Frequency response analysis of the in-vivo human skull
by Gerald Martin Grammens

A thesis submitted in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE in Mechanical Engineering
Montana State University
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Abstract:
In this study, a finite element model of the human skull is presented. The intent is to explore the feasibility of determining in-vivo material properties of the human skull by frequency response analysis.

Qualitative results are obtained by approximating material and geometrical parameters for the skull. Initially, boundary conditions were chosen to prevent rigid body translation by securing nodes about the mid-plane. The axisymmetric mode shapes obtained reveal resonance of the fundamental bending mode at 3700 Hz.

To account for this disparity of results compared with those obtained by previous investigators, an end-mounted support at a single node was imposed to replace the mid-mount boundary condition. This forced boundary condition was of the type employed in these previous studies. For the same skull with the end-mount boundary condition, the first resonance occurred at 440 Hz. The corresponding fundamental bending mode shape can best be described as a rigid body motion of the bulk of the skull with a local deformation at the fixed node. Recognition of the nature of this mode shape raises serious implications concerning past interpretations of such results.

The qualitative results of this study suggest that either of the two mode shapes may prove helpful in determination of skull material properties, but not without major experimental and analytical difficulties.
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FREQUENCY RESPONSE ANALYSIS OF THE IN-VIVO HUMAN SKULL

by

GERALD MARTIN GRAMMENS

A thesis submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

in

Mechanical Engineering

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Bozeman, Montana
March, 1977
ACKNOWLEDGMENT

The author wishes to express his appreciation to Dr. Dennis Blackketter for the suggestion of this thesis topic and for his assistance throughout its evolution.
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Qualitative results are obtained by approximating material and geometrical parameters for the skull. Initially, boundary conditions were chosen to prevent rigid body translation by securing nodes about the mid-plane. The axisymmetric mode shapes obtained reveal resonance of the fundamental bending mode at 3700 Hz.

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The qualitative results of this study suggest that either of the two mode shapes may prove helpful in determination of skull material properties, but not without major experimental and analytical difficulties.
INTRODUCTION

A host of investigations have been carried out to determine the response of the human head to external excitation. Most of these investigations have as their objective advancing the understanding of the events which take place during the short period subsequent to impact of the skull. The impetus frequently cited for these studies is the statistically impressive rate of occurrence of death or injury due to impact sustained by the head. This is a result, primarily, from automotive and related accidents.

Such studies assume an element of uncertainty when a particular set of rheological parameters is assigned to the analytical models of in-vivo (live) human bone and associated tissues. The non-availability of this in-vivo information constitutes an obstacle to obtaining meaningful analytical results. In previous studies this impasse has been readily circumvented by taking as in-vivo those properties determined from, or extrapolated from, in-vitro (or ex-vivo) studies. The importance of these parameters in quantitative prediction of in-vivo behavior of the human head, along with the emergence of a wide range of values in recent literature, combine to substantiate the need for further refinement of in-vivo rheological properties.

A method to determine in-vivo material properties of human bone has been advocated by Garner and Blackketter [13]. This calls for an experimental procedure to measure the response of the system to an externally applied harmonic force over a range of frequencies. Similarly,
a frequency response analysis is performed by utilizing an analytical model of the physiological structure under investigation. The input material parameters for the analytical model are altered so that the experimental and analytical responses most closely correspond. Those optimized material parameters are then said to be the in-vivo material properties.

The long bones, such as the tibia and ulna, are typically chosen for such studies because they can be characterized as having relatively simple geometry. However, experimental results and analytical modeling become complicated with the presence of attached and surrounding soft tissues. Also, slight changes in orientation and support conditions can significantly affect experimental response characteristics, which in turn adversely affect reproducibility.

Because the properties differ significantly between the long bones and those of the skull, it is imperative that similar determinations be attempted for the skull itself. In addition to the desirability of determining the in-vivo properties of the skull, such a study will provide further indication of the usefulness of frequency response analyses for these purposes.

One can surmise that the skull is less subject to the difficulties encountered in obtaining reproducible experimental results. Damping due to the brain and its surrounding tissue can be considered constant over periods of time such as might elapse between one experimental sampling.
and the next. In addition, one might expect little variation in brain properties from one individual to another. Positioning of the skull and accelerometer transducers can be easily reproduced, and, skull support conditions have minimal effects for properly chosen modes. Inherent with the skull is a greater complexity of geometry, which tends to detract from its desirability as the subject of dynamic analysis.

Because cranial bone properties cannot be equated to those of bone serving other functions [31], a method is needed to determine in-vivo bone properties specifically for cranial bone. The question at hand, then, addresses the feasibility of using the human skull to determine the material properties (or changes in material properties) of in-vivo bone. Success in this venture is contingent upon the ability of both the experimental and analytical investigations to accurately detect and predict at least one bending mode of deformation excited by a specific driving force. It is also essential to convincingly demonstrate that indeed the same modes are being described by both segments of the procedure. The emphasis in this study is placed on an analytical determination of the frequency response of the human skull, and, an attempt to determine which mode is likely to aid in the establishment of in-vivo skull (bone) properties.
Chapter I
LITERATURE REVIEW

The bulk of work done in determining the mechanical properties of bone has been carried out in the last ten years. For earlier work the emphasis had been placed on determining parameters which describe bone as an elastic material. As early as 1965 Sedlin [27] published the results of experimental work in which he demonstrated that, for small deformations, bone behaves as a linear viscoelastic material. Furthermore, Sedlin's experimental results were best modeled by the 3-parameter solid model, also referred to as the standard linear solid model. Sedlin's work was conducted using in-vitro compact femoral bone.

Curry [5] conducted experimental investigations on the microscopic level dealing with the effect of mineral content of bone and its relationship to measured engineering properties. Curry studied the effect of mineralization or calcification on the elastic parameter—Young's Modulus. As will be seen in the following chapter on bone structure, one might have expected that Curry's work would verify the high causal-effect relationship between mineral content and Modulus of Elasticity.

Because of requirements for non-destructive testing, most information on bone, particularly human, is the direct result of in-vitro experimentation. Values used in studies for which in-vivo bone properties are necessary input parameters are usually taken directly from or are extrapolated from available in-vitro data. The validity of using
in-vitro properties to describe in-vivo behavior is dependent upon the changes incurred by bone following death.

Past studies apparently are not in concurrence as to the effect of post-mortem age on bone properties. The following studies demonstrate this point.

McElhaney [22] indicates that the mechanical properties of embalmed bone do not differ "significantly" from immediate post-mortem properties. McElhaney's properties refer to the elastic properties—Young's Modulus and Poisson's Ratio.

Evans [8] modeled bone as a viscoelastic material and states that the elastic parameter is not affected by embalming.

Tennyson [28] investigated the stress-strain characteristics of beef femur bone as a function of post-mortem age (PMA). Tennyson initially modeled bone using the 3-parameter linear viscoelastic model. His results indicated that one of the stiffness parameters was small enough to allow him to neglect its effect. This reduced the model to the 2-parameter or Voight model. Further results based on the Voight model produced little change in the stiffness parameter with increased PMA, however, the viscous parameter showed radical change during early PMA. Here, early PMA refers to the first 24 hours of post-mortem time. It is necessary to extrapolate Tennyson's data to speculate on immediate post-mortem effect on bone properties. With decreasing PMA, approaching zero, the viscous parameter appears to increase exponentially for at
least as far as Tennyson's data is shown (beginning approximately at .5 days PMA). Tennyson obtained his experimental data from the split-Hopkinson-bar technique.


At this point the literature seems to indicate that bone's transition from the in-vivo state to an in-vitro state involves small changes in the elastic characteristics and marked changes in the viscous or damping characteristics. The inconclusiveness of previous work establishes the need for meaningful in-vivo results. Furthermore, in the literature reviewed here, values taken for Young's modulus for the skull range from $6.0 \times 10^5$ psi to $2.29 \times 10^6$ psi, while the majority of investigators use an intermediate value of $2.0 \times 10^6$ psi. The following discussion can account for a portion of the variation in this one parameter.

Bone properties vary from species to species, from one individual to another, and from one functional physiological location to another within the same specimen. These facts are borne out by the following studies.

