Abstract:
In this thesis we have obtained a solution of the partial differential equations governing the flow of electricity in a transmission line of finite length l (miles) with constant parameters (L henries/mile, R ohms/mile, C farads/mile, G mhos/mile). This solution satisfies preassigned boundary conditions at the sending and receiving end.

The analytic expression for the voltage drop e(x,t), x miles from the sending end and t seconds after the line has been energized, is given in the form of an infinite series. This series is found by applying the superposition theorem to two elementary solutions of the transmission line equation which satisfy unit boundary conditions. The current i(x,t) is determined from e(x,t) by the use of standard methods and is also expressed as an infinite series.

No attempt has been made to use the results obtained for numerical calculations.

ON THE SOLUTION OF THE TRANSMISSION LINE EQUATIONS

by

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ABSTRACT

In this thesis we have obtained a solution of the partial differential equations governing the flow of electricity in a transmission line of finite length l (miles) with constant parameters (L henries/mile, R ohms/mile, C farads/mile, G mhos/mile). This solution satisfies preassigned boundary conditions at the sending and receiving end.

The analytic expression for the voltage drop $e(x,t)$, x miles from the sending end and t seconds after the line has been energized, is given in the form of an infinite series. This series is found by applying the superposition theorem to two elementary solutions of the transmission line equation which satisfy unit boundary conditions. The current $i(x,t)$ is determined from $e(x,t)$ by the use of standard methods and is also expressed as an infinite series.

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1. INTRODUCTION

The basic equations for the transmission line are

(1.1)
$$-\frac{\partial e}{\partial x} = Ri + L\frac{\partial i}{\partial t}$$

(1.2)
$$-\frac{\partial i}{\partial x} = Ge + C\frac{\partial e}{\partial t}$$

where $e(x,t)$ and $i(x,t)$ are the voltage and current respectively; the variable x (miles), measured from the sending end, is restricted to the interval $0 \leq x \leq l$, the variable t (seconds) to $t \geq 0$. The line parameters (R, L, C, and G) are considered constant throughout the problem.

By differentiating equations (1.1) with respect to x and (1.2) with respect to t, we can eliminate $i(x,t)$ and obtain

(2)
$$\frac{\partial^2 e}{\partial x^2} = RGe + (RC + LG)\frac{\partial e}{\partial t} + LC\frac{\partial^2 e}{\partial t^2} .$$

We shall solve equation (2) under the boundary conditions

(Set A)
$$e(0,t) = E_{m1} \sin wt$$
$$e(l,t) = E_{m2} \sin (wt + \alpha),$$

i.e., we shall prescribe the sending and receiving end voltages. This will enable us to determine the performance of the line for a given sending end voltage, such that the receiving end voltage is $E_{m2} \sin(wt+\alpha)$. Futhermore, we assume the line to be energized at $t=0$, i. e., for all values of $t \leq 0$ the line is dead. This yields the two initial conditions

$$e(x,0) = 0,$$

$$\left.\frac{\partial e}{\partial t}\right|_{t=0} = 0.$$

We shall first solve two subsidiary problems.

1. Problem I.

We shall obtain a solution of (2) satisfying the boundary conditions

(Set B) $$U(0,t) = E_{m1} \sin wt,$$
$$U(l,t) = 0.$$

This solution will be of the form

(3) $$U(x,t) = F(0)\, u(x,t) + \int_0^t F'(\tau)\, u(x,t-\tau)\, d\tau ,$$

where $F(t) = E_{m1} \sin wt$ and $u(x,t)$ is a solution of (2) satisfying the conditions

(Set C)

$$u(0,t) = 1,$$
$$u(l,t) = 0,$$
$$u(x,0) = 0,$$
$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0.$$

Equation (3) is commonly referred to as the superposition theorem.

