A method for the computer-aided dynamic analysis of spatial mechanisms
by Derrick Wayne Johnson

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Mechanical Engineering
Montana State University

Abstract:
This thesis presents a method for the dynamic analysis of spatial mechanisms. A vector model of the mechanism using vector operations such as vector loops, dot products, and cross products is used to describe the kinematics of the mechanism. Newton's second law was applied to each element to describe the kinetics. The method was described such that a computer program could be written and used to generate and solve a complete set of kinematic and dynamic equations for a given mechanism. The method is limited to rigid elements. Two examples are given.

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APPROVAL

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May 25 1984 — Dennis O. Blackketter

Date — Chairperson, Graduate Committee

Approved for the Major Department

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Date — Head, Major Department

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Date May 25, 1984

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ABSTRACT

This thesis presents a method for the dynamic analysis of spatial mechanisms. A vector model of the mechanism using vector operations such as vector loops, dot products, and cross products is used to describe the kinematics of the mechanism. Newton's second law was applied to each element to describe the kinetics. The method was described such that a computer program could be written and used to generate and solve a complete set of kinematic and dynamic equations for a given mechanism. The method is limited to rigid elements. Two examples are given.

CHAPTER I

INTRODUCTION

There are several methods available for the dynamic analysis of mechanisms, each of which often involves the formulation and solution of a large set of equations. It is this nature of dynamic analysis that makes the utilization of computer assisted methods advantageous.

Historically, kinematic and dynamic analysis of mechanisms has been done through the use of graphical methods. Recently, however, the complexity of mechanical manipulators requires that analytical methods be used in the areas of both design and control. Hermani, Jaswa, and McGhee [3]* have presented and compared several methods available for the solution of manipulator dynamics.

In the area of control, much of the present work is based on the symbolic notation and matrix transformation method developed by Denavit and Hartenberg [2]. Great advances have been made in applying both Lagrangian formulations [10,4] and Newton-Euler equations [6] to this notation. The formulations developed by Hollerbach [4] and Luh, Walker, and Paul, R. P. C. [6] were developed for the

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* Numbers in brackets indicate references listed in the Literature Cited section.

real-time control of mechanical manipulators. In addition, Luh and Lin [5] have developed an algorithm for the computer generation and simplification of the Newton-Euler equations used by Luh, Walker, and Paul. Walker and Orin [14] then compared several schemes for the solution of these equations.

Thomas and Tesar [11] have used a slightly different notation from that of Denavit and Hartenberg and applied principles of virtual work to develop the dynamic equations. By doing this they have increased the number of parameters and equations required to describe a mechanism but have also made it easier to generate the equations.

Another class of methods involves the use of vectors and vector loops to model the mechanism. Paul, B. and Krajcinovic [7,8] used this scheme in combination with Lagrangian formulations to develop dynamic equations for planar machinery. Paul, B. [7], later, summarized this work, compared it to graphic methods, and presented some variations to the overall approach. The use of vectors and vector loops was also used to analyze the kinematics of spatial mechanisms by Townes, Blackketter, and Lowell [12]. Since then, Townes and Blackketter [13] have enhanced this method by describing kinematic constraints with vector dot products and cross products.

In this thesis, the method of kinematic analysis by Townes and Blackketter [13] will be expanded to include the dynamic analysis of mechanisms. Emphasis will be placed on describing the method such that a computer program can be used to generate and solve a complete set of kinematic and dynamic equations for a given mechanism. Newton's second law will be used to describe the dynamics and the method will be limited to mechanisms consisting of rigid elements.

CHAPTER II

GENERAL DESCRIPTION

The general procedure for this method consists, first, of building a kinematic vector model of the mechanism. Next, kinematic models of each element are formulated and added to the mechanism model. A kinematic analysis at this point will yield the geometry and velocities necessary to generate the kinematic acceleration equations and the kinetics equations. Elements are then defined in terms of their mass characteristics and the nodes at which forces and moments act. Finally, the kinetics equations are constructed by applying Newton's second law to each element.

A kinematic analysis is often performed on just the mechanism before considering the addition of the element models to be sure to obtain the correct mechanism model. The geometry and velocities of the mechanism are independent of the forces acting upon it. Therefore, this information is the same from the first kinematic analysis throughout the rest of the dynamic analysis.

