Partial-betweenness convexity
by John Richard Ellefson

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE in Mathematics
Montana State University

Abstract:
Five postulates are given which are used to define a betweenness relation on a set. Examples are given which show the relation is a generalization of the betweenness usually associated with real vector space on the one hand and lattices on the other. Convex subsets are then defined on the set and shown to be of finite character.

The extreme points of a convex subset and maximal convex sets are next defined and some of their properties developed. The convex hull of a subset is proved to be equal to the intersection of the maximal convex subsets.

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ABSTRACT

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INTRODUCTION

In order to define convexity in a set, it is necessary to have some means of determining when a given element is between two other elements. Conversely, if we are able to determine when a point is between two other points, we can define convexity. Exactly what is meant by "between" in the above statements depends on the set containing the given elements. If the set is a real vector space, betweenness is usually taken as being a member of the line segment determined by two vectors. If the set is a lattice, betweenness can be taken as greater than the join and less than the meet of two elements. It is also possible to postulate certain properties for the betweenness relation and let these determine what betweenness is to mean. The latter procedure is followed in our case.

Once we have a betweenness, we are able to define what is meant by a convex set. Convex sets are usually assumed to have certain properties, and it is necessary to demonstrate these properties in order to earn the full right to the name. In Sections 2, 3, and 4 these properties are developed without commentary, which makes it easier to refer to the theorems, propositions, etc., when they are mentioned in Section 5.

We will express "the element b is between a and c" as <u>abc</u>. The union of two sets will either be written out or we will use $\cup$ placed between the sets. Small Roman letters will be used for elements of sets; and capital Roman letters will be used for sets with the exception of single-element sets, which will be designated by small Roman letters with bars over them.

In Section 1 an example is given which contains numbers, and no attempt is made to distinguish between the numbers and vectors. There is little danger of confusion, since it is obvious what is to be proved and how it is to be done.

1. THE POSTULATES

Let E be a non-empty set of elements for which a betweenness relation satisfying the following postulates is defined.

I. *If* a, b, *and* c *are elements of* E, *abc implies cba*.

II. *If* a, b, *and* c *are elements of* E, *abc and acb if and only if* b = c.

III. *If* a, b, c, *and* d *are elements of* E, *abc and adb imply dbc*.

IV. *If* a, b, c, x, *and* y *are elements of* E, *axc*, *byc*, a $\neq$ y, *and* b $\neq$ x *imply there exists an element* s *in* E *such that asy and bsx*.

V. *If* a, b, c, t, x, *and* y *are elements of* E, *axc*, *byc*, *xty imply there exists an element* s *in* E *such that stp and asb*.

Some of the immediate consequences of the postulates which will be needed in developing the properties of convex sets defined by the betweenness relation will be derived next.

1.1 *If* a *and* b *are elements of* E, *aba if and only if* a = b.

From II we see that a = a implies *baa*, which in turn implies *aab*. The latter, combined with the given *aba*, implies a = b and establishes the necessity. The sufficiency follows immediately from II.

1.2 *If a, b, t, x, and y are elements of E, axb, ayb, and xty imply atb.*

Using V, we see that the hypothesis implies the existence of an element s such that stb and asa. By 1.1, a = s and the conclusion follows.

1.3 *If a, b, t, and y are elements of E, ayb and aty imply atb.*

Let a = x in 1.2 and the conclusion follows immediately.

The postulates and the above properties are the only ones we will use in the subsequent development. It would be interesting to investigate the implications of the postulates more thoroughly; but the central purpose is to investigate subsets having "convexity properties," and for this goal we will not need further elaboration of the postulates.

It is necessary, however, to assure ourselves that we are not dealing with an empty theory. To this end it seems appropriate to construct examples satisfying the postulates. Two examples will be given which show that the convex sets to be defined correspond to well-known convex sets under suitable assumptions. A third example satisfying the postulates will be given which indicates that these convex sets are more general than those in the other examples.

Example 1. Let E be a real or complex vector space, and define abc as meaning there are non-negative real numbers

m and n such that $b = ma + nb$ and $m + n = 1$. The verification of the postulates in this example is messy but essentially trivial. We will not verify them here, but will pass on to a more interesting example.

<u>Example 2</u>. Let E be a distributive lattice, and define <u>abc</u> as meaning ordinary lattice betweenness (6). The verification of the first four postulates is again trivial. However, the fifth one depends on constructing the element s and, while not difficult, is not immediately obvious. We will, therefore, sketch the verification of Postulate V.

