Effect of plastic flow on flexural stresses in reinforced concrete beams
by Louis A Divras

A THESIS Submitted to the Graduate Committee in partial fulfillment of the requirements for the
degree of Master of Science in Civil Engineering
Montana State University
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Abstract:
In this thesis, the fact is discussed that the stress variation in the compression side of the beam is not
actually a straight line as it is assumed by most designers, but it has a shape of a curve, which is
somewhat close to a parabola.

Most designers, while designing certain beams for different loads, base their design on the straight line
theory, which in turn considers only the instantaneous unit deformation produced by the application of
a load and which is known as elastic strain.

Concrete, however, is a plastic material and additional deformations occur immediately after the
application of a load. Even though the load remains constant, these additional strains increase as time
passes. These strains, which are usually called plastic strains, vary with the amount of load and with the
time. Considering this, it is shown that modulus of elasticity, as far as concrete design is concerned, is
meaningless. In the experimental work, two beams were tested. One beam was designed by the plastic flow method and one was designed by the straight line stress variation method, in which the
modulus of elasticity of concrete must be used.

The test results demonstrate that the strains do not vary as a straight line from the neutral axis to the
extreme fiber; consequently the stress does not vary as a straight line from the neutral axis to the
extreme fiber.
EFFECT OF PLASTIC FLOW ON FLEXURAL STRESSES IN REINFORCED CONCRETE BEAMS

by

LOUIS A. DIVRAS

A THESIS

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Chairman, Examining Committee

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In this thesis, the fact is discussed that the stress variation in the compression side of the beam is not actually a straight line as it is assumed by most designers, but it has a shape of a curve, which is somewhat close to a parabola.

Most designers, while designing certain beams for different loads, base their design on the straight line theory, which in turn considers only the instantaneous unit deformation produced by the application of a load and which is known as elastic strain.

Concrete, however, is a plastic material and additional deformations occur immediately after the application of a load. Even though the load remains constant, these additional strains increase as time passes. These strains, which are usually called plastic strains, vary with the amount of load and with the time. Considering this, it is shown that modulus of elasticity, as far as concrete design is concerned, is meaningless.

In the experimental work, two beams were tested. One beam was designed by the plastic flow method and one was designed by the straight line stress variation method, in which the modulus of elasticity of concrete must be used.

The test results demonstrate that the strains do not vary as a straight line from the neutral axis to the extreme fiber; consequently, the stress does not vary as a straight line from the neutral axis to the extreme fiber.
Symbols Used

$A_c$ = Effective concrete area.
$a$ = Depth of equivalent stress diagram.
$A$ = Overall area of concrete in a column.
$A_s$ = Area of tensile steel.
$b$ = Width of beam.
$C$ = Total compressive force.
$T$ = Total tensile force.
$c$ = Arm of resisting couple.
$f_c$ = Tensile stress in concrete.
$f_s$ = Compression stress in steel.
$f^c$ = Compressive stress of a concrete cylinder.
$f_y$ = Yield point stress of steel.
$d$ = Effective depth of beam.
$M$ = Resisting moment.
$n$ = Modular ratio (which is equal to $\frac{E_s}{E_c}$).
$E_s$ = Modulus of elasticity of steel.
$E_c$ = Modulus of elasticity of concrete.
$p$ = Steel ratio ($A_s/bd$).
$S$ = Shortening of a non-reinforced concrete block due to shrinkage.
$S_E$ = Shortening of reinforced concrete beam due to shrinkage.
$f's$ = Allowable stress in steel.
$f''c$ = Allowable stress in concrete.
$jd$ = Moment arm of the internal couple.
$kd$ = Position of the neutral axis.
V = Total shear force at the section.

ν = Intensity of vertical shearing stress.

ε_t = Total concrete strain represented in the trapezoidal diagram at final rupture.

βε_t = Plastic strain represented by the horizontal portion of the trapezoidal diagram.

β = Plasticity ratio.

ε_o = Total concrete strain minus plastic strain.

f_{st} = Computed theoretical stress of steel.

f_{ct} = Computed theoretical stress of concrete.

S_s = Computed actual stress of steel.

S_c = Computed actual stress of concrete.

ε_s = Actual strain of steel.

ε_c = Actual strain of concrete.
Introduction

Experience has shown that the actual stress distribution in a reinforced concrete beam does not agree with the accepted theory. It is also well known that this usual theory, when used with properly selected unit stresses gives results that are safe in spite of the fact that the actual unit stresses may be entirely different.

It has long been recognized that concrete changes in volume when subjected to stress (elastic deformation), sustained load (plastic flow), temperature changes, and changes in moisture content (shrinkage), and many investigators have worked upon the determination of the magnitude of these volume changes.

In recent years, the criticisms of the ordinary straight line theory of reinforced concrete have given rise to proposals for the abandonment of the modular ratio $\frac{E_p}{E_c}$ in favor of formulas based on various theories of plastic action. In all these theories, emphasis is placed upon the load at final rupture of the beam. While the wisdom of such emphasis is open to question, there can be no doubt that it is desirable to be able to predict, within reasonable limits, the ultimate capacity of a beam. Since concrete is not truly elastic, the stress variation across the beam section usually is to be assumed a parabola. The writer does not feel satisfied with such an assumption. Since its deformations under load increases with time and because of shrinkage and plastic flow, the elastic or straight line theory has long been recognized as approximate. Since shrinkage and flow are impossible of prediction with any accuracy for a given structure, it appears to be a hopeless task to determine exactly the
stresses under a given load. It is possible, however, to derive a simple relationship which will predict with considerable accuracy the ultimate strength of a reinforced concrete beam, a value little affected by shrinkage and flow since there is a redistribution of stress with the large strains previous to failure. It thus becomes possible to design on the basis of ultimate strength applying a definite factor of safety suitable to the conditions.

The writer, in the experiments conducted in the Civil Engineering Laboratory of Montana State College, attempted to find the actual stress distribution in two full size concrete beams.

The undersigner wants to express his deep gratitude to Professor R. C. DeHart for his kind assistance in carrying on the work. He also wants to thank Mr. J. R. Arboe and Mr. Drummond for their kind assistance in offering the facilities of the Mechanical Department necessary for the completion of the work.
Historical Development

It is probable that a number of observers had noticed the plastic flow phenomenon in concrete before 1907, when Professor W. K. Hatt published the results of a series of deflection tests on beams. This appears to be the first publication on the subject. According to Professor Hatt, these results taken together show a sort of plasticity in concrete by which it yields under the action of a load applied for a long time or applied a number of times.

