Characterization and tabulation of a density giving significance levels in variance testing for the bivariate normal case

by Raymond P Hitchcock

A THESIS Submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

Montana State University

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This thesis is concerned with an examination of significance levels in variance testing in the case of sampling from a bivariate normal density with a nonzero correlation coefficient. The density related to this situation is derived from the Wishart distribution. Using this basic density, a characterization is made with the emphasis being placed on the behavior of the density for special cases. A tabulation of the density follows, and conclusions are drawn as to the effect of the correlation coefficient on significance levels in testing for homogeneity of variance. The numerical analysis techniques, tables, and IBM 650 programs necessary for the numerical examinations are included.

CHARACTERIZATION AND TABULATION OF A DENSITY GIVING SIGNIFICANCE LEVELS IN VARIANCE TESTING FOR THE BIVARIATE NORMAL CASE

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ABSTRACT

This thesis is concerned with an examination of significance levels in variance testing in the case of sampling from a bivariate normal density with a nonzero correlation coefficient. The density related to this situation is derived from the Wishart distribution. Using this basic density, a characterization is made with the emphasis being placed on the behavior of the density for special cases. A tabulation of the density follows, and conclusions are drawn as to the effect of the correlation coefficient on significance levels in testing for homogeneity of variance. The numerical analysis techniques, tables, and IBM 650 programs necessary for the numerical examinations are included.

I. INTRODUCTION AND STATEMENT OF THE PROBLEM

An important phase of statistical inference is testing for homogeneity of variance. The assumption of homogeneity of variance is necessary in analysis of variance techniques for tests of hypotheses, e.g., $\mu_1 = \mu_2 = \ldots = \mu_k$; and, if there is question as to the validity of this assumption, the test for homogeneity should be made before proceeding to the tests of hypotheses concerning location parameters. There are also many times when the test for homogeneity of variance is itself the principal object of the experiment.

In the general case of k variances, an approximate test due to Bartlett is available. In the case of two variances, an exact test is made using the F distribution. In the usual variance testing situation, because of the assumption of stochastic independence, correlation coefficients are irrelevant in null and alternative hypothesis distributions. If, however, sampling is from a multivariate normal distribution, the various correlation coefficients appear in the joint distribution of the sample variances and covariances. In the case of sampling from a bivariate population, there is a single correlation coefficient, ρ, to be considered; and we no longer have two independent samples with the resulting independent estimates of two variances, but rather a single bivariate sample from which we obtain two estimates of possibly different variances whose joint distribution involves ρ.

A test for equality of variances is sometimes required when we have a single sample from a bivariate normal distribution. The question then arises as to the effect of ρ upon the true significance levels if the usual test is made for equality of variances taking the critical points from the F distribution, which, of course, is not the appropriate one in this case. The particular distribution required is that of the quotient of the ratio of each unbiased sample variance to the corresponding population variance; and this distribution involves ρ, the ratio of the true variances, and n (the bivariate sample size minus one). Investigation of significance levels implies examination of the null hypothesis distribution; i.e., the case of equal variances, so that the ratio of the true variances becomes unity, and only ρ and the sample size appear as parameters in the distribution.

The problem is then (i) that of tabulating the applicable distribution for various values of ρ and n in order to obtain percentage points and probabilities associated with percentage points of the F distribution, and (ii) examining analytically and characterizing the distribution. The distribution is not a simple function, and its tabulation was a task of a magnitude sufficient to require extensive use of an electronic digital computer.

II. ANALYTIC EXAMINATIONS OF THE DENSITY FUNCTION

A. Derivation of the Density

We wish to find the joint distribution of r, the sample correlation coefficient, and $v = \frac{s_1/\sigma_1}{s_2/\sigma_2}$ where s_{11} and s_{22} are the unbiased sample variances from the bivariate normal population and σ_{11} and σ_{22} are the corresponding population variances. This distribution will be one in which the population correlation coefficient is not necessarily zero. Upon simplification, the distribution will be reduced to a function of v alone.

We shall begin with the Wishart distribution in the form taken from Anderson [1]. This form is

$$(1) \quad \text{e.p.}(v_{11}, v_{12}, v_{22}) = \frac{(\lambda_{11}\lambda_{22} - \lambda_{12}^2)^{\frac{n}{2}}}{4\pi\Gamma(n-1)} \; (v_{11}v_{22} - v_{12}^2)^{\frac{n-3}{2}}$$

$$\cdot \exp\left[-1/2(\lambda_{11}v_{11} + 2\lambda_{12}v_{12} + \lambda_{22}v_{22})\right]$$

$$\cdot \, dv_{11}\, dv_{12}\, dv_{22}$$

where the v_{ij} denote the sample variances and covariances, and λ_{ij} denote the elements of $\sum^{-1}$ and $\sum$ is the dispersion matrix, i.e., the matrix of the population variances and covariances.

From the definition of sample and population correlation coefficients, we have

$$(2)\quad r = \frac{v_{12}}{(v_{11}v_{22})^{\frac{1}{2}}}$$

$$\rho = \frac{-\lambda_{12}}{(\lambda_{11}\lambda_{22})^{\frac{1}{2}}}$$

and the density may be simplified by the change of variables

$$(3)\quad v_{11} = \frac{u_1}{\lambda_{11}} ; \quad v_{22} = \frac{u_2}{\lambda_{22}} ; \quad v_{12} = \frac{w}{(\lambda_{11}\lambda_{22})^{\frac{1}{2}}}$$

where the ranges of the new variables are

$$(4)\quad 0 \leq u_1 < \infty$$

$$0 \leq u_2 < \infty$$

$$-\infty < w < \infty \quad .$$

Upon substituting these values into the Wishart distribution, we obtain

$$(5)\quad \text{e.p.}(u_1, u_2, w) = (1 - \rho^2)^{\frac{n}{2}} (u_1u_2 - w^2)^{\frac{n-3}{2}} / 4\pi\Gamma(n-1)$$