Let ab and $a + b$ be the meet and join of a and b in the lattice. We wish to show that there exists an s such that $ab \leq s \leq a + b$ and $sp \leq t \leq s + p$ when we are given $ap \leq x \leq a + p$, $bp \leq y \leq b + p$, and $xy \leq t \leq x + y$. We will take $s = ab + at + bt$. This rather obviously satisfies $ab \leq s \leq a + b$. To check the other relationship, notice that $abp \leq t$ because $ap \leq x$ and $bp \leq y$. We have, therefore, $sp = abp + atp + btp \leq t$. Now $(s + p)t = (ab + a + b + p)t = (a + b + p)t = t$ because $t \leq x + y \leq a + b + p$. Thus Postulate V is verified.

<u>Example 3</u>. Let E be a filter on a distributive lattice, and define <u>abc</u> as meaning $ac \leq b \leq a$ or $ac \leq b \leq c$, where the notation is the same as in Example 2. We note that the dual filter, or ideal, might as easily have been used. The

first three postulates are again easy to verify; and the verification of the fourth is quite similar to that of the fifth, which we shall verify.

We wish to show that there exists an s such that $ab \leq s \leq a$ or $ab \leq s \leq b$; and $sp \leq t \leq s$ or $sp \leq t \leq p$ when we are given that $ap \leq x \leq a$ or $ap \leq x \leq p$, $bp \leq y \leq b$ or $bp \leq y \leq p$, and $xy \leq t \leq x$ or $xy \leq t \leq y$.

Assume $ap \leq x \leq a$ and $bp \leq y \leq b$ and $xy \leq t \leq x$. Let $s = ab + t$, $s = ab + t \leq ab + x \leq ab + a = a$ and $sp = abp + tp \leq xy + tp \leq t$. The first chain assures us that s is an element of the filter, and the second that $sp \leq t$. Collecting our results, we see that $ab \leq ab + t = s \leq b$ and $sp \leq t \leq ab + t = s$.

Assume $ap \leq x \leq p$ and $bp \leq y \leq p$ and $xy \leq t \leq y$. If we let $s = ab$, we see that s is an element of the filter and that $ab = s \leq a$, $sp = abp \leq ab \leq ab = s$; and once again Postulate V is verified.

The remaining cases are quite similar to one or the other of the above verifications and will be assumed to avoid excessive repetition.

The use of distributive lattices in Examples 2 and 3 is necessary because the betweenness relation satisfies 1.2. A theorem by M. F. Smiley and E. Pitcher (6) proves that this is necessary and sufficient for E to be a distributive lattice if E is a lattice. This puts somewhat of a restriction on the sets E in which our convex sets can be found, but not an overly strong one. In Example 3 we have a set which is not a lattice even though it is a subset of a lattice, which is adequate for the postulates.

It is very easy to construct examples showing the independence of postulates on finite sets. However, Postulate IV can be derived from V in a real vector space, and this raises the question of the independence of IV in a sufficiently "rich" set E. The question seems to be far from trivial and is unanswered at present.

Examples of finite sets which illustrate the independence of Postulates IV and V are Figures 1 and 2 below. Figure 3 is a finite set satisfying all the postulates and shows that E need not have infinitely many points.

In the following figures an element x is said to be between a and b if it lies on the line segment determined by a and b.

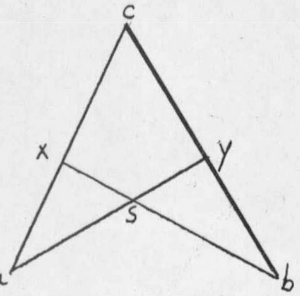

Figure 1.

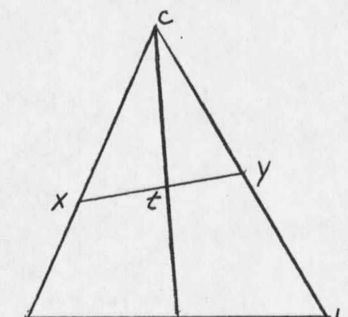

Figure 2.

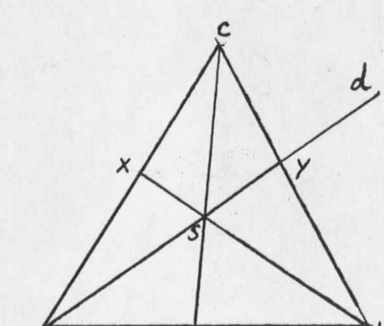

Figure 3.