Wood [31] tested in-vitro cranial bone samples from 30 subjects ranging in age from 25 to 95 years, and performed tensile testing vs. strain rate. For cranial bone, no regional variation in the properties
measured was noted, and no variation of properties with respect to tangential direction was found. Wood also states that cranial bones have properties intermediate of those for the longitudinal and transverse properties of long bones.

Hubbard [18] used beam testing techniques to determine the flexural stiffness and strength of layered cranial bone. Test specimens were obtained from embalmed calvaria (skull less jaw and facial bones). From these, the elastic properties were determined using a three-point flexure test. An equivalent modulus of elasticity was determined for the actual cross-section. Hubbard noted no significant effect resulting from the test sample extraction site, nor due to orientation, normal or inverted (i.e. convex or concave) of the test specimen in the test apparatus.

McElhaney [21] set out to determine the mechanical properties of cranial bone which are relevant to the study of biomechanics of head injury. Bone specimens were obtained from embalmed cadavers, craniotomies, and autopsy. In addition to the human cranial bone tested, specimens were also obtained from monkeys and tested. It is interesting to note that there were significant differences in resulting elastic parameters between the two species. Testing was done on the Tinius Olsen Electromatic Testing Machine with the bone samples maintained in "wet" condition. McElhaney noted that variations in the thickness of the porous middle layer of bone (diploe) resulted in a high standard
deviation for material properties such as energy absorption and gross stiffness. It was also shown that the tensile properties for the inner and outer layers (tables) of compact cortical bone display no significant differences. Again, McElhaney is modeling cranial bone as an elastic material.

Research inspired by a high occurrence rate of head injuries largely due to automotive accidents has been conducted in which various approaches have been undertaken to attempt to increase understanding of the biomechanics of the human skull.

Gordon et. al. [14] used a 2-dimensional model of the human head to determine the response to impact. The results were in terms of a pressure distribution through the brain. The work employed a somewhat more sophisticated model of the head than that used in previous analyses, in that a finite wave speed was used for the brain and skull.

Engin [6] determined the response for a thin elastic spherical shell filled with an inviscid compressible fluid. In a similar analysis Kenner and Goldsmith [20] worked with wave propagation through a liquid filled sphere for impacts of greater duration than those which can be represented by Dirac's delta function.

The above three analyses represent an approach to modeling the human skull and brain which involves the use of closed form solutions. These methods are useful in demonstrating qualitatively phenomena such as wave propagation, however, the closed form solutions require
simplifying assumptions such as a spherical geometry for the skull, constant thickness of the cranium, and an inviscid irrotational fluid for the brain. An alternate approach incorporates the use of finite element techniques, which allows consideration of irregular geometries and variation in skull thickness.

Hardy and Marcal [16] hypothesized in their investigation of brain damage (particularly that resulting from automotive accidents) that, even though it is the brain itself which limits the impact absorption capacity of the human head, this limit may be best measured in terms of skull response. This prompted Hardy and Marcal to model the skull using doubly curved triangular shell finite elements. The skull was modeled as an elastic material, and, only static deflections were studied.

Nickell and Marcal [24] carried the preceding analysis one step further. The same finite element model was used, and the essential aspect for any impact study—the dynamic response of the skull—was considered. Three different types of support boundary conditions (frontal, rear, and base mounted) were used to determine natural frequencies and associated mode shapes which (it was hoped) might be relatively independent of the arbitrary support conditions. The lower mode shapes are the result of the particular imposed boundary conditions. In general, these can be described as rotation (mostly rigid body) about the prescribed support. The third mode in each case appears to exhibit relative motion of points at the support and those diametrically opposite,
and occurs in the frequency range of 400-700 Hz. Nickell and Marcal hypothesize that this mode shape is a result of deformations due to the variations in thickness and curvatures of the particular skull modeled.

The above study used a modulus of elasticity approximately one third of that found in literature of similar investigations. This would cause the observance of lower frequencies for given mode shapes. No reference is given to support the value taken for this parameter.

Hickling [17], as was the case with most other investigators, had as his goal the determination of the mechanics of brain and skull damage during and after impact, particularly that associated with automotive accidents. He modeled the skull and brain as linear viscoelastic and isotropic materials and used a spherical geometry for the head. For his study and indeed applicable to other such studies, Hickling laments,

"Unfortunately, the ability of the present model to predict tolerances is greatly hampered because adequate data on the material properties of the skull and brain are lacking and because appropriate damage criteria are not known".

Having examined some of the models of the skull and noting inherent difficulties and inadequacies, our attention is now directed to some of the experimental work which has been performed on the skull.

Early experimental work on the dynamic response of the skull was performed by Franke [10] and updated by Gurdjian et. al. [15].

Franke frontally loaded a dry preparation (skull) with a small support at the occiput (rear of skull) and found resonance at 820 Hz.
The same loading, when applied to a live human skull, produced no measurable response. This failure was described as the result of insufficient coupling between the force piston and the skull due to the layer of skin. Franke then performed his experiment on cadavers with the skin removed at the forcing location. The first flexural mode was found at 610 Hz. Without substantiation, this mode was taken as the dampened version of that found in the dry preparation at 820 Hz.

Gurdjian frontally mounted cadaver calvaria to a source of a harmonic force and measured the response at the left side, vertex, and occiput of the skull. The calvaria were tested both empty and filled with silicon gel. Resonant frequencies were detected at approximately 300 and 900 Hz. The first, an anti-resonant mode, resulted in an occiput acceleration amplitude three to four times greater than that of the frontal location where the harmonic load was applied. The second resonance exhibited relatively large frontal excursion as opposed to that of the occiput. With response monitored at only three points, little information is presented to give indication of actual mode shape.

In addition to the cadaver studies, human volunteers were used in a similar experimental set up, and response to a frontal force was measured. Gurdjian states that for frequencies below 150 Hz, only local deflections at the point of application of the force are detectible.

Prior and subsequent work has been done to determine resonance of biological structures. Those most applicable to this study are work
done by Jurist [19] and Garner and Blackketter [13]. Jurist's work is mostly clinical in nature, and concentrates on the long bones of the outer extremities (ulna, radius, tibia, and fibula). Jurist modeled these as hollow cylinders of constant cross section and proceeded to determine the first bending frequencies.

Garner's work, which is the antecedent to this study, represents a more sophisticated analytical-experimental approach to correlate low frequency response to parameter determination. Garner used a finite element model of the forearm in which the bones (radius and ulna) along with the soft tissues were modeled as viscoelastic materials. Garner varied the rheological parameters in his analytical model until the response most closely coincided with that obtained from his experimental work, and thus defined the rheological properties.
Chapter 2
THE HUMAN SKULL.

In order to more fully justify the use of simplifying assumptions necessitated by our analytical model of the skull, the present chapter is included. We will first examine the material of which the skull, the subject of our investigation, is composed—bone. Our next concern will be with the biological structure, the skull itself.

Bone

Bone is a highly specialized form of connective tissue. On a microscopic level, bone is composed of a soft branching matrix of cells around which are deposited mineral salts. These salts are composed primarily of calcium and phosphate, which results in bone's characteristic hardness. It is this hardness which most differentiates bone from other connective tissues such as cartilage and ligaments. In addition to the inorganic bone salts and water, the interstitial substance includes a fibrous structure comprised of collagen. These collagenous fibers tend to orient themselves in a longitudinal fashion in long bones so as to maximize strength in directions of greatest tensile stress.

Most bones (particularly the long bones) are modeled during early development as cartilage, and only in later development are they re-modeled as cortical bone. It is this remodeling process which results in the presence of haversian systems (or osteons). Consider a hollow
core (many times greater in length than diameter) called the haversian canal. Around this haversian canal, the bone matrix and interstitial materials are deposited in concentric layers or lamellae. Lamellae of a haversian system and the outer lamellae of adjoining haversian systems become connected along what are termed cement lines. Thus the thickness of say a long bone is composed of many haversian systems.

**Macroscopic Properties**

In macroscopic perspective there are two basic types of bone: 1) hard compact cortical bone such as that found in the shafts of long bones, and 2) spongy or cancellous bone which consists of many fine partitions called trabeculae, which form cavities containing red or fatty marrow. Cancellous bone is found in the vertebrae, majority of flat bones, and at the end of long bones. Thus cancellous bone is porous and is characterized by intricate architecture.

