau x$. These sets are called the τ-open sets and for each $x \in X$, they generate a neighborhood filter $\mathcal{V}(x)$. So for each $x \in X$, $\bar{\omega}\tau x = \{\mathcal{F} \in F(X) \mid \mathcal{F} \geq \mathcal{V}(x)\}$. It is clear that for each convergence structure τ on X we have the relations $\bar{\omega}\tau \leq \psi\tau \leq \tau$.

If $f\colon X \to Y$ is an arbitrary mapping between the non-empty sets X and Y, then for any $\mathcal{F} \in F(X)$, $\{f(F) \mid F \in \mathcal{F}\}$ is a filter base on Y. Hence, let $f(\mathcal{F})$ denote the filter that this filter base generates. A function $f\colon (X,\tau) \to (Y,\tau')$ between convergence spaces is said to be continuous if for each $x \in X$, $f(\tau x) \subseteq \tau' f(x)$. A bi-continuous bijection between convergence spaces is called a homeomorphism.

If $\mathcal{F}$, $\mathcal{F}'$ are filters on X, X' respectively, $\mathcal{F} \times \mathcal{F}'$ is the filter on $X \times X'$ generated by the filter base $\{F \times F' \mid F \in \mathcal{F}, F' \in \mathcal{F}'\}$. If (X, τ) and (X', τ') are two convergence spaces and $\mathcal{G}$ is a filter on $X \times X'$, the product convergence structure $\tau \times \tau'$ on $X \times X'$ is given by:

$$\mathcal{G} \in (\tau \times \tau')(x,x') \iff \begin{cases} p_1(\mathcal{G}) \in \tau x, \text{ and} \\ p_2(\mathcal{G}) \in \tau' x'. \end{cases}$$

where p_1 and p_2 are the projection mappings. It is easy to see that the continuity of a mapping $f\colon (X \times X', \tau \times \tau') \to (X'', \tau'')$ is equivalent to the condition:

$$\left.\begin{array}{l} \mathcal{F} \in \tau x \\ \mathcal{F}' \in \tau' x' \end{array}\right\} \Rightarrow f(\mathcal{F} \times \mathcal{F}') \in \tau'' f(x,x') \text{ for all } (x,x') \in X \times X'.$$

The usual properties of products, valid for topologies, hold also in this more general setting. For a more detailed analysis see [7, p.290-291].

If (X,τ) and (Y,τ') are convergence spaces and Y^X denotes the family of all continuous functions from X to Y, then for any $H \subseteq Y^X$, the convergence structure of continuous convergence γ_c is defined on H by $\mathcal{F} \in \gamma_c f \iff$ for all $x \in X$ and for all $\Phi \in \tau x$, $\mathcal{F}(\Phi) = \omega(\mathcal{F} \times \Phi) \in \tau' f(x)$ where $\omega\colon H \times X \to X$ is the evaluation mapping defined by $\omega(f,x) = f(x)$. It should be noted that inherent in this definition is the minimum of conditions to guarantee the continuity of ω.

For an extensive analysis of the properties of γ_c see [4] and [2].

Let X be a non-empty set. If Δ represents the set $\{(x,x) \mid x \in X\}$ then $[\Delta]$ is the filter of all supersets of Δ in $X \times X$. If F and G are subsets of $X \times X$, let $F^{\Delta} = \{(y,x) \mid (x,y) \in F\}$ and also, $F \circ G = \{(x,y) \mid$ there exists a $z \in X$ such that $(x,z) \in F$ and $(z,y) \in G\}$ (note that composition reads left to right). Then for filters $\mathcal{F}$ and $\mathcal{G}$ on $X \times X$, let $\mathcal{F}^{\Delta} = [\{F^{\Delta} \mid F \in \mathcal{F}\}]$ and also $\mathcal{F} \circ \mathcal{G} = [\{F \circ G \mid F \in \mathcal{F}, G \in \mathcal{G}\}]$. For this latter filter to exist it is clear that $F \circ G$ must be non-empty for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$.