Care should be taken not to confuse the illustrations with the set E in general. For instance, the intersection of non-parallel lines need not define a unique point; there may be other points common to the two line segments.

2. CONVEX SETS

Definition. A line segment determined by two elements of E is defined to be the set of all elements of E which are between the two given elements. If a and b are the elements, the line segment determined by a and b will be denoted by L(a,b).

2.1 Remark. For any a and b in E, L(a,b) is not empty and is equal to L(b,a).
This follows immediately from the definition and Postulates I and II.

2.2 Proposition. For any a and b in E, if t belongs to L(a,b), then the union of L(a,t) and L(t,b) is a subset of L(a,b).

Proof. Let x be any element of L(a,t). By definition we have atb and axt. Applying 1.3 we get axb. Since x was arbitrary in L(a,t), L(a,t) is a subset of L(a,b). In a similar manner we get that L(t,b) is also a subset of L(a,b), whence the proposition.

2.3 Proposition. For any a and b in E, if x belongs to L(a,b), then the intersection of L(a,x) and L(x,b) is equal to $\overline{x}$.

Proof. Let t be any element in the intersection. This means t belongs to both L(a,x) and L(x,b), which implies by definition xtb and axb. Applying 1.3 we get atb, which combined with axb implies axt using Postulate III. Since t belongs to $L(a,x)$, we have <u>atx</u>, which implies $x = t$ when we use II. Thus we see that any element in the intersection is equal to x, which is to say $\overline{x}$ equals the intersection.

2.4 Proposition. <u>For any</u> a, b, c, d, <u>and</u> x <u>in</u> E, <u>if</u> a <u>is an element of</u> $L(b,x)$ <u>and</u> c <u>is an element of</u> $L(d,x)$, <u>then the intersection of</u> $L(a,d)$ <u>and</u> $L(b,c)$ <u>is not empty</u>.

Proof. First consider the case $b = c$ or $a = d$. If $b = c$, then <u>bax</u> and <u>dbx</u>, which implies <u>dba</u> by III and I. By definition we have b in $L(a,d)$ and $L(b,b)$ so the intersection is not empty. Similarly, $a = d$ implies the intersection is not empty.

Next consider the case $b \neq c$ and $a \neq d$. By definition we have <u>bax</u> and <u>dcx</u>, which implies that there exists an s such that <u>asd</u> and <u>bsc</u>. Therefore, s is contained in the intersection of $L(a,d)$ and $L(b,c)$, which means that it is not empty.

The line segments correspond to the closed intervals in the real numbers as can be seen from properties 2.1 through 2.4. In fact, since the usual definition of betweenness in the real numbers satisfies our postulates, if $E = R$, the closed intervals will be the line segments.

<u>Definition</u>. A convex subset of E is defined to be a

set which contains the line segment determined by any pair of its points. Rather than use the phrase, "A is a convex subset of E," we will frequently shorten this to "A is convex" or speak of "convex A".

2.5 Remark. E, ∅, <u>and any single-element set of</u> E <u>are all convex</u>.

2.6 Proposition. <u>For any</u> a <u>and</u> b <u>in</u> E, L(a,b) <u>is convex</u>.

Proof. Let x and y be any elements of L(a,b) and t any element of L(x,y). From the definition of a line segment we have <u>axb</u>, <u>ayb</u>, and <u>xty</u>. Applying 1.2 we get <u>atb</u>, which implies that t belongs to L(a,b). This implies that L(x,y) is a subset of L(a,b); and, since x and y are arbitrary, this means that L(a,b) is convex.

2.7 Proposition. <u>The intersection of any family of convex sets of</u> E <u>is convex</u>.

Proof. Let J be the family of convex sets and x and y any elements in the intersection. Since x and y are in every member of J and the members of J are convex, L(x,y) is a subset of every member of J and is, therefore, contained in the intersection demonstrating its convexity.

<u>Definition</u>. The convex hull of any given subset of E is defined to be the intersection of all convex sets which contain the given set. If A is a subset of E, the convex hull

of A will be denoted by A^c.

2.8 Remark. $\emptyset^c = \emptyset$, $E^c = E$; <u>and, if</u> x <u>belongs to</u> E, $\bar{x}^c = \bar{x}$.

2.9 Remark. <u>If</u> A <u>is a subset of</u> E, A <u>is a subset of</u> A^c.

This follows immediately from the definition.

2.10 Proposition. <u>If</u> A <u>is a subset of</u> E, A^c <u>is convex</u>.

Proof. By definition, A^c is the intersection of the family of convex sets which contain A. 2.7 implies the convexity of A^c.

