The forced vibration of viscous damped, N-beam structures
by Daniel Franklin Prill

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of
MASTER OF SCIENCE in Aerospace and Mechanical Engineering
Montana State University
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Abstract:
The equation of motion for a linear beam with viscous damping and dynamic loading is presented. The
particular solution for a uniformly distributed load which varies sinusoidally in time is presented for a
beam with general boundary conditions. It is recognized that the boundary conditions are functions of
the first three spatial derivatives and thus recursion formulas in matrix form are developed.

For a system of beams which are interconnected, knowledge of the boundary conditions allows
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particular solution for each beam of the system. Thus a method, including the appropriate computer
program, is presented which results in the exact particular solution for a structural system composed of
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Also, the method is such that any number of the natural frequencies and mode shapes for the undamped
structural system can be approximated with a high degree of accuracy.

Example solutions for structural systems are presented for a cantilever beam and for a three beam
configuration.
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Date  [11-24-70]
THE FORCED VIBRATION OF VISCOUS
DAMPED, N-BEAM STRUCTURES

by

DANIEL FRANKLIN PRILL

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of
MASTER OF SCIENCE
in
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Approved:

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TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>TITLE PAGE</td>
<td>i</td>
</tr>
<tr>
<td>VITA</td>
<td>ii</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENT</td>
<td>iii</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>v</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>vi</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>vii</td>
</tr>
<tr>
<td>CHAPTER I - INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>CHAPTER II - SOLUTION OF THE FORCED VIBRATION OF THE VISCOUS DAMPED BEAM EQUATION</td>
<td>2</td>
</tr>
<tr>
<td>Equation of Motion</td>
<td>2</td>
</tr>
<tr>
<td>Nondimensional Equation of Motion</td>
<td>4</td>
</tr>
<tr>
<td>Solution</td>
<td>5</td>
</tr>
<tr>
<td>CHAPTER III - APPLICATION TO AN N-BEAM STRUCTURE</td>
<td>11</td>
</tr>
<tr>
<td>Method</td>
<td>11</td>
</tr>
<tr>
<td>Procedure</td>
<td>11</td>
</tr>
<tr>
<td>CHAPTER IV - EXAMPLES</td>
<td>16</td>
</tr>
<tr>
<td>The Cantilever Beam</td>
<td>16</td>
</tr>
<tr>
<td>A Three Beam System</td>
<td>23</td>
</tr>
<tr>
<td>CHAPTER V - DISCUSSION</td>
<td>36</td>
</tr>
<tr>
<td>APPENDICES - Appendix A, Computer Program</td>
<td>39</td>
</tr>
<tr>
<td>LITERATURE CONSULTED</td>
<td>62</td>
</tr>
</tbody>
</table>
**LIST OF TABLES**

<table>
<thead>
<tr>
<th>Number</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Three-Beam System Properties</td>
<td>25</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Number</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Sign Convention</td>
<td>3</td>
</tr>
<tr>
<td>4.1</td>
<td>Cantilever Beam</td>
<td>17</td>
</tr>
<tr>
<td>4.2</td>
<td>Magnification Factor vs Frequency Ratio: $\frac{C}{C_c} = 0$</td>
<td>21</td>
</tr>
<tr>
<td>4.3</td>
<td>Phase Angle vs Forcing Frequency</td>
<td>22</td>
</tr>
<tr>
<td>4.4</td>
<td>Three-Beam System</td>
<td>24</td>
</tr>
<tr>
<td>4.5</td>
<td>End Elements</td>
<td>27</td>
</tr>
<tr>
<td>4.6</td>
<td>Nondimension Deflection at $z_1=1$ vs Forcing Frequency: $C=0$</td>
<td>29</td>
</tr>
<tr>
<td>4.7</td>
<td>Beam One Mode Shape: First Natural Frequency</td>
<td>30</td>
</tr>
<tr>
<td>4.8</td>
<td>Beam Two Mode Shape: First Natural Frequency</td>
<td>31</td>
</tr>
<tr>
<td>4.9</td>
<td>Beam Three Mode Shape: First Natural Frequency</td>
<td>32</td>
</tr>
<tr>
<td>4.10</td>
<td>Beam One Mode Shape: Second Natural Frequency</td>
<td>33</td>
</tr>
<tr>
<td>4.11</td>
<td>Beam Two Mode Shape: Second Natural Frequency</td>
<td>34</td>
</tr>
<tr>
<td>4.12</td>
<td>Beam Three Mode Shape: Second Natural Frequency</td>
<td>35</td>
</tr>
</tbody>
</table>
ABSTRACT

The equation of motion for a linear beam with viscous damping and dynamic loading is presented. The particular solution for a uniformly distributed load which varies sinusoidally in time is presented for a beam with general boundary conditions. It is recognized that the boundary conditions are functions of the first three spatial derivatives and thus recursion formulas in matrix form are developed.