2. Problem II.

We shall obtain a solution of (2) satisfying the boundary conditions

(Set D)

$$V(0,t) = 0,$$
$$V(l,t) = E_{m2} \sin (wt + \alpha).$$

This solution will be of a form identical with equation (3) where $F(t) = E_{m2} \sin (wt+\alpha)$ and $u(x,t)$ is replaced by a solution, $v(x,t)$, of (2) which satisfies the conditions

(Set E)

$$v(0,t) = 0,$$
$$v(l,t) = 1,$$
$$v(x,0) = 0,$$
$$\left.\frac{\partial v}{\partial t}\right|_{t=0} = 0.$$

Upon adding $U(x,t)$ and $V(x,t)$ we obtain

(4) $$e(x,t) = U(x,t) + V(x,t),$$

a solution of (2) satisfying the conditions of (Set A).

Substitution of equation (4) into (1.2) yields

$$-\frac{\partial i}{\partial x} = Ge + C\frac{\partial e}{\partial t} = -\varphi(x,t),$$

whence, upon partial integration with respect to x,

(5) $$i(x,t) = \int \varphi(x,t)\,dx + f(t) = g(x,t) + f(t).$$

The arbitrary function $f(t)$ is determined by making use of the preassigned initial conditions and substitution of (4) and (5) into (1.1) which reduces to

$$L\frac{df}{dt} + Rf(t) = 0.$$

2. SOLUTION OF PROBLEM I

In order to simplify calculations we rewrite (2) as

(6) $$\frac{\partial^2 u}{\partial x^2} + a\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} + cu = 0,$$

where $a = -LC$, $b = -(RC + LG)$, and $c = -RG$. We assume $u(x,t) = u_1(x) + u_2(x,t)$ where $u_1(x)$ shall satisfy the boundary conditions

(7) $$u_1(0) = 1,$$

$$u_1(l) = 0.$$

Then, by virtue of the conditions of (Set C), $u_2(x,t)$ must be determined according to the conditions

(8)
$$u_2(x,0) = -u_1(x),$$
$$u_2(0,t) = 0,$$
$$u_2(l,t) = 0,$$
$$\left.\frac{\partial u_2}{\partial t}\right|_{t=0} = 0.$$

Upon substitution of $u(x,t)$ into (6) we have two differential equations to solve:

(9.1)
$$\frac{d^2u_1}{dx^2} + cu_1 = 0,$$

(9.2)
$$\frac{\partial^2 u_2}{\partial x^2} + a\frac{\partial^2 u_2}{\partial t^2} + b\frac{\partial u_2}{\partial t} + cu_2 = 0.$$

The general solution of (9.1) is

$$u_1(x) = A\cosh\sqrt{RG'}\,x + B\sinh\sqrt{RG'}\,x.$$

By applying the boundary conditions (7) we obtain the particular solution

(10)
$$u_1(x) = \frac{\sinh\sqrt{RG'}\,(l-x)}{\sinh\sqrt{RG'}\,l}.$$

To solve equation (9.2) we assume

$$u_2(x,t) = \varphi(x)\ \psi(t).$$

Upon substitution (9.2) becomes

(11) $$\frac{1}{\varphi}\frac{d^2\varphi}{dx^2} = -\frac{1}{\psi}\left(a\frac{d^2\psi}{dt^2} + b\frac{d\psi}{dt}\right) - c = -m^2,$$

in which the variables are separated. Hence

(12.1) $$\frac{d^2\varphi}{dx^2} + m^2\varphi = 0,$$

(12.2) $$a\frac{d^2\psi}{dt^2} + b\frac{d\psi}{dt} + (c-m^2)\psi = 0,$$

where m is an arbitrary constant to be determined later.

The general solution of (12.1) is

$$\varphi(x) = A\cos mx + B\sin mx.$$

Imposing the boundary conditions (8), we find that any one of the $\varphi_n(x) = \sin m_n x$, $m = \frac{n\pi}{l}$ where $n = 1, 2, \cdots$, will be a particular solution. The functions $\varphi_n(x)$ are the eigen functions associated with the set of eigen values $m = \frac{n\pi}{l}$, $n = 1, 2, \cdots$.