2.11 Proposition. <u>If</u> A <u>is a subset of</u> E, $(A^c)^c = A^c$.

Proof. From 2.9 we know that A^c is a subset of $(A^c)^c$. From 2.10 we see that A^c is a convex subset containing A^c and, therefore, contains $(A^c)^c$, hence the proposition.

2.12 Proposition. <u>If</u> A <u>is a subset of</u> E, <u>then</u> A <u>is convex if and only if</u> $A^c = A$.

Proposition 2.10 shows the necessity. A being convex implies that A^c is a subset of A, which shows the sufficiency.

2.13 Proposition. <u>If</u> A <u>and</u> B <u>are any subsets of</u> E, <u>the following statements are equivalent:</u>

a) <u>The convex hull of the intersection of</u> A <u>and</u> B <u>is a subset of the intersection of</u> A^c <u>and</u> B^c.

b) <u>The union of</u> A^c <u>and</u> B^c <u>is a subset of the convex</u>

hull of the union of A and B.

c) If A is a subset of B, A^c is a subset of B^c.

Proof. First we shall prove a) implies b). Since A and B are any subsets of E, let the B in a) stand for $A \cup B$. Set Theory then implies that A^c is a subset of $(A \cup B)^c$. Statement a) is symmetrical in A and B, hence Statement b).

Next we shall prove that b) implies c). If A is a subset of B, Statement b) implies that A^c is a subset of $(A \cup B)^c$ and that the latter is equal to B^c.

Last we prove that c) implies a). The intersection of A and B is contained in both A and B. Statement c), therefore, implies that the convex hull of the intersection of A and B is contained in both A^c and B^c and thus is in their intersection.

2.14 Theorem. If A is a convex subset of E, A is equal to the union of the convex hulls of all its finite subsets.

Proof. Let F be any finite subset of A. 2.9 and 2.11 imply that F^c is a subset of A. This is true for any finite subset of A; therefore, it is true for their union. If we can now show the union of the convex hulls of all finite subsets of A is convex, we shall have completed the proof.

Let x and y be any two elements of the union. There exist finite subsets F and G of A such that x and y are contained in F^c and G^c, respectively. F and G are subsets of the union of F and G, so F^c and G^c are subsets of $(F \cup G)^c$ according to 2.13. Therefore, x and y are elements of $(F \cup G)^c$, which implies that $L(x,y)$ is a subset of $(F \cup G)^c$. Since F and G are finite, their union is finite; and we have shown that $L(x,y)$ is a subset of the union of convex hulls of finite subsets of A whenever x and y are elements of this union, whence the property.

2.15 Theorem. <u>If</u> A <u>is any subset of</u> E <u>and</u> p <u>is any element of</u> E, <u>then the convex hull of</u> $A \cup \overline{p}$ <u>equals the union of all</u> $L(a,p)$ <u>where</u> a <u>ranges over</u> A^c.

Proof. For any a in A^c, $L(a,p)$ is a subset of $(A \cup \overline{p})^c$ since a and p are elements of $(A \cup \overline{p})^c$, which is convex. Therefore, the union of all $L(a,p)$ is a subset of $(A \cup \overline{p})^c$. Inclusion the other way is demonstrated by taking an arbitrary x in $(A \cup \overline{p})^c$. A is certainly a subset of the union of the $L(a,p)$'s since it is a subset of A^c and each element of A^c shows up in an $L(a,p)$. Obviously p is an element of the union since it is in each $L(a,p)$. If we can now show that the union of the $L(a,p)$'s is convex, we will have inclusion the other way and, therefore, equality.

Let x and y be any two elements in the union of all the $L(a,p)$'s. There exist a and b in A^c such that x

and y are elements of L(a,p) and L(b,p), respectively. We want to show that L(x,y) is a subset of the union. If t is any element of L(x,y), we have axp, byp, and xty by the definition of a line segment. Application of V guarantees an element s in E such that stp and asb. The latter means that s is an element of L(a,b) which is a subset of A^c; therefore, s is an element of A^c. stp implies that t is an element of L(s,p) and is, therefore, an element of the union of the L(a,p)'s where a ranges over A^c. Since t was arbitrary, we have demonstrated convexity and proved the theorem.

3. EXTREME POINTS

Definition. An element x of a subset A in E is an extreme point of A if and only if $x = a$ or $x = b$ whenever x is in $L(a,b)$ for any a and b in A. The set of extreme points of A will be denoted by $e(A)$. It may happen that $e(A)$ is empty.