The inorganic mineral is the major component of bone. McLean and Urist [23] cite for Bovine bone the following weight percentages:

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<tr>
<td>Mineral</td>
<td>72%</td>
</tr>
<tr>
<td>Organic Matter</td>
<td>24%</td>
</tr>
<tr>
<td>Water</td>
<td>4%</td>
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When the fatty material is removed, 90-96% of the remaining organic matter of bone is collagen.
Bone Dynamics

Changes occur in bone by way of the normal physiological mechanisms of growth, aging, maintenance and repair, by pathological mechanisms such as osteomalacia, and by trauma such as impact with a foreign object culminating in fracture.

Three types of specialized cells account for the physiological dynamics of bone. The first of these is the osteoblast. Osteoblasts appear on the surface of growing or developing bone and provide the biological mechanism for the formation of new bone. Osteocytes are osteoblasts which have become surrounded by the matrix, which subsequently calcifies, and provide for the maintenance of surrounding bone. The last of the three types of cells is the osteoclast. Osteoclasts are usually found on the bone surface near areas of active remodeling and perform the physiological function of resorption of existing bone.

Changes in bone are apparent during growth, but even when the geometric parameters are constant, as in periods during adulthood, the process of remodeling (replacement of existing bone with new) is active. By way of example, McLean and Urist [23] state that for a typical 57 year old human, the bone mass turnover in one day is 0.036% for the femur and fibula and 0.012% for the tibia.

The physiological mechanism of aging and the pathological mechanisms can cause an upset in the balance of osteogenic activity, resulting
in changes in bone's microscopic structure and therefore, the macroscopic properties. McLean and Urist discuss the condition of atrophy which refers to a loss of substance of bone without a corresponding loss in the external dimensions (or gross volume) of the bone. This may advance to a state where as much as 75% of the original bone mass has been lost.

Little is known of the effect of early post-mortem time on the macroscopic material properties of bone. Some researchers, such as McElhaney [21] indicate that there is little effect on bone properties due to post-mortem age. Others, such as Garner and Blackketter [13], cite indications of marked changes. Tennyson [28], using Split-Hopkinson-bar testing techniques, plots changes in an elastic and viscous parameter for bone with respect to post-mortem time. His results indicate little change in the elastic parameter but great effect on the viscous parameter immediately after death and through the first few days post-mortem.

The Skull

The human skull consists of the calvarium, the jaw (mandible), and the facial bones. The calvarium is an ellipsoidal shaped enclosure which serves to protect the brain. The one major opening in the
calvarium, through which the medulla oblongata (anterior portion of the brain which connects to the spinal cord) passes, is the foramen magnum located at the base of the skull. The calvarium is composed of eight different bones which are joined by cartilage and a fibrous tissue in early development. By adulthood these bones are joined in a much more rigid manner at the boundaries along connective formations referred to as sutures. Examination of cadaver skulls reveals great variations in the degree to which the cranial bones have fused at the sutures. The range is from interlocking, but still separate, bones to cases in which fusion is so complete that the suture lines are hardly discernible.

The bones which comprise the calvarium are sandwich structured through the thickness. The bones are composed of inner and outer tables of compact cortical bone separated by a porous inner layer of bone called the diploë. The thickness of the diploë layer varies within the constituent bones from as much as half of the total skull thickness to no thickness at the sutures. The thickness of the calvarium varies with location. Typical variations might be from 0.12 inches in the temporal region to 0.25 inches in the frontal region.

The regional nomenclature for a typical skull is shown in Figure 2-1.
Having discussed briefly the physiological nature of bone and the skull itself, we have the basis on which evaluation of the validity of simplifying assumptions can be made. We must also consider the cases for which specific conditions are requisite to the acceptance of a particular simplifying assumption.

Earlier it was noted that, with the exception of the collagenous fibers, bone appears to be homogeneous on a macroscopic level. The effect of the collagenous fibers is minimal in the flat bones such as those which comprise the skull. Along these same lines, it was previously reported from an experimental study by Hubbard [18] that no significant variation in bone properties was measured in directions
tangential to the skull surface. Thus, we have the basis for assuming bone to be homogeneous in in-plane directions.

Variations of thickness of the inner and outer tables, and the diploe layer, along with slightly altered properties in a normal (to the surface) direction suggest that inclusion of these characteristics is necessary for quantitative results. This is of even greater significance for bending deformation, where stiffness increases as the cube of the distance from the central fiber. A constant thickness and the use of gross properties to compensate for diploe thickness can only be expected to produce qualitative results.

The homogeneous shell model will suffice to predict bending response for a skull where sutures are sufficiently fused so as to provide continuity between component bones. It must be cautioned that in-vivo skulls with merely interlocking or cartilaginous sutures present a physical situation which is not likely to be adequately modeled quantitatively by the homogeneous shell.

The calvarium, being nearly a complete enclosure, can be modeled in simplest form as a spherical shell of constant thickness. Though the sphere is a first generation model of the skull, its response is indicative of the resonant frequencies and mode shapes of the skull. A more accurate geometrical representation is the ellipsoid. The typical calvarium, however, deviates somewhat from this geometry. In some specimens irregularities even result in a lack of symmetry about the
sagittal plane (i.e. non-symmetry between the left and right sides of the calvarium). A truly quantitative model would take into account even these irregularities.

Other difficulties in modeling the skull which must be considered or for which proper assumptions must be made are: the effects of attached or surrounding facial bones, jaw bone (mandible), brain and fleshy materials, and the physiological support system.
Chapter 3
MODELING CONCEPTS

The analytical model of the skull presented in this study was developed so that, if necessary, consideration can be given to the viscoelastic nature of bone, irregularities in geometry, and regional variations in thickness and material properties. Initially, however, the skull is modeled as a thin wall homogeneous shell of constant thickness and of axisymmetric geometry. The intent here is to obtain qualitative insight into skull response characteristics.

The use of an axisymmetric geometry permits the generation of nodal coordinates and assumed modes to be completed with minimal computational effort. Of greater significance is the capacity to effect a coordinate reduction of the original 336 degree of freedom system to one described by 28 generalized coordinates, while maintaining the capability of identifying any axisymmetric mode.

Geometry Generation

The skull was modeled in this study as an ellipsoidal shell for ease in generating nodal locations and to fill the requirements of axial symmetry. The equation for an ellipsoid in cartesian coordinates is

\[ 1 = \frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} \]  

(3-1)
where $X$, $Y$, and $Z$ are cartesian coordinate values in the $X$, $Y$, and $Z$ coordinate directions respectively, and $A$, $B$, and $C$ are constants which define the shape of the ellipsoid. The sphere is a special case of an ellipsoid where the constants $A$, $B$, and $C$ are all numerically equal to the radius.

Finite element nodal locations are expressed in terms of the global cartesian coordinates $X$, $Y$, and $Z$. These global coordinate values become the geometric input parameters for finite element mass and stiffness generation.

The generation of node locations is most easily carried out in spherical coordinates and then transformed into cartesian coordinates. The spherical coordinates $\theta$, $\phi$, and $\rho$ will be employed as shown in Figure 3-1 below.
The expressions relating cartesian coordinates to spherical coordinates are

\[
\begin{align*}
X & = \rho \sin(\theta) \cos(\phi) \\
Y & = \rho \sin(\theta) \sin(\phi) \\
Z & = \rho \cos(\theta)
\end{align*}
\] (3-2)

Substituting equations (3-2) into (3-1) and simplifying gives

\[
\rho = \left( \frac{\sin^2(\theta) \cos^2(\phi)}{A^2} + \frac{\sin^2(\theta) \sin^2(\phi)}{B^2} + \frac{\cos^2(\theta)}{C^2} \right)^{-\frac{1}{2}} \] (3-3)

For the ellipsoid defined by constants A, B, and C, \( \rho \) is the distance from the origin to the node positioned at angular coordinates \( \theta \) and \( \phi \). The cartesian values for \( X \), \( Y \), and \( Z \) are obtained by inserting this value for \( \rho \) into equation (3-2).

To obtain an adequate grid size (i.e. elements small enough so as to closely approximate shell curvature), nodes were generated for each change in \( \theta \), and \( \phi \) of 0.1\( \pi \) radians. Taking further advantage of symmetry, the ellipsoid need only be generated for \( 0 \leq \phi \leq \frac{\pi}{2} \).