For a system of beams which are interconnected, knowledge of the boundary conditions allows development of algebraic equations which can be solved for the integration coefficients of the particular solution for each beam of the system. Thus a method, including the appropriate computer program, is presented which results in the exact particular solution for a structural system composed of a large number of beams.

Also, the method is such that any number of the natural frequencies and mode shapes for the undamped structural system can be approximated with a high degree of accuracy.

Example solutions for structural systems are presented for a cantilever beam and for a three beam configuration.
CHAPTER I

INTRODUCTION

The method of solution to the free-vibration of linear beams is well known [1]. When damping and a forcing function are considered, the complexity of the solution increases and, in some cases, an approximation technique has been used to determine a solution [2].

If an undamped, multi-element structure is to be analyzed for free-vibrations, the eigenvalue problem may become complex [3].

This paper presents an exact solution to the problem of a viscous damped beam with a distributed load, forcing function. The solution presented is similar to that of Stanek [4]. The solution, as presented, is in matrix form and is applied to an N-beam structure.

Examples are worked for a cantilever beam and for a three-beam structure. Since the computer is used to obtain numerical results, Fortran-4 level programs are presented.

A method of finding natural frequencies and the mode shapes of these frequencies is also presented.

---

1Numbers in brackets refer to literature consulted.
SOLUTION OF THE FORCED VIBRATION OF
THE VISCIOUS DAMPED BEAM EQUATION

Equation of Motion

The derivation of the linear beam equation for dynamic motion is
well known [1]. When viscous damping and a forcing function are included
in the derivation, the resulting equation of motion has two terms which
are not in the linear equations. These terms are the damping term,
\( \frac{C}{L} \frac{\partial y}{\partial t} \), and the forcing function, \( F_0 \sin \omega t \).

Adopting the sign convention of Figure 2.1, the equation of motion
is

\[
\frac{W}{gL} \frac{\partial^2 y}{\partial t^2} + \frac{C}{L} \frac{\partial y}{\partial t} + E I \frac{\partial^4 y}{\partial x^4} = F_0 \sin \omega t \tag{2.1}
\]

where:
- \( W \) = Total weight of beam (lb)
- \( g \) = Acceleration due to gravity (in/sec^2)
- \( L \) = Length of beam (in)
- \( C \) = Total damping constant (lb-sec/in)
- \( y \) = Lateral deflection of beam (in)
- \( x \) = Axial coordinate of beam (in)
- \( E \) = Modulus of Elasticity of material of beam (lb/in^2)
- \( I \) = Moment of Inertia of beam cross-section (in^4)
Figure 2.1. Sign Convention
\( F_0 \) = Maximum load on beam (lb/in)  
\( \omega \) = Circular frequency of load (rad/sec)  
\( t \) = time (sec)  

**Nondimensional Equation of Motion**

The solution of Equation 2.1 may be presented in a more efficient fashion in nondimensional form. The equation of motion is put into nondimensional form by use of the following quantities:

\[
z = \frac{x}{L} \\
Y = \frac{Y}{\frac{F_0 L^4}{C_0 EI}} \\
\theta = \omega t
\]

where:  
z = Nondimensional axial coordinate  
Y = Nondimensional lateral deflection  
\( \theta \) = Nondimensional time  
\( C_0 \) = Nondimensional constant

\( C_0 \) is a numerical nondimensional constant which is chosen for convenience. The constant may be any number, but is chosen to be eight for the cantilever beam example of this paper and is chosen to be one for the three-beam example. See Chapter IV for reasons for these choices.
The nondimensional equation of motion is

\[
\beta^4 \frac{\partial^2 y}{\partial \phi^2} + \alpha \beta^4 \frac{\partial y}{\partial \phi} + \frac{\partial^4 y}{\partial z^4} = C_0 \sin \theta
\]  
(2.3)

where: \( \beta^4 = \frac{W L^3 \omega^2}{gEI} \)  
(2.4)
and: \( \alpha = \frac{Cg}{W \omega} \)

Solution

For this study, only the steady-state solution is of interest. The solution will, therefore, be a particular solution. Thus, assume a solution of the form:

\[
Y = Z^S_0 \sin \theta + Z^C_0 \cos \theta
\]  
(2.5)

where the Z-functions (\( Z^S_0 \) and \( Z^C_0 \)) are functions of only the nondimensional axial coordinate, \( z \). The subscripts and superscripts of Equation 2.5 are explained as follows:

1) The superscript on the Z-function in question will be either an "s" or a "c". An "s" will designate that the Z-function is a coefficient of the term \( \sin \theta \); and a "c" will designate that the Z-function is a coefficient of the term \( \cos \theta \).

