On the geometry of affine tangent bundles with multiple and single vector fields as fibres
by Samuel Ralph Thompson

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Mathematics
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Abstract:
ABSTRACT Recently (2) T. Okubo has shown that the frame (or principal, (1)p82) bundle F(Xn) of a differentiable manifold Xn of class Cr(r≥4) admits a tensor field Φ of type (1,1), rank nR, satisfying Φ3=-E , In this thesis I generalize the notion of a frame bundle and consider a fibre bundle Tm(Xn) having m (1< m< n) linearly independent tangent vectors as fibre. Following the method of T.Okubo it is shown that the bundle Tm (Xn) also admits such a tensor fieldΦ A symmetric affine connection is then assigned to Tm(Xn) and the differential geometry of Tm(Xn) is developed on the basis of this tensor structure and of the extended lift of a vector field in Xn. .

The latter portion of this thesis considers the case m=1; the so-called tangent bundle T(Xn) = T (Xn) , It is now well-known that the tangent bundle T(Xn), with Xn a Riemannian manifold, admits a tensor field F of type (1,1), rank 2n, satisfying F2= -E. It is shown here that T(Xn) also admits such a tensor field F when the base manifold is only an affinely connected space, A symmetric affine connection is introduced in T(Xn) on the same principle as before and the geometry of T(Xn) similarly discussed. In particular it is shown that the natural lift C (7),(3) of a path C in Xn is again a path in T(Xn).

Finally, an interesting aspect of this thesis from a technical point of view is the use made of the "adapted frames" as originated by K, Yarn and T. Okubo (6) as our reference frames instead of the usual "natural frames",, This choice of frames results in a number of simplifications in the calculations.

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ON THE GEOMETRY OF AFFINE TANGENT BUNDLES WITH MULTIPLE AND SINGLE VECTOR FIELDS AS FIBRES

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VITA

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ABSTRACT

Recently (2) T. Okubo has shown that the frame (or principal, (1)p82) bundle $F(X_n)$ of a differentiable manifold X_n of class C^r $(r \geq 4)$ admits a tensor field Φ of type (1,1), rank n^2, satisfying $\Phi^3 = -E$. In this thesis I generalize the notion of a frame bundle and consider a fibre bundle $T_m(X_n)$ having m ($1 < m < n$) linearly independent tangent vectors as fibre. Following the method of T. Okubo it is shown that the bundle $T_m(X_n)$ also admits such a tensor field Φ. A symmetric affine connection is then assigned to $T_m(X_n)$ and the differential geometry of $T_m(X_n)$ is developed on the basis of this tensor structure and of the extended lift of a vector field in X_n.

The latter portion of this thesis considers the case $m=1$; the so-called tangent bundle $T(X_n) \equiv T_1(X_n)$. It is now well-known that the tangent bundle $T(X_n)$, with X_n a Riemannian manifold, admits a tensor field F of type (1,1), rank 2n, satisfying $F^2 = -E$. It is shown here that $T(X_n)$ also admits such a tensor field F when the base manifold is only an affinely connected space. A symmetric affine connection is introduced in $T(X_n)$ on the same principle as before and the geometry of $T(X_n)$ similarly discussed. In particular it is shown that the natural lift $\bar{C}$ (7),(3) of a path C in X_n is again a path in $T(X_n)$.

Finally, an interesting aspect of this thesis from a technical point of view is the use made of the "adapted frames" as originated by K. Yano and T. Okubo (6) as our reference frames instead of the usual "natural frames". This choice of frames results in a number of simplifications in the calculations.

INTRODUCTION

Until recently (2) it had been an open problem as to the tensor structures admitted by the frame bundle $F(X_n)$ of a differentiable manifold X_n, that is, a kind of tangent bundle whose fibre is composed of n linearly independent vectors. This thesis has generalized the notion of a frame bundle and considers a fibre bundle $T_m(X_n)$ having m $(1 < m < n)$ linearly independent vectors as fibre and following the method of T. Okubo it is shown that $T_m(X_n)$ admits a tensor structure H of type (1,1), rank n^2, satisfying $H^3 - H = 0$, or equivalently a structure Φ satisfying $\Phi^3 = -E$. Moreover, H decomposes the unit tensor E of rank $n+mn$ into two complementary tensor fields P and Q of type (1,1) and of ranks n and mn, respectively; and H acts on P as an annihilator and on Q as an almost product structure.

