Network theory for optimal pulse design via Pontryagin's maximum principle
by Merle Edward Ross

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Electrical Engineering
Montana State University

Abstract:
The subject of this thesis is the development of concepts and techniques of optimal input signal design for linear, lumped-element, time-invariant electrical networks.

The primary purpose of the thesis project is to develop a general optimal input signal design theory for linear, lumped-element, time-invariant electrical networks that realistically accounts for input signal energy and magnitude limitations, The content of the thesis is summarized as follows: First, a topologically-based procedure for representing networks by state variable equations is given. This representation of networks assures correspondence of mathematical variables and measurable quantities within the network. Second, energy relations are derived from the state variable equations representing the network. These energy relations are expressed in terms of the state variables and matrices that are fundamental to the state variable representation of the network. Third. a generalized augmented performance criterion utilizing the network energy relations is introduced. Fourth, application of Pontryagin's maximum principle to the optimal input signal design problem leads to the conclusion that the optimal input signal is in general composed of both bang-bang and singular segments ---- it is shown that for some cases the singular segments satisfy sufficient conditions for an optimum. The equations representing the necessary conditions for both bang-bang and singular segments are given, Fifth, examples are given that 1) demonstrate the nonnecessity of the sufficient conditions for a singular segment, and 2) illustrate a technique of solving the necessary condition equations for simple networks.

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Approved:

Paul E. Uhlrich
Head, Major Department

Donald A. Pierre
Chairman, Examining Committee

K J Goering
Graduate Dean

MONTANA STATE UNIVERSITY
Bozeman, Montana

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TABLE OF CONTENTS

LIST OF FIGURES

Page

Page

Page

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CHAPTER 1

INTRODUCTION

1.1 HISTORICAL BACKGROUND

Throughout the history of Electrical Engineering, one of the prime functions of the engineer has been to design systems for the transmission of energy from point to point, whether for power, communication of information, or routing of signals within a single piece of equipment. Even as early as World War II, the growing complexity of electronic equipment had led to work directed at optimizing the transmission of signals. The earliest of these efforts was by North (38); he showed that in order to maximize the ratio of peak signal to the root-mean-square noise in a pulse communication system, the filter characteristic should be the complex conjugate of the Fourier transform of the pulse. This was perhaps the first context in which the concept of matching a filter and waveform appeared. For an excellent bibliography of papers pertinent to matched filter theory, see Turin (59).

A different approach was presented in 1950 when Chalk (10) effected the minimization of interchannel interference and showed a method of selecting pulse shapes which give optimum available energy for detection in a pulse-communication system by maximizing the ratio of energy inside an allocated frequency band to the total energy of the pulse, which is equivalent to maximizing the ratio of output power to total power over all time. This technique leads to an integral equation that can be solved analytically only for relatively simple systems.

The problem of selecting a signal waveform that produces a desired output at a given instant of time while minimizing the energy of the signal waveform was solved by Beattie (5) in 1958. Historically, this was the first optimal signal design problem in which an energy constraint was realistically handled by use of admittance parameters; that is, Beattie accounted for nonconstant input impedances, but the result of his variational approach is a cumbersome integral equation. In 1961, Mostov, et al., (36) considered selecting the shape of a time-limited pulse which, when applied to a series RLC network, maximizes the voltage across the capacitor at a given instant of time, subject to the constraint that the energy of the input pulse equal a constant value, K. The integral, over the pulse period, of the instantaneous power dissipated in the resistor served as the energy constraint; the energy constraint and performance criteria were combined by use of a Lagrange multiplier technique, and the solution was derived by use of classical calculus of variation. The solutions in (36) are for relatively simple networks and are obtained in the time domain.

Also during 1961, Gerst and Diamond (19) addressed themselves to the problem of determining input wave shapes that result in outputs, of a linear time-invariant system, that are zero after a finite length of time. The objective of their approach was to completely eliminate intersymbol interference in pulse communication systems. The conditions characterizing these pulse input-output pairs are given in terms of their Laplace transforms $E_i(s)$ and $E_o(s)$ and the system transfer function F(s). The solution technique, in essence, calls for selecting an $E_i(s)$ such that some of the zeros of $E_i(s)$ cancel the poles of F(s), and if $E_i(s)$ has poles, these must be cancelled by zeros of F(s). No consideration is given to either energy or efficiency; hence, the use of pulse-shapes which are selected on the basis of (19) result in a relatively low energy transfer efficiency, a fact which is noted by Hancock, et al., (22). An interesting observation along these lines results from a series of publications beginning in 1961 by Tufts (55, 56, 57). Tufts recognized that transmission without intersymbol interference may not be the most desirable criterion and considered the problem of optimizing transmitted signal waveforms and receiver filters under the criterion of minimizing the mean-square error in the receiver's reproduction of the message sequence.

Concerned with minimizing a linear combination of relevant performance measures, Franks (17) in 1964, expressed these performance measures as quadratic forms and by using linear transformations, formulated the problem so that a solution by variational methods leads to a homogeneous Fredholm integral equation. At this time, Pierre (42, 43) considered two problems: first, the problem of selecting the input waveform to a general linear time-invariant system such that the output at a predetermined instant of time is maximum subject to the constraint that the input energy is fixed; and second, the problem of selecting the input waveform to a general linear time-invariant system so the output energy is maximized over a given time interval. It is noted that the first problem is a generalization of the specific problem treated by Mostov, et al., (36) and that realistic energy constraints are incorporated in the problem; the solution of the problem is obtained through use of a novel extension of the Wiener-Hopf spectrum-factorization technique (37). The second problem of Pierre (43) is a generalization of the work of Chalk (10); unfortunately, in order to obtain a reasonable solution, the input admittance is assumed to be a constant during the pulse interval.