A <u>uniform convergence structure</u> $\mathcal{J}$, is a $\wedge$-ideal of filters on $X \times X$ satisfying:

1. $[\Delta] \in \mathcal{J}$,

2. $\mathcal{F} \in \mathcal{J} \Rightarrow \mathcal{F}^{\Delta} \in \mathcal{J}$, and

3. $\mathcal{F}, \mathcal{G} \in \mathcal{J} \Rightarrow \mathcal{F} \circ \mathcal{G} \in \mathcal{J}$, whenever this latter filter exists. The uniform convergence structure $\mathcal{J}$ induces a convergence structure $\tau_{\mathcal{J}}$ on X by the definition:

$$\mathcal{F} \in \tau_{\mathcal{J}} x \Leftrightarrow \mathcal{F} \times \overset{\circ}{x} \in \mathcal{J}.$$

$(X, \tau_{\mathcal{J}})$ is then said to be a <u>uniform convergence space</u>. The family of uniform convergence structures on X forms a complete lattice where $\mathcal{J} \leq \mathcal{J}' \Leftrightarrow \mathcal{J} \subseteq \mathcal{J}'$. If $\{\mathcal{J}_\alpha\}$ is a family of uniform convergence structures on X, then $\tau_{\sup_\alpha \mathcal{J}_\alpha} = \sup_\alpha \tau_{\mathcal{J}_\alpha}$. Finally, a function $f: (X, \tau_{\mathcal{J}}) \to (Y, \tau_{\mathcal{J}'})$ is said to be <u>uniformly continuous</u> if $(f \times f)\, \mathcal{J} \subseteq \mathcal{J}'$.

If (X,τ) is a convergence space and $(Y,\tau_{\mathcal{J}})$ a uniform convergence space with $\tau_{\mathcal{J}}$ induced from the uniform convergence structure $\mathcal{J}$, then for any non-empty $F \subseteq Y^X \times Y^X$ and C some non-empty subset of X, let

$[F,C] = \{(f(x),f'(x)) \mid (f,f') \in F,\ x \in C\}$.

Moreover, for any filter $\mathcal{F} \in F(Y^X \times Y^X)$, let $[\mathcal{F},C]$ be the filter in $Y \times Y$ that is generated by $\{[F,C] \mid F \in \mathcal{F}\}$. Now let μ_C be the family of filters on $Y^X \times Y^X$ defined by

$\mathcal{F} \in \mu_C \iff [\mathcal{F},C] \in \mathcal{J}$.

μ_C is a uniform convergence structure on Y^X and is called the <u>uniform convergence structure of uniform convergence on C</u>. If $\mathcal{C}$ is a non-empty family of non-empty subsets of X, let $\mu_{\mathcal{C}} = \sup_{C \in \mathcal{C}} \mu_C$. $\mu_{\mathcal{C}}$ is then called the <u>uniform convergence structure of uniform convergence on the collection $\mathcal{C}$</u>. If $\mathcal{C} = \{\{x\} \mid x \in X\}$, $\mu_{\mathcal{C}}$ is called the <u>uniform convergence structure of simple convergence</u>. Here $\mathcal{F} \in \tau_{\mu_{\mathcal{C}}} f \iff$ $\mathcal{F}(x) = \omega(\mathcal{F} \times \{x\}) \in \tau_{\mathcal{J}} f(x)$ for all $x \in X$.

For more detail on the subject of uniform convergence structures, see [5].

CHAPTER II

We first observe some basic results concerning the lattice of topologies on function sets that are also valid in the lattice of convergence structures.

Let X and Y be convergence spaces and let Y^X denote the set of all continuous functions from X to Y. A convergence structure on Y^X is said to be conjoining if for every convergence space Z, the continuity of $\hat{\alpha}: Z \to Y^X$ implies the continuity of $\alpha: Z \times X \to Y$ where $\alpha(z,x) = \hat{\alpha}(z)(x)$. Secondly, a convergence structure on Y^X is said to be splitting if for every convergence space Z, the continuity of $\alpha: Z \times X \to Y$ implies the continuity of $\hat{\alpha}: Z \to Y^X$. A convergence structure on Y^X is said to be admissible if the evaluation mapping $\omega: Y^X \times X \to Y$ ($\omega(f,x) = f(x)$) is continuous.