In 1915, Professor F. R. MacMillan reported tests on beams and slabs in which he measured fiber deformation changes as well as deflections under conditions where the effect of shrinkage due to drying out were separated from the effect of plastic flow. In some of his tests on slabs, though the deflection continued to increase, the fiber stress measurements on the steel indicated compression after one year. Apparently the loading was not great enough to form any cracks in the region of the steel and the total shortening of the beam due to shrinkage was great enough to cause compression in the steel and the combined effect of shrinkage and plastic flow on the compression side were sufficient to allow the increased deflection. The concrete on the tension side was apparently taking the full bending plus the compression, which the shrinkage caused in the steel. These tests illustrated the effect of shrinkage better than that of plastic flow.

In 1916, two other important papers, one by E. B. Smith and another

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1. Superior figures throughout this thesis refer to bibliography.
by A. T. Goldbeck and E. B. Smith and a major discussion by F. R. MacMillan were published. The first of these papers brought out the fact that the plastic recovery after a load that had been sustained for some time was removed with considerably less than the plastic flow under the sustained load. It also showed that concretes under water flow less than those in air. The second paper showed that the plastic flow is comparatively rapid at first and appears to undergo a progressive slowing down as time goes on. Tests on beams and slabs in which fiber deformations were measured brought out the information that the deformations at six months were roughly three times the initial deformations and at two years, four times. MacMillan made calculations and reported the plastic flow in terms of a reduced modulus of elasticity for the concrete or increased modular ratio of (modulus of elasticity of steel to modulus of elasticity of concrete).

In 1917, two other important papers appeared in the Proceedings of the American Concrete Institute. One, a report by A. R. Lord of measurements made during a year on a flat slab panel, confirmed previous reports and emphasized the effect of shrinkage in reducing the tensile stresses in steel. Lord reported that the plastic flow and shrinkage tended to reduce the negative bending resistance, thus producing greater deflections in regions of positive moment. He also found that the "straight line theory" of bending for reinforced concrete did not apply for the conditions of the test even with low stresses in the concrete and the steel.

E. B. Smith, in 1917, found that the law of plastic flow of concrete tended toward a limit and that the amount of plastic flow depends
somewhat upon the kind of aggregate used. He found that the coarser the aggregate, the greater the plastic flow. He also found that the action of plastic flow is to lower the neutral axis of a beam, which happens when the modulus of elasticity is reduced. It follows then that the compression area is increased, thus reducing the resultant couple arm and consequently, increasing the unit stress in steel.

In 1927, Oscar Faber, an authority on mechanics, published in England the results of tests he had made on beams. The effect of shrinkage and plastic flow was brought into the theory of mechanics by way of change of the modular ratio with time.

Professor R. E. Davis of the University of California, was commissioned to make tests so that proper adjustments and corrections for plastic flow could be made. The results of these tests are the most comprehensive that have so far been made and they are published in the Proceedings of the American Concrete Institute, first in 1928, and later in 1931, as well as in the Proceedings of the American Society for Testing Materials in 1930 and 1934.

A number of papers have lately appeared in which theoretical aspects of plastic flow have been discussed. Among them is one by G. S. Whitney, in which he shows how the plastic flow property of concrete eliminates many of the secondary stresses in arches, a problem that has concerned designers considerably and has been the cause of very special procedures and devices introduced into design and construction. The effect of plastic flow figures largely in the conclusions drawn from the comprehensive series of tests on columns lately carried out by the University of Illinois.
Lorenz G. Straub 15 made some tests and reports that a repeated application loads produce effects similar to a sustained load because of the difference between the plastic flow and plastic recovery. Results in a series of rigid frames constructed and tested at the University of Illinois 16 serve to check the theoretical calculations.
Plastic Flow of Concrete

Ordinary calculations for stresses and stress distribution are based on the theory that the material is elastic, that the deformation is proportional to stress, and no term is introduced for time. It is now known that some structural materials are affected by time and that strains or deformations continue to change after the stresses have become constant. This property is called "time yield" or "creep" or "plastic flow". None of these terms apply to steel at low temperatures so long as the stresses are within the elastic limit. The designer of steel parts for stress at high temperatures where the elastic properties are not so definite is concerned with the property usually known as "creep" and rather good studies have been made of this property of metals. Other materials such as timber, concrete and resinous substances as celluloid have this plastic property, at ordinary temperatures and at any degree of stress. The nature of this property is different for different materials. The term "plastic flow" as used in this thesis is that property of a material which is evidenced by the long time continuation of a varying increase of deformation or strain under sustained load or decrease of stress during sustained strain.

A plain concrete specimen in compression under a constant load will continue to shorten after loading. This shortening begins immediately after the load is applied. The change in length at constant stress is quite rapid at first and slows down as time goes on. The shape of the curve of time plotted against deformation is that of a power or exponential function not unlike that of a cubic parabola. When steel and concrete are both present, as in a building column, the plastic flow is lessened because
of the presence of steel which is not plastic. The steel must take on an increasing proportion of the load as time goes on. In beams, the deflection continues after the load has been applied, and the stress is increased in the tension steel because of the reduction of the resultant internal couple arm. The straight line representing variations of the fiber deformations continues to change and an increasing depth of the beam is brought into compression.

The concept of plastic flow as indicated by the term itself is readily applied to conditions of sustained load or stress but the action of plastic flow under sustained deformation or strain is somewhat more difficult to understand, although it is little, if any, less important. If a plain concrete specimen is placed in a testing machine and a certain deformation is imposed and held, the scale beam will drop. If an attempt is made to keep the beam poised, the indicated load will have to be reduced, rapidly at first and then progressively more slowly. If a beam is loaded to a certain elastic deformation and this deformation is held, the load will have to be reduced in the same way as it would have to be reduced for the specimen on the testing machine. If a beam is subjected to secondary stress, the plastic flow in the concrete will cause the fiber stresses in compression in the concrete to be reduced and as a consequence, the stresses in the steel will have to reduce also. Thus the action of plastic flow on secondary stresses is generally beneficial.
Plastic Flow and Volume Changes of Concrete

The best picture of what goes on inside a piece of concrete when shrinkage or plastic flow takes place seems to be that given by R. E. Davis, H. E. Davis and Hamilton of the University of California. In tests which they made under mass curing conditions, the stresses developed due to thermal changes in large concrete cylinders under complete axial restraint were measured. It was found that the degree to which flow takes place has an important influence upon the stresses developed. They have also found that in a period of 10 years, 95 percent of the plastic flow occurring in 10 years took place in the first 5 years and the remaining 5 percent in the last five years. Tests to determine the effect of water cement ratio and of aggregate cement ratio upon plastic flow indicated that the more the cement used per unit volume of concrete, the less the flow and that this variable was very important. Within the range of normal concretes tested, it was observed that with pastes of the same water cement ratio, the flow was practically proportional to the amount of the paste in the concrete.