All of the element characteristics, mass, mass moments of inertia, and geometry, are defined in terms of an element coordinate system. And since the mechanism model is defined in terms of system coordinates, the element charactersistics are transferred to system coordinates.

The description of a mechanism is accomplished by specifying a set of known parameters and calculating the unknown parameters. Often, it is difficult to determine the right set of known parameters and thus obtain a correct dynamic model of the mechanism. One procedure to simplify the analysis of a complicated mechanism is to first solve the dynamic equations in their uncoupled form. That is, the parameters can be specified such that the accelerations can be calculated independent of the forces. Once a satisfactory solution can be obtained from the uncoupled equations, the parameters can be redefined in a stepwise fashion until the desired form of the problem is reached.

CHAPTER III

KINEMATIC ANALYSIS

Geometry

In the kinematic analysis, vectors are used to represent the function of each physical link in the mechanism, locate any special points of interest, and to constrain certain elements of the mechanism. These vectors are relative to a Newtonian reference frame, or system coordinates, and are referred to as system vectors. In matrix form the system vector R_n is expressed as

$$\{R_n\} = P_{n,4}\{P_n\} \tag{1}$$

where $P_{n,4}$ is the length of the vector and $\{P_n\}$ is a unit vector in the direction of R_n. The components of the unit vector, $P_{n,\beta}$, are the directional cosines of R_n in the Newtonian reference frame X_β, $\beta = 1,2,3$. In Figure 1 a system vector is shown.

In three-dimensional space there are three parameters required to describe a vector, namely the length and two of the three directional cosines. Only two of the cosines are independent in that the sum of the squares of the cosines must equal one, or in matrix form

$$\lfloor P_n \rfloor \{P_n\} = 1 \tag{2}$$

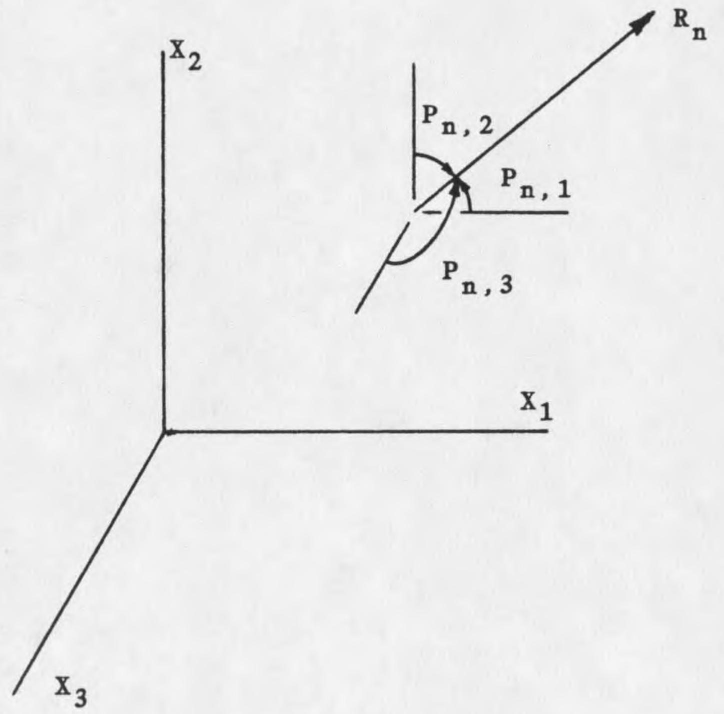


FIGURE 1. - System vector.

This is referred to as the "cosine squared constraint".

Equations can be developed to describe the geometry of the mechanism by specifying which vectors form closed loops. The sum of the vector components in each coordinate direction around the loop must equal zero. Therefore, each vector loop results in one equation for the sum in each of the three coordinate directions.

Besides vector loops and cosine squared constraints, two additional constraints available to describe the geometry of the mechanism are dot products and cross products. A dot product is used to specify the known angle between any two vectors. The dot product constraint between vectors R_n and R_m is written as

$$\lfloor P_n \rfloor \{P_m\} = d_i \qquad (3)$$

where $\lfloor P_n \rfloor$ and $\{P_m\}$ are unit vectors in the directions of R_n and R_m, respectively, and d_i is the cosine of the included angle. Note that the dot product constraint is independent of the vector lengths. A dot product can also be added to the set of geometry equations to find an unknown angle between two vectors.