2) The subscripts of the Z-functions will designate the order
of the derivative of the Z-functions with respect to the axial coordinate, \( z \). For example, the nondimensional slope of the beam, \( \frac{\partial Y}{\partial z} \), would be written as:

\[
Y' = Z_l^s \sin \theta + Z_l^c \cos \theta \tag{2.6}
\]

Differentiating Equation 2.5, substituting the results into Equation 2.3 and equating coefficients of \( \sin \theta \) and \( \cos \theta \) yields

\[
-\beta^4 Z_s^o - \alpha \beta^4 Z_c^o + Z_c^s = C_o \tag{2.7}
\]

\[
-\beta^4 Z_c^o + \alpha \beta^4 Z_s^o + Z_s^c = 0 \tag{2.8}
\]

Equation 2.5 becomes a solution of Equation 2.3 if the Z-functions satisfy Equations 2.7 and 2.8 simultaneously. Combining Equations 2.7 and 2.8 to eliminate \( Z_s^o \) yields the following eighth-order nonhomogeneous differential equation.

\[
Z_c^c - 2 \beta^4 Z_c^4 + \beta^8 Z_c^o (1 + \alpha^2) = -C_o \alpha \beta^4 \tag{2.9}
\]

The following symbols are introduced for writing the homogeneous solution of Equation 2.9.

\[
\mu = \beta(1 + \alpha^2)^{1/8}
\]

\[
\phi = \frac{1}{4} \tan^{-1} \alpha \tag{2.10}
\]

\[
a = \mu \cos \phi
\]
$b = \mu \sin \phi$ \hfill (2.10 cont')

The complete solution of Equation 2.9 is

$$Z^c_o = \begin{bmatrix} \cosh az & \sinh az \end{bmatrix} M^c_o \begin{bmatrix} \cos b z \\ \sin b z \end{bmatrix} + \begin{bmatrix} \cosh b z & \sinh b z \end{bmatrix} N^c_o \begin{bmatrix} \cos az \\ \sin az \end{bmatrix} + Z^c_p$$ \hfill (2.11)

where $Z^c_p$, the particular solution of Equation 2.9, is

$$Z^c_p = -\frac{C_o \alpha \beta^4}{\mu^8}$$ \hfill (2.12)

The eight integration constants are the elements of the $M^c_o$ and $N^c_o$ matrices where

$$M^c_o = \begin{bmatrix} A & D \\ C & E \end{bmatrix}^c_o$$

and

$$N^c_o = \begin{bmatrix} E & H \\ G & F \end{bmatrix}^c_o \hfill (2.13)$$

The subscripts and superscripts of Equation 2.13 have the same meanings as those of the $Z$-functions.

The form of the $Z^s_o$ function is the same as that for the $Z^c_o$ function.

There are eight additional integration constants for the $Z^s_o$ function.

Since the $Z$-functions contain sixteen integration constants, the eight integration constants of one $Z$-function must be dependent upon the
integration constants of the other Z-function. To find this relationship the first four derivatives of the Z-functions must be developed. These derivatives are

\[ Z_i^j = \begin{bmatrix} \cosh az & \sinh az \end{bmatrix} M_i^j \begin{bmatrix} \cos bz \\ \sin bz \end{bmatrix} \]

\[ + \begin{bmatrix} \cosh bz & \sinh bz \end{bmatrix} N_i^j \begin{bmatrix} \cos az \\ \sin az \end{bmatrix} \]

(2.14)

i = 1,2,3,...n
j = s or c

The difference between successive derivatives will be the elements of the \( M_i^j \) and \( N_i^j \) matrices. When derivatives of the Z-functions are taken, recursion relationships for the \( M_i^j \) and \( N_i^j \) matrices become evident. These relations are

\[ M_i^j = a J M_{i-1}^j + b M_{i-1}^j K \]

\[ N_i^j = b J N_{i-1}^j + a N_{i-1}^j K \]

i = 1,2,3,...n
j = s or c

(2.15)

where the J and K matrices are defined as

\[ J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]

(2.16)

Expansion of Equations 2.15 enables one to write the \( M_i^j \) and \( N_i^j \) matrices which occur in the first four derivatives of the Z-function as

\[ M_i^j = \mu (\cos \phi J M_0^j + \sin \phi M_0^j K) \]

(2.17)
\[ M_2^j = \mu^2 (\cos 2\phi M_o^j + \sin 2\phi J M_o^j K) \]
\[ M_3^j = \mu^3 (\cos 3\phi J M_o^j + \sin 3\phi M_o^j K) \]
\[ M_4^j = \mu^4 (\cos 4\phi M_o^j + \sin 4\phi J M_o^j K) \]

