Abstract:
The following paper is a further application of the method of suc-cessive corrections. The analysis of thin-walled cylindrical vessels is shown to be easily made by an application of the Hardy Cross moment and shear distribution procedures.

The paper is divided into two sections. The first section consists of an analysis of thin-walled cylindrical shells and flat-plate heeds Which is made by separating the membrane stresses from the bending stresses. such a separation results in the deflection equation for thin-walled shells being (Formula not captured by OCR) where r' is the membrane deflection, and the remainder of the expression is deflection due to bending.

Tbe second auction of the paper shows the method and resulting ex- pression for the fixed-end moments and shears, the carry-over factor and the distribution factors which are necessary for the application of successive corrections.

Examples are worked by both methods to demonstrate the practicability of the method introduced.

THE ANALYSIS OF THIN-WALLED PRESSURE VESSELS
BY
RELAXATION METHODS

by

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Appendix

ABSTRACT

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The paper is divided into two sections. The first section consists of an analysis of thin-walled cylindrical shells and flat-plate heads which is made by separating the membrane stresses from the bending stresses. Such a separation results in the deflection equation for thin-walled shells being

$$v = e^{Bx}(A_1 \cos Bx + B_1 \sin Bx) + e^{-Bx}(C_1 \cos Bx + D_1 \sin Bx) + r'$$

where r' is the membrane deflection, and the remainder of the expression is deflection due to bending.

The second section of the paper shows the method and resulting expression for the fixed-end moments and shears, the carry-over factor and the distribution factors which are necessary for the application of successive corrections.

Examples are worked by both methods to demonstrate the practicability of the method introduced.

INTRODUCTION

During the past twenty years a great deal of interest has been shown in numerical methods of analysis, particularly for engineering problems which do not lend themselves readily to rigid mathematical analysis. This interest was stimulated in this country almost entirely by Professor Hardy Cross[1] who presented the procedures for moment distribution. Numerical methods of analysis had been used before the presentation of this work but a new trend, of accenting the physical rather than the mathematical aspects, was introduced by this method.

The numerical methods of successive corrections has now been applied to such engineering fields as hydraulics, electricity, vibrations, thermodynamics and stress analysis. The wide use of the method attests to its practicability.

This thesis is a further application of the relaxation method. The work is based on the Hardy Cross methods of moments and shear distribution and results in a simplification of the conventional method of thin-walled pressure vessel analysis.

The general equation for the deflection at any point in a thin-walled cylindrical shell is found to be

$$v = e^{Bx}(A_1 \cos Bx + B_1 \sin Bx) + e^{-Bx}(C_1 \cos Bx + D_1 \sin Bx) + r'$$

where A_1, B_1, C_1 and D_1 are constants resulting from the solution of a differential equation. These constants are shown to be dependent only on the manner of supporting the ends of the shell.

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(1) Hardy Cross, Analysis of Continuous Frames by Distributing Fixed-End Moments, Transactions A. S. C. E., 1932, pp 1-156.

The conventional method of analysis consists of equating the deflection of the shell to that of its restraining members at the joints. Since there at least four unknown constants in these expressions, it is necessary to equate also the moment or shearing force equations. This means that it is necessary to solve at least four simultaneous equations, and since the restraint on the shell is dependent on the stiffness of the restraining member it is necessary to solve these four equations for each vessel.

By applying artificial restraints to the ends of the shell to eliminate any rotation or deflection of these points, there is only one set of boundary conditions for all shells. This eliminates a major portion of the work from the solution. The artificial restraints are then removed by successive corrections.

In the analysis of thin-walled cylindrical shells the membrane stresses are separated from the bending stresses. This is necessary since the balancing of moments and shears is independent of the membrane stresses.

SECTION I

ANALYSIS OF STRESSES IN THIN-WALLED PRESSURE VESSELS

INTRODUCTION - A thin-walled vessel is one in which the wall thickness is small in comparison with the diameter of the vessel. The degree to which this condition must be satisfied is such that the assumption - the unit stresses within the material, neglecting those produced by bending, are uniform across the longitudinal cross-section of the vessel - can be made with little error.

A thin-walled shell, free from the restraining effects of heads or stiffeners and subjected to an internal fluid pressure, expands uniformly in the radial and longitudinal directions and the stresses induced are uniformly distributed over any cross-section. These stresses are the membrane stresses and are the major stresses to be considered in a pressure vessel at points where there are no discontinuities. For instance, a vessel with a neutral surface in the form of a sphere and with no discontinuities such as riveted or reinforced joints would have no stresses other than membrane stresses. A soap bubble is a perfect example of such a stress state.