To solve (12.2) we first determine the roots of the characteristic equation

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4a(c-m^2)}}{2a}.$$

We see that since $b^2 - 4ac = (RC - LG)^2 > 0$, the discriminant $\Delta = b^2 - 4ac - 4am^2 = (RC - LG)^2 - 4LC\frac{n^2\pi^2}{l^2}$ will be negative provided $(RC - LG)^2 < 4LC\frac{n^2\pi^2}{l^2}$. For a given R, C, L, and G it is evident that there always exists a positive integer, n_o, such that this inequality holds for all $n \geq n_o$. We shall assume throughout that n is restricted to these values in order to obtain solutions containing sine and cosine terms. Otherwise, it can be shown that no solutions exist satisfying conditions (8).

Putting $\Delta = -4(LC)^2\gamma_n^2$ we obtain as a general solution of (12.2)

$$(13) \qquad \psi_n(t) = (A_n' \sin \gamma_n t + A_n'' \cos \gamma_n t)e^{-\frac{bt}{2a}}.$$

Therefore

$$u_{2,n}(x,t) = e^{-\frac{bt}{2a}} (A_n' \sin\gamma_n t + A_n'' \cos \gamma_n t) \sin\frac{n\pi x}{l},$$

and

$$(14) \quad u_2(x,t) = e^{-\frac{bt}{2a}} \sum (A_n' \sin \gamma_n t + A_n'' \cos \gamma_n t) \sin\frac{n\pi x}{l}.$$

All summations used in this thesis will be from $n = n_o$ to $n = \infty$. To determine A_n' and A_n'' we recall that

$$u(x,t) = \frac{-\sinh\sqrt{RG'}\,(l-x)}{\sinh\sqrt{RG'}\,l},$$

Making use of the well known formulas for the coefficients in a Fourier series we obtain

$$A_n'' = \frac{-2}{l}\int_0^l \frac{\sinh\sqrt{RG}\,(l-x)}{\sinh\sqrt{RG}\,l}\,\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{-2n\pi}{RGl^2 + n^2\pi^2}, \qquad n \geq n_o.$$

and

$$A_n' = \frac{b}{2a\gamma_n}\,A_n''$$

$$= \frac{-2bn\pi}{2a\gamma_n(RGl^2 + n^2\pi^2)}, \quad n \geq n_o.$$

If we define

$$\tan\,\delta_n = \frac{b}{2a\gamma_n},$$

and

$$B_n' = \frac{1}{\cos\delta_n}\,A_n'',$$

then the solution of (6) will be

$$(15) \qquad u(x,t) = \frac{\sinh\sqrt{RG}\,(l-x)}{\sinh\sqrt{RG}\,l} + e^{-\frac{bt}{2a}}\sum B_n'\cos(\gamma_n t - \delta_n)\sin\frac{nx\pi}{l}.$$

Applying the superposition theorem to $u(x,t)$, where $F(t) = E_m \sin wt$, we will obtain $U(x,t)$ as defined. Putting

$$p_n + iq = \left(\frac{b}{2a} + iw\right)^2 + \gamma_n^2 ;$$

$$p_n = \left(\frac{b}{2a}\right)^2 + \gamma_n^2 - w^2 = \frac{c}{a} - w^2 + \frac{1}{a}\left(\frac{n\pi}{l}\right)^2 ,$$

$$q = \frac{bw}{a} ,$$

we find, after considerable simplification,

$$(16) \quad U(x,t) = E_{m1} \sin wt \, \frac{\sinh\sqrt{RG'}(l-x)}{\sinh\sqrt{RG'}\,l} + wE_{m1} e^{-\frac{bt}{2a}} \sum \frac{\sin\frac{n\pi x}{l}}{p_n^2+q^2} B_n' \Big[e^{\frac{bt}{2a}} \Big\{ (p_n \cos wt + q \sin wt) \frac{\gamma_n}{\cos\delta_n} \sin 2\delta_n - wp_n \sin wt \cos\delta_n + wq \cos wt \cos\delta_n \Big\} - \Big\{ p_n \frac{\gamma_n}{\cos\delta_n} \sin(2\delta_n - \gamma_n t) + wq \cos(\gamma_n t - \delta_n) \Big\} \Big] .$$