(2.17 con't)

\[ M^j = (\cos \phi M^j + \sin \phi J M^j K) \]

\[ j = s \text{ or } c \]

and:

\[ N_1^j = \mu (\sin \phi J N_o^j + \cos \phi N_o^j K) \]
\[ N_2^j = \mu^2 (-\cos 2\phi N_o^j + \sin 2\phi J N_o^j K) \]
\[ N_3^j = -\mu^3 (\sin 3\phi J N_o^j + \cos 3\phi N_o^j K) \]
\[ N_4^j = \mu^4 (\cos 4\phi N_o^j - \sin 4\phi N_o^j K) \]

(2.18) \[ j = s \text{ or } c \]

Substitution of Equations 2.17 and 2.18 into Equation 2.17 gives the \( Z_o^s \) function, in terms of the integration constants of the \( Z_o^c \) function, as

\[ Z_o^s = - \left[ \coshaz \sinhaz \right] J M_o^c K \begin{bmatrix} \cosbz \\ \sinbz \end{bmatrix} \]
\[ \left. + \left[ \coshbz \sinhbz \right] J N_o^c K \begin{bmatrix} \cosaz \\ \sinaz \end{bmatrix} - \frac{C_o^4}{8\mu} \right. \]

(2.19)

The relationships between the integration constants of the \( Z^\text{functions are:} \)

\[ M_o^s = - J M_o^c K \quad , \quad N_o^s = J N_o^c K \]

(2.20)
or in expanded form as

\[
\begin{bmatrix}
A & D \\
C & B
\end{bmatrix}_S = \begin{bmatrix}
-B & C \\
-D & A
\end{bmatrix}_C
\]

\[
\begin{bmatrix}
E & H \\
G & F
\end{bmatrix}_S = \begin{bmatrix}
F & -G \\
H & -E
\end{bmatrix}_C
\]

The particular solution of Equation 2.3 is now complete.
CHAPTER III

APPLICATION TO AN N-BEAM STRUCTURE

Method

The solution to the equation of motion developed in Chapter II is adaptable to an N-beam structure. A structure is first divided into substructures, each being a beam. Each beam has a separate equation of motion and a separate solution. The determination of the integration constants of each solution is done through the use of equations which identify the boundary conditions of deflection, slope, moment and shear. The system will be defined such that the boundary conditions will be evaluated at the origin of each coordinate system, \( z = 0 \), or at the point where \( z = 1 \) in the coordinate system. Note that these two points locate the ends of the beam in the coordinate system. The number of boundary conditions required for a solution is \( 4N \).

Procedure

General boundary conditions for deflection, slope, moment, and shear will be developed such that a general N-beam structure may be analyzed. Twisting of beams will not be allowed.

In order to insure that the boundary conditions are satisfied, each \( Z \)-function will be required to satisfy an appropriate set of boundary
conditions.

The following quantities are introduced for brevity:

\[ S_1 = \cosh \alpha z \quad T_1 = \cosh \beta z \]
\[ S_2 = \sinh \alpha z \quad T_2 = \sinh \beta z \]
\[ S_3 = \cos \beta z \quad T_3 = \cos \alpha z \]
\[ S_4 = \sin \beta z \quad T_4 = \sin \alpha z \]  \hspace{1cm} (3.1)

and:

\[ S_{1,3} = S_1 S_3 \quad T_{1,3} = T_1 T_3 \]
\[ S_{1,4} = S_1 S_4 \quad T_{1,4} = T_1 T_4 \]
\[ S_{2,3} = S_2 S_3 \quad T_{2,3} = S_2 S_3 \]
\[ S_{2,4} = S_2 S_4 \quad T_{2,4} = T_2 T_4 \]  \hspace{1cm} (3.2)

The boundary conditions may now be written in general for the
Z-functions. They are, starting with nondimensional deflection,

\[ Z_o^j = S_{1,3} A_o^j + S_{2,4} B_o^j + S_{2,3} C_o^j + S_{1,4} D_o^j + T_{1,3} E_o^j \]
\[ + T_{2,4} F_o^j + T_{2,3} G_o^j + T_{1,4} H_o^j + Z_p^j \]  \hspace{1cm} (3.3)

where:

\[ Z_p^c = - \frac{C_o \alpha 8}{\mu} \quad (3.4) \]

and:

\[ Z_p^s = - \frac{C_o \beta 8}{\mu} \quad (3.5) \]

for nondimensional slope as
for nondimensional moment as:

\[ Z_2^j = (S_{1,3} \mu^2 \cos^2 \phi - S_{2,4} \mu^2 \sin^2 \phi) A_o^j \]

\[ + (S_{2,4} \mu^2 \cos^2 \phi + S_{1,3} \mu^2 \sin^2 \phi) B_o^j \]

\[ + (S_{2,3} \mu^2 \cos^2 \phi - S_{1,4} \mu^2 \sin^2 \phi) C_o^j \]

\[ + (S_{1,4} \mu^2 \cos^2 \phi + S_{2,3} \mu^2 \sin^2 \phi) D_o^j \]

\[ + (T_{2,3} \mu \sin \phi - T_{1,4} \mu \cos \phi) E_o^j \]

\[ + (T_{1,4} \mu \sin \phi + T_{2,3} \mu \cos \phi) F_o^j \]

\[ + (T_{1,3} \mu \sin \phi - T_{2,4} \mu \cos \phi) G_o^j \]

\[ + (T_{2,4} \mu \sin \phi + T_{1,3} \mu \cos \phi) H_o^j \]