$$\cdot \exp\left[-1/2(u_1 - 2\rho w + u_2)\right] du_1du_2dw$$

From (2) and (3) it is apparent that

$$(6) \quad r = w/(u_1u_2)^{\frac{1}{2}}$$

$$dw = (u_1u_2)^{\frac{1}{2}}\, dr.$$

Upon making these substitutions in (5), the Wishart becomes

$$(7) \quad e.p.(u_1, u_2, r) = (1-\rho^2)^{\frac{n}{2}} (1-r^2)^{\frac{n-3}{2}} (u_1u_2)^{\frac{n-2}{2}} / 4\pi\,\Gamma(n-1)$$

$$\cdot \exp\left\{-\frac{1}{2}\left[u_1 - 2\rho r(u_1u_2)^{\frac{1}{2}} + u_2\right]\right\}$$

$$\cdot\, du_1\, du_2\, dr\ .$$

It is convenient at this time to recall the relationships

$$(8) \quad s_1 = (v_{11}/n)^{\frac{1}{2}}$$

$$s_2 = (v_{22}/n)^{\frac{1}{2}}$$

which gives the value of v as

$$(9) \quad v = (v_{11})^{\frac{1}{2}} (v_{22})^{-\frac{1}{2}} \sigma_{22}/\sigma_{11}\ .$$

Also, from the matrix relationship

$$(10)\quad \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{\lambda_{22}}{D} & -\frac{\lambda_{12}}{D} \\ -\frac{\lambda_{12}}{D} & \frac{\lambda_{11}}{D} \end{bmatrix}$$

where $D = \lambda_{11}\lambda_{22} - \lambda_{12}^2$ and from the definition of equality of matrices we have

$$(11)\quad \sigma_2/\sigma_1 = (\sigma_{22})^{\frac{1}{2}} (\sigma_{11})^{-\frac{1}{2}} = (\lambda_{11}/D)^{\frac{1}{2}} (\lambda_{22}/D)^{-\frac{1}{2}} = (\lambda_{11})^{\frac{1}{2}} (\lambda_{22})^{-\frac{1}{2}}$$

so that

$$(12)\quad v = (\lambda_{11}v_{11})^{\frac{1}{2}} (\lambda_{22}v_{22})^{-\frac{1}{2}} = (u_1)^{\frac{1}{2}} (u_2)^{-\frac{1}{2}}$$

or

$$(13)\quad u_1 = u_2 v^2$$

$$(u_1 u_2)^{\frac{1}{2}} = u_2 v$$

$$du_1 = 2u_2 v \, dv .$$

Making these substitutions in (7) yields

$$(14)\quad \text{e.p.}(v, u_2, r) = (1 - \rho^2)^{\frac{n}{2}} (1 - r^2)^{\frac{n-3}{2}} v^{n-1} u_2^{n-1} / 2\pi \Gamma(n-1)$$

$$\cdot \exp\left[-\frac{u_2}{2}(v^2 - 2\rho\, r\, v + 1)\right] dv\, du_2\, dr$$

and upon integrating u_2 over its range the joint distribution of v and r is obtained.

If a change of variable is made by letting

$$(15)\quad t = u_2(v^2 - 2\rho\, r\, v + 1)/2 \qquad 0 \le t < \infty$$

$$du_2 = 2\, dt \,/\, (v^2 - 2\rho\, r\, v + 1)$$

and substituting into (14) gives

$$(16)\quad e.p.(v, t, r) = (1 - \rho^2)^{\frac{n}{2}} (1 - r^2)^{\frac{n-3}{2}}\, 2^{n-1}\, v^{n-1}\, t^{n-1}$$

$$\cdot \left[\pi\, \Gamma(n-1)(v^2 - 2\rho r v + 1)^n\right]^{-1} e^{-t}\, dv\, dr\, dt$$

The integral over t is $\Gamma(n)$, hence the joint density of v and r becomes

$$(17)\quad e.p.(v,r) = (n-1)2^{n-1}(1-\rho^2)^{\frac{n}{2}}(1 - r^2)^{\frac{n-3}{2}}\, v^{n-1}$$

$$\cdot \left[\pi(v^2 - 2\rho r v + 1)^n\right]^{-1} dv\, dr\, .$$

To find the distribution of v, r must be integrated out of (17). Making the substitutions

$$(18)\quad r = \frac{u(1 + 2\rho\, v + v^2) - (1 - u)\,(1 - 2\rho v + v^2)}{u(1 + 2\rho\, v + v^2) + (1 - u)\,(1 - 2\rho v + v^2)}$$

$$a = 1 + v^2$$
$$b = 2\rho v$$
$$k = 2^{n-1}(n-1)(1-\rho^2)^{\frac{n}{2}}\pi^{-1}$$

the transformation becomes

$$(19)\quad r = (2au - a + b)(2bu + a - b)^{-1}$$
$$dr = 2(a^2 - b^2)(2bu + a - b)^{-2}\, du$$

where the range on r is $-1 \leq r \leq 1$.

When $r = -1$, we have $2u(a + b) = 0$ which yields the possibilities that $u = 0$ if $a + b \neq 0$ since $a + b = 0$ is a trivial case. That is, $v^2 + 2\rho v + 1 = 0$ and upon examining the discriminant $(4\rho^2 - 4)^{\frac{1}{2}}$ we see that it is real only if $|\rho| = 1$. When $r = 1$, we have after simplification $2u(a-b) = 2(a-b)$. Since $u = 1$, if $a - b \neq 0$, we have a case amenable to the same argument as above; i.e., the discriminant is the same in this case.