3. SOLUTION OF PROBLEM II

The solution of problem II proceeds along the same lines as problem I. We obtain

$$(17) \qquad v_1(x) = \frac{\sinh\sqrt{RG'}\,x}{\sinh\sqrt{RG'}\,l} ,$$

and

$$(18) \qquad v_2(x,t) = e^{-\frac{bt}{2a}} \sum (\bar{A}_n' \sin\gamma_n t + \bar{A}_n'' \cos\gamma_n t) \sin\frac{n\pi x}{l} ,$$

where

$$\bar{A}''_n = \frac{-2}{l}\int_0^l \frac{\sinh\sqrt{RG'}\,x}{\sinh\sqrt{RG'}\,l}\,\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2n\pi(-1)}{RG\,l^2 + n^2\pi^2},$$

and

$$(-1)^n B'_n = -B_n = -\bar{A}''_n\,\frac{1}{\cos\delta_n}.$$

Therefore

$$(19)\quad v(x,t) = \frac{\sinh\sqrt{RG'}\,x}{\sinh\sqrt{RG'}\,l} + e^{-\frac{bt}{2a}}\sum B_n\cos(\gamma_n t - \delta_n)\sin\frac{n\pi x}{l}.$$

By means of the superposition theorem we find V(x, t) to be

$$(20)\quad V(x,t) = E_{m2}\sin(\omega t+\alpha)\,\frac{\sinh\sqrt{RG'}\,x}{\sinh\sqrt{RG'}\,l} + E_{m2}\,\omega\, e^{-\frac{bt}{2a}}\sum B_n\,\frac{\sin\frac{n\pi x}{l}}{p_n^2+q^2}\Big[e^{\frac{bt}{2a}}\Big\{[p_n\cos(\omega t+\alpha) + q\sin(\omega t+\alpha)]\,\gamma_n\,\frac{\sin 2\delta_n}{\cos\delta_n} + \omega q\cos(\omega t+\alpha)\cos\delta_n - \omega p_n\cos\delta_n\sin(\omega t+\alpha)\Big\} - \Big\{(p_n\cos\alpha + q\sin\alpha)\frac{\gamma_n}{\cos\delta_n}\sin(2\delta_n - \gamma_n t) + \omega q\cos\alpha\cos(\gamma_n t-\delta_n) - \omega p_n\sin\alpha\cos(\gamma_n t-\delta_n)\Big\}\Big] + E_{m2}\sin\alpha\, e^{-\frac{bt}{2a}}\sum B_n\cos(\gamma_n t-\delta_n)\sin\frac{n\pi x}{l}.$$

4. DETERMINATION OF e(x, t) AND i(x, t)

The expression for the voltage $e(x,t)$ is obtained by adding $U(x,t)$ (16) and $V(x,t)$ (20),

$$e(x,t) = U(x,t) + V(x,t).$$

Noticing that some of the terms contain the damping factor $e^{-\frac{bt}{2a}}$, we shall separate the expression for $e(x,t)$ into the steady state part, $e_s(x,t)$, and the transient part, $e_t(x,t)$.