\( j = s \text{ or } c \) \( (3.7) \)
and for nondimensional shear as:

\[ Z_3^j = (S_{2,3} \mu^3 \cos^3 \phi - S_{1,4} \mu^3 \sin^3 \phi) A_0^j \]
\[ + (S_{1,4} \mu^3 \cos^3 \phi + S_{2,3} \mu^3 \sin^3 \phi) B_0^j \]
\[ + (S_{1,3} \mu^3 \cos^3 \phi - S_{2,4} \mu^3 \sin^3 \phi) C_0^j \]
\[ + (S_{2,4} \mu^3 \cos^3 \phi + S_{1,3} \mu^3 \sin^3 \phi) D_0^j \]
\[ + (T_{1,4} \mu^3 \cos^3 \phi - T_{2,3} \mu^3 \sin^3 \phi) E_0^j \]
\[ + (-T_{2,3} \mu^3 \cos^3 \phi - T_{1,4} \mu^3 \sin^3 \phi) F_0^j \]
\[ + (T_{2,4} \mu^3 \cos^3 \phi - T_{1,3} \mu^3 \sin^3 \phi) G_0^j \]
\[ + (-T_{1,3} \mu^3 \cos^3 \phi - T_{2,4} \mu^3 \sin^3 \phi) H_0^j \]

\[ j = s \text{ or } c \quad (3.8) \]

Matching of dimensional boundary conditions is necessary, thus the nondimensional boundary conditions developed in Equations 3.3 through 3.8 must be multiplied by an appropriate constant to be changed to dimensional form. These constants (in parentheses) are:

\[ y' = Y' \left( \frac{F L^3}{C_O EI} \right) \]
\[ y'' = Y'' \left( \frac{F L^2}{C_O EI} \right) \]
\[ EIy''' = Y''' \left( \frac{F}{C_O} \right) \quad (3.9) \]
Similarly for the fourth derivative:

\[ EI y^{IV} = y^{IV} \left( \frac{F_c L}{C_0} \right) \]  

(3.10)

The equations necessary for determining the 16N integration constants for an N-beam structure may now be developed. Eight-N of the relationships are given by Equations 2.20; recall that these equations relate the integration constants of the Z-functions. The remaining equations come directly from boundary conditions. Since both Z-functions are required to satisfy boundary conditions, each boundary condition will yield two equations. The resulting equations are linear, hence the integration constants may be solved for using matrix methods.
CHAPTER IV

EXAMPLES

The Cantilever Beam

The analysis of a cantilever beam is presented to illustrate the method of formulating the matrix of boundary condition equations. A method of illustrating the results of the analysis is also presented.

As stated in Chapter II, \( C_q \) is chosen to be eight. The reason for this is that the nondimensional deflection can then be referred to as the magnification factor. Anderson [5] uses this term in his analysis of the forced vibration of a single degree of freedom system. Thus, in this example, the magnification factor is the deflection, \( y(x,t) \), divided by the static deflection of the end of the beam under the load \( F_0 \) [6].

The boundary conditions for the cantilever beam of Figure 4.1 are:

\[
\begin{align*}
y(0) &= 0 \\
y'(0) &= 0 \\
Ei y''(L) &= 0 \\
Ei y'''(L) &= 0
\end{align*}
\]

(4.1)

These boundary conditions are insured when:

\[
\left( \frac{F_o L^4}{8EI} \right) z_j^0(0) = 0
\]

(4.2)
Figure 4.1. Cantilever Beam
The equations relating integration constants are written in the matrix form: $\{A\} \{B\} = \{C\}$. These are Equations 4.3.

The first eight rows of the A matrix are derived by choosing the appropriate conditions from Equations 3.3 through 3.8. The second eight rows of this matrix are Equations 2.20. Note that the first eight columns are associated with the $Z^c_i$ functions ($i=0,1,2,3$), and the last eight columns are associated with the $Z^s_i$ functions ($i=0,1,2,3$).

The B matrix is a column matrix, the elements of which are the sixteen integration constants. The C matrix is a column matrix whose elements are either zero or the constants $z^p_j$ ($j=s$ or $c$).