In assignment of affine connections to $T_m(X_n)$ we have a pattern due to K. Yano and E. T. Davies (7) when the base space X_n is Riemannian. However, since the tensor structures considered in this investigation are coherent to the affine structure of the base manifold X_n; that is, independent of the metric structure, the geometry of $T_m(X_n)$ should be most profitably discussed in a framework free from metric considerations. The difficulty in the theory of connections of $T_m(X_n)$ lies partially in the fact that X_n is always reduced to a trivial affine space when the connection is given so that $T_m(X_n)$ is integrable, and mostly in the fact that a basic symmetric connection is required in our case to satisfy two criteria: that the parallel displacement of the horizontal lift (2), (7) of a vector field in X_n coincide with the parallelism of Levi-Civita, and that the natural lift of a path in X_n should be a path in $T_1(X_n)$.

This thesis gives a symmetric affine connection to the bundle $T_m(X_n)$ fulfilling the above requirements and discusses the geometry of $T_m(X_n)$ on the basis of the tensor structure H and of the extended lift of a vector field in X_n. The notion of the extended lift in $T_m(X_n)$ of a vector field in X_n being a generalization of what S. Sasaki (3) has called the extension of a vector field in the case of a bundle with a single tangent vector as fibre, that is, the case $m = 1$; and what K. Yano and E. T. Davies (7) have called the complete lift, again for the case $m = 1$.

As for a tangent bundle whose fibre is a single tangent vector it is now well-known that when the base manifold is Riemannian the bundle admits a tensor field F of type (1,1). rank 2n, satisfying $F^2 = -E$; (4) and F transforms the vertical distribution of $T_1(X_n)$ onto the horizontal distribution at each point of $\pi^{-1}(U)$, $U \subset X_n$. It is shown here that any tangent bundle $T_1(X_n)$ admits such a tensor field F when the base manifold is only an affinely connected space. A symmetric affine connection is introduced in $T_1(X_n)$ on the same principle as before and the geometry of $T_1(X_n)$ similarly discussed. In particular, it is shown that the natural lift $\bar{C}$ of a path C in X_n is again a path in $T_1(X_n)$ which generalizes results obtained by S. Sasaki (3) and by K. Yano and E. T. Davies (7).

Finally, it should be remarked that a systematic use of the "adapted frames" as originated by K. Yano and T. Okubo (6) is made throughout this thesis, and it is the author's opinion that the work required to do the same calculations with respect to the so-called "natural frames" would be prohibitive.

CHAPTER I

THE BUNDLE $T_m(X_n)$ WITH MULTIPLE TANGENT VECTORS $X^h_{(\alpha)}$ AS FIBRE

1. The existence of a tensor field H of type (1,1) in $T_m(X_n)$ satisfying $H^3 - H = 0$.

We adopt the following conventions for indices:

$$\begin{aligned} &a, b, c, d, h, i, j, k = 1, 2, \ldots, n \\ &\alpha, \beta, \gamma, \delta, \epsilon = 1, 2, \ldots, m \\ &\left.\begin{matrix} A, B, C, D, E \\ \lambda, \mu, \nu, \rho, \tau, \omega \end{matrix}\right\} = 1, 2, \ldots, n, n+1, \ldots, n+mn. \end{aligned}$$