Appearing in the density are the quantities $(1 - r^2)$ and $(a - br)$, and upon evaluating these it is found that

$$(20)\quad 1 - r^2 = 4u(1-u)(a^2 - b^2)(2bu + a - b)^{-2}$$
$$a - br = (a^2 - b^2)(2bu + a - b)^{-1}.$$

Substituting these values in (17) yields

$$(21)\quad e.p.(v, u) = k\, 2^{n-2} v^{n-1} (a^2 - b^2)^{-\left(\frac{n+1}{2}\right)}$$

$$\cdot \left\{ 2bu^{\frac{n-1}{2}} (1-u)^{\frac{n-3}{2}} + (a-b)\left[u^{\frac{n-3}{2}} (1-u)^{\frac{n-3}{2}} \right] \right\}$$

$$\cdot\ du\, dv\ .$$

Upon integrating u over its range, the Beta function is obtained, and the density becomes

$$(22)\quad e.p.(v) = k\, 2^{n-2} v^{n-1} (a^2 - b^2)^{-\left(\frac{n+1}{2}\right)}$$

$$\cdot \left[2\, b\, B\left(\frac{n+1}{2}, \frac{n-1}{2}\right) + (a-b)\, B\left(\frac{n-1}{2}, \frac{n-1}{2}\right) \right] dv\ .$$

Simplifying the quantity in brackets in (22) gives

$$(23)\quad a\Gamma^2\left(\frac{n-1}{2}\right)\Gamma^{-1}(n-1)$$

and upon substituting for k, e.p.(v) becomes

$$(24)\quad e.p.(v) = \frac{2^{n-1}(n-1)(1-\rho^2)^{\frac{n}{2}}\, 2^{n-2}\, v^{n-1}\, a\, \Gamma^2\left(\frac{n-1}{2}\right)}{\pi (a^2 - b^2)^{\frac{n+1}{2}}\, \Gamma(n-1)}\ dv\ .$$

Using Euler's relation , we find that

$$(25)\quad \Gamma^2\left(\frac{n-1}{2}\right) = \frac{\pi\, 2^{4-2n}\, \Gamma^2(n-1)}{\Gamma^2\left(\frac{n}{2}\right)}\ .$$

Replacing the values of a and b by those of (18), and making use of the relationship between the beta and gamma functions, we have

$$(26)\quad e.p.(v) = \frac{2(1-\rho^2)^{\frac{n}{2}}\, v^{n-1}}{B(\frac{n}{2},\frac{n}{2})\,(1+v^2)^n}\left[1 - \frac{4\rho^2 v^2}{(1+v^2)^2}\right]^{-\left(\frac{n+1}{2}\right)} dv \,.$$

The formula (26) is a density having the correlation coefficient, ρ, as a factor. The quantity $v^2 = \dfrac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2}$ is of little use since the population variances are generally not known. Since we are concerned with the null hypothesis distribution in testing $\sigma_1^2 = \sigma_2^2$, we let $\sigma_1^2 = \sigma_2^2$ and then v^2 becomes the ratio of two sample variances.

In order to simplify the form of (26), let

$$(27)\quad u = v^2$$
$$du = 2v\,dv = 2u^{\frac{1}{2}}\,dv$$

and upon substituting these values into (26) we have

$$(28)\quad e.p.(u) = L\,u^{\frac{n-2}{2}}(1+u)^{-n}\left[1 - 4\rho^2 u(1+u)^{-2}\right]^{-\left(\frac{n+1}{2}\right)} du$$

where

$$(29)\quad L = \frac{(1-\rho^2)^{\frac{n}{2}}}{B(\frac{n}{2},\frac{n}{2})} = \frac{(1-\rho^2)^{\frac{n}{2}}\,\Gamma(n)}{\Gamma^2(\frac{n}{2})}\,.$$

With some minor simplifications, we have the density that is to be considered in this thesis, i.e.,

$$(30)\quad f(u)\,du = \text{e.p.}(u) = L\,u^{\frac{n-2}{2}}\,(1+u)\left[u^2 + (2-4\rho^2)\,u + 1\right]^{-\left(\frac{n+1}{2}\right)} du\,.$$

B. Verification of the Reciprocal Relationship for Percentage Points

As stated in the numerical methods section, only the 90, 95, 97.5, 99, and 99.5 percentage points are isolated since the values of the lower percentage points are simply the reciprocals of the corresponding upper percentage points. For example, suppose that, for some particular values of the parameters n and ρ, the value of u which has 97.5 percent of the area under the curve to the left of it is $u = 5$. Then the value of u that has 2.5 percent of the area to the left of it is equal to 1/5 or .20 .

The nature of the statistic considered indicates that its reciprocal should have the same density as the statistic itself. That this is actually the case is easily verified by making the change of variable $w = u^{-1}$ in (30) of Section II, Part A. The differential becomes $du = w^{-2}\,dw$, and substituting these values into (30) of Section II, Part A, we have

$$\text{(1)}\quad e.p.(w) = \frac{L\,w^{-\left(\frac{n-2}{2}\right)}\,(1+w^{-1})\,w^{-2}}{\left[w^{-2} + (2+4\rho^2)\,w^{-1} + 1\right]^{\frac{n+1}{2}}}\,dw$$

Simplifying the denominator yields the form

$$e.p.(w) = \frac{Lw^{-\frac{n}{2}+1}\,(1+w^{-1})\,w^{-2}\,w^{n+1}}{\left[w^2 + (2-4\rho^2)w + 1\right]^{\frac{n+1}{2}}}\,dw$$

$$\text{(2)}\quad = Lw^{\frac{n}{2}-1}\,(1+w)\left[w^2 + (2-4\rho^2)\,w + 1\right]^{-\left(\frac{n+1}{2}\right)}\,dw$$

The form of (2) is the same as that of (30) of Section II, Part A, and this verifies the fact that the lower percentage points are the reciprocals of the upper percentage points for a fixed n and ρ.

C. An Equation for the Mode

Since the density is unimodal, the nonzero value of u for which the first derivative of the density equals zero is the modal value. In this part, we will examine the first derivative and find an equation for the mode.

Taking the derivative of (30) Section II, Part A, we have

$$(1) \quad f'(u) = R(Au^3 + Bu^2 + Cu + D)$$

where

$$(2) \quad R = \frac{(1-\rho^2)^{\frac{n}{2}} \, u^{\frac{n-4}{2}}}{\left[u^2 + (2-4\rho^2)u + 1\right]^{\frac{n+1}{2}}}$$

$$A = -\frac{(n+2)}{2}$$

$$B = \frac{4\rho^2 - 6 - n}{2}$$

$$C = -3 + 6\rho^2 + \frac{n}{2}$$

$$D = \frac{n-2}{2} \quad .$$

Upon setting $f'(u)$ equal to zero, it is seen that $R = 0$ only if $u = 0$ or $\rho = \pm 1$. The values of $\rho = \pm 1$ represent the degenerate cases.