The cross product constraint is used to define a mutually perpendicular vector to two other vectors. The cross product between unit vectors $\{P_n\}$ and $\{P_m\}$ is written as

$$\{R_h\} = [C_n]\{P_m\} \qquad (4)$$

where

$$[C_n] = \begin{bmatrix} 0 & P_{n,3} & -P_{n,2} \\ -P_{n,3} & 0 & P_{n,1} \\ P_{n,2} & -P_{n,1} & 0 \end{bmatrix} \qquad (5)$$

and R_h is the resulting vector. Note that while $\{P_n\}$ and $\{P_m\}$ are unit vectors, R_h is a unit vector only if the included angle between $\{P_n\}$ and $\{P_m\}$ is 90 degrees.

In Figure 2, a piston-crank is modeled using vectors, a vector loop, two dot products, and the four cosine squared constraints. The set of geometry equations has the form

Loop Equations

$$P_{1,4}P_{1,1} + P_{2,4}P_{2,1} - P_{3,4}P_{3,1} = 0 \qquad (6)$$

$$P_{1,4}P_{1,2} + P_{2,4}P_{2,2} - P_{3,4}P_{3,2} = 0$$

$$P_{1,4}P_{1,3} + P_{2,4}P_{2,3} - P_{3,4}P_{3,3} = 0$$

Dot Products

$$P_{1,1}P_{4,1} + P_{1,2}P_{4,2} + P_{1,3}P_{4,3} = d_1 = \cos 90^o$$

$$P_{2,1}P_{4,1} + P_{2,2}P_{4,2} + P_{2,3}P_{4,3} = d_2 = \cos 90^o$$

Cosine Squared

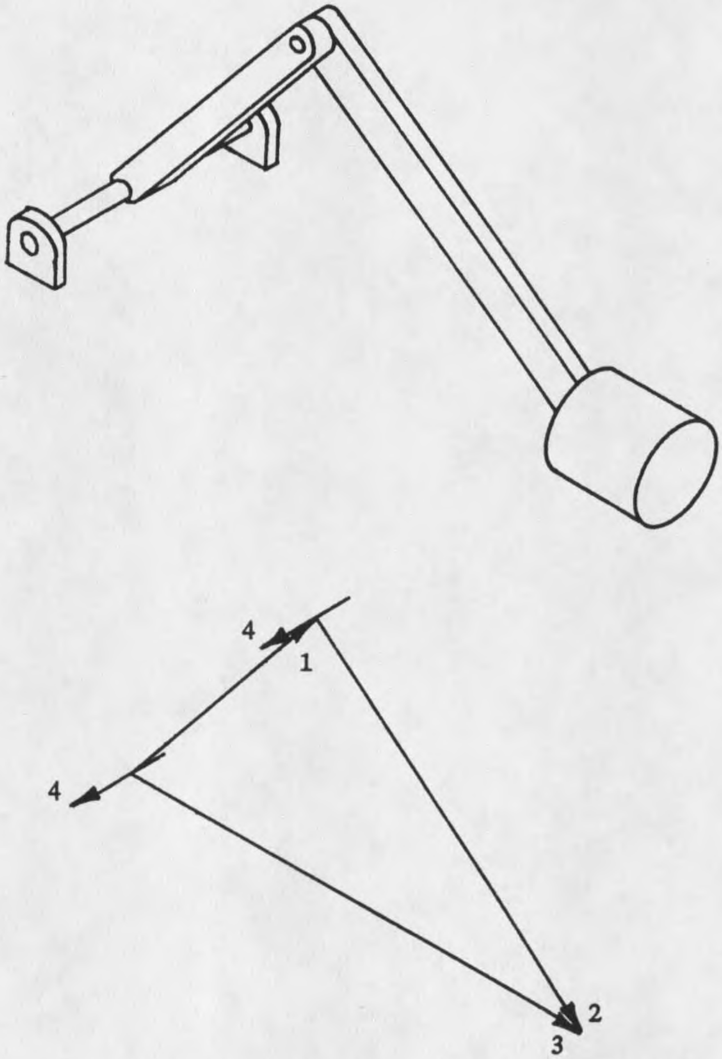
$$P_{1,1}P_{1,1} + P_{1,2}P_{1,2} + P_{1,3}P_{1,3} = 1$$