The use of certain properties of the free-vibration solution to the viscous damped beam of Stanek [4] and the free vibration solution presented by Anderson [5] results in the derivation of a frequency ratio and a damping ratio for the beam of Figure 4.1. The frequency ratio is

$$\omega \overline{\omega n_1} = \frac{B^2}{\lambda^2}$$

(4.4)
<table>
<thead>
<tr>
<th>( (F_o L^4/8EI)(Z^c_o(0) - Z^c_p) )</th>
<th>( (F_o L^4/8EI)(Z^s_o(0) - Z^s_p) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (F_o L^3/8EI) Z^c_1(0) )</td>
<td>( (F_o L^3/8EI) Z^s_1(0) )</td>
</tr>
<tr>
<td>( (F_o L^2/8) Z^c_2(1) )</td>
<td>( (F_o L^2/8) Z^s_2(1) )</td>
</tr>
<tr>
<td>( (F_o L/8) Z^c_3(1) )</td>
<td>( (F_o L/8) Z^s_3(1) )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

* Blanks Indicate Zeros.
where: \( \omega_n \) = First natural frequency
\( \lambda \) = First eigenvalue

and the damping ratio is:

\[
\frac{C}{C_c} = \frac{\alpha \beta^2}{2\lambda^2}
\]  \hspace{1cm} (4.5)

where: \( C_c \) = Critical damping constant.

Specification of the frequency ratio and the damping ratio allows \( \beta \) and \( \alpha \) to be determined. The results may then be presented in the same manner as Anderson [5] presents the results of a single degree of freedom system. Figure 4.2 is a plot of magnification factor versus frequency ratio for the end of the beam.

When the magnification factor increases to infinity, Figure 4.2, the beam is being forced at a natural frequency. Also, when the forcing frequency is very near a natural frequency, the mode shape associated with the frequency becomes dominant. Thus the analysis results in an approximation of the natural frequencies and the associated mode shapes. As the system becomes more complex, the value of plots similar to Figure 4.2 increases in importance.

The phase angle between the forcing function and motion is defined as

\[
\psi = \tan^{-1} \left( \frac{-Z^c}{Z^g} \right)
\]  \hspace{1cm} (4.6)
Figure 4.2. Magnification Factor vs Frequency Ratio: $\frac{C}{C_C} = 0$. 
Figure 4.3. Phase Angle vs Forcing Frequency
The normalized phase angle \( \left( \frac{\phi}{\sigma} \right) \) versus the forcing frequency is plotted in Figure 4.3.

The curves of Figures 4.2 and 4.3 show a similarity to the curves presented by Anderson [5] in his analysis of the single degree of freedom lumped-mass system.

A Three Beam System

Consider the three beam system of Figure 4.4. The properties of the three beam system are given in Table 4.1. The boundary conditions of the system are expressed as follows. The deflection at each origin is zero; this requires

\[
\left( \frac{F_0^i L_i^4}{E_i I_i} \right) z_{o,i}^j (0) = 0 \quad i = 1, 2, 3 \quad j = s, c \quad (4.7)
\]

The slope of each beam at its origin is zero; this requires:

\[
\left( \frac{F_0^i L_i^3}{E_i I_i} \right) z_{1,i}^j (0) = 0 \quad i = 1, 2, 3 \quad j = s, c \quad (4.8)
\]

The deflections of the ends of the beams, \( z_i = 1 \), are equal; this requires:

\[
\text{---}
\]

\text{3The introduction of the new subscript indicates beam number.}
Figure 4.4. Three-Beam System.
### Table 4.1. Three Beam System Properties

<table>
<thead>
<tr>
<th>Property</th>
<th>Beam Number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td><strong>Length (in)</strong></td>
<td>100</td>
</tr>
<tr>
<td><strong>Height (in)</strong></td>
<td>6</td>
</tr>
<tr>
<td><strong>Width (in)</strong></td>
<td>3</td>
</tr>
<tr>
<td><strong>Moment of Inertia (in^4)</strong></td>
<td>54</td>
</tr>
<tr>
<td><strong>Material</strong></td>
<td>Steel</td>
</tr>
<tr>
<td><strong>Weight per Unit Length (Ib/in)</strong></td>
<td>5.112</td>
</tr>
<tr>
<td><strong>Modulus of Elasticity (Ib/in^2)</strong></td>
<td>30x10^6</td>
</tr>
<tr>
<td><strong>Damping Coefficient (lb-sec/in)</strong></td>
<td>0</td>
</tr>
<tr>
<td><strong>Maximum Load (Ib/in)</strong></td>
<td>40</td>
</tr>
<tr>
<td><strong>Forcing Frequency (Rad/sec)</strong></td>
<td>0-140</td>
</tr>
</tbody>
</table>
26

\[
\left( \frac{F_{01} L_1^4}{E_1 I_1} \right) z_{0,1}^j (1) - \left( \frac{F_{02} L_2^4}{E_2 I_2} \right) z_{0,2}^j (1) = 0
\]

and:

\[
\left( \frac{F_{02} L_2^4}{E_2 I_2} \right) z_{0,2}^j (1) - \left( \frac{F_{03} L_3^4}{E_3 I_3} \right) z_{0,3}^j (1) = 0 \quad (4.9)
\]