For the following four theorems, convergence structures apply to the set Y^X.

Theorem 1: A convergence structure is conjoining if and only if that structure is admissible.

Theorem 2: A convergence structure which is finer than a conjoining structure is conjoining.

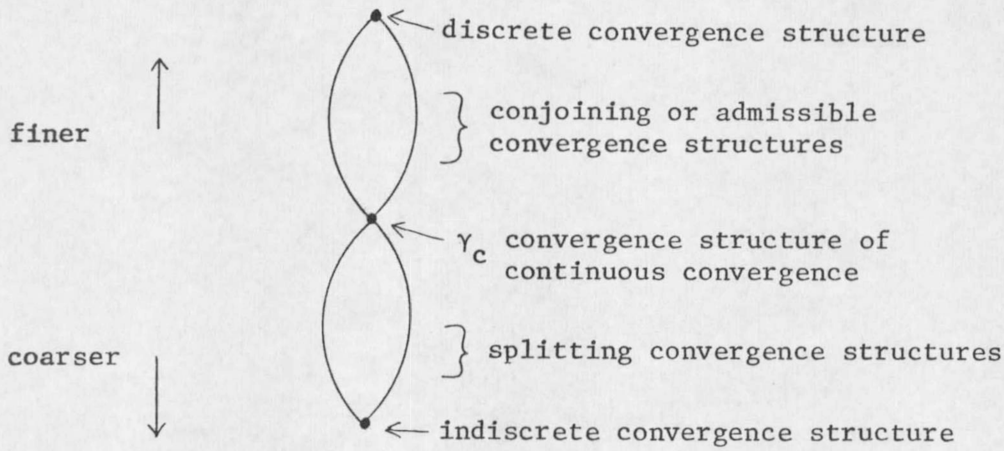
Theorem 3: A convergence structure which is coarser than a splitting structure is splitting.

Theorem 4: Any conjoining convergence structure is finer than any splitting structure.

The proofs of the above theorems are virtually identical to the proofs given in [6,p.274-275]. We demonstrate the technique for Theorem 1 only:

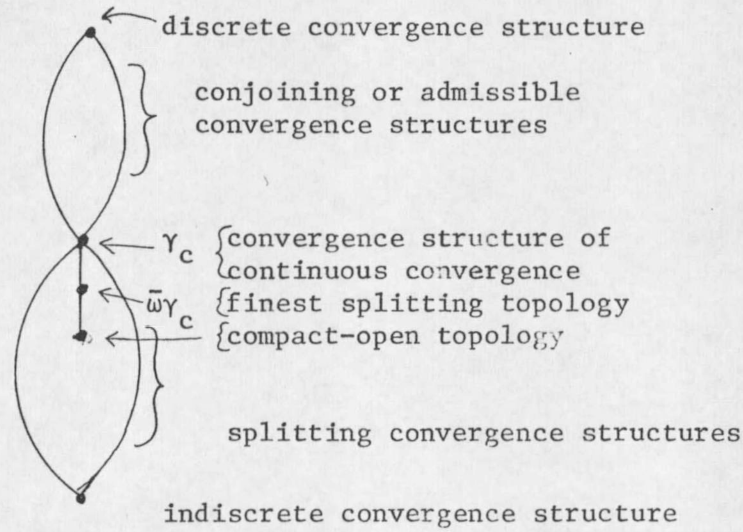
Proof: As $\omega: Y^X \times X \to Y$ is given by $\omega(f,x) = f(x)$, we have $\hat{\omega}: Y^X \to Y^X$ as the identity mapping since $\hat{\omega}(f)(x) = \omega(f,x) = f(x)$. Assuming then that τ is a conjoining convergence structure on Y^X, the continuity of $\hat{\omega}:(Y^X, \tau) \to (Y^X, \tau)$ as the identity mapping implies the continuity of $\omega: Y^X \times X \to Y$. Conversely, if ω is continuous and if $\hat{\alpha}: Z \to Y^X$ is continuous for some convergence space Z, then the composition of $Z \times X \xrightarrow{\hat{\alpha} \times id} Y^X \times X \xrightarrow{\omega} Y$ is continuous. But this composition is precisely $\alpha: Z \times X \to Y$ showing that τ is a conjoining structure.