$$P_{2,1}P_{2,1} + P_{2,2}P_{2,2} + P_{2,3}P_{2,3} = 1$$

$$P_{3,1}P_{3,1} + P_{3,2}P_{3,2} + P_{3,3}P_{3,3} = 1$$

$$P_{4,1}P_{4,1} + P_{4,2}P_{4,2} + P_{4,3}P_{4,3} = 1$$

This set of equations must be reorganized before a solution can be found by placing all of the known terms on the right hand side. The resulting equations are typically nonlinear so, Newton's method is used to solve for the unknown parameters.



$$R_1 + R_2 - R_3 = 0$$

$$R_1 \cdot R_4 = \cos 90^{\circ}$$
$$R_2 \cdot R_4 = \cos 90^{\circ}$$

$$\lfloor P_1 \rfloor \{P_1\} = 1$$
$$\lfloor P_2 \rfloor \{P_2\} = 1$$
$$\lfloor P_3 \rfloor \{P_3\} = 1$$
$$\lfloor P_4 \rfloor \{P_4\} = 1$$

FIGURE 2. - Piston-crank and vector model.

<u>Velocities</u>

The velocities of the mechanism are described using the first time derivative of a vector,

$$\{\dot{R}_n\} = P_{n,4}[C_n]\{\Omega_n\} + \dot{P}_{n,4}\{P_n\} \tag{7}$$

where the angular velocity vector is expressed as

$$\{\Omega_n\} = \begin{Bmatrix} \omega_{n,1} \\ \omega_{n,2} \\ \omega_{n,3} \end{Bmatrix} \tag{8}$$

Note that the first term in Equation 7 is just the cross product of the vector R_n and the angular velocity $\{\Omega_n\}$. The equations used to describe the velocities of a mechanism can be found by replacing the vectors in the loop equations with vector velocities and taking the first time derivative of the dot and cross product constraints. The first time derivative of the cosine squared constraint, however, results in a trivial equation. So, in the velocity analysis it is replaced with an equation which controls the spin of a vector about its own axes.

In this method of kinematic analysis, the spin of a vector about its own axis is arbitrary and can be specified as any value, including zero, and not affect the structure of the mechanism. Most often, this spin is set to zero and is written as

$$\lfloor P_n \rfloor \{\Omega_n\} = 0 \tag{9}$$

This expression is referred to as the "no spin constraint".

The equation set for the velocities of the piston-crank is

Loop Equations

$$\dot{R}_{1,1} + \dot{R}_{2,1} - \dot{R}_{3,1} = 0 \qquad (10)$$

$$\dot{R}_{1,2} + \dot{R}_{2,2} - \dot{R}_{3,2} = 0$$

$$\dot{R}_{1,3} + \dot{R}_{2,3} - \dot{R}_{3,3} = 0$$

Dot Products

$$(\omega_{1,2}P_{1,3} - \omega_{1,3}P_{1,2})P_{4,1} + (\omega_{1,3}P_{1,1} - \omega_{1,1}P_{1,3})P_{4,2} + (\omega_{1,1}P_{1,2} - \omega_{1,2}P_{1,1})P_{4,3} + (\omega_{4,2}P_{4,3} - \omega_{4,3}P_{4,2})P_{1,1} + (\omega_{4,3}P_{4,1} - \omega_{4,1}P_{4,3})P_{1,2} + (\omega_{4,1}P_{4,2} - \omega_{4,2}P_{4,1})P_{1,3} = 0$$

$$(\omega_{2,2}P_{1,3} - \omega_{2,3}P_{1,2})P_{4,1} + (\omega_{2,3}P_{1,1} - \omega_{2,1}P_{1,3})P_{4,2} + (\omega_{2,1}P_{1,2} - \omega_{2,2}P_{1,1})P_{4,3} + (\omega_{4,2}P_{4,3} - \omega_{4,3}P_{4,2})P_{2,1} + (\omega_{4,3}P_{4,1} - \omega_{4,1}P_{4,3})P_{2,2} + (\omega_{4,1}P_{4,2} - \omega_{4,2}P_{4,1})P_{2,3} = 0$$