This leaves only the possibility of the cubic expression

$$(3) \quad Au^3 + Bu^2 + Cu + D = 0$$

This equation is one which, when solved for u, yields the modal value for any fixed ρ and $n \geq 3$. The cases $n < 3$ will be considered in a later discussion.

D. Behavior of the Density for Small n

In this part of the thesis we will examine the behavior of the density for small values of n as we let the variable u become small. We shall consider three cases for small u.

The first case is, when n = 1, what the limit of f(u) will be as $u \to 0$. That is, since

$$(1)\quad f(u) = \frac{(1-\rho^2)^{\frac{n}{2}}\,\Gamma(n)\left(u^{\frac{n}{2}} + u^{\frac{n-2}{2}}\right)}{\Gamma^2\left(\frac{n}{2}\right)\,(u^2 + Pu + 1)^{\frac{n+1}{2}}}$$

where $P = 2 - 4\rho^2$, we wish to examine

$$(2)\quad \lim_{u \to 0} f(u) = \frac{(1-\rho^2)^{\frac{1}{2}}\,\Gamma(1)}{\Gamma^2\left(\frac{1}{2}\right)} \lim_{u \to 0} \frac{u^{\frac{1}{2}} + u^{-\frac{1}{2}}}{(u^2 + Pu + 1)}$$

Since this limit becomes infinite as $u \to 0$, we see that for the case when n = 1, f(u) approaches infinity as $u \to 0$, i.e., the density roughly resembles the upper half of the hyperbola xy = A.

The second case is when n = 2. Again, we are interested in the limit of f(u) as $u \to 0$. Setting n = 2 in (1), we have

$$(3)\quad \lim_{u \to 0} f(u) = \frac{(1-\rho^2)\,\Gamma(2)}{\Gamma^2(1)} \lim_{u \to 0} \frac{(1+u)}{(u^2 + Pu + 1)^{3/2}}.$$

Since the limit approaches 1 as $u \to 0$, we see that for the case n = 2, f(u) approaches $1 - \rho^2$ as u becomes small.

The third case to be considered is what the limit of f(u) will be when $u \to 0$ for $n \geq 3$. Putting $n = 3$ in (1), we have

$$(4)\quad \lim_{u \to 0} f(u) = \frac{(1 - \rho^2)^{\frac{3}{2}} \Gamma(3)}{\Gamma^2(\frac{3}{2})} \lim_{u \to 0} \frac{u^{\frac{3}{2}} + u^{\frac{1}{2}}}{(u^2 + Pu + 1)^2}$$

Since u appears as a factor in each term in the numerator of the fraction, we see that for $n \geq 3$, f(u) approaches zero as u becomes small.

In (2) Section II, Part C, the quantity $u^{\frac{n-4}{2}}$ appears as a factor in the expression for the first derivative of the function. This raises the question as to the method of approach of the function to the origin. As noted above, for the case $n = 1$, the function becomes infinite as $u \to 0$ so that in this case the first derivative is meaningless. For $n = 2$, the function approaches $1 - \rho^2$ and $f'(u)$ approaches infinity; hence, we have tangency to the f(u) axis. For $n = 3$, the function approaches zero as $u \to 0$, but again $f'(u)$ approaches infinity which implies we have tangency to the f(u) axis. For $n = 4$, the function approaches zero, and $f'(u)$ is finite and positive implying intersection with both axes at the origin. Finally, for $n \geq 5$, the function approaches zero as $u \to 0$; and $f'(u)$ is zero, implying that the function is tangent to the u axis with the order of contact increasing as n increases.

This completes the discussion for the various cases as $u \to 0$, and we arrive at the conclusion that the density exhibits the same characteristics for small n as the F distribution.

E. Moments of the Density

Since all moments exist for the F distribution, the questions arises as to whether the moments exist when $\rho \neq 0$. We will show that after a certain point, f(u) of (30), Section II, Part A, is less than g(u) where g(u) is the F distribution, and hence that moments of all orders exist.

Recalling

$$(1)\quad f(u) = \frac{(1-\rho^2)^{\frac{n}{2}}\,\Gamma(n)\left(u^{\frac{n-2}{2}} + u^{\frac{n}{2}}\right)}{\Gamma^2\left(\frac{n}{2}\right)\left[u^2 + (2-4\rho^2)u + 1\right]^{\frac{n+1}{2}}}$$

$$g(u) = \frac{\Gamma(n)\,(u^{\frac{n-2}{2}} + u^{\frac{n}{2}})}{\Gamma^2\left(\frac{n}{2}\right)\left(u^2 + 2u + 1\right)^{\frac{n+1}{2}}} ,$$

we see that the ratio f(u) / g(u) is

$$(2)\quad \frac{f(u)}{g(u)} = \frac{(1-\rho^2)^{\frac{n}{2}}\,(1+u)^{n+1}}{\left[u^2 + (2-4\rho^2)\,u + 1\right]^{\frac{n+1}{2}}} .$$

Taking the limit as u becomes large gives

$$(3)\quad \lim_{u\to\infty} \frac{f(u)}{g(u)} = (1-\rho^2)^{\frac{n}{2}} \lim_{u\to\infty} \frac{(1+u)^{n+1}}{\left[u^2 + (2-4\rho^2)u + 1\right]^{\frac{n+1}{2}}}$$

$$= (1-\rho^2)^{\frac{n}{2}}$$

since the highest power of the polynomial in the numerator is the same as the highest power of the polynomial in the denominator.

The quantity $(1 - \rho^2)^{n/2}$ is always less than one, and this implies that f(u) is bounded by g(u) for all u greater than some value say a. It is apparent that the introduction of the factor u^r in f(u) and g(u) of (1) will produce the same result in the limit.