Coordinate Reduction

The finite elements were generated such that rings of nodes appear in planes which are parallel to the global XY plane. For axisymmetric geometry, these nodes which are designated by the subscript \( r \) have the same spherical angle \( \theta \). See Figure 3-2 for nomenclature. In addition,
the nodes are generated in vertical rings which are coplanar along with the Z-axis. These planes are situated at an angle ($\phi$) from the X-axis, and, are designated by the subscript (s).

![Polar node](image)

**FIGURE 3-2.** Nodal Nomenclature.

For axisymmetric displacements (about the Z-axis) all nodes in ring (r) will experience the same excursion in the Z-direction. The sense of the components of displacement parallel to the XY plane is directed
perpendicular to the Z-axis, so that for a unit displacement of a node in this direction

\[ U_{rs} = \cos(\phi_s) \text{ inches} \]

and \[ V_{rs} = \sin(\phi_s) \text{ inches} \] (3-4)

where \( U_{rs} \) and \( V_{rs} \) are the displacements of node \((rs)\) in the \(X\) and \(Y\) coordinate directions respectively.

Similarly, angular deformations, expressed on a unit basis, for the nodes in ring \((r)\) are

\[ \theta_{xrs} = \sin(\phi_s) \text{ radians} \]

and \[ \theta_{yrs} = -\cos(\phi_s) \text{ radians} \] (3-5)

where \( \theta_{xrs} \) and \( \theta_{yrs} \) are the angular deformations at node \((rs)\) about the \(X\) and \(Y\) axes respectively. For axisymmetric geometry, there is no angular displacement about the Z-axis. The deformation \( U_{rs} \) and \( V_{rs} \) are not mutually independent, nor are \( \theta_{xrs} \) and \( \theta_{yrs} \). The displacements for the nodes in ring \((r)\) can then be expressed as

\[ W_{rs} = p_{r1} \cdot 1 \]

\[ U_{rs} = p_{r2}\cos(\phi_s) \text{ and } V_{rs} = p_{r2}\sin(\phi_s) \] (3-6)

\[ \theta_{xrs} = p_{r3}\sin(\phi_s) \text{ and } \theta_{yrs} = -p_{r3}\cos(\phi_s) \]

Thus, three reduced coordinates, \( p_{r1}, p_{r2}, \) and \( p_{r3} \) will completely describe the displacement of each ring \((r)\). The three corresponding assumed modes for ring \((r)\) are then generated such that the first, when multiplied by \( p_{r1} \), yields only deformations of all nodes in that ring in the \(Z\)-direction. The second, when multiplied by \( p_{r2} \), gives the
deformations of only nodes in ring \( r \) in the \( X \) and \( Y \) coordinate direction, and the third assumed mode, when multiplied by \( p_{r3} \), results in angular deformations about the \( X \) and \( Y \) axes. All of this is in accordance with equations (3-6).

Only one reduced coordinate is necessary to describe the motion of each of the two polar nodes, which are allowed translatory motion along the \( Z \)-axis only for axisymmetric deformation.

According to methods described by Anderson [1], the previous results are combined in the form of

\[
\{ \delta \} = [\gamma] \{ \mathbf{P} \} \tag{3-7}
\]

where

\[
\{ \delta \} = \text{global displacement vector}
\]

\[
\{ \mathbf{P} \} = \text{reduced coordinate vector}
\]

and

\[
[\gamma] = \text{transformation matrix}
\]

The columns of the transformation matrix \([\gamma]\) are not assumed modes in the usual sense. By limiting the response to axial symmetry, the coordinate reduction is executed without loss of generality. Only computational round-off error will produce discrepancies from the axisymmetric response had it been obtained instead from the total system.

Implementation of the coordinate reduction process is discussed in the chapter dealing with frequency response methods. The actual coordinate reduction was performed on each element upon generation and then
superimposed in a global reduced mass and stiffness matrix. This was
done so as to avoid generation and storage of the entire mass and stiff-
ness matrices.

Input Parameters

Measurements were taken from a cadaver skull to obtain approximate
geometrical parameters. For the spherical model the constants
\[ A = B = C = 3.25 \text{ in.} \]
which produce a sphere with a radius of 3.25 inches, were chosen to
represent the skull.

The ellipsoid defined by
\[
\begin{align*}
A &= 3.0 \text{ in.} \\
B &= 3.0 \text{ in.} \\
C &= 3.5 \text{ in.}
\end{align*}
\]
has a circular cross-section at the XY plane of 3.0 inches radius, and,
a distance along the Z-axis from pole to pole of 7.0 inches. This
geometry was used for the approximation of the skull as an ellipsoid.

A constant thickness \( t = .15 \text{ inches} \), which includes consider-
ation of the diploe layer, was used as an average overall value.

The bone of the skull was modeled as an elastic material, due
primarily to the absence of reliable viscoelastic properties. The
parameters used for this qualitative model are taken from Wood [31],
and, appear to be the most widely used in current literature. These are
Boundary Conditions

For the dynamic system, it becomes necessary to impose boundary conditions which preclude the possibility of unrestrained rigid body motion. Two different sets of boundary conditions were imposed for the skull in this study at the coordinate reduction level. A mid-mounted condition was accomplished by prohibiting translation in the Z-direction of all the nodes which lie in the fifth ring (i.e. in the XY plane). This is done by exclusion, from the γ matrix, of the assumed mode which would otherwise have provided the only mechanism for Z-direction translation of all nodes in the fifth ring (i.e. those lying in the XY plane). Similarly, an end-mounted boundary condition was imposed by removing, from the particular set of assumed modes discussed earlier, that assumed mode, whose sole purpose was to provide the degree of freedom allowing translation of one of the polar nodes along the Z-axis. For the end-mounted boundary condition, the assumed mode which is devoted to Z-direction translation of the nodes in ring 5, must be reinstated. In effect, this procedure results in the generation of non-singular global reduced mass and stiffness matrices.
Other Aspects

The coordinate transformations from global to local coordinates, for finite element generation, and back to global coordinates were executed in usual fashion [32] and [25].

The coordinate reduction was employed on an element by element basis. This produced elemental mass and stiffness matrices which were over-layed to form the global reduced mass and stiffness matrices [32].
Chapter 4

FINITE ELEMENT THEORY

For the development of finite element theory, the state of a small but finite portion of the system under study is considered. This allows for the description of the element state to be approximated in simplified and useful form. The finite elements utilized in this analysis are derived using assumed displacements. In addition, the shell element is the result of the superposition of a plate bending element and a planar (or membrane) element derived within the confines of small displacement and thin shell theory [32].

The general procedure for this type of finite element development calls first of all for a statement of the assumed displacement field in terms of coordinate location with yet unspecified coefficients. For the plane-stress element shown in local coordinates in Figure I-1, the assumed displacement field is

\[ u(x, y) = a_1 x + a_2 y + a_3 x^2 + a_4 y + a_5 x y \\
\text{and} \quad v(x, y) = a_4 x + a_5 y + a_6 x y \]  

(4-1)

where \((u)\) is the unit displacement in the \(X\) coordinate direction and \((v)\) is the unit displacement in the \(Y\) coordinate direction. Because this displacement field gives a constant stress distribution, the element is sometimes referred to as the constant stress triangular element. The six \(a\) coefficients can be expressed in terms of the six degrees of freedom for each element—displacement in the \(X\) and \(Y\) directions at
By taking the appropriate derivatives, the strain can be written in terms of displacements, and, using the constitutive relationship, and integral expression for strain energy is obtained. Minimization of potential energy yields an expression for the element stiffness. A similar procedure involving minimization of kinetic energy results in the expression for the consistent mass matrix [12] and [32]. The mass matrix is "consistent" in the sense that it is derived from the same assumed displacements (often referred to as shape functions) which were used in the stiffness derivation.

The same procedure is applicable to the development of the bending portion of the shell element [32]. The assumed normal displacement field of the element shown in Figure 1-3 is taken as the third order polynomial expansion of the independent coordinate locations x and y. This expansion involves ten terms, however, only nine degrees of freedom (displacement in the z direction and angular rotation about the x and y axes at each node) are available to determine the ten coefficients for bending. This difficulty is readily circumvented by arbitrarily ignoring one of the polynomial expansion terms. The result of this is a non-conforming element. By removing the $x^2y$ term, we maintain conformity of displacement and rotation at coincident nodes of adjacent elements, but are not guaranteed continuity of slope along one side of each element. A variety of alternatives are available to produce conforming elements which generally are of much greater complexity and require significant additional computation effort.
The apparent deficiencies of the non-conforming elements do not necessarily manifest themselves in application. Both Zienkiewicz [32] and Gallagher [12] cite examples (both static and dynamic) in which the results from the simpler non-conforming elements are superior to those of the much more sophisticated conforming elements. In any event, the non-conforming element used in this study is rapidly convergent and produces excellent results in displacement for "adequate" grid sizes.