No Spin

$$\omega_{1,1}P_{1,1} + \omega_{1,2}P_{1,2} + \omega_{1,3}P_{1,3} = 0$$

$$\omega_{2,1}P_{2,1} + \omega_{2,2}P_{2,2} + \omega_{2,3}P_{2,3} = 0$$

$$\omega_{3,1}P_{3,1} + \omega_{3,2}P_{3,2} + \omega_{3,3}P_{3,3} = 0$$

$$\omega_{4,1}P_{4,1} + \omega_{4,2}P_{4,2} + \omega_{4,3}P_{3,3} = 0$$

where

$$\dot{R}_{n,1} = P_{n,4}(P_{n,3}\omega_{n,2} - P_{n,2}\omega_{n,3}) + \dot{P}_{n,4}P_{n,1}$$

$$\dot{R}_{n,2} = P_{n,4}(P_{n,1}\omega_{n,3} - P_{n,3}\omega_{n,1}) + \dot{P}_{n,4}P_{n,2} \qquad (11)$$

$$\dot{R}_{n,3} = P_{n,4}(P_{n,2}\omega_{n,1} - P_{n,1}\omega_{n,2}) + \dot{P}_{n,4}P_{n,3}$$

Note that all of the velocity equations are linear with respect to velocities so, a routine, such as Gauss-Elimination, can be used to solve this set of equations.

Accelerations

The second time derivative of a vector, used in the acceleration analysis, is

$$\{\ddot{R}_n\} = P_{n,4}[C_n]\{\dot{\Omega}_n\} + \ddot{P}_{n,4}\{P_n\} + P_{n,4}[\dot{C}_n]\{\Omega_n\} + 2\dot{P}_{n,4}[C_n]\{\Omega_n\} \quad (12)$$

The terms in Equation 12 are, respectively, the tangential, radial, centripetal, and Coriolis accelerations. The form of the no spin constraint for accelerations is

$$\lfloor P_n \rfloor \{\dot{\Omega}_n\} = 0 \quad (13)$$

Note that in this equation the angular acceleration of the vector about its own axis is controlled and the equation is not just the time derivative of Equation 9.

The loop equations for the piston-crank can be found by replacing the velocity terms in the first three equations of Equation Set 10 with the appropriate acceleration terms The dot products can be found by taking the time derivative of the fourth and fifth equations of Equation Set 10 and the no spin equations take the form of Equation 13.

CHAPTER IV

SOLID ELEMENTS

As stated in Chapter II, the first kinematic analysis is often performed on the mechanism before considering the effects of the solid elements which make up the mechanism. Only the function of the physical link is modeled in the first analysis. Once a correct solution is obtained from this analysis, solid element models are added to introduce the inertia effects necessary for a dynamic analysis.

To simplify the calculation of the element characteristics, such as the mass moments of inertia, a solid element is defined in terms of a local coordinate system. While the orientation of this coordinate system can be arbitrary, a scheme must be established in order to relate the element coordinate system to the Newtonian coordinate system.

The element coordinate system is defined relative to a plane formed by the intersection of two element vectors, namely the element principal vector and the element plane vector. The element principal vector corresponds to the system vector used to describe the element's function in the kinematic analysis and the element plane vector corresponds to a system vector which is most often used to locate the center of mass of the element relative to the connection to the preceding element in the mechanism. A vector in the same direction as the principal vector defines the x_1-axis in element coordinates and the plane formed by the intersection of the principal vector and the element plane vector is the x_1-x_2 plane. The x_3-axis is defined as the direction of the principal vector cross the element plane vector and, finally, the direction of the x_2-axis is recovered by crossing the x_3-axis with the x_1-axis. Figure 3 demonstrates a general solid element and its local coordinate system.