Referring to Widder [4] we have as a theorem on the convergence of improper integrals

(4) Theorem

1. $f(u), g(u) \in C \qquad a \leq u < \infty$
2. $0 \leq f(u) \leq g(u) \qquad a \leq u < \infty$
3. $\int_a^\infty g(u)\, du < \infty$

$$\longrightarrow \int_a^\infty f(u)\, du < \infty$$

To show that condition one holds, let us examine f(u) and g(u) for continuity. Since the ratio of two polynomials is continuous at all points except where the denominator is equal to zero, let us solve the quadratic in the denominator of f(u). Solving by the quadratic formula, we see that

(5) $u = (2\rho^2 - 1) \pm 2\rho(\rho^2 - 1)^{\frac{1}{2}}$

are the only possible roots. However, the quantity $(\rho^2 - 1)^{\frac{1}{2}}$ is imaginary in the interval $|\rho| < 1$, so all of the roots are imaginary. Similarly, $u = -1$ is the only root of the denominator of $g(u)$; and, since u is defined to have the range $0 < u < \infty$, there are no points where $g(u)$ becomes undefined. Thus, $f(u)$ and $g(u)$ are continuous.

Since $f(u)$ is a density function, it is greater than zero; and we have shown in (3) above that, for u sufficiently large, $f(u) \leq g(u)$. Thus, condition 2 is satisfied. As the moments of the F distribution are known to exist, we have satisfied all of the hypotheses of the theorem; therefore, we may conclude that moments of all orders of the density exist.

III. NUMERICAL ANALYSIS

A. Formula for Programming the IBM 650

Due to the complexity of the form of e.p.(u) in (30) of Section II, Part A, a closed form could not be found for the integral of the density, and numerical methods are used to find the critical percentage points of the cumulative distribution.

To facilitate the programming of the IBM 650, logarithms to the base 10 of the density are taken. This form of the density is

$$(1)\quad \log\left[f(u)\right] = (n/2)\log(1-\rho^2) + \log\Gamma(n) + \log(1+u) - 2\log\Gamma(n/2) + \left(\frac{n-2}{2}\right)\log u - \left(\frac{n+1}{2}\right)\log\left[u^2 + (2-4\rho^2)u + 1\right].$$

The integration program of the Appendix uses this formula as the integrand.

As a check on the validity of the derivations preceding (1), we may substitute the value of $\rho = 0$ into (30) of Section II, Part A. The e.p.(u) should reduce to e.p.(F) where F is the ratio of two independent chi-square variables.

Proceeding with $\rho = 0$, we find that

$$(2)\quad \text{e.p.}(u) = L\, u^{\frac{n-2}{2}} (1+u)^{-n}\, du$$

where

(3) $L = \frac{\Gamma(n)}{\Gamma^2(\frac{n}{2})}$

for this special case. The F distribution is

(4) $e.p.(F) = \frac{\Gamma\left(\frac{n_1 + n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} n_1^{\frac{n_1}{2}} n_2^{\frac{n_2}{2}} F^{\frac{n_1}{2}-1} (n_2+n_1F)^{-\left(\frac{n_1+n_2}{2}\right)} dF$

where

(5) $0 < F < \infty$

$n_1 + 1$ = number of observations in sample 1

$n_2 + 1$ = number of observations in sample 2

$F = s_1^2/s_2^2$

Recalling the fact that we have let $\sigma_1^2 = \sigma_2^2$ in our null hypothesis distribution, it is seen that $F = u$ and $n = n_1 = n_2$. Making these substitutions in (4) gives the relationship

(6) $e.p.(F) = e.p.(u) = \frac{\Gamma(n)}{\Gamma^2 \frac{n}{2}} u^{\frac{n-2}{2}} (1 + u)^{-n} du$

Therefore, for the special case $\rho = 0$, we have that e.p.(u) = e.p.(F).

B. Numerical Methods

An important portion of the problem is to evaluate the integral of the density in (30) of Section II, Part A. Due to the complexity of the density, numerical methods are employed in the isolation of critical points of the cumulative distribution.

Since numerical analysis techniques are employed, errors are introduced into the tabulated values; however, by taking increments of u to be .01, it is believed that reasonably accurate results are obtained. In order to assess the accuracy of the tabulation, a table of values of the cumulative distribution calculated when $\rho = 0$ has been included. It is seen that these values agree with the values of the cumulative F distribution.

For the numerical integration, Simpson's rule has been selected because of the simple coefficients involved. An h value of .01 was chosen, and cumulative probabilities are tabulated. Values of the cumulative distribution are obtained for n = 6, 8, 10, 12, 15, 20, 25, 30 with $\rho = 0.0(0.1)0.5(0.3)0.8$ for each n. Since ρ appears only as ρ^2 in (30) of Section II, Part A, only positive values of ρ are considered, and the tabulated values are applicable for negative ρ.

The program to accomplish the integration is Program A of the Appendix. The formula programmed is Simpson's rule with (1) of Section III, Part A, being used to evaluate e.p.(u).

After the tabulation of the cumulative areas for a given n and ρ, the next step is to isolate the critical percentage points of the cumulative distribution. The 90, 95, 97.5, 99, and 99.5 points are relevant in the testing of hypotheses. As shown in Section II, Part B, the density of $w = u^{-1}$ is of the same form as that of u; hence, only the upper percentage points are tabulated. The lower points are found by taking the reciprocal of the tabulated upper percentage points.

To isolate the percentage points, it is necessary to use a numerical technique for inverse interpolation. Lagrange's interpolation formula has been selected with the primary reason, again, being the ease in the programming of the IBM 650. A third degree polynomial is used with the formula being taken from Kunz [2]. The formula is

$$(1)\quad x = \frac{(y - y_1)(y - y_2)(y - y_3)}{(y_0 - y_1)(y_0 - y_2)(y_0 - y_3)} x_0 + \frac{(y - y_0)(y - y_2)(y - y_3)}{(y_1 - y_0)(y_1 - y_2)(y_1 - y_3)} x_1$$

$$+ \frac{(y - y_0)(y - y_1)(y - y_3)}{(y_2 - y_0)(y_2 - y_1)(y_2 - y_3)} x_2 + \frac{(y - y_0)(y - y_1)(y - y_2)}{(y_3 - y_0)(y_3 - y_1)(y_3 - y_2)} x_3$$

$$= \frac{A}{B} x_0 + \frac{C}{D} x_1 + \frac{E}{F} x_2 + \frac{G}{H} x_3$$

where the correspondence is:

(2) y = percentage point desired

x = u value of percentage point desired

$$y_i = \int_0^{u_i} f(u)\, du$$

x_i = u values corresponding to the y_i

A, B, . . . H = notation used in IBM 650 program.

