A method of graphical analysis for unsymmetrical three-phase circuits
by Armin John Hill

A THESIS Submitted to the Graduate Committee in partial fulfillment of the requirements for the degree of Master of Science in Electrical Engineering
Montana State University

Abstract:
This thesis presents a method of graphical analysis for electric circuits which is based upon some of the simple aspects of tensor analysis as recently applied in electrical problems* A development of the method is made in such a manner as to be understandable by undergraduate engineering students. The relationship 'between the homogenous linear transformation used in teaser analysis, and the ordinary system of three simultaneous linear equations found • in -elementary analytic geometry is shorn? and the graphical representation of this transformation is used as a basis of the later developments# It becomes possible through such a presentation to solve a system of three simultaneous linear equations graphically* by applying1 some principles Of descriptive geometry® A brief development of the tensor notation is included in order that this notation can be used in the more complex developments of the later parts® The possibility of the study of electric circuits through tx graphical analysis based upon such a presentation is discussed briefly* and a few of the fundamental methods of procedure for such an analysis are presented® An extension, of the principles to include equations with complex quantifies, is made* and these are applied to the study of alternating current circuits® General circuit problems in this form are found to be very complicated* but most of the actual problems are simplified enough that they can be handled on a practicable basis® The method is found to be particularly useful in handling unbalanced three-phase systems* either directly or by means of symmetrical components® With the latter* the transformation to such components can be made by aeons of a chart which has the acme form for all problems of this nature.

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Abstract

This thesis presents a method of graphical analysis for electric circuits which is based upon some of the simple aspects of tensor analysis as recently applied in electrical problems. A development of the method is made in such a manner as to be understandable by undergraduate engineering students. The relationship between the homogenous linear transformation used in tensor analysis, and the ordinary system of three simultaneous linear equations found in elementary analytic geometry is shown, and the graphical representation of this transformation is used as a basis of the later developments. It becomes possible through such a presentation to solve a system of three simultaneous linear equations graphically, by applying some principles of descriptive geometry. A brief development of the tensor notation is included in order that this notation can be used in the more complex developments of the later parts. The possibility of the study of electric circuits through a graphical analysis based upon such a presentation is discussed briefly, and a few of the fundamental methods of procedure for such an analysis are presented.

An extension of the principles to include equations with complex quantities is made, and these are applied to the study of alternating current circuits. General circuit problems in this form are found to be very complicated, but most of the actual problems are simplified enough that they can be handled on a practicable basis. The method is found to be particularly useful in handling unbalanced three-phase systems, either directly or by means of symmetrical components. With the latter, the transformation to such components can be made by means of a chart which has the same form for all problems of this nature. In the commonly occurring cases where the zero-phase components are absent, this chart takes the form of a convenient slide rule, on which the two remaining sets of components can be obtained from two settings of the slide.

As with the tensor analysis which it parallels, this method makes possible a simultaneous analysis of an entire network. It also offers a graphical introduction to tensor methods for the student who has had a limited mathematical background. Extensions of the graphical method to other fields are indicated, and it is hoped that such indications will lead to an increased application and interest in the valuable and important concepts given the engineer by tensor analysis.

A METHOD OF GRAPHICAL ANALYSIS FOR UNSYMMETRICAL THREE-PHASE CIRCUITS

"The key to the simplest analysis of alternating currents lies in the use of graphical methods."[1] This statement made by Dr. Kennelly in 1898 has been well verified since that time for graphical methods have been widely developed and applied in practically every branch of electrical engineering. In fact, in most cases the graphical development has paralleled closely the application of the analytical mathematics, and often has proved so valuable that it has become the accepted method of proceedure.

For instance, we find the concept of the time vector used by Steinmetz[2] as a pictorial representation of the complex number which had been so successfully introduced by Kennelly[3] and Steinmetz[4] to represent alternating current quantities. A study of the effect on such vectors of the changes which take place under certain operating conditions led to the discovery of the invaluable circle diagram[5] which since has come into general use in the analysis of the operating characteristics of alternating current machinery. Likewise charts and graphs proved themselves indispensable in the study of magnetic circuits when the corresponding mathematical equations were found to be too complicated for practical use. A survey of the field of power transmission reveals numerous applications of graphical proceedures, some of which, such as the Mershon diagram[6], and the Dwight[7], and the Thomas[8] charts, have become standard equipment in the handling of many of the problems in this field. Similar applications are to be found in practically every branch of the electrical engineering field.