and let X_n be an n-dimensional differentiable manifold of class C^r with $r \geq 4$. If we let $X^h_{(\alpha)}$ be the components of m ($1 < m < n$) linearly independent vectors of X_n with respect to a local coordinate neighborhood $U(u^h)$, then we can consider the $(n + mn)$-dimensional differentiable manifold $T_m(X_n)$ with local coordinates $(u^h, X^h_{(\alpha)}) \equiv (u^h, X^h_{(1)}, \ldots, X^h_{(m)})$ in the neighborhood $U \times E^n \times \cdots \times E^n \equiv U \times E^{mn}$. Let $U(u^h)$ and $U'(u^{h'})$ be two overlapping coordinate neighborhoods of X_n. Then $U \times E^{mn}$ and $U' \times E^{mn}$ are overlapping coordinate neighborhoods in $T_m(X_n)$ and corresponding to the transformation $u^{h'} = u^{h'}(u^h)$ in $U \cap U'$ we have the transformation

$$(1.1)\quad \begin{cases} u^{h'} = u^{h'}(u^h) \\ X^{h'}_{(\alpha)} = \partial_h u^{h'} X^h_{(\alpha)} \quad \text{where } \partial_h \overset{\text{def}}{=} \dfrac{\partial}{\partial u^h} \end{cases}$$

In the future we shall write $h_\alpha = n\alpha + h$ so that we can describe the $n + mn$ coordinates in $T_m(X_n)$ as ξ^A where $\xi^h = u^h$ and $\xi^{h_\alpha} = X^h_{(\alpha)}$. (1.1) is now written

$$(1.2)\quad \begin{cases} \xi^{h'} = \xi^{h'}(\xi^{h}) \\ \xi^{h'_\alpha} = \partial_h \xi^{h'} \xi^{h_\alpha} \end{cases}$$

and for convenience we write (1.2) as $\xi^{A'} = \xi^{A'}(\xi^{A})$. Now

$$(1.3)\quad \left[\partial_A \xi^{A'}\right] \equiv \begin{pmatrix} \partial_h \xi^{h'} & \partial_{h_\alpha} \xi^{h'} \\ \partial_h \xi^{h'_\beta} & \partial_{h_\alpha} \xi^{h'_\beta} \end{pmatrix}$$

where $\partial_A = \dfrac{\partial}{\partial \xi^A}$; and in particular $\partial_{h_\alpha} = \dfrac{\partial}{\partial \xi^{h_\alpha}} = \dfrac{\partial}{\partial X^h_{(\alpha)}}$. Nows

$$\partial_h \xi^{h'} = \partial_h U^{h'}, \quad \partial_{h_\alpha} \xi^{h'} = 0$$

$$\partial_h \xi^{h'_\beta} = \partial_h(\partial_j U^{h'} X^j_{(\beta)}) = X^j_{(\beta)} \partial_h \partial_j U^{h'}$$

$$\partial_{h_\alpha} \xi^{h'_\beta} = \partial_{h_\alpha}(\partial_j U^{h'} X^j_{(\beta)}) = \partial_j U^{h'} \partial_{h_\alpha} X^j_{(\beta)} = \delta^\beta_\alpha \delta^j_h \partial_j U^{h} = \delta^\beta_\alpha \partial_h U^{h'}$$

and consequently (1.3) becomes

$$(1.4)\quad \left[\partial_A \xi^{A'}\right] = \begin{array}{cc} & \begin{array}{cccc} (n) & (n) & & (n) \end{array} \\ & \left(\begin{array}{cccc} \partial_h U^{h'} & 0 & \cdots & 0 \\ X^j_{(1)} \partial_h \partial_j U^{h'} & \partial_h U^{h'} & & 0 \\ X^j_{(2)} \partial_h \partial_j U^{h'} & & \ddots & \\ \vdots & 0 & & \\ X^j_{(m)} \partial_h \partial_j U^{h'} & & & \partial_h U^{h'} \end{array}\right) \begin{array}{c} (n) \\ (n) \\ \vdots \\ \\ (n) \end{array} \end{array}$$

implying that

(1.5) $$\det\left[\partial_\Lambda \xi^{\Lambda'}\right] = \left(\det\left[\partial_h u^{h'}\right]\right)^{m+1}$$

From which we have the following (5, p. 214)

Theorem 1.1. If m is even, then the bundle $T_m(X_n)$ has the same orientation as X_n ; and if m is odd, $T_m(X_n)$ is always orientable.