Unit vectors in each element coordinate direction, for element k, are signified by $e_{k,\alpha}$ ($\alpha = 1,2,3$), whereas, the unit vectors in the Newtonian coordinate system are signified by i_β ($\beta = 1,2,3$). The relation between these two coordinate systems is found to be

$$\{e_k\} = [E_k]\{i\} \tag{14}$$

where $[E_k]$ is the set of directional cosines, $E_{k,\alpha,\beta}$, between the vectors $e_{k,\alpha}$ and the vectors i_β for element k. For example, $E_{1,2,3}$ is the directional cosine between $e_{1,2}$ and i_3. From this relation any vector in element coordinates can be converted to system coordinates through the transformation

$$\{PS_{k,m}\} = [E_k]^T\{PE_{k,m}\} \tag{15}$$

where $\{PE_{k,m}\}$ is a unit vector in element coordinates in the

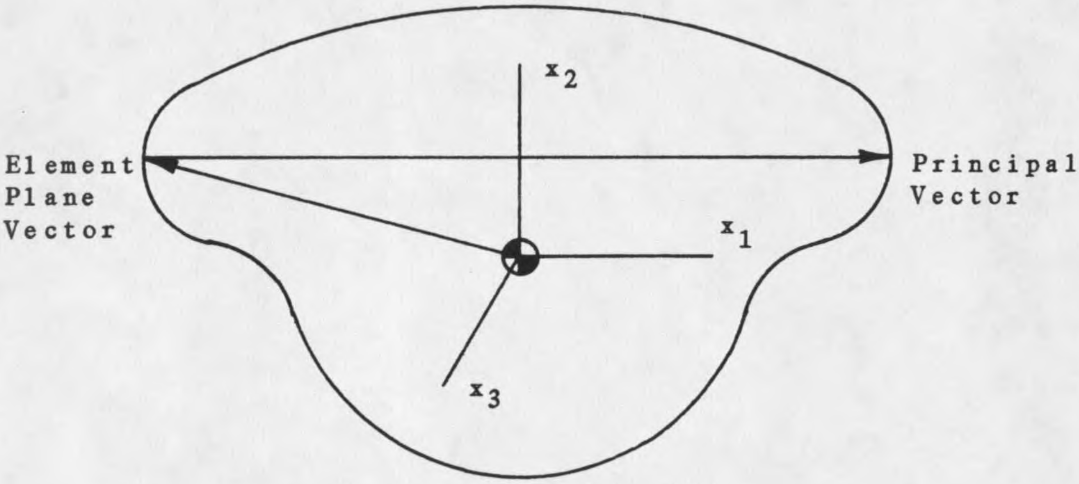


FIGURE 3. - General element and element coordinate system.

same direction as the element vector and $\{PS_{k,m}\}$ is the corresponding system unit vector. The magnitudes, $PS_{k,m,4}$ and $PE_{k,m,4}$, are the same in each coordinate system. Element k is illustrated in Figure 4 with the complete element and Newtonian coordinate systems.

Once all of the elements have been modeled, the second kinematic analysis is performed to determine the motion of each element. This time, however, the spin of an element plane vector and its corresponding principal vector should be equal to represent the motion of a solid body. To describe this constraint, the no spin constraints for each vector must be replaced with two equations which set the X_1 and X_2 angular velocities equal for the two vectors. Since there is a third possible angular velocity in the X_3 direction, another equation must also be replaced to constrain the spin of the two vectors. This other equation is the dot product constraint specifying the known angle between the element plane vector and the principal vector. Note that if all three angular velocities are known for any vector, the no spin equation is no longer a valid equation and can not be used in the system of equations. Therefore, all three angular velocities can not be specified for either the element plane vector or the principal vector.