The moment at the end of beam one, \( z_1 = 1 \), is zero; this requires:

\[
\left( \frac{F_{01} L_1^2}{E_1 I_1} \right) Z_{2,1}^j (1) = 0 \quad j = s \text{ or } c \quad (4.10)
\]

Figure 4.5 shows a free-body diagram of the end elements of the three beams, \( z_1 = 1 \). The moment relationship at the ends of beams two and three (\( z_2 = z_3 = 1 \)) are given as:

\[
\left( \frac{F_{02} L_2^2}{E_2 I_2} \right) Z_{2,2}^j (1) + \left( \frac{F_{03} L_3^2}{E_3 I_3} \right) Z_{2,3}^j (1) = 0 \quad j = s \text{ or } c \quad (4.11)
\]

The slope relationship at the ends of these beams (\( z_2 = z_3 = 1 \)) is given as:

\[
\left( \frac{F_{02} L_2^3}{E_2 I_2} \right) Z_{1,2}^j (1) + \left( \frac{F_{03} L_3^3}{E_3 I_3} \right) Z_{1,3}^j (1) = 0 \quad j = s \text{ or } c \quad (4.12)
\]

The shears at the ends of the beams (\( z_1 = 1 \)) are related as follows:

\[
\left( \frac{F_{01} L_1}{E_1 I_1} \right) Z_{3,1}^j (1) + \left( \frac{F_{02} L_2}{E_2 I_2} \right) Z_{3,2}^j (1) + \left( \frac{F_{03} L_3}{E_3 I_3} \right) Z_{3,3}^j (1) = 0 \quad j = s \text{ or } c \quad (4.13)
\]
Figure 4.5. End Elements.
The boundary condition equations may now be put into matrix form. The A matrix will be a 48 x 48 matrix. The B matrix will be a 48 x 1 matrix, as will be the C matrix. The method of writing the equations is identical to the method presented for the cantilever beam. The digital computer programs presented in Appendix A can be used to determine numerical results for an N beam system.

Figure 4.6 is a plot of the nondimensional deflection at the end of beam one, $z_1 = 1$, versus the forcing frequency. The first two natural frequencies may be closely approximated by inspection of this curve. Approximations for the mode shapes of the system associated with the first two natural frequencies are shown in Figures 4.7 through 4.12.
Figure 4.6. Nondimension Deflection at $z_1 = 1$ vs Forcing Frequency: $C = 0$
Figure 4.7. Beam one mode shape: First natural frequency.
Figure 4.8. Beam Two Mode Shape: First Natural Frequency.
Figure 4.9. Beam Three Mode Shape: First Natural Frequency.
Figure 4.10. Beam One Mode Shape: Second Natural Frequency.
Figure 4.11. Beam Two Mode Shape: Second Natural Frequency.
Figure 4.12. Beam Three Mode Shape: Second Natural Frequency.
CHAPTER V

DISCUSSION

The formulation presented is applicable to complex systems; limitations in complexity are primarily a result of computer capacity. It is estimated that the XDS Sigma 7 at Montana State University has the capacity to perform the numerical analysis for a 14 beam system with the present core storage available; use of additional equipment can greatly increase this capacity.

Since the deflection of a system becomes large when it is forced to vibrate at a natural frequency, the natural frequencies may be approximated using a plot similar to Figure 4.2. Approximations of the natural mode shapes are also found by forcing with a frequency very near a natural frequency.

Persons involved with approximate methods of analysis may also find use of this solution. The Galerkin Method, for example, requires that deflection shapes be assumed. This method of solution can be used to determine an infinite number of deflection shapes which satisfy a specified set of boundary conditions. The parameters which affect the deflection shape include the forcing frequency, damping coefficients, and the time for which a solution is selected.

It is recognized that the solution as presented is applicable to structures with elements in various planes; however, the fact that
neither axial loading nor twisting effects are included will be a limitation. There is need for a development which will allow these two effects to be included.
Appendix A