T. Okubo (2) has told me that the frame bundle $F(X_n)$ of an n-dimensional differentiable manifold of class C^r, $r \geq 4$, admits a tensor field of type (1,1) whose cube is -E, the identity. Here we intend to show that $T_m(X_n)$ also admits such a tensor field. Following T. Okubo define a field of quantities $\mathbb{F}^A_B$ in $T_m(X_n)$ at a point $\xi^A(u^h, X^h_{(\alpha)})$ by

(1.6) $$\mathbb{F}^A{}_B \equiv \begin{pmatrix} \mathbb{F}^h_j & \mathbb{F}^h_{j\alpha} \\ \mathbb{F}^{h_\beta}_j & \mathbb{F}^{h_\beta}_{j\alpha} \end{pmatrix} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array} \begin{array}{cc} \begin{array}{c} (n) \quad \overbrace{(n) \quad \cdots \quad (n) \quad (n)}^{(m)} \end{array} & \\ \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \Gamma^h_{ji} X^i_{(m)} & & & & \delta^h_j \\ \Gamma^h_{ji} X^i_{(m-1)} & & 0 & \delta^h_j & \\ \vdots & & \cdot^{\cdot^{\cdot}} & & \\ \vdots & & & 0 & \\ \Gamma^h_{ji} X^i_{(1)} & \delta^h_j & & & 0 \end{pmatrix} & \begin{array}{l} (n) \\ \left.\begin{array}{l} (n) \\ (n) \\ \vdots \\ \vdots \\ (n) \end{array}\right\} (m) \end{array} \end{array}$$

where we now assume that X_n is an affine space with a symmetric linear connection whose components with respect to the local coordinates (u^h) are $\Gamma^h_{ji} = \Gamma^h_{ij}$. If $U(u^h)$ and $U'(u^{h'})$ are two overlapping coordinate neighborhoods, then Γ transforms according to the law (cf 1, p. 79)

(1.7) $$\Gamma^{h'}_{j'i'} = \partial_h u^{h'}\left\{ \partial_{j'} u^j \partial_{i'} u^i \Gamma^h_{ji} + \partial_{j'}\partial_{i'} u^h \right\}.$$

We denote by $\nabla_j v^h$ the covariant derivative of a vector field v:

$$\nabla_j v^h = \partial_j v^h + \Gamma_{ji}^h v^i$$

Theorem 1.2.

(1) $\mathbb{H}_B^A \mathbb{H}_C^B \mathbb{H}_D^C = \mathbb{H}_D^A$

(2) $\mathbb{H}_B^A$ are the components of a tensor field of type (1,1) in $T_m(X_n)$ with respect to the local coordinates (ξ^A).

Proof: For brevity we write Γ_α for the $n \times n$ matrix $\left[\Gamma_{ji}^h X_{(\alpha)}^i\right]$ and E for the $n \times n$ matrix $\left[\delta_j^h\right]$. Then

$$\mathbb{H}_B^A = \begin{array}{c} \begin{array}{ccccc} (n) & & \overbrace{(n) \quad \cdots \quad (n) \quad (n)}^{(m)} & & \end{array} \\ \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \Gamma_m & & & & E \\ \Gamma_{m-1} & & 0 & E & \\ \vdots & & \iddots & & \\ \Gamma_1 & E & & 0 & \end{pmatrix} \begin{array}{l} (n) \\ (n) \\ (n) \\ \vdots \\ (n) \end{array} \end{array} \Bigg\} (m)$$

$$\mathbb{H}_B^A \mathbb{H}_C^B = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \Gamma_m & & 0 & & E \\ \Gamma_{m-1} & & & E & \\ \vdots & & \iddots & 0 & \\ \Gamma_1 & E & & & \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \Gamma_m & & 0 & & E \\ \Gamma_{m-1} & & & E & \\ \vdots & & \iddots & & 0 \\ \Gamma_1 & E & & & \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \Gamma_1 & E & & & \\ \Gamma_2 & & E & 0 & \\ \vdots & & & \ddots & \\ \Gamma_m & 0 & & & E \end{pmatrix}$$