In addition to the element plane vector and the principal vector, element vectors exist to locate every node, except the center of mass, on the element. A node is

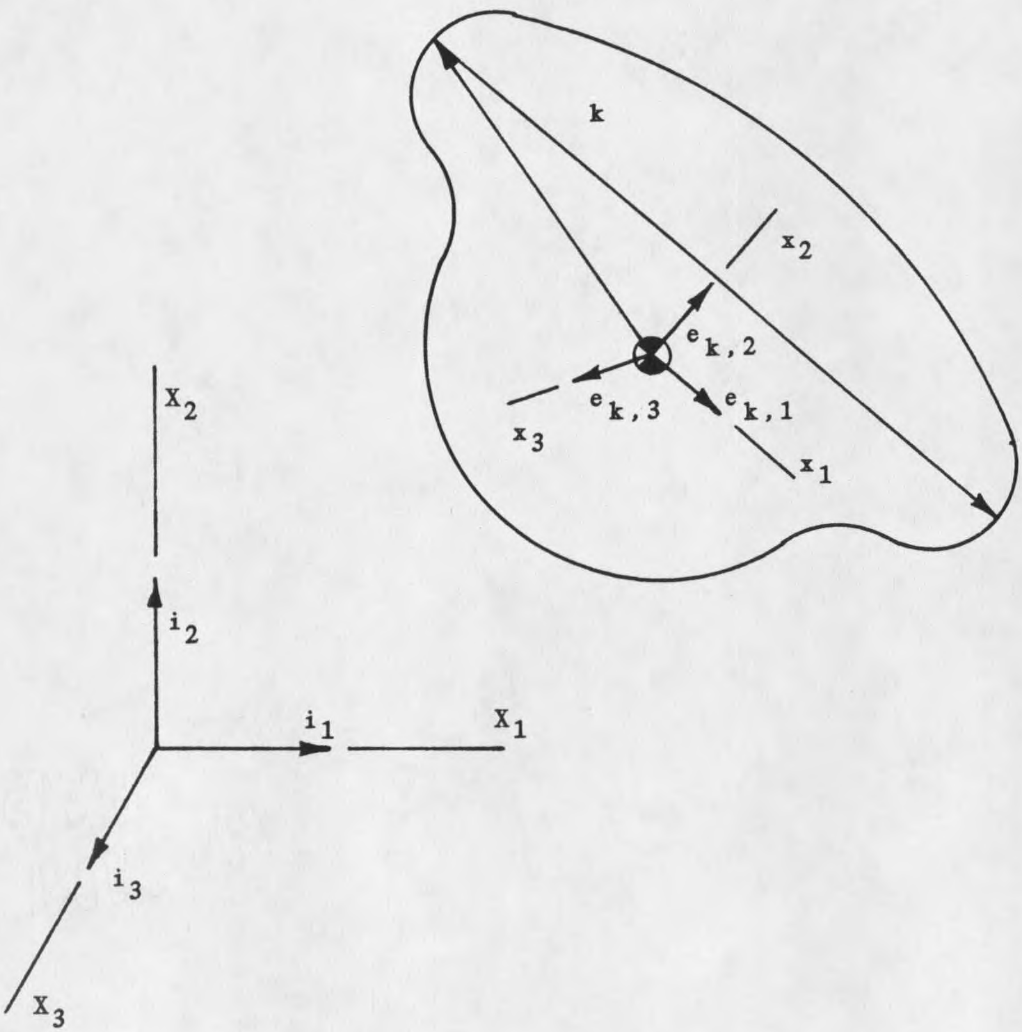


FIGURE 4. - Element with element and Newtonian coordinate systems.

any point of connection to another element, the center of mass of an element, and points of externally applied forces. Element vectors always extend from the center of mass to a node, so a set procedure can be used to calculate the moments due to the forces at that node, and are numbered the same as the node. The nodes are numbered such that the center of mass will always be the highest numbered node on the element and thus, will also be the total number of nodes on the element. A sample element is shown in Figure 5. Note that the principal vector is numbered the same as the center of mass. This will not always be true, especially in the case where connections are made to the center of mass. Figure 6 demonstrates an element to which connections were made to the center of mass. In this case, an arbitrary element vector was used as the principal vector.

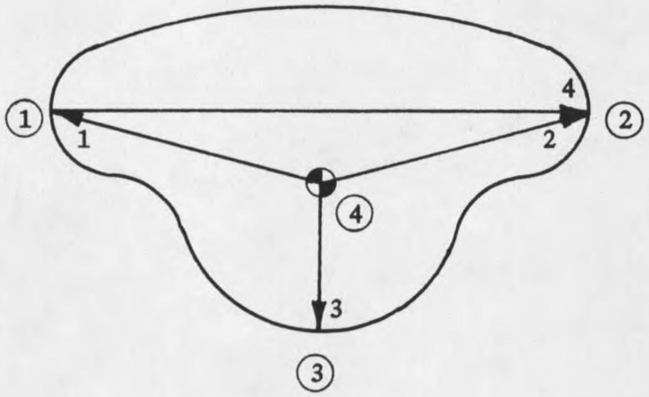


FIGURE 5. - Element numbering system (numbers inside circles indicate node numbers).