The complete derivation for the stiffness matrix of the triangular shell element is given in Appendix I.
The dynamic viscoelastic problem [13] can be expressed in Laplace transform space as

\[ (s^2[M] + R_1(s)[\bar{K}_h] + R_2(s)[\bar{K}_d]) \{\vec{\delta}\} = \{F\} \]  \hspace{1cm} (5-1)

where \( R_1 \) describes the hydrostatic viscoelastic stress-strain relationship such that

\[ \bar{s}(s) = R_1(s) \bar{e}(s) \]

and \( R_2 \) similarly describes the deviatoric viscoelastic stress-strain relationship such that

\[ \bar{S}(s) = R_2(s) \bar{E}(s) \]

where

- \( s \) = Laplace variable
- \( \bar{s}(s) \) = hydrostatic stress in Laplace space
- \( \bar{e}(s) \) = hydrostatic strain in Laplace space
- \( \bar{S}(s) \) = deviatoric stress in Laplace space
- \( \bar{E}(s) \) = deviatoric strain in Laplace space
- \( \delta \) = nodal displacement vector
- \( F \) = external force vector
- \( M \) = consistent mass matrix
- \( K_h \) = hydrostatic stiffness
\[ K_d = \text{deviatoric stiffness} \]

and, the over-bar implies "the Laplace transform of".

Equation (5-1) can be solved in its present form, however, for the problem at hand, frequency response determination requires inverting a prohibitively large system of equations for each frequency sampled. For the grid size shown in Figure 6-3, the entire system would involve a set of 1092 equations. To circumvent this difficulty, a coordinate reduction is employed as outlined by Anderson [1], and discussed in a previous chapter.

We choose to approximate the actual motion of the system with a reduced set of coordinates \( \{P\} \) such that

\[ \{\delta\} = [\gamma]\{P\} \]

where the columns of \( [\gamma] \) are the "assumed modes" for the system. If we make this substitution prior to the minimization of total energy we can write equation (5-1) in reduced coordinates as

\[ (\dot{\omega}^2 [RM] + R_1(\omega)[RK_h] + R_2(\omega) [RK_d])\{P\} = \{FR\} \quad (5-2) \]

where the reduced mass, hydrostatic stiffness, and deviatoric stiffness are

\[ [RM] = [\gamma]^T[M][\gamma] \]

\[ [RK_h] = [\gamma]^T[K_h][\gamma] \]

\[ [RK_d] = [\gamma]^T[K_d][\gamma] \]
The reduced force becomes

$$\{FR\} = [γ]^T \{F\}$$

To transform equation (5-2) into the frequency domain [26] we effect the substitution

$$s \rightarrow j\omega$$

where

$$j = \sqrt{-1}$$

$$\omega = \text{frequency of forcing function}$$

The expression for the frequency response then becomes

$$(-\omega^2 [RM] + R_1(j\omega)[RK_h] + R_2(j\omega)[RK_d]) \{P(j\omega)\} = \{FR(j\omega)\}$$

The frequency response, being a measure of the amplitude ratio \(A_r\) of displacement to input force, can then be expressed as

$$A_r = \left| \frac{P(j\omega)}{FR(j\omega)} \right| = \left| (-\omega^2 [RM] + R_1(j\omega)[RK_h] + R_2(j\omega)[RK_d]) \right|$$
In order to establish a level of confidence in our finite element model, the dynamic performance is compared to results obtained from closed form solutions for geometries where such a solution is readily available. This, it was felt, was necessary to preclude any possibility of the finite element model being the source of apparent inconsistencies with previous work.

Two test geometries—the circular ring and the spherical shell—will be modeled using the triangular shell finite element. The circular ring is useful in that the two modes of deformation, in-plane and bending, can be evaluated separately. The extensional mode of vibration, in which all points are moving uniformly in a radial direction, involves only in-plane deformation. The first bending mode, in which the ring assumes an "elliptical" shape at the instant of maximum excursion, is characterized primarily by bending (i.e. increased or decreased curvature).

Comparison of the finite element results for the circular ring shown in Figure 6-1 to the closed form solution given by Timoshenko [29] is presented in Table 6-1.
Table 6-1.
Comparison of finite element results to closed form solution for a circular ring.

Resonant Frequency (Hz.)

<table>
<thead>
<tr>
<th></th>
<th>Finite Element Model</th>
<th>Timoshenko</th>
<th>% Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Extensional Mode</td>
<td>4908</td>
<td>4897</td>
<td>.2</td>
</tr>
<tr>
<td>First Bending</td>
<td>243</td>
<td>233</td>
<td>4.3</td>
</tr>
</tbody>
</table>

As was previously noted, the skull's geometry can be approximated as an ellipsoidal shell. However, due to the non-availability of a closed form solution for an ellipsoidal shell, we must revert to the thin wall spherical shell to test our finite element model. Finite element results for the lowest axisymmetric bending mode of a sphere are compared to closed form solutions obtained by Baker [2]. In Figure 6-2 the difference in resonant frequencies obtained from finite elements vs. those from closed form solution expressed as a percentage of the closed form frequency is plotted against sphere radius while thickness is held constant. The finite element frequency, as must be the case, is always greater than the theoretical resonant frequency.
**Figure 6-1.** Grid Pattern for Circular Ring Test Geometry.

**Figure 6-2.** Comparison of Finite Element to Closed-form Results for the Fundamental Bending Mode of a Thin Spherical Shell.
It is important to note that Baker's development is based on membrane theory only. As the shell diameter decreases, the thickness/diameter ratio increases, and, the effect of the bending stiffness becomes more pronounced. Thus the divergence of results at small diameters is not necessarily attributable to finite element deficiencies, but instead can be viewed as the manifestation of actual deviation from total membrane deformation. It can be seen that, at large diameters, the finite element model frequencies converge to a level approximately 2.5% greater than theoretical.

Skull Results

The grid size, along with the geometrical and rheological parameters shown in Figure 6-3 are used to generate qualitative information pertaining to skull resonance. The assumed modes are chosen such that any axisymmetric mode shape can be synthesized. Included is the possibility of local deformations at the "poles" of our model.

To obtain dynamic results resembling the fundamental bending mode for a sphere, boundary conditions are imposed such that nodes on the mid-plane are restrained from translatory motion parallel to the axis. An analytic harmonic force is applied at the poles. Modeled as an elastic material, the frequency response for the spherical skull is shown in Figure 6-4. Resonance occurs at 3700 Hz. and, the mode shape which results is shown in Figure 6-5. The first bending mode is shown
Calvarium to be modeled.

Grid pattern for the spherical model.

Grid pattern for the ellipsoidal model.

FIGURE 6-3.
FIGURE 6-4. Frequency Response for Mid-mounted Sphere.

* - undeformed skull
+ - mode shape

FIGURE 6-5. Mode Shape for Mid-mounted Sphere at 3700 Hz.
only. Continuation of the frequency response procedure yields resonant
mode shapes at higher frequencies which closely correspond to those
predicted by the closed form solution. In the search for lower frequency
modes, the frequency response was computed for input frequency step
sizes of 5 Hz. None was found. Thus, our mid-mounted spherical model
produces a fundamental bending mode at a frequency which is nearly an
order of magnitude greater than that indicated by previous studies.

In an attempt to simulate conditions which could plausibly explain
the experimental and analytical results obtained by previous investi­
gators, an alternate set of boundary conditions was imposed on the
spherical model. The node at one pole was held stationary, and, a
harmonic force, axially directed, was applied at the opposite pole. The
frequency response is shown in Figure 6-6, and the mode shape at re­
sonance (440 Hz.) is shown in Figure 6-7. The mode shape appears to be
the superposition of a combined local deformation of the sphere proximal
to the point of support, along with rigid body motion of the remaining
portion of the skull. The additional effective mass, due to the bulk of
the skull moving in rigid body fashion, accounts for a resonance which
is an order of magnitude less than previous results for the same sphere.