The steady state part, $e_s(x,t)$, is found to be

$$(21)\quad e_s(x,t) = \frac{1}{\sinh\sqrt{RG}\,l}\Big[E_{m1}\sin\omega t\,\sinh\sqrt{RG}(l-x) + E_{m2}\sin(\omega t+\alpha)\sinh\sqrt{RG}\,x\Big] + \omega\sqrt{\left(\frac{b}{a}\right)^2+\omega^2}\;\sum\frac{(-1)^n 2n\pi}{RGl^2+n^2\pi^2}\sqrt{\frac{E_{m1}^2+E_{m2}^2+2(-1)^{n+1}E_{m1}E_{m2}\cos\alpha}{p_n^2+q^2}}\;\sin(\omega t+\epsilon_n)\sin\frac{n\pi x}{l}$$

where

$$\tan\epsilon_n = \frac{(-1)^{n+1}E_{m1}\sin\theta_n + E_{m2}\sin(\alpha+\theta_n)}{(-1)^{n+1}E_{m1}\cos\theta_n + E_{m2}\cos(\alpha+\theta_n)},$$

$$\tan\theta_n = \frac{2\gamma_n p_n\sin\delta_n + \omega q\cos\delta_n}{2\gamma_n q\sin\delta_n - \omega p_n\cos\delta_n} = \frac{b}{\omega}\;\frac{c-\frac{n^2\pi^2}{l^2}}{b^2+a^2p_n}$$

and the transient part, $e_t(x,t)$, is

$$(22)\quad e_t(x,t) = e^{-\frac{bt}{2a}} \sum \frac{(-1)^n 2n\pi}{RGl^2+n^2\pi^2} \sin\frac{n\pi x}{l} \left\{ \frac{E_{m2}}{\cos\delta_n} \Big[\sin\alpha \right.$$

$$\cos(\gamma_n t-\delta_n) - \frac{\omega}{\sqrt{p_n^2+q^2}} \left[(p_n\cos\alpha + q\sin\alpha)\cos\alpha + \omega^2\right]^{\frac{1}{2}}$$

$$\left.\cos(\gamma_n t + \Delta_n(\alpha) - \delta_n)\Big] + \frac{(-1)^n \omega E_{m1}}{\cos\delta_n} \sqrt{\frac{e^{-\frac{n^2\pi^2}{l^2}}}{a(p_n^2+q^2)}}\right.$$

$$\left.\cos(\gamma_n t + \Delta(0) - \delta_n)\right\}$$

where

$$\tan\Delta_n(\alpha) = \frac{\gamma_n(p_n\cos\alpha + q\sin\alpha)}{\frac{b}{2a}(p_n\cos\alpha + q\sin\alpha) + \omega(q\cos\alpha - p_n\sin\alpha)},$$

$$\tan\Delta(0) = \frac{b\omega}{2a}.$$

To obtain $i(x,t)$ we substitute $e(x,t) = e_s(x,t) + e_t(x,t)$ into equation (1.2) and integrate the resulting expression partially with respect to x. This yields, after simplification,

$$(23)\quad i(x,t) = \frac{\sqrt{RG}}{\sinh\sqrt{RG}\,l} \Big\{ E_{m1}(G\sin\omega t + \omega C\cos\omega t)\cosh\sqrt{RG}(l-x) -$$

$$E_{m2}[G\sin(\omega t+\alpha) + \omega C\cos(\omega t+\alpha)]\cosh\sqrt{RG}\,x -$$

$$\omega\sqrt{\left(\frac{b}{a}\right)^2+\omega^2} \sum \frac{(-1)^n 2l}{RGl^2+n^2\pi^2} \sqrt{\frac{E_{m1}^2+E_{m2}^2+2(-1)^{n+1}E_{m1}E_{m2}\cos\alpha}{p_n^2+q^2}}$$

$$[G\sin(\omega t+\epsilon_n) + \omega C\cos(\omega t+\epsilon_n)]\cos\frac{n\pi x}{l} -$$

$$\left(\frac{LG-RG}{2L}\right) e^{-\frac{bt}{2a}} \sum \frac{(-1)^n 2l}{RGl^2+n^2\pi^2} \cos\frac{n\pi x}{l} \left\{ \frac{E_{m2}}{\cos\delta_n} \left[\sin\alpha \right.\right.$$