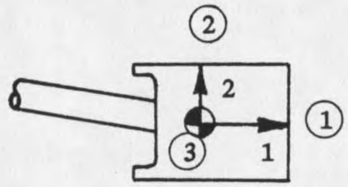


FIGURE 6. - Element with connection to center of mass.

CHAPTER V

ELEMENT DYNAMICS

Newton's second law for an element comprised of N nodes is written as

$$\sum_{j=1}^{N} \{f_{k,j}\} = \{\dot{W}_k\} \tag{16}$$

where $\{f_{k,j}\}$ is the set of resultant forces and moments acting on the element due to the forces and moments at node j, and $\{\dot{W}_k\}$ is the time rate of change of momentum of the element. The set of resultant forces and moments is expressed in terms of the six force and moment components at node j by

$$\{f_{k,j}\} = \begin{bmatrix} [U] & 0 \\ PS_{k,j,4}[C_{k,j}] & [U] \end{bmatrix} \{F_{k,j}\} \tag{17}$$

where [U] is a 3x3 unit matrix and $\{F_{k,j}\}$ is the set of forces and moment components. For $\gamma = 1,2,3$, $F_{k,j,\gamma}$ represents a force component in the Newtonian coordinate system and when $\gamma = 4,5,6$, $F_{k,j,\gamma}$ represents a moment. The term $PS_{k,j,4}[C_{k,j}]$ is the moment due to the the element vector which locates the node, in system coordinates, crossed with the force vector at the node.

The time rate of change of momentum is expressed as

$$\{\dot{W}_k\} = \begin{Bmatrix} M_k \sum_{n=1}^{L} \{\ddot{R}_n\} \\ \\ \{\dot{H}_k\} \end{Bmatrix} \tag{18}$$

where M_k is the mass of the element, $\sum_{n=1}^{L} \{\ddot{R}_n\}$ is the sum of the acceleration vectors from the origin of the Newtonian coordinate system to the center of mass of the element, and

$$\{\dot{H}_k\} = [\dot{I}_k]\{\Omega_k\} + [I_k]\{\dot{\Omega}_k\} \tag{19}$$

Here, the angular velocity of element k, $\{\Omega_k\}$, is the angular velocity of its principal vector. The inertia tensor, of Equation 19, is in system coordinates and should be expressed in terms of the inertia tensor in element coordinates which is readily calculated and constant with respect to time. Using the transformation from Equation 14 this is written as

$$[I_k] = [E_k]^T [IE_k][E_k] \tag{20}$$

where $[IE_k]$ is the inertia tensor in element coordinates and is known as

$$[IE_k] = \begin{bmatrix} IE_{k,1,1} & -IE_{k,1,2} & -IE_{k,1,3} \\ -IE_{k,2,1} & IE_{k,2,2} & -IE_{k,2,3} \\ -IE_{k,3,1} & -IE_{k,3,2} & IE_{k,3,3} \end{bmatrix} \tag{21}$$

Substituting Equation 20 into Equation 19, the result is

$$\{\dot{H}_k\} = [\dot{E}_k]^T[IE_k][E_k]\{\Omega_k\} + [E_k]^T[IE_k][\dot{E}_k]\{\Omega_k\} + [E_k]^T[IE_k]][E_k]\{\dot{\Omega}_k\} \qquad (22)$$

where $[\dot{E}_k]$ is given by Equation 23.

Combining the kinematic acceleration equations with the equilibrium equations results in one matrix equation used to describe the dynamics of the mechanism. This matrix equation has the form of Equation 24. Note that if the number of kinematic acceleration equations equals the number of unknown accelerations, the accelerations can be solved for independently of the forces and moments. This is referred to as the uncoupled form of the equations.