Two values of y_i are chosen on each side of the desired percentage point. The program to accomplish the interpolation is Program B of the Appendix. The values of x_i and y_i necessary for the program are obtained from Program A of the Appendix.

The maximum values of the density are also the values of the mode for each n and ρ. To facilitate graphing and to aid in the study of the characteristics of the density, the first derivative of the density must be found. Upon examination of the values obtained in the tabulation of the density, it is seen that there is only one maximum for a given n and ρ. Therefore, it is sufficient to find the value of u for which the first derivative changes sign.

The formula selected is taken from Nielsen [3] and was derived by taking the first derivative of Newton's Forward Interpolation formula. The simplified form is

(3) $y'(u) = 1/h \sum_i D_i \Delta^i y_0$

where

(4) $D_{i+1} = \frac{1}{i+1} \left[(w - 1) D_i + (w - i + 1) b_i \right]$

$$b_{i+1} = \frac{1}{i+1}\left[b_i \; (w - i + 1)\right]$$

$$b_1 = D_1 = 1$$

and the correspondence is:

(5) $w = \dfrac{u - u_0}{h}$; u = initial guess; u_0 = value corresponding to y_0;

$h = .1$; $y_i = f(u_i)$.

Three values of y_i are chosen on each side of the estimated value of $f(u)$ where the first derivative is equal to zero. The program to accomplish the differentiation is given as Program C of the Appendix. The values of y_i are the evaluation of $f(u)$ as given in (30) of Section II, Part A.

IV. NUMERICAL EXAMINATIONS OF DATA AND CONCLUSIONS

In Table I, the percentage points for n = 6, 8, 10, 12, 15, 20, 25, 30 with $\rho = 0.0(0.1)0.5(0.3)0.8$ are tabulated. As n increases for a fixed ρ and a fixed percentage point, the value of u required in a test of a hypothesis becomes smaller. For a fixed n and percentage point, the value of u decreases as ρ increases.

This implies that, if one is testing at the α level of significance, the upper critical point for a two-tailed test decreases. For example, suppose one was making a test at the 5 percent level of significance and had taken samples of size 7. Realizing that the tables are tabulated for the sample size minus 1, it is seen in Table I that for $\rho = 0$ and $n = 6$, the upper critical point is 5.82. However, if there is reason to believe that $|\rho|$ actually is equal to 0.5, then the critical point for the test should be 4.78.

For small values of ρ, the differences in the values of u for a fixed n and percentage point are very small. However, as ρ increases the differences become quite appreciable. When a test is made at a higher level of significance, the relative differences in u become greater. If there is reason to believe that $|\rho|$ is as great as 0.4, then the critical points for rejection are smaller than those for the case when $\rho = 0$.

As n takes on integral values beginning with one, the shape of the graphs of the density show considerable variation. As pointed out in

Section II, Part D on the analytic examination of the density, for $n = 1$ the density become infinite as $u \to 0$ and for $n = 2$ the density assumes the value of $1 - \rho^2$ as $u \to 0$. For values of $n \geq 3$, the density approaches zero as $u \to 0$.

These facts have been represented graphically in Figure 1 at the end of this section. For $n = 1$ and $n = 2$, the graph of the density would show that there is no maximum value. For $n \geq 3$, the maximum value, or mode of the density, is given by the equation (3) of Section II, Part C. It is further observed that for large values of n the dispersion appears to be decreased. For the values of n and ρ tabulated, the mode and its abscissa increase as n and ρ increase.

Table II shows the effect of ρ upon the cumulative distribution for a given u and n. The first two columns of the body of the table give u and the percentage points it represents for the F distribution. The remainder of the table gives the associated cumulative probabilities for the indicated value of ρ.

Upon examination of these values, it is seen that if one uses the value of u for the case when $\rho = 0$, the test is actually at a lower level of significance; i.e., the test is more stringent, when there is reason to believe that $\rho \neq 0$. For example, if a sample of size 7 is taken, so that $n = 6$, the value of u to be used as the upper critical point at the 5 percent level of significance when $\rho = 0$ is 5.82. However, if $|\rho|$ is actually 0.5, the test would be made at the 3 percent level of significance.

These facts have been illustrated by Figure 2 at the end of this section. The values of α have been plotted against varying ρ values for the case $n = 10$. From the graph, it is seen that α decreases as n increases. For example, a 10 percent level test is actually conducted at the 7.6 percent level for $|\rho|$ equal to 0.4.

In Figure 3 at the end of this section, the function $f(u)$ has been plotted for varying values of u when $n = 10$. For a fixed n, the dispersion of the density decreases as ρ increases. Also, as ρ increases, the mode and its abscissa increase. The curve $\rho = 0$ is the F distribution, and for $n = 10$ the curves $\rho = 0.3$ and $\rho = 0.8$ are bounded by the F distribution as u becomes large. This is as expected from our analytic considerations of the density.

In conclusion, this thesis has been concerned with the analytic and tabular examinations of a density where the correlation coefficient between two random variables is not necessarily zero. The following results have been obtained:

1. The density satisfies a reciprocal relationship for percentage points. This requires tabulation of only upper-tail critical points.
2. The density exhibits fundamental changes in character for small values of n. For the case $n = 1$, the function becomes infinite as $u \to 0$; for $n = 2$, the function approaches $(1 - \rho^2)$ as $u \to 0$; and for $n \geq 3$, the function approaches zero as $u \to 0$.

3. The median of the cumulative distribution is at $u = 1$, regardless of the value of n and ρ. As n and ρ increase, the value of the mode and its abscissa both increase. The equation for the mode is a cubic, and the density has no mode for $n = 1$ and $n = 2$. In general, the variance of the density decreases as ρ increases for a fixed n.