A pressure vessel with a thin-walled cylindrical shell such as is to be considered here, must have at least one end closed. Whether the closure is formed by hydraulic pressure and a junction with some other structure or by a head, a discontinuity will exist at the joint. This discontinuity in the uniformity of the structure will produce shearing and bending stresses which are not uniformly distributed over the longitudinal cross-section.

For the application of successive approximations it is necessary to consider the stresses in the vessel as being membrane stresses produced by the internal fluid pressure, superimposed on bending stresses produced by restraining moments and shears.

MEMBRANE STRESSES - The magnitude of the unit stress acting on a longitudinal cross-section of a cylindrical vessel subjected to an internal fluid pressure p is

$$S_1 = \frac{p\,r}{t} \qquad (a)$$

where S_1 is the unit stress acting normal to the longitudinal cross-section, r is the radius of the vessel and t is the wall thickness.

The magnitude of the unit stress acting on a transverse cross-section is given by the expression

$$S_2 = \frac{p\,r}{2t} \qquad (b)$$

where S_2 is the unit stress acting normal to the transverse cross-section of the vessel.

For convenience the stresses S_1 and S_2 will hereafter be referred to as the circumferential membrane stress and the longitudinal membrane stress respectively.

DEFORMATIONS DUE TO MEMBRANE STRESSES - The membrane stresses found above will be accompanied by deformations of the heads and shell of the vessel. In the heads the stresses will produce a change in the diameter at the junction with the shell. However, in a vessel with flat heads as will be considered here, this change will be negligible in comparison with the other deformations and may be considered zero without introducing any appreciable errors. The deformations of the shell consist of an elongation in the longitudinal direction and a change in the shell radius. As the deformation in the longitudinal direction has no bearing on the work to follow, it may be omitted and only the radial deflection found.

THE CHANGE IN SHELL RADIUS DUE TO MEMBRANE STRESSES - The membrane stresses act on a differential particle of the cylindrical shell in the manner shown in Figure 1. The X-axis and Y-axis represent the longitudinal and transverse directions respectively and the Z-axis represents the direction normal to the shell surface.

By Poisson's Ratio

$$S_y = S_1 - uS_2$$

where u is Poisson's ratio for the material of the shell and S_y is the total stress on the circumferential direction.

Also by Young's Modulus

$$S_y = E\, e_y$$

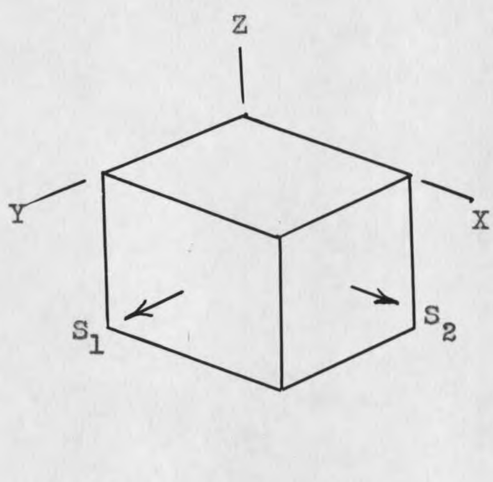

Figure 1

where e_y is the unit distortion in the circumferential direction produced by the stress S_y. Equating these expressions and solving for e_y gives

$$e_y = \frac{S_1 - uS_2}{E}$$

or, using the values for S_1 and S_2 given by equations (a) and (b)

$$e_y = \frac{p\,r}{2Et}(2 - u)$$

Now by letting ΔC be the change in the length of the circumference of the shell and denoting the change in the radius of the neutral surface of the shell by r', it is evident that

$$\Delta C = 2\,\pi r e_y = \frac{\pi p r^2}{Et}(2 - u)$$

and

$$r' = \frac{\Delta C}{2\pi}$$

or

$$r' = \frac{pr^2}{2Et}(2 - u) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

This expression is derived using the stresses defined by equations (a) and (b) which are based on the assumption that the pressure is uniformly distributed within the vessel. Equation (1) must therefore be limited to vessels with uniform pressure distribution.

<u>MEMBRANE STRESSES AND DEFORMATIONS IN VESSELS WITH NON-UNIFORM PRESSURE DISTRIBUTION</u> - Vessels such as storage tanks and standpipes which contain fluids of relatively high densities have membrane stresses which not only depend on the pressure but also on the manner of support of the vessel. For this reason it is not feasible to write a general equation for the change in radius in a vessel of this type. Hereafter in this paper when it is desirable to consider a vessel containing a non-uniform pressure, the change in radius will be referred to as $f(x)$ rather than r'.