The γ_c convergence structure of continuous convergence is both splitting and conjoining [2,Satz 2, p.7], so with the theorems above, γ_c is characterized as the unique finest splitting and coarsest conjoining or admissible convergence structure on Y^X. As the discrete convergence structure (topology) is conjoining and the indiscrete convergence structure (topology) is splitting, we can schematically represent these results as follows:



In the case X and Y are topological spaces we know first that the compact-open topology is always splitting and secondly, there is always a finest splitting topology for Y^X. This gives rise to the following diagram:

X,Y topological spaces



It is well known that when X is locally compact then $\gamma_c = \bar{\omega}\gamma_c$ = compact-open topology [6,p.275]. In general, γ_c is not a topology when X and Y are topological spaces.

CHAPTER III

Let H(X) represent the group of homeomorphisms of the convergence space (X,τ).

Definition: For each f in H(X) let σf consist of all filters $\mathcal{F}$ on H(X) such that:

1) for all $x \in X$ and for all $\Phi \in \tau x$, $\mathcal{F}(\Phi) \in \tau f(x)$, and
2) for all $x \in X$ and for all $\Phi \in \tau x$, $\mathcal{F}^{-1}(\Phi) \in \tau f^{-1}(x)$.

Here $\mathcal{F}(\Phi) = \omega(\mathcal{F} \times \Phi)$ where $\omega\colon H(X) \times X \to X$ is the evaluation mapping, and $\mathcal{F}^{-1}$ is the filter with filter base $\{F^{-1} \mid F \in \mathcal{F}\}$ where $F^{-1} = \{f^{-1} \in H(X) \mid f \in F\}$.

Theorem 5: σ is a convergence structure on H(X) making $(H(X),\sigma)$ a convergence group.

Proof: A: $\overset{o}{f} \in \sigma f$.

1) $\overset{o}{f}(\Phi) = f(\Phi) \in \tau f(x)$ as f is continuous.

2) $(\overset{o}{f})^{-1}(\Phi) = f^{-1}(\Phi) \in \tau f^{-1}(x)$ as f^{-1} is continuous.

B: $\mathcal{G} \geq \mathcal{F} \in \sigma f \Rightarrow \mathcal{G} \in \sigma f$

1) $\mathcal{G} \geq \mathcal{F} \in \sigma f \Rightarrow \mathcal{G}(\Phi) \geq \mathcal{F}(\Phi) \in \tau f(x) \Rightarrow \mathcal{G}(\Phi) \in \tau f(x)$.

2) $\mathcal{G} \geq \mathcal{F} \Rightarrow \mathcal{G}^{-1} \geq \mathcal{F}^{-1} \Rightarrow \mathcal{G}^{-1}(\Phi) \geq \mathcal{F}^{-1}(\Phi) \in \tau f^{-1}(x)$
$\Rightarrow \mathcal{G}^{-1}(\Phi) \in \tau f^{-1}(x)$.

C: $\mathcal{F},\mathcal{G} \in \sigma f \Rightarrow \mathcal{F} \wedge \mathcal{G} \in \sigma f$

1) $\mathcal{F}(\Phi), \mathcal{G}(\Phi) \in \tau f(x) \Rightarrow \mathcal{F}(\Phi) \wedge \mathcal{G}(\Phi) \in \tau f(x)$ and as

$\mathcal{F}(\Phi) \wedge \mathcal{G}(\Phi) = (\mathcal{F} \wedge \mathcal{G})(\Phi)$ it follows $(\mathcal{F} \wedge \mathcal{G})(\Phi) \in \tau f(x)$.

2) $\mathcal{F}^{-1}(\Phi), \mathcal{G}^{-1}(\Phi) \in \tau f(x) \Rightarrow \mathcal{F}^{-1}(\Phi) \wedge \mathcal{G}^{-1}(\Phi) \in \tau f^{-1}(x)$ and similarly $\mathcal{F}^{-1}(\Phi) \wedge \mathcal{G}^{-1}(\Phi) = (\mathcal{F}^{-1} \wedge \mathcal{G}^{-1})(\Phi) = (\mathcal{F} \wedge \mathcal{G})^{-1}(\Phi) \Rightarrow (\mathcal{F} \wedge \mathcal{G})^{-1}(\Phi) \in \tau f^{-1}(x)$.