Similar qualitative results are obtained for the ellipsoid shown in
Figure 6-3. The frequency response for mid-plane support and for end
support are shown in Figures 6-8 and 6-10 respectively, and, mode shapes
at resonance are shown in Figures 6-9 and 6-11. Again, note that no
FIGURE 6-6. Frequency Response for End-mounted Sphere.

FIGURE 6-7. Mode Shape for End-mounted Sphere at 440 Hz.
**FIGURE 6-8.** Frequency Response for Mid-mounted Ellipsoid.

**FIGURE 6-9.** Mode Shape for Mid-mounted Ellipsoid at 4160 Hz.
FIGURE 6-10. Frequency Response for End-mounted Ellipsoid.

FIGURE 6-11. Mode Shape for End-mounted Ellipsoid at 530 Hz.
low frequency local deformations materialize for the mid-supported sphere. Comparison reveals results which are similar to those derived from the spherical model. The mode shape at 530 Hz. for the end-mounted ellipsoid exhibits the same rigid body-local deformation characteristic that was seen in the results obtained from the spherical model.

In a further attempt to exhaust the search for possible local deformation (without rigid body motion), the same analysis was employed for an extremely shallow ellipsoid. This is to check conjecture that low frequency responses are the result of local deformations of relatively flat portions of the skull. Again, the presence of a local deformation at a resonance lower than that of the first bending mode is not indicated.
A finite element model of the human skull has been presented. The results for a thin wall spherical shell closely agree in both resonant frequency and corresponding mode shape to that predicted by a closed form solution. This model, when supplied with the geometrical and rheological parameters approximating those for a human skull, indicates resonance of the fundamental bending mode at approximately 3700 Hz. Further insight into this apparent inconsistency with previous studies can be gained by subjecting the same skull to an end mounting as described earlier. This results in a significantly different mode shape, best described as a rigid body-local deformation (RBLD) mode. The following discussion demonstrates that the RBLD mode interpretation is consistent with the results of previous related investigations, and the ramifications of this interpretation are considered.

Comparison to Previous Studies

We will first examine an experimental study in which a cadaver skull is subjected to an external harmonic force. The intent of the particular study is to infer the response of the human head subjected to similar external stimulus.

Gurdjian [15] frontally mounted a cadaver skull to a force generator. A force of constant amplitude was imparted to the skull over a specified frequency range. Accelerations were measured at and diametri-
cally opposite the loading, at the apex, and at the side of the skull. Gurdjian reports that an anti-resonant and resonant condition occur at approximately 300 Hz. and 900 Hz. respectively. The anti-resonant mode at 300 Hz. is of particular interest in that its description closely corresponds to that of the RBLD mode described in this study. The measurement of activity at only four locations over the entire skull in Gurdjian's study makes conclusive comparison of the two modes unlikely.

The anti-resonant mode is described by stating, "the mode of vibratory motion of the head is such that very little movement occurs at the driven point compared to the occiput." Viewed in a slightly different perspective, this anti-resonant mode corresponds to the RBLD mode shown in Figure 6-7, where the resulting loading at the fixed node is analogous to Gurdjian's excitation. As noted previously, the RBLD mode exhibits large relative deformations of the occiput vs. the frontal region.

A large mechanical impedance (also appearing in Gurdjian's measurements) is the consequence of the RBLD mode. This impedance occurs as the result of a large portion of the skull translating in rigid body fashion. The strain energy becomes concentrated at the fixed node—a condition which is the result of the particular boundary condition chosen. Because of the absence of complete experimental information, verification of the coincidence of the RBLD mode and Gurdjian's anti-resonant mode must be the product of further experimental work. The
determination of such a correspondence would mandate re-examination and possible re-interpretation of the response of the human head to external disturbance. Specific areas of re-evaluation will be addressed later in this chapter.

Analytical Comparison

The following discussion will concentrate on the analytic behavior of the present axisymmetric model compared to that described in a similar study by Nickell and Marcal [24]. In that study, the skull is modeled using doubly curved triangular finite elements, and the lowest mode shapes were obtained for three different sets of prescribed boundary conditions. It is the choice of these boundary conditions and the manner in which they influence the effective system which raises the case for an alternative and perhaps more precise interpretation of resultant mode shapes. In each case one node (located at the front, rear, or at the base) is restrained so as to exclude the possibility of translatory motion. A second arbitrary stiffness provides "sufficient restraint at an adjacent point to prevent rotation."

For each set of boundary conditions, four mode shapes were obtained. The first two will be discussed only briefly because they are in essence rigid body rotation (rocking) about the fixed point. These modes are not axisymmetric and, therefore, do not afford comparison to the results of our axisymmetric study. The first and second mode shapes are
reproduced in Figures 7-1 and 7-2. In each figure the mode shapes are also shown superimposed on the undeformed skull to illustrate distortion. The first is nearly a pure rigid body rotation about the fixed node. Only a slight distortion of the skull is detectable, this occurring at the frontal base of the skull. The frequency associated with this mode shape is then primarily dependent upon the support stiffness prescribed by the investigator.

The second mode is similar to the first in that the predominant characteristic is a rigid body or rocking movement about the fixed node. In addition to the rigid body motion, a significant distortion occurs at the rear and the frontal base of the skull. Again, the resonant frequency will be greatly influenced by the nature and magnitude of the artificial restraining stiffness which has been applied as a boundary condition.

Nickell and Marcal's third mode is described as exhibiting "large local skull distortion in the frontal and occipital regions." Figure 7-3a shows the third mode and the undeformed skull taken from the study by Nickell and Marcal. In Figure 7-3a the displacements are shown such that the fixed node is coincident in the undeformed and third mode representation. Figure 7-3b shows the same two shapes superimposed so as to illustrate relative distortion. With the exception of the deformation occurring at the frontal base of the skull, the mode shape appears to be the rigid body-local deformation mode obtained from the axisymmetric study presented in this paper. See Figure 6-7.
FIGURE 7-1. Nickell & Marcal's rear mounted skull (a.) with the 1st mode & undeformed shape coincident at the fixed node, and (b.) with the undeformed skull and mode shape superimposed to illustrate relative distortion.
FIGURE 7-2. Nickell & Marcal's rear mounted skull (a.) with the 2nd mode & undeformed shape coincident at the fixed node, and (b.) with the undeformed skull and mode shape superimposed to illustrate relative distortion.
FIGURE 7-3. Nickell & Marcal's rear mounted skull (a.) with the 3rd mode & undeformed shape coincident at the fixed node, and (b.) with the undeformed skull and mode shape superimposed to illustrate relative distortion.
The area of local deformation shown in Figure 7-3 is somewhat larger than that indicated by the axisymmetric model (Figure 6-7). Variations in skull thickness and geometrical anomalies which were included in the Nickell-Marcal model can at least partially account for differences in the area of local deformation. The greatest thickness was ascribed to the occiput while a substantially thinner zone was modeled in the proximal region where maximum bending appears.

The boundary conditions imposed by Nickell and Marcal are the same as those employed in the axisymmetric model with the additional freedom to allow rocking about the fixed node. This option gives license to the lowest modes obtained. The third mode displays the essential characteristics which point to interpretation as a RBLD mode, but, not without the irregularities emanating from a lack of geometrical and structural symmetry.

Implications

Some immediate questions arise concerning previous interpretations of skull response in these and similar investigations. A common assumption in such skull (in-vacuo) studies is that response of the skull to external excitation is indicative of that for the human head subjected to the same excitation. How this premise withstands the RBLD mode interpretation is the subject of the first query. Secondly, can the RBLD mode be excited in the in-vivo human head? The implication here is
that this particular mode may be inherent to the system which has been modified by the use of artificial experimental and analytic boundary conditions. And, third, how does the above effect previous conclusions pertaining to the mechanisms of brain damage?

Use of the response of the skull alone to imply the nature of head injury necessitates the assumption that the brain acts in a passive manner—contributing to damping and producing only secondary effects in response characteristics. In our axisymmetric study, it has been noted that the change in boundary conditions from mid-mounted to end-mounted results in a reduction of the first bending mode frequency by nearly an order of magnitude. This is the consequence of an increased effective mass producing larger inertial forces which must be countered at the end support. It follows that inclusion of the mass of the brain will further increase the effective mass of the system, particularly with rigid body related modes. Somewhat less evident is the effect that the brain has on the overall stiffness of the head. Failure to incorporate the effects of the brain where predominately rigid body motion of the skull prevails, will produce results of questionable validity.

The RBLD mode appears in analytical and experimental studies in which a node at (or diametrically opposite to) the external loading point is held stationary. To maintain this support during frequency response testing, large forces are necessary to counter the inertial effects of rigid body motion. For in-vitro experimental work, the
specimen is rigidly attached to the test apparatus. Non-invasive in-vivo testing does not allow so convenient a solution. A pressure support can maintain the boundary condition for only the compression half of the cycle. Whether this support condition can adequately be synthesized for in-vivo experimentation emerges as a question which must be resolved.

An in-depth discussion of the mechanisms of brain damage is beyond the scope of this paper, so that the third question will go largely unanswered. Let it suffice to state that extreme caution must be practiced in the interpretation of results from studies similar to those discussed here. The case for the existence and the role of the RBLD mode must be considered fully.

Feasibility for Determination of Rheological Properties

Returning our discussion to the original purpose of this paper we consider the feasibility of determining the rheological properties of the human skull using low frequency vibrations. Based on the results of the axisymmetric model, two approaches are available to achieve our objective. The first involves the use of the fundamental bending mode which is a response characteristic of the free-body (or mid-mounted) skull. Because the resonant frequency associated with this mode shape is quite high (3700 Hz. for the axisymmetric skull modeled in this study), the following enumeration of requirements, upon which success is contingent, is presented.
1. Transducers and experimental equipment must be available with intrinsic sensitivities capable of measuring extremely small amplitude motion of the skull at high frequencies (2 to 5 KHz.). Along these same lines, a system to impart the necessary harmonic force to the skull must also be available.

2. An effective means to contend with the damping effect of the layer of skin covering the skull must be developed. The difficulty is particularly troublesome at the locations of applied load and response measurements.

3. At elevated frequencies the internal damping of the bone and the contents of the skull will further reduce the probability of measuring response.

4. The succession of resonant modes for a spherical shell are separated by a frequency difference which is small relative to the frequency itself. When allowed insight into the response of only a few nodes, the characteristics which differentiate one mode from another may become extremely subtle. As the geometry increases in complexity, so does the task of defining with a reasonable degree of accuracy resultant mode shapes. The concern here is for the quantity and quality of instrumentation required for such a determination.
An alternative approach is to utilize the RBLD mode as the fundamental bending mode. The lower frequencies alleviate some of the distressing requirements posed above, but, in turn, the use of this mode warrants its own list of requirements. These follow.

1. The question relating to the capability or need to reproduce the end-mount boundary condition for in-vivo studies must be resolved. One would expect that an experimental procedure utilizing a simplified geometry might produce a satisfactory answer.

2. Response, involving a large portion of the skull moving as a rigid body, intimately involves the physiological support for the head, i.e., the spinal column along with the muscular and fleshy tissues of the neck. The analytical modeling of the neck may constitute as formidable a task as that involved with modeling the head itself. Appropriate simplifying assumptions or an adequate model of the support structure must be included in the analytical portion of the investigation. Reproducibility of experimental results will also be adversely affected due to inconsistencies in the state of relaxation of the neck.

3. The damping and inertial effects of the brain must be included in the analysis, or the determination made that such effects are negligible.

4. For the RBLD mode, the experimental instrumentation of the skull must produce definitive information regarding the resultant mode.
shape. An important and perhaps vital measurement for the in-vivo skull is the extent of the area of local deformation. Again, this requires, sophisticated experimental capabilities.

Summary

A finite element model of the in-vivo skull indicates a fundamental free-body bending resonance at approximately 3700 Hz. A modified system with an end-mounted boundary condition produces a rigid body with local deformation mode shape at approximately 450 Hz. This mode appears to be consistent with the results of similar studies and suggests a reconsideration of previous interpretations.

Neither of the two mode shapes offers particular encouragement to the investigator interested in utilization of these mode shapes for determination of rheological properties. The former becomes unattractive primarily because of the high frequencies involved, and the later because of the entrance of the neck as a modeling problem. The need to arrive at material properties for the in-vivo human skull remains. Despite the above mentioned shortcomings, the procedures discussed may still be the best means available to determine rheological properties of the human skull.
REFERENCES


APPENDIX I

SHELL ELEMENT

The finite element derivation for the triangular shell element closely follows that described by Zienkiewicz [32]. Deviations from Zienkiewicz's development involve the use of a right hand cartesian coordinate system, and the independent development of the hydrostatic and deviatoric stiffness. This is done so that the Correspondence Principle can be applied directly [9] for solution of the viscoelastic problem. The individual treatment of the hydrostatic and deviatoric stiffness allows for algebraic simplifications which increase computational efficiency.

The shell element is subjected to both in-plane and bending loads. The small deformation assumption is utilized so that the two deformation modes (bending and in-plane) are taken as mutually independent. The two elements are then combined by linear superposition to form the shell element.

The In-Plane Element

The displacements (two in-plane deformations and one rotation) and the corresponding external loads for a typical node are shown in Figures (I-1) and (I-2).
The in-plane deformations \((u \text{ and } v)\) are allowed to vary linearly with respect to coordinate location such that

\[
\begin{align*}
  u(x,y) &= \alpha_1 + \alpha_2 x + \alpha_3 y \\
  v(x,y) &= \alpha_4 + \alpha_5 x + \alpha_6 y
\end{align*}
\]  

\(\text{(I-1)}\)

The result is six unknown \(\alpha\) coefficients and six degrees of freedom (two at each node) from which these can be determined.

The displacements in the \(x\) coordinate direction at the nodes can be written
\[ u_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i \]
\[ u_j = \alpha_1 + \alpha_2 x_j + \alpha_3 y_j \]  \hspace{1cm} (I-2)
\[ u_k = \alpha_1 + \alpha_2 x_k + \alpha_3 y_k \]

where subscripts \( i, j, \) and \( k \) identify the respective node.

Equations (I-2) can be written in matrix form as

\[
\begin{bmatrix}
  u_i \\
  u_j \\
  u_k
\end{bmatrix} =
\begin{bmatrix}
  1 & x_i & y_i \\
  1 & x_j & y_j \\
  1 & x_k & y_k
\end{bmatrix}
\begin{bmatrix}
  \alpha_1 \\
  \alpha_2 \\
  \alpha_3
\end{bmatrix}
\]  \hspace{1cm} (I-3)

By inverting the 3X3 nodal coordinate matrix, the following expression for the first three alpha coefficients can be obtained.

\[
\begin{bmatrix}
  \alpha_1 \\
  \alpha_2 \\
  \alpha_3
\end{bmatrix} =
\frac{1}{2\Delta}
\begin{bmatrix}
  a_i & a_j & a_k \\
  b_i & b_j & b_k \\
  c_i & c_j & c_k
\end{bmatrix}
\begin{bmatrix}
  u_i \\
  u_j \\
  u_k
\end{bmatrix} \hspace{1cm} (I-4)
\]

where \( 2\Delta \) is the determinant of the previous 3X3 matrix and is numerically equal to twice the area of the triangular element. The \( a, b, \) and \( c \) constants are elements of the adjoint of the nodal coordinate matrix. A similar expression can be written for the remaining three alpha coefficients using the nodal displacements in the \( y \)-coordinate direction.
Plane strain for small deformations is defined

\[
\{\varepsilon\} = \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_{xy}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\end{bmatrix}
\]  
(I-5)

Substituting for \( u \) and \( v \) from equation (I-4), the strain becomes

\[
\{\varepsilon\} = \begin{bmatrix}
\alpha_2 \\
\alpha_2 \\
\alpha_3 + \alpha_5
\end{bmatrix}
\]  
(I-6)

and, again substituting from equation (I-4) for the alpha coefficients give the following expression relating the strain field to nodal displacements.

\[
\{\varepsilon\} = [B] \{\delta_e\}  
\]  
(I-7)

where

\[
[B] = \frac{1}{2\Delta} \begin{bmatrix}
b_i & 0 & b_j & 0 & b_k & 0 \\
0 & c_i & 0 & c_j & 0 & c_k \\
c_i & b_j & c_j & b_k & c_k & b_k
\end{bmatrix}
\]

and

\[
[\delta_e]^T = [u_i \ v_i \ u_j \ v_j \ u_k \ v_k]
\]

The stress-strain relationship is
\[
\begin{pmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{pmatrix} = \frac{E}{(1-v^2)} \begin{pmatrix}
1 & v & 0 \\
v & 0 & 0 \\
0 & 0 & \frac{1-v}{2}
\end{pmatrix} \begin{pmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_{xy}
\end{pmatrix}
\] (I-8)

or, in terms of hydrostatic and deviatoric stress and strain

\[
\begin{pmatrix}
s_1 \\
s_2 \\
0
\end{pmatrix} = \frac{E}{2(1-v)} \begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
e_1 \\
e_2 \\
0
\end{pmatrix}
\] (I-9)

where the hydrostatic stress \(s\) is defined

\[
s = \frac{1}{2}(\sigma_x + \sigma_y)
\] (I-10)

and the hydrostatic strain \(e\) is

\[
e = \frac{1}{2}(\varepsilon_x + \varepsilon_y)
\] (I-11)

Let the bulk modulus \(\kappa\) be defined as

\[
\kappa = \frac{E}{2(1-v)}
\] (I-12)

We can then write

\[
s = \kappa \cdot 2e
\]

or

\[
\{s\} = \kappa[D_h] \{e\}
\] (I-13)
Similarly for the deviatoric stress-strain relationship we have

\[
\begin{align*}
\{S_x\} &= \frac{E}{2(1+\nu)} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \{E_x\} \\
\{S_y\} &= \frac{E}{2(1+\nu)} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \{E_y\} \\
\{S_{xy}\} &= \frac{E}{2(1+\nu)} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \{E_{xy}\}
\end{align*}
\]

where

\[
S_x = \sigma_x - s
\]

and

\[
E_x = \varepsilon_x - e \quad \text{etc.}
\]

The shear modulus \(G\) is defined

\[
G = \frac{E}{2(1+\nu)}
\]

and, we can then write

\[
\{S_{xy}\} = G \cdot \{E_{xy}\} \quad (I-15)
\]

or, in matrix form

\[
\{S\} = G[D_d] \{E\} \quad (I-16)
\]

where

\[
[D_d] = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
To formulate the hydrostatic and deviatoric stiffness we write the expression for the potential energy ($\Pi_e$) of the element.

$$\Pi_e = 1/2 \int_V [s] \{\varepsilon\} dv + 1/2 \int_V [S] \{\varepsilon\} dv - [F] \{\delta_e\}$$ (I-17)

The first term is the strain energy due to the hydrostatic stress, the second term is the strain energy due to the deviatoric stress, and the final term is the contribution of the work done by the external forces.

Substituting equations (I-7), (I-13), and (I-16) into equation (I-17) and minimizing potential energy by taking the first variation of displacement gives

$$0 = \left\{ \kappa \int_V [B]^T[D_h][B] dv + G \int_V [B]^T[D_d][B] dv \right\} \{\delta_e\} - \{F\}$$ (I-18)

In both integrals the integrand is independent of coordinate location within the element so that equation (I-18) can be written

$$\{F_e\} = \left\{ [K_h] + [K_d] \right\} \{\delta_e\}$$ (I-19)

where


and


Here $t$ is the thickness and $A$ is the area of the element.
Bending Element

For the bending element, the displacements and external forces are as shown in Figure (I-3) and (I-4).

\[ \begin{bmatrix} w_j \\ \phi_x \phi \end{bmatrix} = \begin{bmatrix} w_j \\ \frac{\partial w_j}{\partial y} \\ \frac{\partial w_j}{\partial x} \end{bmatrix} \] (I-22)
The displacement field is allowed to be fit by a third order polynomial expansion over the area of the triangle, i.e.

\[ w(x,y) = 1 + x + y + x^2 + xy + y^2 + x^2 y + xy^2 + y^3 \{a\} \quad (1-23) \]

where \( \{a\} \) is the vector of nine coefficients which are to be determined by the nine (three at each node) nodal displacements.

Then

\[ \{\delta\} = \{A\}\{a\} \quad (I-24) \]

where the elements of \( \{A\} \) are determined by differentiating equation (I-23) according to equation (I-22) at nodal locations \( i, j, \) and \( k \).

Substitution of this result back into equation (I-23) gives

\[ w(x,y) = \begin{bmatrix} N(x,y) \end{bmatrix} \{\delta\} \quad (I-25) \]

where the shape function \( N(x,y) \) becomes

\[ \begin{bmatrix} N(x,y) \end{bmatrix} = \begin{bmatrix} P(x,y) \end{bmatrix} \begin{bmatrix} A \end{bmatrix}^{-1} \quad (I-26) \]

The strain for small bending deformations is approximated as

\[ \{\varepsilon\} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{bmatrix} = -z \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \quad (I-27) \]
Substituting equations (I-25) and (I-26) into equation (I-27) gives

\[ \{e\} = -z[B^-]\{\delta_e\} \quad (I-28) \]

where

\[ [B^-] = \begin{cases} [P,xx] \\ [P,yy] \\ 2[P,xy] \end{cases} \quad [A]^{-1} \quad (I-29) \]

Here the subscripts \( x \) and \( y \) indicate partial differentiation with respect to the \( x \) and \( y \) independent variables respectively.

Using the hydrostatic and deviatoric stress-strain relationship we can write the potential energy of bending.

\[ \Pi_e = [\delta_e] \int_0^Z \left( \kappa[B^-]^T[D_h][B^-] + 2G[B^-][D_d][B^-] \right) d\{\delta_e\} - [F]\{\delta_e\} \quad (I-30) \]

Minimizing potential energy gives

\[ \{F\} = ([K^-_h] + [K^-_d])\{\delta_e\} \quad (I-31) \]

where

\[ [K^-_h] = \int_A [B^-]^T[D_h][B^-] \, dA \]

\[ [K^-_d] = \int_A [B^-]^T[D_d][B^-] \, dA \]

and

\[ I = \frac{t^3}{12} \]
The integrand for both the hydrostatic and deviatoric bending stiffnesses contain elements dependent upon area coordinates $x$ and $y$, and, therefore, must be included in the integration over the area of the element.

The consistent mass matrix for a viscoelastic material is the same as that for the elastic material. The mass matrices for both the in-plane and the bending element are presented in Zienkiewicz [32], and will not be repeated here.

Shell Element

The hydrostatic stiffness, deviatoric stiffness, and mass for the plane stress and bending elements are superimposed to form the respective stiffness and mass matrices for the shell element.

An arbitrary fictitious stiffness is supplied to complete the relationship between the angular deformation $\phi_z$ and loading $M_z$. Zienkiewicz [32] provides the form for this matrix. When adjacent elements are co-planer, the identity $0 \equiv 0$ arises, which can produce erroneous results. By introducing an arbitrarily large stiffness the possibility of this identity is eliminated and only small error is injected.

The development for the shell element has been carried out in local coordinates as shown in Figure (I-1) and (I-2). Transformation into global coordinates is necessary for consideration of the total problem [32].
Appendix II

FLOW DIAGRAM FOR REDUCED MASS & STIFFNESS GENERATION

```
Initialize

CALL for generation of assumed modes.

Generate node identification.

Generate global nodal coordinates for next element.

CALL for stiffness and mass generation.

Perform coordinate reduction of mass and stiffness matrices.

Overlay reduced matrices in global reduced mass and stiffness matrices.

Output reduced mass, hydrostatic stiffness, and deviatoric stiffness matrices.

SUBROUTINE GAMGEN
Generate assumed modes.

SUBROUTINE STIFF
Transform from global to local coordinates.

CALL for membrane mass & stiffness gen.

CALL for bending mass & stiffness generation

Generate the fictitious mass & stiffness matrices.

Superimpose the bending and membrane matrices.

Transform from local to global coordinates.

RETURN

SUBROUTINE STIFFP
Generate in-plane mass, hydrostatic stiffness, & deviatoric stiffness for the element.

SUBROUTINE STIFFB
Generate bending mass, hydrostatic stiffness, & deviatoric stiffness for the element.

END
```
Grammens, Gerald M
G762 Frequency response
cop.2 analysis of the in-vivo
human skull

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Angel Williams 11-24-68

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