$$\cos(\gamma_n t-\delta_n) - \frac{\omega}{\sqrt{p_n^2+q^2}} \left[(p_n \cos\alpha + q\sin\alpha)\cos\alpha + \omega^2\right]^{\frac{1}{2}}$$

$$\left.\cos(\gamma_n t+\Delta_n(\alpha)-\delta_n)\right] + \frac{(-1)^n \omega E_{m1}}{\cos\delta_n} \sqrt{\frac{c-\frac{n^2\pi^2}{l^2}}{a(p_n^2+q^2)}}$$

$$\left.\cos(\gamma_n t+\Delta(0)-\delta_n)\right\} + Ce^{-\frac{bt}{2a}} \sum \frac{(-1)^n 2l\gamma_n}{RGl^2+n^2\pi^2} \cos\frac{n\pi x}{l} \left\{\right.$$

$$\frac{E_{m2}}{\cos\delta_n} \left[\sin\alpha \sin(\gamma_n t-\delta_n) - \frac{\omega}{\sqrt{p_n^2+q^2}}\left[(p_n\cos\alpha+q\sin\alpha)\cos\alpha + \right.\right.$$

$$\left.\omega^2\right]^{\frac{1}{2}} \sin(\gamma_n t+\Delta_n(\alpha)-\delta_n)\Big] + \frac{(-1)^n \omega E_{m1}}{\cos\delta_n} \sqrt{\frac{c-\frac{n^2\pi^2}{l^2}}{a(p_n^2+q^2)}}$$

$$\left.\sin(\gamma_n t+\Delta(0)-\delta_n)\right\} + f(t).$$

The function $f(t)$ is determined by substitution of $e(x,t)$ and $i(x,t)$ into equations (1.1). Upon inspection we note that $f(t)$ will be of the form

$$f(t) = Ke^{-\frac{Rt}{L}}.$$

Hence the steady state part of the current $i(x,t)$ is of the form

$$(24)\quad i_s(x,t) = \frac{\sqrt{RG}}{\sinh\sqrt{RG}\,l}\Big\{E_{m1}(G\sin\omega t + \omega C\cos\omega t)$$

$$\cosh\sqrt{RG}(l-x) - E_{m2}[G\sin(\omega t+\alpha) + \omega C\cos(\omega t+\alpha)]\cosh\sqrt{RG}\,x -$$

$$\omega\sqrt{\left(\frac{b}{a}\right)^2+\omega^2}\;\sum\frac{(-1)^n\,2l}{RGl^2+n^2\pi^2}\sqrt{\frac{E_{m1}^2+E_{m2}^2+2(-1)^{n+1}E_{m1}E_{m2}\cos\alpha}{p_n^2+q^2}}$$

$$[G\sin(\omega t+\epsilon_n) + \omega C\cos(\omega t+\epsilon_n)]\cos\frac{n\pi x}{l}\Big\}.$$

5. DISCUSSION OF RESULTS AND CONCLUSIONS

The expressions for the voltage and current distribution along a transmission line, commonly found in text books, are

$$(25)\qquad E(x) = E_s\cosh\sqrt{yz}\,x - I_sZ_o\sinh\sqrt{yz}\,x,$$

$$I(x) = I_s\cosh\sqrt{yz}\,x - E_sY_o\sinh\sqrt{yz}\,x,$$

where $z = R + jwL$, $y = G + jwC$, $Z_o = \sqrt{\frac{z}{y}}$, and $Y_o = \sqrt{\frac{y}{z}}$.