<u>BENDING STRESSES IN CYLINDRICAL THIN-WALLED SHELLS</u> - Figure 2 shows a

thin-walled cylindrical shell subjected to an externally applied bending moment of M_o pound inches per unit arc length at one end. Since there are no direct loads, this is a case of pure bending of the shell.

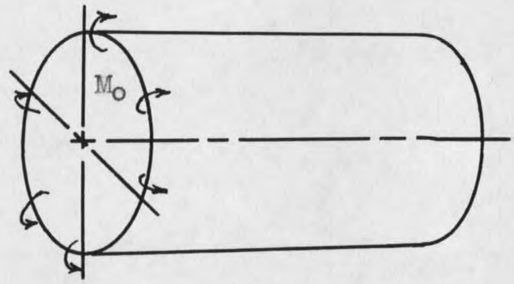

Figure 2

The following analysis will be confined to a longitudinal strip of the shell a unit arc length in width and x units in length. This may be done since both the shell and the loading are symmetrical about the longitudinal axis. This strip is shown in Figure 3. An arbitrary differential length dx, bounded in the unstressed state by planes a and b and in the stressed state by planes a' and b', is represented by the shaded areas.

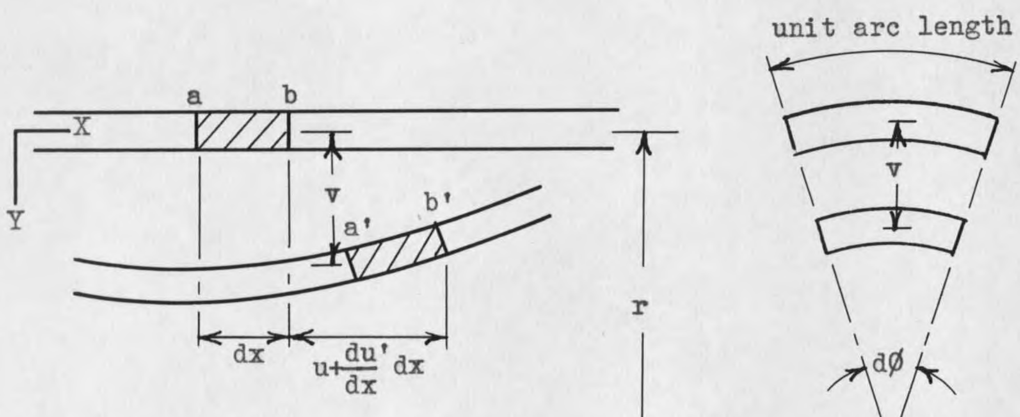

Figure 3

The notation used in the discussion is as follows:

u' = displacement in the X direction

v = displacement in the Y direction

e_x = unit strain in the X direction

e_y = unit strain in the circumferential direction

e_{yx} = unit strain in the X direction caused by the circumferential stress

e_{tx} = unit strain in the X direction caused by the bending stresses

Using this notation it is possible to write the expression for the unit strain in the circumferential direction as

$$e_y = \frac{r d\phi - (r - v) d\phi}{r d\phi} = \frac{v}{r} \qquad (c)$$

The differential section dx is shown again in Figure 4. The plane c'c" is passed parallel to plane a'a" through the intersection of b'b" and the neutral axis. The distance between planes c'c" and a'a" is then the original length of the fibers. Assuming a plane cross-section before bending will be a plane cross-section after bending the final length of the fibers is the distance between planes a'a" and b'b". It is then evident that the change in length of a fiber is the distance between planes b'b" and c'c" measured at the fiber.

Let y be the distance from the neutral axis to any fiber. Then

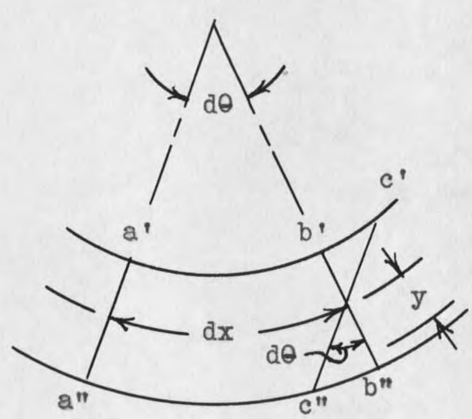

Figure 4