In tests to compare the flow of concretes in axial tension and compression, it was observed that at least during the early periods, the flow under tensile stress was greater than that under compressive stress. In the later stages, the rate of flow was less under tensile than under compressive stress.

In technical books for about a decade, there was frequent reference to the strength and elasticity of concrete but only an occasional reference to shrinkage due to drying and almost no reference to the gradual deformation under the action of sustained load, which has been called "creep".
"time yield", and "plastic flow". As a result, however, of the research of recent years, there has been developed a general conception of the effect of shrinkage and plastic flow upon the behavior of concrete structures. It is now believed that shrinkage and plastic flow are closely related phenomena, each being primarily due to changes in the amount of absorbed water in the cement gel and being but little directly influenced by the free water occupying the pore spaces within the concrete mass.

Many researches were made to find the magnitude of plastic flow and shrinkage under the conditions surrounding a given concrete structure. However, they were not able to succeed.

The property of shrinkage of concrete due to drying is altogether undesirable. Shrinkage can never be entirely eliminated, yet there seems to be the possibility through the proper selection of materials and methods of reducing shrinkage to a point where it will not be a factor for serious consideration. On the whole, plastic flow does not seem to be an undesirable property. In certain reinforced concrete members, it tends to make possible more efficient use of steel, such as thin structures subjected to drying, as well as in mass structures subjected to thermal changes due to the hydration of cement.

Plastic Flow Under Long Time Loading:

The plastic flow of plain and reinforced concrete cylinders, which have been under load for 10 years, are shown in figure 1. One load of specimens was loaded at the age of 28 days and the other at the age of 3 months. Prior to loading, the concrete was moist cured and after loading, it was stored in air at 70 percent relative humidity and 70°F.
Fig. 4.—Flow Under Long-Sustained Stress

Fig. 2.—Effect of Aerogel-Cement Ratio and Water-Cement Ratio Upon Flow.
sustained compressive stresses were 300, 600, 900, and 1200 pounds per square inch. For the concrete loaded at the age of 28 days to a stress of 900 pounds per square inch, about 80 percent of the flow took place within the first year and about 97 percent within 5 years. An approximate proportionality of stress to deformation appears to have been maintained over the loading period. For the same sustained stress, the age at loading does not appear to have a large effect upon the rate of change in length after the first few months under load. As the flow, however, was progressing, more and more load was transferred from the concrete to the steel and the rate of flow decreased in much greater proportion than would be the case for plain concrete sustaining the same initial stress.

Effect of Aggregate-Cement Ratio and Water-Cement Ratio upon Plastic Flow.

In order to determine the effects of these factors, a series of tests were made in which concretes of three different cement contents and two different water-cement ratios were used. All specimens were subjected to a sustained compressive stress of 800 pounds per square inch at the age of 28 days. The concrete was cured under standard moist conditions until time of loading, after which the tests were carried out in air at 50 percent relative humidity and 70°F. The results of these tests are shown on Figure 2. At the end of these tests, it was found that concretes having the same water-cement ratio, the richer mixes have more flow than the lean mixes. This is what might be expected if the major portion of the flow is considered to be due to the movement in the hardened paste. Statements also have been made that: (1) lean concretes flow more than rich concretes, (2) weak concretes flow more than strong concretes. For the same consistency,
a lean concrete requires a higher water cement ratio than a rich concrete. It appears reasonable to suppose that the more open the pore structure, the less will be the areas of contact between neighboring hydrated cement grains and so under a given load, the higher will be the stresses over such areas of contact. There appears the interesting possibility that the particle size distribution of cement, through its effect upon the structure of the hardened cement paste, may have an important influence upon the rate and magnitude of volume changes in general.

**Flow in Axial Tension and Compression:**

The results of a series of tests to study the flow of concrete under sustained tension and compression are shown in figure 3. Two cements, a low heat portland and a normal portland are compared. Prior to the time of loading, the specimens were maintained under mass curing conditions from a starting temperature of 40° F. The load was applied at the age of 28 days. During the period of test under sustained load, the temperature was 80° F.

The tension specimens were subjected to constant sustained stresses of 50 and 100 pounds per square inch and the compression specimens were subjected to constant sustained stresses of 200 and 400 pounds per square inch. At the end of these tests, it was observed that at the early ages, the flow under tensile stress was greater than that under compressive stress. However, after a few weeks under load, the rate of tensile flow was considerably less than the rate of compressive flow. While this property of flowing more in tension than in compression may not be characteristic of all concretes, it obviously is a desirable property, since it tends
Fig. 4 — Effect of Composition and Fineness of Cement upon Flow.

Fig. 3 — Flow in Compression and Tension — Mass-Cured Concretes
to reduce cracking either due to drying or thermal changes.

Fiber Strains Under Sustained Flexural Load

Almost the same results as those in determining the flow in axial tension and compression were found in tests to determine the change of strain with time in both plain and reinforced concrete beams under constant sustained bending moment. Obviously, whether the structural element be plain or reinforced, it is desirable that the concrete of which it is composed be of quality, which, under drying conditions, will exhibit maximum flexural strength under sustained moment. In other words, it is desirable that the concrete be one in which the effects of plastic flow and shrinkage due to drying, one tending to offset the other, would combine to produce the most favorable distribution of stress.

Effect of Fineness and Composition of Cement Upon Plastic Flow:

At the end of these tests, it was observed that the coarsely ground low heat cement exhibited much the greater plastic flow at all ages, and that its rate of flow was greatest not only at the early ages, but also at the late ages. Referring to figure 4, it is seen that in the case of the low heat portland cement, the flow is greater for the coarser cement. In the case of the normal portland cement, the reverse is true. The reason for the reversal is not evident, but it seems possible that it may be due to variations in the particle size distribution of the cements. Results of later series of tests covering a wider range in specific surface of cements exhibit similar peculiarities again pointing to the possibility that the particle size distribution of a cement may exert an influence upon the flow characteristic of cement.
The shrinkage of concrete is a very complex action and is effected by many factors, such as temperature, moisture conditions, amount of reinforcement and size and shape of the concrete mass as well as by the characteristics of the concrete itself.