The development presented in this thesis is an attempt to give a graphical interpretation of a few of the simpler applications which have recently been made of tensor analysis in the study of electric circuits. "For many years the concepts introduced by vector analysis were sufficient for handling the types of electromagnetic phenomena encountered in electrical circuits and apparatus; but with the later increasing complexities involved in machine design, Steinmetz's complex numbers became inadequate for universal application."[9] Consequently, Gabriel Kron of the General Electric Company issued in 1932 a series of mimeographed articles dealing with the applications of tensor analysis to electrical machinery[10] and gave an informal paper on the subject before the winter convention of the AIEE in January, 1933.[11] This is apparently the first attempt at such an application in the field of electrical engineering, but almost immediately a widespread interest in this new tool was apparent, and many articles began to appear concerning various applications of tensor and matrix methods to electrical problems.[12] Kron revised and enlarged his original work, publishing it as a series of articles in the General Electric Review[13], and in this form it is the most comprehensive treatment of the applications of tensor analysis to electric circuit problems found in available literature. It is primarily on some of the elementary parts of this work that the material of this thesis is based.

Tensor analysis, from the very beginning stages of its development demonstrated itself to be an extremely powerful and useful mathematical concept. It quickly proved its worth in the field of geometry, and the demands of relativistic and quantum physics showed that its methods were capable of a wide variety of applications. Now these recent attempts to apply it to the type of problems with which engineers are primarily concerned have clearly shown that here at last is a tool, powerful and versatile enough to cope with the increasingly complex problems of the engineering field.

The rapidly increasing interest brought about by the success of these applications has made it imperative that engineers who wish to keep pace with present literature acquire a working knowledge of tensor principles. In fact, it is safe to predict that before long a thorough knowledge of tensor analysis will be an indispensable requisite of the well trained engineer. Up to the present time, however, such an understanding is the special privilege of the few who have had an opportunity to study a considerable amount of advanced mathematics. Now it has been found that many of the applications presented in this thesis can be based directly upon the mathematical forms encountered in elementary algebra and analytic geometry, and therefore should be within the grasp of the undergraduate engineering student, or of the engineer whose mathematical background is limited. For this reason care has been taken to present the material in such a way that it can be understood by one who has had no more than the equivalent of one year of college mathematics. It is hoped that in this way, such a presentation may help to bring about a more general understanding of this powerful mathematical concept.

Therefore, while the primary purpose of this thesis is to present a graphical parallel of the application of tensor analysis to electric circuits, insofar as physical limitations will permit, it is also hoped that the form of presentation will accomplish two other results. First, the graphical presentation offers an excellent introduction to tensor methods. One of the chief obstacles in the path of a general understanding of tensors is the difficulty of obtaining a clear mental picture of the concepts involved.[14] Since some of the more elementary of these are here developed graphically, and are thus given a physical interpretation, it is hoped this will provide a ground work of such a nature that the mental hazards of the more advanced concepts are materially reduced. In the second place, when such an approach is made, the close connection between the simpler aspects of tensor analysis and the forms encountered in algebra and in analytic geometry is stressed.

With these points in mind, care has been taken to develop the material from the standpoint of one not acquainted with tensor methods. Ordinary algebraic notation is used for the first developments, with the tensor notation introduced for handling the more complex applications. Also an attempt is made at all times to keep the close connection between the graphical, the algebraic, and the tensor concepts in mind. For instance a system of simultaneous linear equations is shown graphically as a transformation from one set of coordinates to another, and this in turn is shown to be the equivalent of a fundamental transformation used in tensor analysis.

As might be expected, since it is based upon a different form of mathematics, the graphical analysis presented here has little in common with methods now in accepted use. Many more or less successful methods of graphical analysis for electric circuits have been developed, among which may be mentioned the one by Eddy[15], which is particularly applicable to variable frequency circuits, those presented by Lee[16], which give the effect of the variation of any circuit constant upon the other circuit values, and a recently developed method of handling graphically impedances in parallel, given by Boening.[17] However, all of these are based upon the equations for a single electric circuit, or portion of a circuit. As with the corresponding mathematics, the circuit constants are inextricably mixed with the values impressed upon the circuit, with the result that for each new condition a new equation must be set up, and likewise a new graphical plot must be made.

The strength of the tensor method lies in the fact that all the conditions within a complex machine or network can be represented by a single equation[18], and this equation not only is unchanged in form when a change takes place within the circuit, but it is similar in form for similar problems involving different networks or machines. Likewise the graphical application sets up a space structure which is useful in the analysis of all problems of a certain type. Variation of individual quantities within a given problem can be studied directly as a shift in the position of certain lines or points, in most cases not affecting many of the other values, and in no case affecting the general form of the problem. Further, within certain physical limitations, this method allows an analysis of an entire network at one time, a feature which could not be expected of a method based upon the mathematics of a single circuit only.

Applications in this thesis will be confined to the analysis of electric circuits only, though the same principles may easily be applied to other types of electrical problems as well as to problems in other fields of engineering where the corresponding tensor transformations are applicable. Steady state conditions only have been considered, though again there seems to be a possibility of an extension to cover some types of transient conditions. Extensions have been made to include complex quantities, however, making possible a study of alternating currents.