D: Continuity of composition: If $\mathcal{F} \in \sigma f$ and $\mathcal{G} \in \sigma g$, we want to show that $\mathcal{F} \circ \mathcal{G} \in \sigma(f \circ g)$.

1) $(\mathcal{F} \circ \mathcal{G})(\Phi) = \mathcal{F}(\mathcal{G}(\Phi)) \in \tau(f(g(x))) = \tau(f \circ g)(x)$.

2) $(\mathcal{F} \circ \mathcal{G})^{-1}(\Phi) = (\mathcal{G}^{-1} \circ \mathcal{F}^{-1})(\Phi) \in \tau(g^{-1}(f^{-1}(x))) = \tau(f \circ g)^{-1}(x)$.

E: Continuity of inversion: We need to show that $\mathcal{F} \in \sigma f \Rightarrow \mathcal{F}^{-1} \in \sigma f^{-1}$.

1) $\mathcal{F}^{-1}(\Phi) \in \tau f^{-1}(x)$ by part 2) of the definition.

2) $(\mathcal{F}^{-1})^{-1}(\Phi) = \mathcal{F}(\Phi) \in \tau f(x) = \tau(f^{-1})^{-1}(x)$ by part 1) of the definition.

<u>Theorem 6</u>: σ is the coarsest admissible convergence structure on H(X) such that H(X) is a convergence group.

Proof: By the definition of σ it is clear that σ is admissible. Let σ' be any convergence structure such that $(H(X), \sigma')$ is an admissible convergence group. We want to show $\sigma \leq \sigma'$, that is, for all $f \in H(X)$, $\sigma' f \subseteq \sigma f$. Let $\mathcal{F} \in \sigma' f$, and let $x \in X$, $\Phi \in \tau x$. As σ' is admissible $\omega(\mathcal{F} \times \Phi) = \mathcal{F}(\Phi) \in \tau f(x)$. Moreover, $\mathcal{F} \in \sigma' f$ implies $\mathcal{F}^{-1} \in \sigma' f^{-1}$, so again using admissibility, $\mathcal{F}^{-1}(\Phi) \in \tau f^{-1}(x)$ implies $\mathcal{F} \in \sigma f$.

<u>Theorem 7</u>: σ on H(X) has the following property:

$$\mathcal{F} \in \sigma f \quad \Rightarrow \quad \begin{cases} \text{for all } x \in X \text{ and for any filter } \Phi \text{ on } X \\ \mathcal{F}(\Phi) \in \tau f(x) \text{ if and only if } \Phi \in \tau x. \end{cases}$$

Proof: Assume $\mathcal{F} \in \sigma f$ and that Φ is some filter on X such that $\mathcal{F}(\Phi) \in \tau f(x)$. By 2) of the definition of σ, $\mathcal{F}^{-1}(\mathcal{F}(\Phi))$ belongs to $\tau f^{-1}(f(x)) = \tau x$. But $\Phi \geq \mathcal{F}^{-1}(\mathcal{F}(\Phi))$ which implies $\Phi \in \tau x$. The reverse implication is immediate from 1) of the definition of σ.

The convergence structure of continuous convergence, γ_c, applied to homeomorphism groups is given by:

$\mathcal{F} \in \gamma_c f$ in H(X) if and only if for all $x \in X$ and for all $\Phi \in \tau x$, $\mathcal{F}(\Phi) \in \tau f(x)$. As γ_c is the coarsest of all admissible convergence structures on H(X), and as σ is admissible, we have $\gamma_c \leq \sigma$. Arens' example [1,p.601], although presented in a different setting shows that $\sigma \neq \gamma_c$. To quickly see the distinctness of σ and γ_c though, consider the following example:

<u>Counterexample</u>: Let Q be the set of rational numbers with the usual topology. Let $\mathcal{F}$ be the filter on H(Q) generated by the filter base of homeomorphisms $f_k: Q \to Q$ defined by: