A study of the state of stress in a two-dimensional solid containing multiple-crack systems
by Dale Edward Nottingham

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of
MASTER OF SCIENCE in Civil Engineering
Montana State University

Abstract:
This study is an analysis of "the state of stress existing around a given crack in a biaxially-stressed, two-dimensional, elastic solid containing a system of cracks. Photoelasticity was used to define the stress condition around both a multiple-crack system used in the major study and around a single crack used in the minor study. This minor study was an experimental and analytical analysis of the stress conditions around a single crack. This information was used to analyze the experimental data obtained from the multiple-crack study.

Several characteristics of the specific multiple-crack systems studied were observed. The largest difference between principal stresses always occurred at the ends of the cracks. These end stresses were probably increased by decreasing the crack's radius of curvature, increasing its length, or orienting the cracks at about 60 degrees with the major applied stress. The end-stress magnitude was a function of both the sum of and the difference between the applied stresses. When the applied stresses were equal, the stress fields were symmetrical about the cracks and the end-stress magnitudes were probably proportional to the applied stresses. Changes in the stress fields due to changing the crack patterns are complex.

The results of the entire study are intended to provide part of the background information necessary for determining stress conditions in real fractured or jointed media.

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ABSTRACT

This study is an analysis of the state of stress existing around a given crack in a biaxially-stressed, two-dimensional, elastic solid containing a system of cracks. Photoelasticity was used to define the stress condition around both a multiple-crack system used in the major study and around a single crack used in the minor study. This minor study was an experimental and analytical analysis of the stress conditions around a single crack. This information was used to analyze the experimental data obtained from the multiple-crack study.

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CHAPTER I

INTRODUCTION

Background

This project was inspired by the very real and complex problems associated with determining the strength of fractured or jointed materials. These types of materials have been dealt with for many years by mining engineers who design tunnels and shafts, but recent developments in open-pit mining and nuclear excavation have greatly increased interest in the subject.

The U.S. Army Engineer Waterways Experiment Station has been conducting a major study to develop a general method for analyzing slopes formed by nuclear explosions under the study name, "Engineering Properties of Nuclear Craters" (1968). This study deals with the problems associated with both loose and intact rock slopes. The Bureau of Mines in conjunction with the Kennecott Copper Corporation made a study of a similar problem, designing an economical slope modification for the Kimbley Pit at Ely, Nevada. (Results of this study consist of four individual reports by Robert H. Merrill, Wilson Blake, Perry N. Halstead, and David W. Wiscarver, all of which were published in 1968 and are available by reprint from the Society of Mining Engineers of AIME.) These projects and others dealing with both slope stability (Terzaghi, 1962) and structural stability (Goodman, 1967) have one common problem, jointed material.

Several different methods are used for finding the stability of a slope or structure, but all require determining the strength of the materials and the mode of failure. The mode of failure commonly assumed is shear. Terzaghi (1962) uses this mode in presenting a technique for evaluating the strength of intact rock slopes that is similar to his method for analyzing soil slopes.

In the present study, however, no mode of failure is chosen because the author is not convinced that any one failure mode controls for all applicable systems. The problem dealt with here is determining the stress conditions imposed upon a material by the presence of cracks. An extension of this information may allow finding the strength of a fractured material based on the properties of the parent material and the crack pattern.

For the case of a real material, the amount, orientation and types of joints and fractures must be known prior to analysis. Methods for determining this information on a statistical basis have been devised by Klaus (1962), Aho (1960), and others; and the parent material's physical properties can be found by conventional laboratory means. This says that methods are presently known for collecting the supporting data to solve the real problem regardless of whether or not good techniques are available for using this information.

Object and Scope

The purpose of this project is to analyze the state of stress in a two-dimensional solid containing multiple-crack systems. A real fractured media is much too complex to lend itself to analysis; so a simplified, well-controlled system must be adopted. Real problems exist in three dimensions, but here, the analysis is done in two dimensions to reduce the complexity of the experimental and analytical studies. The field of research is narrowed still further as each crack considered is entirely surrounded by intact material, and the crack faces are not in contact with each other.

The problem that is left after all the simplifications have been made seems rather simple compared to the real problem. However, when one learns that the plate with a single discontinuity is considered to be a relatively complex problem with few, if any, satisfactory solutions, the multiple-crack problem even in the simplified form is very formidable indeed. Because so little information about the stress conditions around a single crack is available, this study has to be incorporated with the main problem.

The author chose to use a three-phase study of the problem. First, an analytical study of a single crack based on a solution presented by Durelli and Murray (1943) was carried out to define significant stress parameters and to obtain indications of how the parameters influenced the stress field. Second, a photoelastic study was conducted to give experimental indication of how the single-crack parameters behaved. The analytical and experimental conclusions were correlated and used as the single-crack characteristics. Third, a photoelastic study of multiple-crack patterns was conducted, and the results were analyzed in the light of the single-crack characteristics.

CHAPTER II

ANALYTICAL STUDY OF SINGLE CRACK

Method of Approach

An analytical approach to the problem of a single crack in a loaded plate may be made by using the solution to the problem of an elliptical discontinuity in a loaded infinite plate. By allowing the ellipse to approach the dimensions of a real crack, the ellipse solution becomes the crack solution. Several authors present solutions to the ellipse problem (Wang, 1953; Timoshenko and Lessells, 1925; etc.); however, many of these solutions exist in unclear forms or are for cases of uniaxial loading only. Durelli and Murray (1943) present a general solution which is clear and quite adaptable to this type of study; consequently, their form was chosen. The solution itself is credited to an Englishman, C.E. Inglis (1913).

The general solutions for stresses presented (Durelli and Murray, 1943) are complex infinite series that would be very difficult to evaluate. However, if several assumptions are made, a relatively simple equation is obtained that allows study of the most critical regions. Those assumptions are:

1. The largest stress concentrations in the field around a crack occur on the boundary of the crack. (This is confirmed by experimental results.)
2. The crack exists in an infinite plate and is loaded by uniform normal stresses applied on boundaries that are mutually perpendicular.
3. The larger applied stress acts in a direction that makes an angle ϕ with the major axis of the crack.

4. The length of the crack is very large compared to the width.

5. The plate containing the crack acts as a true elastic material.

6. The edges of the crack are not in contact with each other.

7. An assumption that is implied by the choice of the coordinate system is that the stress pattern is symmetrical about the major axis of the ellipse.

No restriction is placed on the type of failure because, at the crack boundary, only normal stresses parallel to the edge exist on a boundary element. The stresses on all other planes may be found by simply multiplying this stress by the appropriate value from a Mohr's circle relationship.

Definition of the Solution

Prior to presentation of the solution to be studied, an explanation of the coordinate system is in order. The following long quotation taken directly from the paper by Durelli and Murray (1943) best describes the coordinate system used:

> The mathematical analysis for the elliptical discontinuity is simplified by the use of elliptical coordinates. According to this system each point in a plane can be represented in terms of the coordinates, α and ν, whose values in the cartesian system correspond to,
>
> $$x = c \cosh \alpha \cos \nu$$
> $$y = c \sinh \alpha \sin \nu \qquad (1)$$
>
> For conditions such that α equals a constant these may

be expressed as

$$\frac{x^2}{c^2 \cosh^2 \alpha} + \frac{y^2}{c^2 \sinh^2 \alpha} = 1 \qquad (2)$$

which represents a family of confocal ellipses with a distance of 2c between the foci.

For conditions such that ν equals a constant there results

$$\frac{x^2}{c^2 \cos^2 \nu} - \frac{y^2}{c^2 \sin^2 \nu} = 1 \qquad (3)$$

which corresponds to a family of confocal hyperbolae, orthogonal to the previous ellipses.

Every point represented by "x" and "y" in the cartesian system is the intersection between an ellipse and hyperbola of the elliptical system whose corresponding coordinates will be α and ν (Figure 1). At first this system may seem unduly complicated, however, it has the advantage that the boundary of an elliptical discontinuity can be given by holding one of the parameters, α, constant.

The parameter ν is also known as the eccentric angle of the ellipse. It can be determined for each ellipse (α equals a constant) in a simple manner by using the major and minor auxiliary circles of the ellipse as indicated in Figure 2.

If one thinks of the detail of the stress distribution and considers conditions at some point in the field then it is an easy matter to visualize a small rectangular particle whose center is on the intersection of a confocal ellipse and hyperbolae and whose sides are parallel to the tangents at the point of intersection. Such an element is shown in Figure 3.

Since this study is concerned with the region bordering the crack, Inglis' general solution takes the particular form shown by Durelli and Murray (1943)

$$\sigma_{\nu} = \frac{(\sigma_1 + \sigma_2) \sinh 2\alpha_o + (\sigma_1 - \sigma_2)\left[\cos 2\phi - e^{2\alpha_o} \cos 2(\phi - \nu)\right]}{\cosh 2\alpha_o - \cos 2\nu} \qquad (4)$$

where

$$\sigma_{\alpha} = \tau_{\alpha\nu} = 0,$$

σ_1 = major applied principal stress,

σ_2 = minor applied principal stress,

$\sigma_\nu, \sigma_\alpha, \tau_{\alpha\nu}$ = stresses on the element defined in Figure 3,

α, ν = the elliptical coordinates of any point,

α_o = the coordinate that is constant along the edge of the crack,

ϕ = angle between the crack and direction of the major applied stress.

Having the solution completely defined, one can now look at the modifications for the problem at hand. First, the ranges of some of the terms in the σ_ν equation may be defined by applying the assumption that the crack's length is large compared to its width. The ellipse information derived in Appendix A provides the relationship between crack length-to-width ratio and α_o.

$$\alpha_o = \sinh^{-1} \frac{b}{\sqrt{a^2 - b^2}} \tag{5}$$

If "a" is much larger than "b", then α_o is approximately equal to $\sinh^{-1} b/a$, where b/a is very small compared to one. Table I shows that for small α_o, α_o is approximately equal to $\sinh \alpha_o$ from which one concludes that α_o is approximately equal to b/a, where b/a is very small compared to one.

Study of the Crack Boundary Solution

As the equation for σ_ν is simplified by applicable assumptions and approximations, one is allowed to look at how σ_ν varies with location on the crack edge, orientation of applied stresses, crack size and crack

TABLE I. VALUES OF FUNCTIONS
FROM HANDBOOK OF MATHEMATICS BY ABRAMOWITZ (1964)

$2\alpha_o$	$\sinh 2\alpha_o$	$\cosh 2\alpha_o$	$e^{2\alpha_o}$
0.0000	0.000000000	1.000000000	1.000000000
0.0001	0.000100000	1.000000005	1.000100005
0.0010	0.001000000	1.000000500	1.001000500
0.0100	0.010000167	1.000050000	1.010050167
0.1000	0.100166756	1.005004168	1.105170918
1.0000	1.175201194	1.543080635	2.718281828

shape. These variables will be studied in the following three phases:

1. Let σ_1, σ_2 and ϕ vary while other variables are treated as constants.
2. Let the ratio a/b vary while all other variables are held constant (vary crack shape).
3. Let "a" vary while the ratio, a/b as well as all other variables, are held constant (vary crack size).

Variation of Crack Orientation and Loads' Magnitudes

Consider the two terms comprising σ_ν in the form

$$\sigma_\nu = \frac{(\sigma_1 + \sigma_2)\sinh 2\alpha_o}{\cosh 2\alpha_o - \cos 2\nu} + \frac{(\sigma_1 - \sigma_2)\left[\cos 2\phi - e^{2\alpha_o}\cos 2(\phi - \nu)\right]}{\cosh 2\alpha_o - \cos 2\nu}$$

where, from Table I for very small $2\alpha_o$

$$\cosh 2\alpha_o \approx 1 \quad \text{(slightly larger)},$$
$$e^{2\alpha_o} \approx 1 \quad \text{(slightly larger)},$$
$$\sinh 2\alpha_o \approx 0 \quad \text{(slightly larger)},$$

such that the first term becomes

$$\frac{(\sigma_1 + \sigma_2)\,(0+)}{(1+) - \cos 2\nu}, \qquad \text{(Term One)}$$

where ν is arbitrary so that $\cos 2\nu$ can have any value between plus one and minus one. In general, this term will be small (excluding the case when $\cos 2\nu$ approaches one). The second term becomes

$$\frac{(\sigma_1 - \sigma_2)\left[\cos 2\phi - (1+) \cos 2(\phi - \nu)\right]}{(1+) - \cos 2\nu} . \qquad \text{(Term Two)}$$

Immediately, two cases have become apparent: when σ_1 equals σ_2, and when σ_1 does not equal σ_2.

If σ_1 equals σ_2 then

$$\sigma_\nu = \frac{(\sigma_1 + \sigma_2)\ (0+)}{(1+) - \cos 2\nu},$$

which says σ_ν will have significant values only if $(1+) - \cos 2\nu$ approaches zero and where, in turn, 2ν approaches zero or 2π. This says σ_ν is independent of ϕ and is maximum at the ends of the crack where ν equals zero or π. Also, σ_ν is directly proportional to $(\sigma_1 + \sigma_2)$ at all points. (Consider only the range of ν from zero to π, because that is the total range of ν according to this coordinate system.)

If σ_1 is not equal to σ_2, the first term behaves as it did above; however, the total σ_ν value includes the second term which does not act as nicely. Looking at Term Two independently from Term One may permit evaluation of the effect of the value of ϕ. Variation of Term Two's value with respect to $(\sigma_1 - \sigma_2)$ is obviously a direct proportionality, but the ϕ and ν effects are not immediately apparent. Consequently, Figure 4 is used to show the nature of these variations.

The curve connecting maximum M (M is defined by Figure 4) values for constant ν's allows prediction of the ϕ value that gives the absolute maximum stress contribution from Term Two. By fitting a curve through calculated points, the equation for this curve was found to be M equals sec 2.00ϕ. The ϕ angle between zero and $\pi/2$ that makes M maximum is 45 degrees. A second conclusion drawn from Figure 4 is that for Term Two, maximum stresses occur at the ends of the crack (ν equals zero and π) for all values of ϕ.

The effects of Terms One and Two can be combined only in the light of certain qualifying remarks. First, compression stresses are considered to be positive and tension stresses negative. The largest compressive stress or most compressive stress is σ_1; the smaller compressive stress or most tensile stress is σ_2. Therefore, by this convention $(\sigma_1 - \sigma_2)$ will always be positive, but $(\sigma_1 + \sigma_2)$ may be of either sign. The Term One coefficient of $(\sigma_1 + \sigma_2)$ is $\sinh 2\alpha_o/(1 - \cos 2\nu)$ which is always positive; and the coefficient of $(\sigma_1 - \sigma_2)$ in Term Two is M which varies in sign. For arbitrary ν, nothing can be concluded about the combination of Term One with Term Two because of the sign variation. However, both these terms independently indicate that the largest stress concentrations occur at the ends of the crack regardless of the value of ϕ. For this reason, the points where ν equals zero and π will be studied in more detail.