The most complete analyses are possible when not more than three independent equations are involved. Therefore the method is particularly adapted to a study of three-phase systems. When the principles are applied in obtaining the symmetrical components of an unbalanced three-phase system, immediate success is apparent for it becomes possible to construct a chart from which these components can be read easily, and in the commonly occurring case where the zero-phase components are absent, this chart takes the form of a very convenient slide-rule.

No attempt is made here to cover the field thoroughly or to exhaust the possibilities of any particular branch, as the subject appears too broad to permit more than a preliminary survey. Some of the possibilities are pointed out, however, and it is hoped that enough material is presented to give an incentive for a more complete study of this apparently useful method of presentation.

PART I: SIMULTANEOUS LINEAR EQUATIONS WITH REAL VALUES ONLY

Explicit and Implicit Forms:

Let us begin by considering a system of three simultaneous linear equations:

$$a_1x + b_1y + c_1z = k_1$$
$$a_2x + b_2y + c_2z = k_2 \qquad (1)$$
$$a_3x + b_3y + c_3z = k_3$$

where the a's, b's, and c's are constant real numbers. The values of x, y, and z which will satisfy these equations may be found by using determinants as follows:

Let D represent the determinant of the a, b, and c values, and define r_1 as the minor of a_1 in D, r_2 as the minor of a_2, etc., with s and t to represent the minors of the b and c numbers respectively, with the added assumption that the symbol include not only the minor, but also the proper plus or minus sign according to the position of the respective a, b, or c value in the determinant D. Such minors with proper sign included are known as the "cofactors" of their respective a, b, or c number. Equations (1) may then be "solved" for x, y, and z in this form:

$$x = \frac{k_1r_1 + k_2r_2 + k_3r_3}{D}$$
$$y = \frac{k_1s_1 + k_2s_2 + k_3s_3}{D} \qquad (2)$$
$$z = \frac{k_1t_1 + k_2t_2 + k_3t_3}{D}$$

These equations may now be put in a form similar to that of equations (1) by setting up a determinant of the coefficients which would be the "inverse transpose" of D. This is done by determining nine sets of values (which we can represent by the letters d, e, and f) such that:

$$d_1 = \frac{r1}{D} \qquad e_1 = \frac{r2}{D} \qquad f_1 = \frac{r3}{D}$$
$$d_2 = \frac{s1}{D} \qquad e_2 = \frac{s2}{D} \qquad f_2 = \frac{s3}{D} \qquad (3)$$
$$d_3 = \frac{t1}{D} \qquad e_3 = \frac{t2}{D} \qquad f_3 = \frac{t3}{D}$$

giving us equations (2) in the form:

$$x = d_1k_1 + e_1k_2 + f_1k_3$$
$$y = d_2k_1 + e_2k_2 + f_2k_3 \qquad (4)$$
$$z = d_3k_1 + e_3k_2 + f_3k_3$$

Equations (1) and (4) are now in the same form, but we can see that the positions of the x, y, and z values and of the k_1, k_2, and k_3 values have been interchanged.

In order that no confusion may result in what follows, we will speak of the x, y, and z values as being in the "implicit" form when involved in the equations as they are in equations (1); and as being in the "explicit" form when in the positions they occupy in equations (4).

<u>Interpretation of the Equations as a Transformation of Coordinates:</u>

Select a system of coordinate linear axes in space and let the position of a point P with respect to them be defined by its coordinate distances, x, y, and z. Now determine another set of values, which may be called x', y', and z', in such a way that:

$$x' = a_1x + b_1y + c_1z$$
$$y' = a_2x + b_2y + c_2z \qquad (5)$$
$$z' = a_3x + b_3y + c_3z$$

where the a's, b's, and c's are once again constant values, i.e. they are not in any way affected by the position of the point P.

Now think of the x', y', and z' as representing values of the coordinates of P with respect to another set of axes, X', Y', and Z' respectively. From equations (5) it will be seen that when the values of x, y, and z are zero, the values of x', y', and z' are also zero. Thus X', Y', and Z' are coordinate axes having a common origin with the original set.

Equations (5) may therefore be said to represent a transformation of the coordinates of P from the original x, y, and z values to the new x', y', and z' values. In most of our applications it will be found more convenient to have x, y, and z expressed explicitly, however, and this can be done by the method shown in the preceeding section, thus:

$$\begin{aligned} x &= d_1x' + e_1y' + f_1z' \\ y &= d_2x' + e_2y' + f_2z' \\ z &= d_3x' + e_3y' + f_3z' \end{aligned} \qquad (6)$$

where the determinants of the d's, e's, and f's is the inverse transpose of that of the a's, b's, and c's as in equations (4).

From the similarity between equations (1) and (5), and between (4) and (6), it may be seen that any system of three simultaneous linear equations can be interpreted as a transformation, more specifically as a homogeneous transformation, from one system of coordinates to another. It will therefore be in order for us to examine this transformation in greater detail.