In a 1964 paper (58), Tufts and Shnidman purportedly found the waveform which, when applied to a linear system, is optimum in the sense that the system output is maximum at a given instant of time, while input energy and magnitude constraints are imposed on the input waveform. In essence, this is a matched filter problem with a magnitude constraint included, and the solution is a clipped version of the previously known matched filter solution. The weakness in the problem formulation is that the energy constraint is given as

$$E \geq \int_{-\infty}^{\infty} [u(t)]^2 \, dt \qquad (1.1)$$

where E is a constant, and u(t) is the input waveform; this is a mathematically tenable formulation but physically represents no more than some form of bound on the input.

In 1965, a flurry of publications building on past works appeared; Smith (52) extended the work of Tufts (57) by optimizing transmitted signal waveforms using a mean-squared error criterion and including an average power constraint; Campbell, et al., (9) extended the work of Gerst and Diamond (19) by designing pulse inputs that correspond to a set of orthogonal pulse outputs for a given system; Schwarzlander and Hancock (51) maximized the ratio of energy received to total energy of a transmitted pulse, as Chalk (10) did, when selecting nonoverlapping pulses that eliminate intersymbol interference; the technique was then modified so that successive received pulses are mutually orthogonal. The first formal application of Pontryagin's maximum principle occurred with DiToro and Steiglitz (14) selecting minimum bandwidth pulses that satisfy constraints of given rms value and peak amplitude in pulse communication systems.

The foregoing history is by no means complete, but is intended to introduce the types of problems that have arisen and their respective solutions. With this historical prospective, the essence of the general requirement for optimal signal design can now be outlined.

1.2 FORMULATION OF PROBLEM

All the above mentioned problems are similar in that an optimum input waveform is selected so that the output of a system satisfies some performance criteria. In some of the above, the optimal input waveform is constrained, and in some, no such consideration is given. The mathematical representation of the system in each case is tailored to the form of the performance criteria and side constraints, and in each problem, the method of solution is one which is most mathematically amenable to system representation, performance criteria, and constraining relationships. The external or side constraints imposed on the input waveform normally arise from physical restrictions placed on the input signal source; for example, if the supply potential is limited, a magnitude bound restricting the input u(t) may be

$$a \leq u(t) \leq b \qquad (1.2)$$

where a and b are, respectively, the lower and upper limit of allowed excursions of u. A second form of input signal constraint results from an energy limitation on the signal source; this type of constraint usually takes the form

$$K \geq \int_{t_1}^{t_2} f[u(t)]\, dt \qquad (1.3)$$

Here K represents the energy available for use during the interval $t_1 \leq t \leq t_2$, and $f[u(t)]$ is a function representing the instantaneous expenditure of energy, that is, instantaneous power.

As stated above, the performance criteria and side constraints not only influence the method of solution, but determine the final form of the optimum input waveform selected. It is imperative, then, that the performance criteria and constraint relationships be formulated, not on the grounds of expediency of mathematical manipulation or elegance, as so often appears to be the case, but on the grounds of a realistic representation of "that which is to be optimized." To this end, a reasonable approach would be as follows: 1) select the most general optimization method available, one that can effectively handle all forms of performance criteria and constraints that may be encountered; 2) determine what mathematical model is most compatible with the optimization theory and the class of systems to which the theory is to be applied; and 3) from this model of the system, generate relations capable of representing constraints on the input waveform resulting from physical limitations of components and equipment.

Of the existing optimization techniques, the one that has been successfully applied to the broadest class of problems is the maximum principle of Pontryagin, et al., (46). Although this maximum principle normally appears in the context of optimal control,* the problems in signal design are often analogous to those of control; for this reason, Pontryagin's maximum principle is the optimization context in which input waveform design will be studied in this thesis. In the formulation of the maximum principle,

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*Tou (54) and Athans and Falb (1) give extensive application of Pontryagin's maximum principle to control problems.

Pontryagin, et al., represent the process dynamics by a normal system of differential equations, which according to Pontryagin (47) result from the reduction of a higher order differential equation or a quite general system of differential equations to the form

$$\dot{x}_i = f_i(t, x_1, x_2, \cdots, x_n); \; i = 1, 2, \cdots, n. \qquad (1.4)$$

This classical vector differential equation has become known to control engineers as the system state equation, and a large body of theory has been developed around this vector differential equation and its solutions. For a comprehensive treatise on the state space approach to linear system theory, see Zadeh and Desoer (67). The existence of these two well-developed techniques and the fruits wrought from their union by control engineers, is a convincing argument for state space representation of systems.

The remaining problem, that of arriving at realistic constraint relations, is complicated by the fact that although an n^{th}-order differential equation representing the dynamic behavior of a system can be reduced to n first-order state equations in n state variables, these state variables in general do not identify in a one-to-one correspondence with measurable quantities within the system. Thus, if state equations are used to describe a system, a special analysis technique is needed to assure correspondence of the state variables and measurable quantities. This brings to the fore an analysis technique introduced by Bashkow (3). In his work, Bashkow gives a new method of lumped-element, linear, time-invariant, passive network analysis that leads naturally to a set of first-order differential equations

$$\dot{X} = AX + B \tag{1.5}$$

in which the vector X is an n x 1 matrix that contains certain of the original voltage and current variables, and the column vector B contains the voltage and current sources. The n x n A matrix consists of scalar elements which are combinations of inductances, capacitances, and resistances of the network. The existence of this technique is indeed fortunate in that the linear, time-invariant, passive network is widely used in communication, control, and electronic applications and, as demonstrated in Cheng (11) and D'azzo and Houpis (12), may be used as an analogous representation of certain mechanical rotational and translational systems as well as certain thermal systems. Thus, the use of the lumped-element, linear, time-invariant, passive network model achieves a