3.1 Remark. If A is a subset of E, $e(A)$ is a subset of A.

This follows immediately from the definition.

3.2 Proposition. If A is a subset of E, $e(e(A)) = e(A)$.

Proof. It follows from 3.1 that $e(e(A))$ is a subset of $e(A)$. To show inclusion the other way, let x be any element of $e(A)$ which is not in $e(e(A))$. Since x is not in $e(e(A))$, there exist a and b in $e(A)$ such that x is in $L(a,b)$ and $x \neq a$ and $x \neq b$. By 3.1 we see that these same elements are in A and, therefore, that x is not in $e(A)$. This contradicts our assumption on x and establishes the proposition.

3.3 Proposition. If A is a subset of E and B is any subset of $e(A)$, the set of elements in A^c which are not in B is still convex.

Proof. Let x and y be any two elements in A^c which are not in B. Since x and y are in A^c, $L(x,y)$ is a subset of A^c. Neither x nor y is in B; and no element between x and y can be in B, so $L(x,y)$ does not intersect B. If it did, there would be an extreme point between elements which are distinct from it, which is a contradition. Thus we have that $L(x,y)$ is a subset of A^c which does not intersect B, and also our proposition.

3.4 Proposition. <u>Let</u> A <u>be a subset of</u> E, x <u>be any extreme point of</u> A^c, <u>and</u> B <u>be any subset of</u> A^c. <u>If</u> x <u>is in</u> B^c, x <u>is in</u> $e(B^c)$.

Proof. Assume that x is not in $e(B)$; and let a and b be two elements of B^c such that $a \neq x$, $b \neq x$, and x is in $L(a,b)$. B is a subset of A, so B^c is a subset of A^c. Combine the results and we can conclude x is not in $e(A)$, which is contrary to the hypothesis. We conclude the proposition.

3.5 Theorem. <u>Let</u> A <u>be a subset of</u> E. <u>Then</u> x <u>is in</u> $e(A^c)$ <u>if and only if</u> x <u>in</u> B^c <u>implies that</u> x <u>is in</u> B <u>where</u> B <u>is any subset of</u> A^c.

Proof. The necessity will be proved first. By theorem 2.14, x in B^c implies that there exists a finite subset F_n of B such that x is in F_n^c where n is the number of elements in F_n. Designate the elements of F_n as $f_1, f_2, \ldots, f_n$, and let F_m be the first m elements of F_n. Theorem 2.14 implies that there exists an element a of F_{n-1}^c such that x is in $L(a,f_n)$. Since x is in $e(A)$, it must be that

$x = a$ or $x = f_n$. We are finished if $x = f_n$, since f_n is in B. If not, then x is in F^c_{n-1}. We can repeat this process $n - 1$ times until we get that x is in $L(f_1, f_2)$, which will imply that $x = f_1$ or $x = f_2$. In either case x is in B and we have proved the necessity.

To prove the sufficiency of the condition, assume that x is not in $e(A^c)$. If this is so, there exist elements a and b distinct from x such that x is in $L(a,b)$. The elements a and b constitute a two-element subset of A^c which contains x in its convex hull. By assumption, x is an element of the two-element set, which implies that $x = a$ or $x = b$. This contradicts x being distinct from a and b and establishes the theorem.

3.6 Corollary. _If_ A _is a subset of_ E, x _is an element of_ $e(A^c)$, _and_ B _is any subset of_ A, _then_ x _in_ B^c _implies that_ x _is in_ B.

Proof. Since B is a subset of A and A is a subset of A^c, B is a subset of A^c. Apply 3.5 and the conclusion follows.

3.7 Corollary. _If_ A _is a subset of_ E, $e(A^c)$ _is a subset of_ A.

Proof. A is a subset of A^c, and any x in $e(A^c)$ is in A^c. 3.5 implies that x is in A.

3.8 Corollary. _Let_ A _be a subset of_ E. _If_ $A = e(A^c)$, *then for any* x *in* A, x *is not in the convex hull of* A' *where* A' *equals* A *minus the element* x.

Proof. Assume that there exists an x in A which is also in A'^c. By hypothesis, x in A implies that x is in e(A). A' is a subset of A and, by assumption, x is in A'^c. We conclude that x is in A' by 3.5; but this is a contradiction, hence the conclusion.

4. MAXIMAL CONVEX SETS

Definition. A convex subset of E is said to be a semiset if and only if its complement is also convex.