When $x = l$, then

$$E_r = E_s\cosh\sqrt{yz}\,l - I_sZ_o\sinh\sqrt{yz}\,l,$$

$$I_r = I_s\cosh\sqrt{yz}\,l - E_sY_o\sinh\sqrt{yz}\,l.$$

If we let $E_r = E_s$ and $I_r = I_s$, i.e., if we assume that the amplitude of the voltage and current waves at the sending and receiving ends are equal, then

$$(\cosh\sqrt{yz}\,l - 1)E_s - I_sZ_o\sinh\sqrt{yz}\,l = 0,$$

$$(\cosh\sqrt{yz}\,l - 1)I_s - E_sY_o\sinh\sqrt{yz}\,l = 0.$$

In order that these equations be consistent we must have

$$\begin{vmatrix} \cosh\sqrt{yz}\,l - 1 & -Z_0 \sinh\sqrt{yz}\,l \\ -Y_0 \sinh\sqrt{yz}\,l & \cosh\sqrt{yz}\,l - 1 \end{vmatrix} = 0 .$$

This gives the condition $\cosh\sqrt{yz}\,l = 1$, which is a condition upon the line parameters R, C, L, and G as well as the length of the line. Putting

$$yz = M^2 e^{2i\varphi}$$

whence

$$\sqrt{yz} = M\cos\varphi + iM\sin\varphi,$$

we obtain

$$\begin{aligned} \cosh\sqrt{yz}\,l &= \cosh(lM\cos\varphi)\cos(lM\sin\varphi) \\ &\quad + i\sinh(lM\cos\varphi)\sin(lM\sin\varphi) \\ &= 1. \end{aligned}$$

Equating real and imaginary parts we find that the two equations

$$\cosh(lM\cos\varphi)\cos(lM\sin\varphi) = 1,$$

$$\sinh(lM\cos\varphi)\sin(lM\sin\varphi) = 0,$$

must be satisfied simultaneously. This leads to the two equations

$$lM\cos\varphi = 0,$$

$$lM\sin\varphi = 2n\pi.$$

One set of values φ, M which would satisfy these equations is $\varphi = \frac{\pi}{2}$, $M = \frac{2n\pi}{l}$, $n = 1, 2, 3, \cdots$.

Since

$$M = [(G^2 + w^2C^2)(R^2 + w^2L^2)]^{\frac{1}{4}},$$

it is apparent that the equation $M = \frac{2n\pi}{l}$ restricts the possible values of R, C, L, and G. We therefore conclude that although it is theoretically possible to construct a transmission line for which the input equals the output such a construction may not be feasible from the practical point of view. Futhermore it is seen that, under this assumption, the equations (25) do not lend themselves very readily to numerical calculations.

If we inspect the steady state part $e_s(x,t)$ of the voltage, which we obtained as

$$(21)\quad e_s(x,t) = \frac{1}{\sinh\sqrt{RG}\,l}\Big[E_{m1}\sin wt \sinh\sqrt{RG}(l-x) + E_{m2}\sin wt\sinh\sqrt{RG}\,x\Big] + \omega\sqrt{\left(\frac{b}{a}\right)^2+\omega^2}\sum \frac{(-1)^n 2n\pi}{RGl^2+n^2\pi^2}\sqrt{\frac{E_{m1}^2+E_{m2}^2+2(-1)^{n+1}E_{m1}E_{m2}\cos\alpha}{p_n^2+q^2}}\sin(wt+\epsilon_n)\sin\frac{n\pi x}{l},$$

we notice that it is of the form

$$e_s(x,t) = A_0(x)\sin(wt+\varphi_0(x)) + \sum A_k(x)\sin(wt+\varphi_k(x)),$$

where

$$A_0(x) = \sqrt{E_{m1}^2\left(\frac{\sinh\sqrt{RG}\,(l-x)}{\sinh\sqrt{RG}\,l}\right)^2 + E_{m2}^2\left(\frac{\sinh\sqrt{RG}\,x}{\sinh\sqrt{RG}\,l}\right)^2}$$

$$= \sqrt{E_{m1}^2 u_1^2(x) + E_{m2}^2 v_1^2(x)}\ ,$$

$$\tan\varphi_0(x) = \frac{E_{m2}\sin\alpha\,\sinh\sqrt{RG}\,x}{E_{m1}\sinh\sqrt{RG}\,(l-x) + E_{m2}\cos\alpha\,\sinh\sqrt{RG}\,x}\ .$$

Futhermore

$$A_k(x) = \omega\sqrt{\left(\frac{b}{a}\right)^2 + \omega^2}\ \frac{(-1)^n 2n\pi}{RGl^2 + n^2\pi^2}\ \sin\frac{n\pi x}{l}$$

$$\sqrt{\frac{E_{m1}^2 + E_{m2}^2 + 2(-1)^{n+1}E_{m1}E_{m2}\cos\alpha}{p_n^2 + q^2}}\ ,$$

and

$$\tan\varphi_k(x) = \tan\ = \frac{(-1)^{n+1}E_{m1}\sin\theta_n + E_{m2}\sin(\alpha + \theta_n)}{(-1)^{n+1}E_{m1}\cos\theta_n + E_{m2}\cos(\alpha + \theta_n)}\ ,$$

$$n = n_0,\ n_0 + 1, \ldots,$$

which is independent of x.

Hence the steady state part $e_s(x,t)$ is expressed as

$$e_s(x,t) = E_m(x)\ \sin(wt + \varphi(x))$$

where the amplitude, $E_m(x)$, and the phase angle, $\varphi(x)$, depend upon x only and, futhermore,

$$E_m(0)\ \sin(wt + \varphi(0)) = E_{m1}\sin wt,$$

$$E_m(l)\ \sin(wt + \varphi(l)) = E_{m2}\sin(wt + \alpha).$$

Note also that the frequency of this steady state voltage is equal to the frequency of the input voltage. This was to be expected as we assumed the line parameters constant along the line.

Upon inspection of the transient part of the voltage

$$(22)\quad e_t(x,t) = e^{-\frac{bt}{2a}} \sum \frac{(-1)^n 2n\pi}{RGl^2+n^2\pi^2} \sin \frac{n\pi x}{l} \left\{ \frac{E_{m2}}{\cos \delta_n} \left[\sin\alpha \right.\right.$$

$$\cos(\gamma_n t - \delta_n) - \frac{\omega}{\sqrt{p_n^2+q^2}} \left[(p_n \cos\alpha + q \sin\alpha)\cos\alpha + \omega^2\right]^{\frac{1}{2}}$$

$$\left.\cos(\gamma_n t + \Delta_n(\alpha) - \delta_n)\right] + \frac{(-1)^n \omega E_{m1}}{\cos\delta_n} \sqrt{\frac{c - \frac{n^2\pi^2}{l^2}}{a(p_n^2+q^2)}}$$

$$\left. \cos(\gamma_n t + \Delta(0) - \delta_n) \right\},$$

we see it to be of the form

$$e_t(x,t) = e^{-\frac{bt}{2a}} \sum B_n(x) \sin(\gamma_n t + \psi_n).$$

It is apparent that $e_t(x,t)$ has a variable frequency depending upon n and the line parameters, and a phase angle dependent upon n but independent of x. The amplitude of the wave is periodic in x with two nodes at the sending and receiving ends of the line. The equations (25), commonly used in transmission line calculations, yield the steady state part of the current and voltage only. This introduces no great error except for a short time immediatly after the line has been energized.

If our results are to be used for numerical calculations the input and desired output voltages are to be assumed; then equations (21), (22), (23), and (24) will give the transient and steady state distribution of current and voltage along the line.

We conjecture that in many instances a fair approximation to the exact values of the steady state part $e_s(x,t)$ will be obtained by neglecting the infinite series, i.e., it will suffice to calculate

$$e_s(x,t) = A_0(x) \sin(wt + \varphi_0(x)),$$

and similarly for the steady state part of the current.

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