Concrete exposed to the atmosphere continues to shrink for a long time, rapidly at first and at a decreasing rate until after several years, it practically ceases. Laboratory investigations indicate that the curve of shrinkage strain plotted against time is similar to that of plastic flow. The amount of shrinkage is greatly affected by temperature and humidity storage. In saturated air or in water, there is a slight increase in length, which continues over a long period. The lower the humidity of air, the greater the shrinkage of exposed concrete. It is apparent that large masses do not shrink as rapidly or as much as small masses exposed to the drying effects of the air. It is also apparent that the concrete near the outside of large masses will shrink more than the interior, setting up unequal stresses which are, at least to some degree, released through plastic flow. If the shrinkage is too rapid, cracks may form due to the restraining effect of reinforcing steel or a damp core of concrete. Davis has measured the shrinkage of laboratory specimens of concrete stored for about 350 days under various controlled humidity conditions. These were 4 x 14 inch cylinders of 1 part cement to 5.67 parts gravel, water-cement ratio of 0.89, stored in fog for 28 days and found the results shown on Table I.
<table>
<thead>
<tr>
<th>Storage Condition</th>
<th>Total Change in Length (in inches)</th>
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<tr>
<td>50 Percent Relative Humidity</td>
<td>0.000907 shrinkage</td>
</tr>
<tr>
<td>70 Percent Relative Humidity</td>
<td>0.000706 shrinkage</td>
</tr>
<tr>
<td>100 Percent Relative Humidity</td>
<td>0.000176 expansion</td>
</tr>
<tr>
<td>Water</td>
<td>0.000142 expansion</td>
</tr>
</tbody>
</table>

**Table I.**
Effect of Moisture on Change in Length of 4" x 14" Concrete Cylinders
Stresses Developed due to Shrinkage

The effect of shrinkage upon the stresses in a reinforced concrete member may be understood by considering figure 5, which shows a member of unit cross sectional area $A$, symmetrically reinforced with steel in amount $pA$. If the piece were without reinforcement, the shortening or shrinkage would be $S$ but the action of steel reduces the actual change of length to $S_a$.

It is obvious from the figure that the interaction of the two materials results in a compressive stress in the steel and a tensile stress in the concrete, the two stresses being equal in magnitude for equilibrium. A plain concrete specimen will shrink as it dries out without producing any stress in the member due to shrinkage. Steel, on the other hand, does not shrink as the concrete dries out so its restraint tends to reduce the volume change of reinforced concrete.

Considering figure 5, we can see that the final length of the steel and concrete must be the same so equating the strains, we have

$$\frac{f_s}{E_s} = S - \frac{f_c}{E_c} \quad (1)$$

or

$$S = \frac{f_s}{E_s} + \frac{f_c}{E_c} \quad (2)$$

where

- $f_c = \text{tensile stress in concrete}$
- $f_s = \text{compression stress in steel}$

As the stresses at a given section must be in equilibrium, equating the compressive force in the steel to the tension stress in the concrete, we have
Solving equation 3 for $fs$,
\[ fs = \frac{fc(1-P)A}{PA} = \frac{fc(1-P)}{P} \]
Substituting this value of $fs$ in equation (1),
\[ \frac{fc(1-P)}{nSEc} = S + \frac{fc}{Ec} \]
Solving for $fc$,
\[ fc = \frac{nP}{HP(n-1)} SEc \]  
(4)
In the same way,
\[ fs = \frac{1-P}{H(n-1)P} SES \]  
(5)
As we don't know exactly the value of the shrinkage coefficient $S$, the use of these equations is at best a crude approximation. Plastic flow acts to reduce the stress set up in the concrete by shrinkage and this may be allowed by using an increased value of $n$ in the computations as will be demonstrated out. To cover the effect of flow in reducing shrinkage, Glanville \textsuperscript{**} states that $n$ is generally about 15 and will usually lie between 10 and 20. Other investigators have used a much larger value, \textsuperscript{***}

\textsuperscript{*} In columns, $P = \frac{As}{4}$. $Ax = PA$. But, $Ac = A - As$, so $Ac = A - PA = (1-P)A$.
\textsuperscript{***} For example, Turnauer and Maurer in "Principles of Reinforced Concrete Construction", 4th ed., p. 351, $n$ is taken as 45 for this case.
Fig. 6a: Idealized stress-strain curves for concrete of different strengths

Fig. 752
The Plasticity Ratio

For the purpose of explaining the behavior of concrete in a flexural member, a typical set of stress strain curves extended to rupture for concrete is assumed as shown in figure 6a. These curves are then further idealized in that they are represented approximately by a series of trapezoids as shown in figure 6b. This series of curves is sufficiently broad in scope to cover all grades of concretes from the highest to the lowest strength. The proportions of the trapezoidal diagram for a given compressive strength of concrete, \( f_c \), depend on two coefficients, namely, \( E_c \), the initial modulus of elasticity of the concrete, and \( \beta \), the ratio of the plastic deformation to the total deformation of the concrete at rupture, defined as the "plasticity ratio". Although a wide scatter is shown by the test data relating to the initial modulus of elasticity of concrete made with gravel or crushed stone, a satisfactory approximation may be made from the following empirical formula for the modular ratio:

\[
\eta = \frac{E_c}{f_c} = 5 \times \frac{1000}{f_c^2} \quad (1)
\]

where \( f_c^2 \) is the numerical value of the compressive strength, expressed in pounds per square inch. The formula is simple and easy to use. The modulus of elasticity of steel, \( E_s \), may be taken as \( 30 \times 10^6 \).

The existence of a property of concrete measured quantitatively by the plasticity ratio may be recognized in the behavior of concrete compression specimens at loads near rupture. The sudden rupture of the specimens of high strength almost immediately after the maximum load is reached, gives proof that the horizontal portion of the idealized stress strain
The gradual breakdown of the specimen of low strength, accompanied by large strains and the maintenance of a considerable proportion of its maximum resistance gives evidence that the horizontal portion of the trapezoidal diagram is relatively large for concretes of low strength. If $\varepsilon_1$ is the total concrete strain represented in the trapezoidal diagram at final rupture, and $\beta \varepsilon_1$ is the plastic strain represented by the horizontal portion of the diagram, then $\beta$ is the plasticity ratio which may have a numerical value within the extreme range between zero and one.

For the purpose of calculating the ultimate resistance of the beams, it is proposed that $\beta$ be determined from the empirical equation

\[
\beta = \frac{1}{1 - \left( \frac{f'c}{40000} \right)^2} \quad (2)
\]

where $f'c$ is again the numerical value of the compressive strength of concrete expressed in pounds per square inch.

The essential feature which is being brought out here is that for a given strength of concrete, there are two characteristics necessary to describe the stress-strain relationship all the way to rupture. These are the modular ratio defined empirically by equation (1) and the plasticity ratio defined empirically by equation (2). The complete trapezoidal stress diagram is thus determined by $f'c$ since $n$, $E_c$ and $\beta$ are then known from equations 1 and 2, and since the strains defined in figure 6b are then obtainable from equations

\[
\varepsilon_o = \frac{f'c}{E_c} \quad \varepsilon_1 = \frac{f'c}{(1 - \beta)E_c}
\]

where

\[
\varepsilon_o = \varepsilon_1 - \beta \varepsilon_1
\]
and where

\[ \varepsilon_f = \text{total concrete strain represented in the trapezoidal diagram at final rupture.} \]

\[ \beta \varepsilon_p = \text{plastic strain represented by the horizontal portion of the trapezoidal diagram.} \]

\[ \beta = \text{plasticity ratio.} \]

\[ \varepsilon_o = \text{total concrete strain minus plastic strain.} \]
Plastic Flow Design of Reinforced Concrete Members Subjected to Flexure

The present method of designing members under bending, combined bending and direct stress is unsatisfactory for several reasons.

The usual formulas based on assumption of a cracked section and straight line variation of stress are far from correct both under working loads when the concrete is not materially cracked and under ultimate loads when the stress in the concrete is not even approximately proportional to the distance from the neutral axis.

The usual flexure formulas are complicated by the use of the value of the modular ratio (the modulus of elasticity of steel to the modulus of elasticity of concrete), which is quite unpredictable under high loads and actually has little effect on the ultimate strength of the beam. The effective value of $n$ is widely different for dead and live loads. In the case of arch ribs, the dead and live load stresses cannot be computed separately with difference values of $n$ and added together, because dead load may produce compression only while the live load moment alone would require assumption of a cracked section. The calculation in this case would be very complicated with two values of $n$.

The present method makes no prediction of the loading which will cause cracking and does not give accurate control over the factors of safety under dead loads and live loads.

It has been pointed out recently by several investigators that while the stress variation in the concrete is approximately linear under very light loads, and parabolic under intermediate loads, as the ultimate load
is approached, it assumes a shape about as shown in figure 7a. The stress increases very rapidly near the neutral axis and is nearly uniform for the greater part of the depth of the compression section, probably decreasing slightly toward the edge of the beam. The usual parabolic formulas have been based on the theory that the stress variation in the beam followed the shape of a parabola.

This evidently is not true because of the greater ultimate strains and the different behavior of the concrete in the beam.

It is, therefore, proposed that a rectangular block of uniform stress, as indicated in figure 7b, be used to represent whatever stress may exist in the concrete. Whatever it actually is, it must have an average intensity, $f_0$, and an effective depth, $a$. The resultant is at the middle of the rectangle. Under ultimate load, Hooke's Law and the theory of elasticity have no significance as far as the internal stresses are concerned. No further theoretical justification of the assumption a rectangular compressive stress block is necessary if the formulas derived therefrom accurately predict the ultimate strength of the member.
Fig. 7a  
Actual Stress Distribution

Fig. 7b  
Assumed Design Stress Distribution

Fig. 8  
Assumed Design Stress Distribution

Reinforced Concrete Members Under Flexure

\[ T = A_s f_y = C \]

\[ C = a b f_c \]
Simple Flexure

The assumed relations for a rectangular beam under simple flexure are shown in figure 8.

It is assumed that in an under-reinforced beam, that is, one which will fail in the tensile steel, the concrete will crack as the steel stretches and the depth of the beam in compression, \( a^* \), will be reduced until the concrete unit stress reaches the ultimate and as \( T = 0 \)

\[ a = \frac{A_{sfy}}{bfc} \quad (1) \]

in which: 
- \( A_s \) = Area of tensile steel,
- \( f_y \) = Yield point stress in steel,
- \( b \) = Width of beam,
- \( f_c \) = Ultimate strength of concrete.

This determines the lever arm of the steel reinforcement since \( C = d - \frac{a}{2} \).

It will be assumed that the ultimate compressive strength of the concrete in the beam is equal to 85 percent of the ultimate compressive strength of a tested cylinder.

The values of \( a \) and \( c \) are derived as follows for any particular bending moment. See figure 9.

\[ M = Tc = Gc = abfc(d - a/2) \quad (2) \]

for which

\[ a = \frac{bd - \sqrt{d^2 - \frac{2M}{bfc}}}{bfc} \]

or

\[ \frac{a}{d} = 1 - \sqrt{1 - \frac{2M}{bfcTc}} \quad (3) \]

* The depth, \( a \), is less than the distance to the neutral axis, \( kd \).
and consequently,

\[ C = d - \frac{a}{2} = \frac{1}{2} \left( d - \sqrt{d^2 - \frac{2M}{E_s}} \right) \]  

These expressions are independent of the area of steel and the value of \( \frac{E_s}{E_c} \). The required steel area will be:

\[ M = T_c = f_y A_s \times C \quad A_s = \frac{M}{f_y C} \]  

Since the assumed compressive stress distribution has no exact theoretical basis, the limiting value of the depth of compression, \( a \), for equal concrete and steel strengths in flexure must be determined experimentally. If the beam has at least sufficient steel to fully develop the strength of the concrete, additional steel does not materially increase the strength of the beam.

The limiting value of \( a \) as computed from tests is reported as 0.537\( d \) *. The value of \( f_c \) assumed in equation (3) is 85 percent of the corresponding cylinder strength of \( f_c = 0.85f_{c1} \).

The flexural strength of a fully reinforced rectangular beam with tensile steel only is then, from equation (2),

\[ M = (d - \frac{a}{2})abf_c = (d - 0.2685d)0.537dbf_c = 0.393bd^2f_c. \]

This is a simple equation by which to determine the section dimensions.

---

* The limiting value of \( a \) as computed from tests reported by Slater and Lyse.
To determine the steel area, we say that

\[ M = T_e, \text{ but } T = f_y A_s \text{ and } C = (d - \frac{a}{2}) = 0.732d \]

therefore,

\[ M = f_y A_s (0.732d) \text{ and } A_s = \frac{M}{0.732f_y d} \]

This balanced steel area is considerably more than has been used generally. The plastic theory tends to give a smaller beam section with more steel than the working-stress straight line theory.

* \( f_y \) is the yield stress of steel.
Experimental Work

As mentioned in the introduction, the purpose of this work was to find the actual stress variation in the compression side of the beams.

For this purpose, two 8 foot long beams were designed. One of the beams was designed by the straight line method and the other by the plastic method. The straight line beam was designed for a load of 4000 pounds and the other for a load of 5000 pounds. The strength of the concrete was taken as 2000 pounds per square inch, with a water cement ratio of 8 gallons per sack of cement * and a cement-fine aggregate-coarse aggregate of 1-2/3-3/4.

In order to find the actual strength of the concrete, five standard cylinders were made to be tested. All of these cylinders were made from the same paste as the one used for the beams.

Design of Beams

**Straight line Method:**

Load 4 kips \( f_s = 20000 \text{ pounds per square inch} \)

\( f''c = 2000 \times 0.45 = 900 \text{ pounds per square inch} \)

Span = 8 feet

\[ n = \frac{Es}{Ec} = \frac{20 \times 10^6}{2 \times 10^6} = 15 \]

\[ M = 2000 \times 4 = 8000 \text{ pound-foot} \]

\[ = 8000 \times 12 = 96000 \text{ pound-inch} \]

\[ K = \frac{1}{1 \times f''c \times n} = 0.403 \]

\[ j = 1 - \frac{K}{3} = 1 - \frac{0.403}{3} = 0.866 \]

\[ R = \frac{f''c}{2} K j = \frac{900}{2} \times 0.866 \times 0.403 = 158 \]

\[ M = R b d^2 \]

\[ d^2 = \sqrt{\frac{M}{Rb}} = \sqrt{\frac{96000}{158 \times 6}} = 10 \text{ inches} \]

**Steel Area required:**

\[ M = Tjd = f_s A_s j d \]

\[ 96000 = 20000 \times 0.866 \times 10 \times A_s \]

\[ A_s = \frac{4.8}{10 \times 0.866} = 0.555 \text{ in}^2 \]

Use 3\( \frac{1}{2} \) 6\( \frac{1}{4} \) bars Area provided 0.60 in\(^2\) > 0.555 o.k.

**Check for shear:**

\[ v = \frac{V}{b j d} = \frac{2000}{6 \times 0.866 \times 10} = 38.4 \text{ pounds per square inch} \]

Using special anchorage (deformed bars)

\[ v = 0.03 f''c = 0.03 \times 2000 = 60 \text{ pounds per square inch} \]

60 > 38.4 No stirrups required.
Check for bond:

\[ u = \frac{V}{2000 \epsilon d} = \frac{2000}{3rd \times 0.866 \times 10} = 42.5 \text{ pounds per square inch} \]

Allowable \( u = 0.04 \frac{f_c}{f} = 0.04 \times 2000 = 80 \) pounds per square inch

\( 80 > 42.5 \text{ o.k.} \)

Maximum Deflection:

\[ \Delta = \frac{PT^3}{48EI} = \frac{4000 \times (6 \times 12)^3}{48 \times 2 \times 10^6 \times 6 \times (10)^3} = 0.0737 \text{ inches} \]

The cross section of this beam is shown in figure 9.

Check of Design for the First Beam:

\[ p = \frac{A_s}{bd} = \frac{3 \times 0.20}{6 \times 10} = 0.01 \]

\[ K = \sqrt{2pn - (pn)^2} - pn = \sqrt{2(0.01) \times 15} - (0.01 \times 15)^2 - 0.01 \times 15 \]

\[ K = \sqrt{0.30} - 0.0225 = 0.15 = 0.527 - 0.15 = 0.377 \]

\[ K = 0.377 \]

\[ j = 1 - \frac{K}{3} = 1 - \frac{0.377}{3} = 0.675 \]

\[ T = \frac{H}{Jd} = \frac{96000}{0.875 \times 10} = 11000 \]

\[ C = \frac{H}{Jd} = \frac{96000}{0.875 \times 10} = 11000 \text{ pounds} \]

\[ M = Tjd = f_s A_s jd - f_s'' = \frac{H}{Asjd} = \frac{96000}{3 \times 0.20 \times 0.875 \times 10} \]

\[ f_s'' = \frac{96000}{5.25} = 18300 \text{ pounds per square inch} \]

\[ M = Cjd = \frac{f_s''}{2} (Kd)(b)jd - f_c = \frac{2M}{Kbd^2} = \frac{192000}{0.377 \times 6 \times 100 \times 0.875} \]

\[ f_c'' = \frac{192000}{196} = 970 \text{ pounds per square inch} \]
Fig. 9
Cross Section of the Beam Designed by the Straight Line Method

Fig. 10
Cross Section of the Beam Designed by the Plastic Flow Method
Plastic Method:

Load 5 kips  Span 8 feet  \( f'c = 2000 \) pounds per square inch

\( f''_c = 900 \) pounds per square inch  \( f'_b = 20000 \) pounds per square inch

\( f_y = 50000 \) pounds per square inch

\[
M = C x c = 0.05f'c (0.537)d x 0.732bd
\]

Assuming \( b = 6 \) inches

\[
d^2 = \sqrt{\frac{120000}{1500}} = 8.15 \text{ inches} \quad \text{Use } d = 10 \text{ inches}
\]

Steel Area required:

\[
M = Tc = f_yA_b c
\]

120000 = 20000 x 10 x 0.732A_b

\( A_b = 0.823 \text{ inch}^2 \)

Use 5\( \frac{1}{2} \) \( \phi \) inch bars which furnish \( 1.00 > 0.823 \text{ in}^2 \)  o.k.

Check for shear:

\[
v = \frac{V}{b x 0.732d} = \frac{2500}{6 x 0.732 x 10} = 58 \text{ pounds per square inch}
\]

Allowable \( v = 0.03 f'c = 60 \) pounds per square inch

\( 60 > 58 \)  o.k.

Check for Bond:

\[
u = \frac{V}{2c x 0.732d} = \frac{V}{5\frac{1}{2} x 0.732 x 10} = 43.5 \text{ pounds per square in.}
\]

Allowable \( u = 0.04f'c = 80 \) pounds per square inch

\( 80 > 43.5 \)  o.k.

Maximum Deflection:

\[
\Delta = \frac{F_0^3}{48EI} = \frac{5000(8 x 12)^3 x 12}{48 x 2 x 10^6 x 6 x 10^3} = 0.0921 \text{ inches}
\]

Cross section of this beam is shown in figure 10.
40.

For the same design loading plastic theory gives smaller sections but uses more steel. Therefore, for countries where there is an abundance of steel, plastic theory is much more economical than the straight line method. On the other hand, countries as Turkey, where there is no steel and an abundance of concrete, the straight line method is very economical because the imported steel is very expensive.

The factor of safety should obviously depend on the nature of the structure and its loading. It may be desirable to provide different factors for different kinds of loads in some cases. As a general proposition, it appears that allowable dead plus live loads could be four tenths of the ultimate loads, and for extreme combinations of dead and live loads, and wind, temperature and shrinkage effects, one half of the ultimate might not be excessive.
Sample Calculation of Calculated Stresses at Failure

**Straight Line Method:**

To find the neutral axis of the above section

\[ 6x \left( \frac{y}{2} \right) = nA_g(10-x) \]

\[ 3x^2 = 90 - 9x \]

\[ 3x^2 - 9x = 90 = 0 \] \hspace{1cm} \text{dividing by 3}

\[ x^2 - 3x - 30 = 0 \]

\[ x = \frac{-3 - \sqrt{9 - 4 \times 1 \times 30}}{2} \]

\[ x_1 = \frac{-3 - 11.5}{2} = -7.2 \]

\[ x_2 = \frac{-3 + 11.5}{2} = 4.25 \]

Therefore, \( kd = 4.25 \) inches

\( jd = 10 - 4.25 \times \frac{2}{3}(4.25) = 8.6 \) inches

To find the steel stress at failure:

\[ M = Tjbd = f_s' A_s jd \]

\[ f_s'' = \frac{M}{A_s j d} \] \hspace{1cm} \text{but} \ M = \frac{F_l}{4} = \frac{P \times 8 \times 12}{4} \]
Substituting back we get \[ f_s'' = \frac{24P}{Ad} = \]
\[ f_s'' = \frac{24 \times 13000}{3 \times 0.20 \times 8.6} = 60500 \text{ pounds per sq. inch} \]

To find the concrete stress at failure
\[ M = Cjd = \frac{f_c''}{2}kd x b x jd \]
\[ f_c'' = \frac{24P}{kd x b x jd} = \text{ but } M = 24P \]
\[ f_c'' = \frac{24P}{kd x b x jd} = \frac{48 \times 13000}{4.25 \times 6 \times 8.6} = 2847 \text{ pounds per square inch} \]

Plastic Flow Method:
To find the steel stress at failure
\[ M = Tc = f_s'' A_s c \]
\[ f_s'' = \frac{M}{A_s c} = \frac{24P}{5 \times 0.20 \times 0.732 \times 10} \]
\[ f_s'' = \frac{24 \times 20000}{1000} = 65600 \text{ pounds per square inch} \]

To find the concrete stress at failure
\[ M = Cc = 0.85f_{cu} x a x b x c \]
\[ f_{cu}'' = \frac{24P}{5.37 \times 7.32 \times 6 \times 0.85} = \frac{24 \times 20000}{201} = 2390 \text{ pounds per square inch} \]

To find the unit strain in steel, we multiply the extensometer reading by
\[ \frac{1}{1222} \]
Sample Calculation of Actual Stresses in Steel

\[ E_S = \frac{S_S}{\varepsilon_S} \quad E_S = 30 \times 10^6 \]

\[ E_C = \text{modulus of elasticity of concrete (pounds per square inch)} \]
\[ E_S = \text{modulus of elasticity of steel (pounds per square inch)} \]
\[ S_S = \text{actual stress in steel (pounds per square inch)} \]
\[ \varepsilon_S = \text{actual strain in steel (inch per inch)} \]
\[ S_C = \text{actual stress in concrete (pounds per square inch)} \]
\[ \varepsilon_C = \text{actual strain in concrete (inch per inch)} \]

\[ S_s = 30 \times 10^6 \times 0.00138 = 41400 \text{ pounds per square inch} \]

Sample Calculation of Actual Stresses in Concrete

\[ E_C = \frac{S_C}{\varepsilon_C} \]
\[ S_C = 1.4 \times 10^6 \times 0.00112 = 1570 \text{ pounds per square inch} \]

The results of straight line and plastic flow methods are shown on Tables IV and V.
### Data for the Beam with 3 Longitudinal Bars - Straight Line Method

<table>
<thead>
<tr>
<th>Load in Pounds</th>
<th>Stress in Steel lbs/in²</th>
<th>Strain in Steel in/in</th>
<th>Concrete Strain in in/in</th>
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Table II
Data for the Beam with 5 Longitudinal Bars—Plastic Flow Method

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<th>Extensometer Reading</th>
<th>Strain in Steel in/in</th>
<th>Strain in Concrete 1&quot; from top</th>
<th>Strain in Concrete 3&quot; from top</th>
<th>Strain in Concrete 5&quot; from top</th>
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Table III
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Table III - continued
Straight Line Method

Beam with 3 Longitudinal Bars

\[ E_s = 30 \times 10^6 \quad E_c = 1.4 \times 10^6 \]

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<tr>
<th>Load in Pounds</th>
<th>Computed theoretical stress of steel in ( \text{lb/in}^2 )</th>
<th>Computed theoretical stress of concrete in ( \text{lb/in}^2 )</th>
<th>Computed actual stress of steel in ( \text{lb/in}^2 )</th>
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* For method of computation, see page 44.
** For method of computation, see page 43.
Plastic Flow Method

Beam with 5 Longitudinal Bars

\[ E_s = 30 \times 10^6 \quad E_o = 1.4 \times 10^6 \]

<table>
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<th>Load in Founds</th>
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<th>Computed theoretical stress of concrete in lb/in(^2) *</th>
<th>Computed actual stress of steel in lb/in(^2) *</th>
<th>Computed Actual Stress of Concrete in lbs/in(^2) **</th>
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**Table V**

* For method of computation, see page 42.
** For method of computation, see page 43.
Discussion and Conclusion

The following discussion is based on Tables IV and V and Figures 11, 12 and 13.

At low loads there is a great difference between the actual and the computed stress in steel for both design methods. This is due to the fact that at low loads, most of the tension is taken by the concrete and a very small amount by the steel. The neutral axis at that time is far below the actual one and is very near to the steel bars. Consequently, the moment arm of the resisting moment is very small as shown in figures 11 and 12. However, as the load is increased, it is seen that the actual stress is catching up with the computed one. This is due to the fact that at high loads steel is taking all the tension and none is left for the concrete. The neutral axis goes up and the moment arm of the resisting moment is increased.

At low stresses, there is a small difference between the computed and actual concrete stress. This difference is probably the effect of shrinkage. The evaporation of water causes shrinkage in the concrete, which in turn produces some tension in the concrete and compression in the steel. Therefore, at low loads, we can say that the concrete has very small stress due to the fact that the compressive stress produced by the application of a small load is overcome by the tension in the concrete produced by shrinkage. See figure 14.
Measured Compressing Stress distribution

Beam with 3 Longitudinal Bars

Scale: 1" = 2400 lb/in²

1" = 4000 lb/in²

24600 lb/in²

41400 lb/in²

Fig. 11
Measured Compressive Stress distribution

Beam with 5 Longitudinal bars

Scale 1" = 200 $\frac{\text{lb}}{\text{in}^2}$

Scale 1" = 800 $\frac{\text{lb}}{\text{in}^2}$

Scale 1" = 1600 $\frac{\text{lb}}{\text{in}^2}$

Scale 1" = 2000 $\frac{\text{lb}}{\text{in}^2}$

15000 $\frac{\text{lb}}{\text{in}^2}$

12000 $\frac{\text{lb}}{\text{in}^2}$

24000 $\frac{\text{lb}}{\text{in}^2}$

45000 $\frac{\text{lb}}{\text{in}^2}$

Fig. 12
Stress-Strain Curve for concrete

(Symbols represent the different cylinders tested.)

Unit Strain in Thousandths

Fig. 13 Stress Strain Curve for Concrete
At high loads, however, if we compare Table IV with Table V, we see that Table IV shows the measured concrete stress to be somewhat less than the computed concrete stress by the straight line method. This indicates that the beam has too much concrete, as was expected.

Table V, on the other hand, shows the measured concrete stress to be greater than the computed stress by the plastic method. This is due to the effect of plastic flow, which produces a redistribution of stresses. This redistribution of stresses raises the neutral axis by a small amount. On the other hand, however, it lowers considerably the point of application of the compressive force (see Figure 12) thus decreasing the moment arm of the resistive moment and increasing the stress in the steel. Also, the strains at high loads are increasing rapidly while the stresses are almost constant. This, of course, occurs somewhere in the vicinity of failure.

* See page 41 for computation of stress in concrete by straight line method.
** See page 42 for computation of stress in concrete by plastic method.
where the stress-strain curve becomes flat.

If we examine table V and compare the concrete stress corresponding to the steel stress at the yield point with the concrete stress corresponding to the steel stress at the yield point in table IV, we see that in Table V, when the steel stress is at its yield point, the concrete stress is very near its ultimate cylinder stress, which indicates that the plastic flow theory gives a balanced design. On the other hand, however, if we look at table IV, we see that although the stress in steel has reached its yield point, the corresponding concrete stress is very low, which proves that the straight line method does not give as balanced a design as the plastic method does.

Another important case which is observed is the fact that the compressive stress is always greater than the tensile one. This is most probably due to the measured concrete strains which were somewhat inaccurate. However, the greatest mistake in this case is where we assume the modulus of elasticity of concrete cylinder to be equal to the modulus of elasticity of the beams tested. It has been shown that the modulus of elasticity of concrete is something very inconsistent and that we should never rely on it.

It is assumed that Hooke's Law applies to concrete also to some definite limit, but this is not true as the stress strain curve of concrete is not a straight line as shown in figure 13. Therefore, it is concluded that a small error in reading the strains multiplied by a wrong modulus of elasticity will give results that are in error.

Comparing the external applied moment with the internal resisting moment, we see that in every case the external moment is greater than the internal
Figure 16 shows the cracks which formed in the beam at failure due to excess deformation in steel.

**Conclusions**

The plastic theory proposed for the design of concrete members is different from the straight line method in that it recognizes the plastic action of concrete instead of attempting to predict the stresses at working loads. The plastic flow method does not involve the modulus of elasticity of concrete.

The application of the plastic theory is much simpler than the straight line stress variation method and agrees better with test results. It can be taught more easily than the present method and will lead designers to a better understanding of the actual structural action.

The design of plastic theory gives smaller sections with greater amount of steel. This shows that the plastic flow method is much more economical than the straight line method for countries which have an abundance of steel. For a given story height, the plastic design method will reduce the required building height since shallower beams are needed.

The present method has been useful during a time when knowledge and experience with reinforced concrete were being accumulated. It provides in some cases what is now an excessively high factor of safety against concrete failure. It leads to uneconomical design and an inconsistent factor of safety. It is based on an empirical value of $\frac{E_e}{E_c}$, which $E_c$ in turn assumes the stress-strain curve of concrete to be a straight line till a definite point, which is not true as Hooke's Law does not apply to concrete at all.
Suggestions

The apparatus used to test the two beams is shown in figure 15. As it is seen from this figure, to measure the concrete strain an Amo's dial reading to 0.0001 inches is used. In order to measure the steel strain, however, an extensometer is used. The extensometer, as was expected, gave very good results. On the other hand, it was very difficult to measure the concrete strains because as the instrument was very sensitive, it was impossible to count the revolutions which tends to lead to mistakes. Therefore, I recommend that if any experiments are to be conducted on the same theory, all apparatus should be used in the same way except in the case of measuring the concrete strains. Instead of using one Amo's dial and moving it up and down, three should be used and they should be fixed somehow on the plates so that the readings will be more accurate.
Fig. 15
Apparatus Used

Fig. 16
Cracks Formed at Failure
Literature Cited and Consulted


8. Plastic Flow, Shrinkage, and Other Problems of Concrete and Their Effect on Design, by Oscar Faber, Min. of Proc. Inst. of Civil Engineers (Great Britain), Vol. 225, 1927-1928, Part I.


Effect of plastic flow on flexural stresses in reinforced concrete beams