4. When making tests of hypotheses on the equality of variances, if the assumption is made that $\rho = 0$ when actually $\rho \neq 0$, then the test is being made at a lower level of significance. Also, the critical point for the rejection of the hypothesis is greater when $\rho = 0$ than when $\rho \neq 0$ for a fixed n.

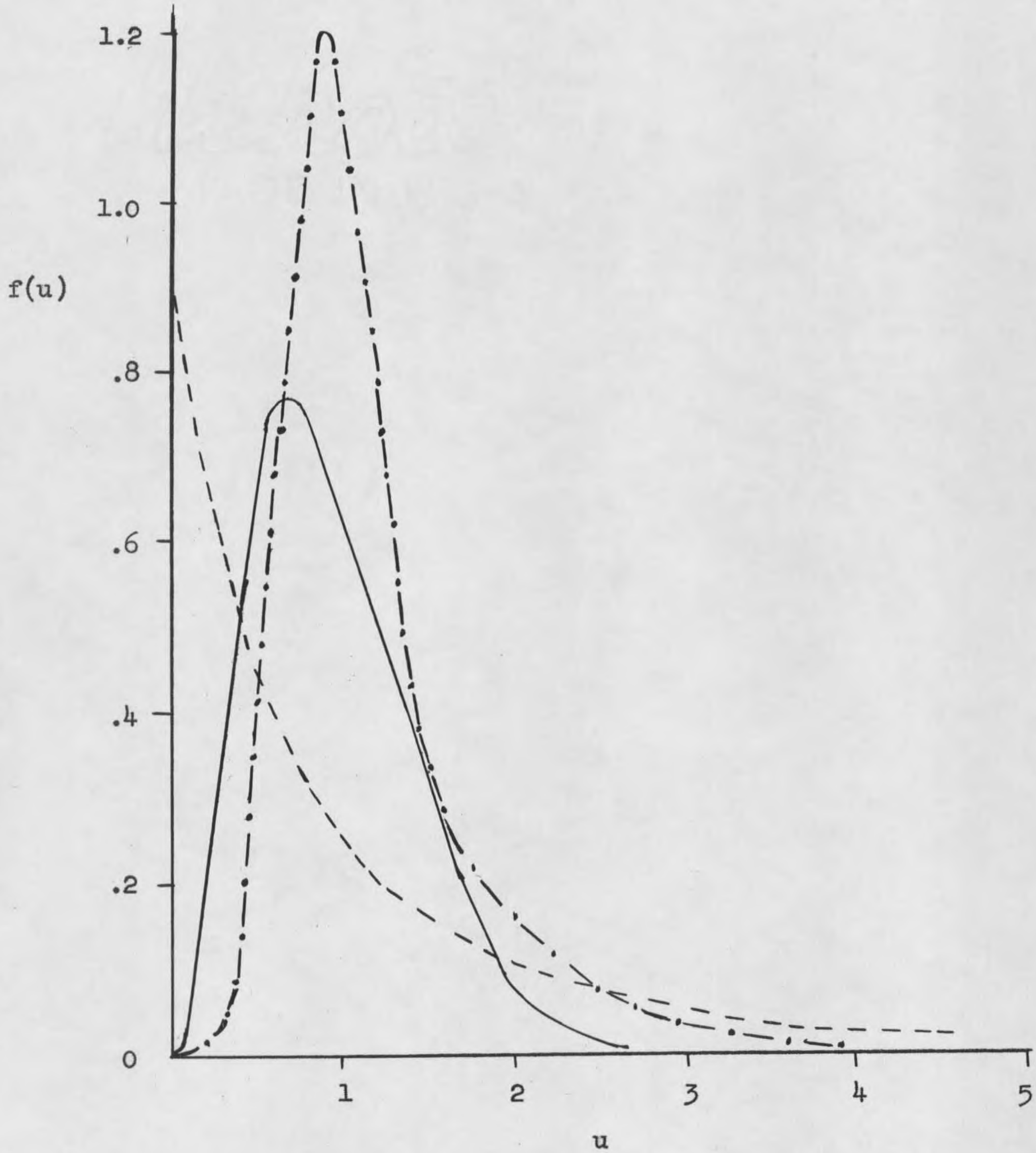

Figure 1

Graph of f(u) for ρ = 0.3

- - - - n = 2 ; ——— n = 10 ; -·-·- n = 30

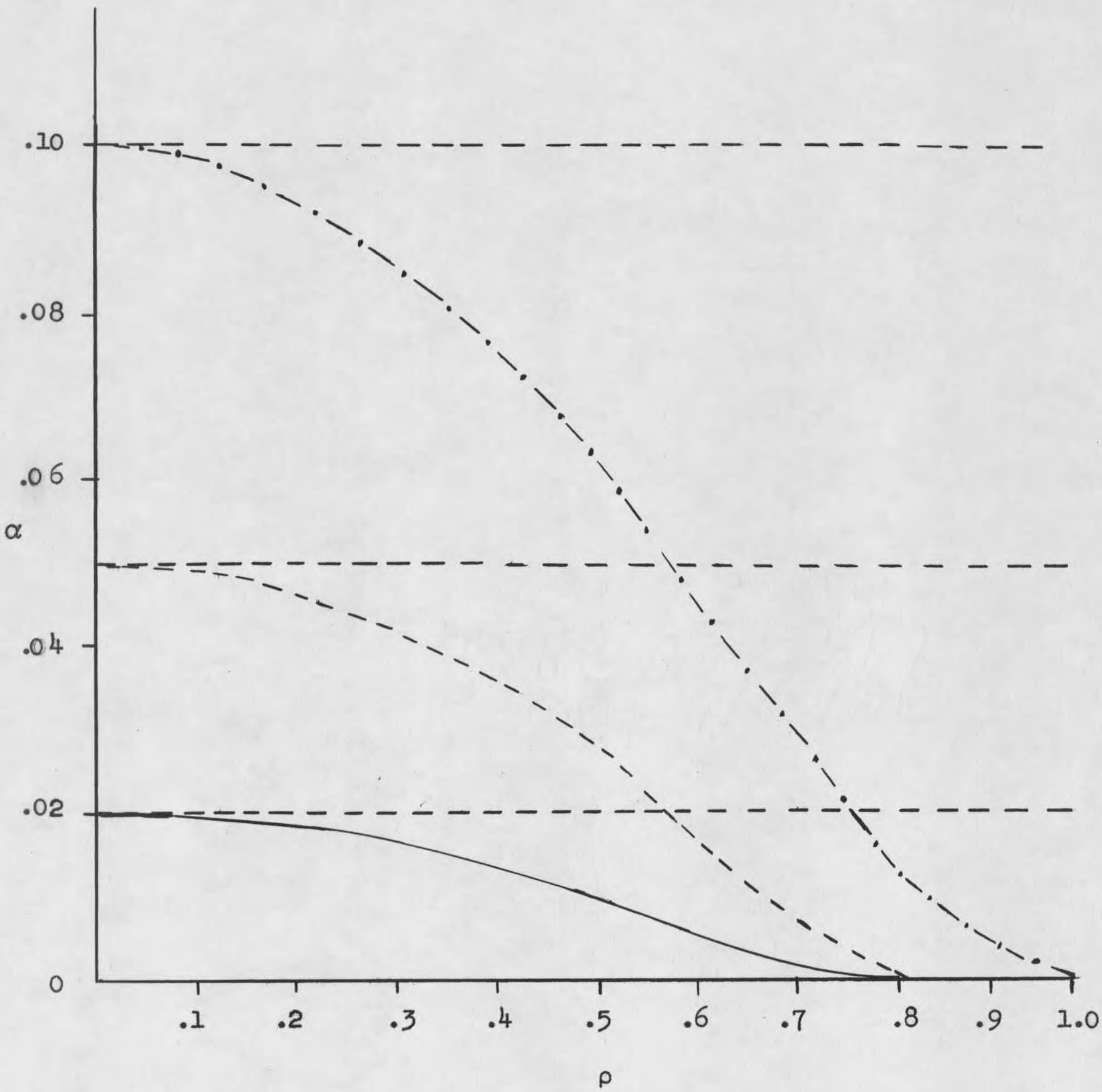

Figure 2

Graph of α versus ρ for $n = 10$

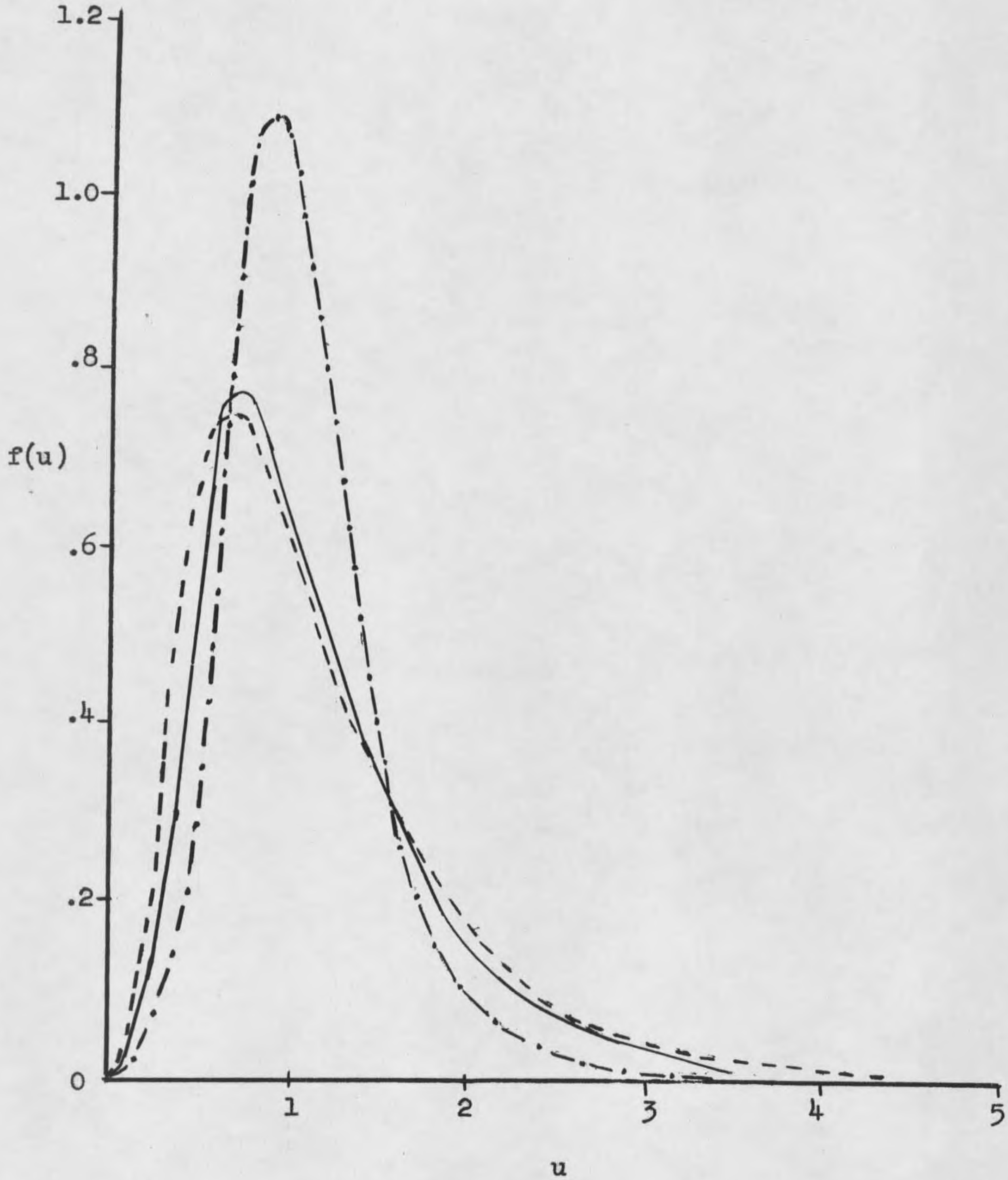

Figure 3

Graph u versus f(u) for n = 10

- - - - $\rho = 0$; ______ $\rho = 0.3$; -·-·- $\rho = 0.8$