4.1 Remark. The complement of a semiset of E is a semiset of E.

4.2 Remark. E and $\emptyset$ are semisets of E.

Definition. A semiset will be said to separate two sets of E if and only if one of the sets is a subset of the semiset and the other is a subset of the complement of the semiset.

4.3 Theorem. (Ellis) If A and B are disjoint convex subsets of E, there exists a semiset C which separates A and B.

Proof. Define M as the set of ordered pairs (G,H) of disjoint convex subsets of E which contain A and B, respectively. Induce a partial order on M as follows: $(G,H) \leq (G',H')$ if and only if G is a subset of G' and H is a subset of H'. Consider a linearly ordered subset N of M, and define n to be the ordered pair obtained by taking the union of all the first components of elements of N as the first component and the union of all the second components of elements of N as the second component. Obviously n is maximal for N, but we need to show that its components are disjoint.

Assume there is an x common to the two components. There exist elements a and b of N such that x is an element of the first component of a, and x is an element of the second component of b. Since N is linearly ordered, we know that $a \leq b$ or $b \leq a$. To be definite, we will take the first case. This means that x is an element of both components of a, but these are disjoint. The contradiction establishes n as an element of M. Since N was arbitrary, we have shown that every linearly ordered subset of the partially ordered set M has an upper bound in M. Of course, M is not empty since (A,B) is an element. Zorn's lemma implies that there is a maximal element (A',B') in M. It remains to be shown that A' and B' are complementary.

Let x be an element in the complement of the union of A' and B'. Assume there is an element b common to $(A' \cup \overline{x})^c$ and B' and an element a common to $(B' \cup \overline{x})^c$ and A'. Using 2.14 we see that there exist a d in A' and a c in B' such that a and b are elements of $L(c,x)$ and $L(d,x)$, respectively. We now apply 2.4 and conclude that the intersection of $L(a,d)$ and $L(b,c)$ is not empty. Since a and d are elements of A' and b and c are elements of B', $L(a,d)$ and $L(b,c)$ are, respectively, subsets of A' and B'. We, therefore, have the contradiction that an empty set contains a non-empty subset. It must be that at least one of the intersections we started with is empty. To be definite, we can take the intersection of $(A' \cup \overline{x})^c$ and

B' as empty. Let $(A' \cup \bar{x})^c = A^*$. A' is a proper subset of A^*, and B' is a subset of itself. Therefore, $(A', B') \leq (A^*, B')$ and $(A', B') \neq (A^*, B')$, which contradicts the maximality of (A', B').

The contradiction tells us that A' and B' are complementary since they are disjoint and their union is the entire set. They are convex and, therefore, semisets. Letting A' equal C proves the theorem.

4.4 Corollary. Let A be any convex subset of E and x be any element in E. If x is not in A, there exists a semiset separating $\bar{x}$ and A.

Proof. If x is not in A, $\bar{x}$ and A are disjoint convex subsets to which we can apply 4.3.

4.5 Corollary. If A is any subset of E and x is an extreme point of A, then there exists a semiset meeting A only at x.

Proof. Let A' equal A minus the extreme point. By 3.3, A' is convex and $\bar{x}$ and A' are disjoint convex sets. Apply 4.3 and the corollary follows.

4.6 Theorem. If A is any subset of E, A^c equals the intersection of all semisets which contain A.

Proof. If A is a subset of any semiset C, A^c is also a subset of C. From this we conclude that A^c is a subset of the intersection of all semisets which contain A.

To show inclusion the other way, we consider an x not in A^c. By 4.3, there is a semiset C such that x is not in C and A^c is a subset of C. A is a subset of A^c, which means that A is also a subset of C. We have found a semiset containing A as a subset which excludes x and can thereby conclude that x is not an element of the intersection of all semisets containing A as a subset.

4.7 Theorem. _Let_ A _be any convex subset of_ E. A _is equal to the convex hull of its extreme points if and only if every semiset of_ E _which intersects_ A _intersects_ $e(A)$.

Proof. The necessity will be proved first. Assume there exists an element y of A and a semiset C in E such that y is in C and $e(A)$ is a subset of the complement of C. Since $e(A)$ is a subset of C, $(e(A))^c$ is also a subset of C by 2.13 and 2.11. From this we can conclude that $A \neq (e(A))^c$. Therefore, the necessity is proved.

The sufficiency is proved by assuming that $A \neq (e(A))^c$. If this is so, we can find an x in A which is not in $(e(A))^c$. Apply 4.4 and we are assured of the existence of a semiset separating x and $(e(A))^c$. Therefore, the sufficiency is proved.