For ν equals zero, the equation for σ_ν (Equation (4)) becomes

$$\sigma_\nu = \frac{(\sigma_1 + \sigma_2)\sinh 2\alpha_o + (\sigma_1 - \sigma_2)\cos 2\phi(1 - e^{2\alpha_o})}{\cosh 2\alpha_o - 1} \qquad (6)$$

From Table I it is seen that for $2\alpha_o$ in the range of 0.001, sinh $2\alpha_o$ is very nearly equal to $(e^{2\alpha_o} - 1)$. This relationship may be shown more analytically by an algebraic manipulation. By the definition of the hyperbolic sine

$$\sinh 2\alpha_o = \frac{e^{2\alpha_o} - e^{-2\alpha_o}}{2} .$$

Forming a least common denominator gives

$$\sinh 2\alpha_o = \frac{e^{4\alpha_o} - 1}{2e^{2\alpha_o}} ,$$

and factoring leaves

$$\sinh 2\alpha_o = \frac{(e^{2\alpha_o} - 1)\,(e^{2\alpha_o} + 1)}{2e^{2\alpha_o}} . \qquad (7)$$

When $2\alpha_o$ becomes very small, Table I shows that $e^{2\alpha_o}$ approaches one. This says the remaining term in Equation (7), $(e^{2\alpha_o} + 1)/2e^{2\alpha_o}$, also approaches one. That leaves

$$\sinh 2\alpha_o = e^{2\alpha_o} - 1. \qquad (8)$$

(See Figure 5 for a graphical comparison of sinh $2\alpha_o$ and $(e^{2\alpha_o} - 1)$ over the range of $2\alpha_o$ shown in Table I.) Making the substitution of sinh $2\alpha_o$ for $(1 - e^{2\alpha_o})$ in Equation (6) gives

$$\sigma_\nu = \frac{(\sigma_1 + \sigma_2) \sinh 2\alpha_o + (\sigma_1 - \sigma_2) \cos 2\phi(-\sinh 2\alpha_o)}{\cosh 2\alpha_o - 1}$$

or

$$\sigma_\nu = \left[(\sigma_1 + \sigma_2) - \cos 2\phi(\sigma_1 - \sigma_2)\right] \left[\frac{\sinh 2\alpha_o}{\cosh 2\alpha_o - 1}\right]. \qquad (9)$$

The variation of σ_ν with respect to ϕ is expressed by

$$(\sigma_1 + \sigma_2) - \cos 2\phi(\sigma_1 - \sigma_2).$$

The remaining term will be explored later to show σ_ν variation with respect to change in crack size and shape.

It has already been shown that for any given σ_1 and σ_2, Terms One and Two, independently, give maximum stress contribution when ν equals zero. Therefore, if their contributions are added with like sign when ν equals zero, the aboslute maximum stress magnitude occurs. The sign will be that of $(\sigma_1 + \sigma_2)$. The Term One contribution is represented by $(\sigma_1 + \sigma_2)$ and the Term Two contribution is represented by $(\sigma_1 - \sigma_2) \cos 2\phi$, as all other terms are factorable. The minimum value of stress at ν equals zero for given σ_1 and σ_2 values will occur when $(\sigma_1 + \sigma_2)$ and $(\sigma_1 - \sigma_2)$ are added with opposite signs. The value for the combination will have the sign of the larger (that is, larger in absolute value). The only way that the sign between $(\sigma_1 + \sigma_2)$ and $(\sigma_1 - \sigma_2)$ can be varied is by picking ϕ such that $\cos 2\phi$ changes sign. Any values of $|\cos 2\phi|$ less than one will not give maximum or minimum values of σ_ν; consequently, for these conditions ϕ must be either zero or $\pi/2$.

Perhaps one of the most important observations is that, if σ_2 is a tensile stress, then it is also possible for σ_ν to be a tensile stress. A stress condition of this type would be particularly important if it occurred in a material such as rock, where the tensile strength is much smaller than the compressive strength. This may mean that in rock any of three types of failure are possible: compressive, tensile or shear.

The other point of interest on the crack is where ν equals π. Equation (4) becomes

$$\sigma_\nu = \frac{(\sigma_1 + \sigma_2)\sinh 2\alpha_o + (\sigma_1 - \sigma_2)\cos 2\phi - e^{2\alpha_o}\cos 2(\phi - \pi)}{\cosh 2\alpha_o - \cos 2\pi},$$

but $\cos 2\pi$ equals one and $\cos(2\phi - 2\pi)$ is equal in sign and magnitude to $\cos 2\phi$; consequently, the argument is reduced to that of ν equals zero. This says the stress condition described by ν equals zero applies to both ends of the crack.

Variation of Crack Shape

When the length of a crack is changed with respect to its width, the shape undergoes a change that may be related by the elliptical parameters, "a" and "b". Consider what will happen to the stress at ν equals zero if the crack becomes longer with respect to its width, but all other factors are held constant. This says, although "b" is already small with respect to "a", it becomes even smaller. Since

$$\alpha_o = \sinh^{-1} b/a,$$

α_o will decrease as this change occurs.

The expression developed previously that isolated the α_o effect upon σ_ν is

$$\sigma_\nu = K \left[\frac{\sinh 2\alpha_o}{\cosh 2\alpha_o - 1}\right] \qquad (10)$$

where K (K is equal to $\left[(\sigma_1 + \sigma_2) - \cos 2\phi(\sigma_1 - \sigma_2)\right]$) is now held constant. Substituting the definition for $\cosh 2\alpha_o$ and $\sinh 2\alpha_o$ into this equation and simplifying, gives an expression that makes obvious the effect of changing α_o.

$$\frac{\sinh 2\alpha_o}{\cosh 2\alpha_o - 1} = \frac{\frac{1}{2}(e^{2\alpha_o} - e^{-2\alpha_o})}{\frac{1}{2}(e^{2\alpha_o} + e^{-2\alpha_o} - 2)}$$

$$\frac{\sinh 2\alpha_o}{\cosh 2\alpha_o - 1} = \frac{e^{4\alpha_o} - 1}{e^{4\alpha_o} + 1 - 2e^{2\alpha_o}}$$

$$\frac{\sinh 2\alpha_o}{\cosh 2\alpha_o - 1} = \frac{(e^{2\alpha_o} + 1)\,(e^{2\alpha_o} - 1)}{(e^{2\alpha_o} - 1)\,(e^{2\alpha_o} - 1)} \qquad (11)$$

As α_o decreases, $e^{2\alpha_o}$ decreases toward one and the value $\sinh 2\alpha_o / (\cosh 2\alpha_o - 1)$ increases (See Figure 6); σ_ν increases.

The shape of this crack may also be defined in terms of the length and the radius of curvature at the ends. William Gerberich (1962) states that for an internal crack the principal stress difference is approximately proportional to the minus one-fourth root of the radius of curvature. A corresponding relationship based upon the assumptions previously stated in this paper will now be derived.

The expression for radius of curvature at the ends of the crack, ρ, as derived in Appendix B, is

$$\rho = b^2/a.$$

Substituting ρ into $\sinh \alpha_o$ equals b/a gives

$$\sinh \alpha_o = \rho/b,$$

and

$$b = \sqrt{\rho a},$$

so that

$$\sinh \alpha_o = \sqrt{\rho a}/a \approx \alpha_o.$$

The expression for σ_ν can now be written

$$\sigma_\nu = K \left[\frac{e^{2\alpha_o} + 1}{e^{2\alpha_o} - 1}\right]$$

where $(e^{2\alpha_o} - 1)$, as shown in Equation (8), is approximately equal to $\sinh 2\alpha_o$ which is in turn approximately equal to $2\alpha_o$. Also, that same section shows that $(e^{2\alpha_o} + 1)$ is approximately equal to two. If the substitution of α_o equals $\sqrt{\rho a}/a$ is carried out, the σ_ν expression may then be written as

$$\sigma_\nu = (K\sqrt{a})\rho^{-\frac{1}{2}}, \tag{12}$$

which says σ_ν is proportional to the minus one-half power of radius of curvature at the ends of the crack. This expression agrees with Gerberich's (1962) conclusion about how the principal stress differences vary, with respect to radius-of-curvature change, around a single-edged crack (external crack).

This last expression for σ_ν (Equation (12)) not only shows how σ_ν varies with respect to radius of curvature (ρ) when length (a) is held constant, but also shows that σ_ν increases in direct proportion to the square root of length as length increases.

Variation of Crack Size

When the crack size is varied without changing its shape, the ratio a/b remains constant while "a" varies. The only term in σ_ν that reflects this type of change is α_o. Since α_o equals $\sinh^{-1}$ b/a and b/a remains constant, α_o remains constant and the stress σ_ν will be unchanged by varying "a". This means that the magnitude of σ_ν is unchanged for all points in the plate; if the crack changes size but not shape and if all other variables are held constant.

Summary of the Analytical Observations

The following observations can be made concerning the variation of σ_ν with variation of σ_1, σ_2 and ϕ:

1. The maximum σ_ν values always occur at the ends of the

crack. (Here, maximum means largest tensile or largest compressive stresses occurring in the region of the crack.)

2. Stress, σ_ν , is directly proportional to $(\sigma_1 + \sigma_2) - \cos 2\phi(\sigma_1 - \sigma_2)$.

3. When the applied stresses are equal, σ_ν is independent of ϕ and is directly proportional to twice the applied stress.

4. The absolute maximum value of σ_ν occurs when $(\sigma_1 - \sigma_2)$ is added with like sign to $(\sigma_1 + \sigma_2)$, and the sign of σ_ν is the same as that of $(\sigma_1 + \sigma_2)$.

5. The minimum values of σ_ν at the ends of the crack occur when $(\sigma_1 - \sigma_2)$ is added with unlike sign to $(\sigma_1 + \sigma_2)$, and the sign of σ_ν will be the same as that of the term having the larger absolute value.

6. Maximum and minimum stresses will occur at the ends of the crack when the crack is either parallel or perpendicular ($\phi = 0$, $\pi/2$) to the direction of the largest applied stress.

7. If one of the applied stresses is a tensile stress, σ_ν may be a tensile stress, depending on the value of ϕ.

The following observations can be made concerning σ_ν as the crack size and shape are varied:

1. Changing the crack's size without changing its shape does not alter the stress magnitude or distribution.

2. If the length of the crack is held constant, σ_ν is directly proportional to the minus one-half power of the radius of

curvature at the ends of the crack.

3. If the radius of curvature of the crack is held constant, σ_{ν} is directly proportional to the square root of the length.

CHAPTER III

LABORATORY RESEARCH

The laboratory approach to this problem was dictated by the available analytical approaches and the assumptions made therein. Consequently, a plane model of linearly elastic material loaded by uniform stresses on mutually perpendicular edges was chosen. Since high stress gradients were expected in some portions of the model and the shape of stress patterns were of interest, a photoelastic study using birefringent plastic was made. This provided continuous stress patterns throughout the model and stress magnitudes at the crack boundary.

Brief Explanation of Photoelasticity

A detailed explanation of photoelasticity and the physical principles involved was found in several sources. Coker and Filon (1957) gave what was probably the most rigorous treatment of the subject, while Frocht (1948) also gave quite a complete explanation. The author found that, for his purposes, Durelli and Riley's Introduction to Photomechanics (1965) presented the simplest, most direct explanation of photoelasticity.

When polarized light passes through a stressed birefringent material and is viewed through an analyzer, fringe patterns may be seen. If the light is circularly polarized and monochromatic, the patterns that appear (called isochromatics) are a function of the difference between the principal stresses. In other words, each line represents a continuous locus of points where the maximum shear stress at each point on the locus is constant. The number of fringes existing between a point of zero maximum shear stress and a point of non-zero maximum shear stress is called the fringe order. If the fringe order is known (found by simply counting the