Computer Program
PROGRAM FOR FINDING THE SOLUTION TO THE FORCED VIBRATION OF AN N-BEAM STRUCTURE

WRITTEN BY DANIEL F. PRILL SUMMER 1970

A5ME DEPT. MONTANA STATE UNIVERSITY

N IS THE NUMBER OF BEAMS IN THE STRUCTURE
ARRAY IS THE MATRIX OF SIZE NX16,NX16+1
PUT IN COMMON STATEMENTS OF THE FORM:
COMMON/ARRAY/F(16XN,16XN+1)
THIS STATEMENT ALSO GOES IN SUBROUTINES BC, RESULTS,
MATHRITE, AND MATINIT
COMMON/PARTIC/ZPC(N),ZPS(N)
COMMON/CONST/H(16XN)
COMMON/PARAM/A(N),B(N),MEW(N),PHI(N)
THESE ALSO GO IN SUBROUTINE RESULTS
NSN DIMENSION THE FOLLOWING WITH A DIMENSION STATEMENT:
ALPHA(N) AMPLITUDE FACTOR
BETA(N) BEAM PARAMETER
CD(N) CONSTANTS BY WHICH NONDIMENSIONAL DEFLECTION MUST BE MULTIPLIED BY TO BE CHANGED TO DIMENSIONAL DEFLECTION
CSL(N) CONSTANTS WITH WHICH NONDIMENSIONAL SLOPE MUST BE MULTIPLIED BY TO BE CHANGED TO DIMENSIONAL SLOPE
CM(N) CONSTANTS WITH WHICH NONDIMENSIONAL MOMENT MUST BE MULTIPLIED BY TO BE CHANGED TO DIMENSIONAL MOMENT
CS(N) CONSTANTS WITH WHICH NONDIMENSIONAL SHEAR MUST BE MULTIPLIED BY TO
FO(N) ********** MAXIMUM LOAD ON EACH BEAM (LB/IN)
L(N) ********** LENGTH OF EACH BEAM (IN)
E(N) ********** MODULUS OF ELASTICITY OF EACH BEAM (PSI)
C(N) ********** VISCOUS DAMPING COEFFICIENT (LB-SEC/IN)
DE(N) ********** EIGHT OF EACH BEAM PER INCH (LB/IN)
MI(N) ********** CROSS-SECTIONAL MOMENT OF INERTIA (IN**4)

EXTERNAL C'H, SH
COMMON/ARRAY/148,49
COMMON/PARTIC/ZPC(3),ZPS(3)/CONST/H(48)/PARAM/A(3),B(3),MEW(3),
1PHI(3)
REAL MEW,MIL

DIMENSION ALPHA(3),BETA(3),CD(3),CS(3),CM(3),CSL(3),F0(3),L(3),
1E(3),C(3),DE(3),MI(3)
C,H(X)=(EXP(X)-EXP(-X))/2.
SH(X)=(EXP(X)-EXP(-X))/2.

GIVE N A VALUE NOW
N=3

NOW READ THE QUANTITIES FO, L, E, MI, DE FOR EACH BEAM
THE READING FORMAT IS: (5F15.0)
READ(105,10)(FO(I),L(I),E(I),MI(I),DE(I),I=1,N)
FORMAT(5F15.0)

NOW READ C FOR EACH BEAM. THE FORMAT IS: (F15.0)
READ(103,20)(C(I),I=1,N)
FORMAT(F15.0)

NOW READ FF, THE FORCING FREQUENCY (RAD/SEC)

C WE NOW CALCULATE AND WRITE ALL CONSTANTS
DO 20 I=1,N
CD(I)=FO(I)*L(I)**4/E(I)/MI(I)
CSL(I)=CD(I)/L(I)
CM(I)=CSL(I)*E(I)/MI(I)/L(I)
CS(I)=CM(I)/L(I).

41
BETA(I) = (DE(I) * L(I) ** 4 * FF * 2/386/MI(I)/E(I)) ** (0.25)
ALPHA(I) = C(I) * 386/FF/(DE(I) ** L(I))
MEW(I) = BETA(I) * (1 + ALPHA(I) ** 2) ** (0.125)
PHI(I) = ATN2(ALPHA(I), 1.0)/4.0
A(I) = MEW(I) * COS(PHI(I))
B(I) = MEW(I) * SIN(PHI(I))
ZPC(I) = ALPHA(I) ** 2/MEW(I) ** 4
ZPS(I) = BETA(I) ** 2/MEW(I) ** 4
WRITE(108, 40) I, BETA(I), ALPHA(I), MEW(I), PHI(I), FO(I), L(I),
1E(I), MI(I), CD(I), CSL(I), CM(I), CS(I), FO(I)
30 FOR'AT(///20X,'THE PROPERTIES OF BEAM',1X,I,1X'ARE:\/
79: 126X,'BETA=',E15.6,/
80: 225X,'ALPHA=',E15.6,/
81: 327X,'MEW=',E15.6,/
82: 427X,'PHI=',E15.6,/
83: 528X,'FO=',E15.6,1X,'LB PER INCH',/
84: 628X,'L=',E15.6,1X,'INCHES',/
85: 729X,'E=',E15.6,1X,'PSI',/
86: 829X,'I=',E15.6,1X,'INCHES**4',/
87: 928X,'CD=',E15.6,/
88: 927X,'CSL=',E15.6,/
89: 928X,'CM=',E15.6,/
90: 928X,'CS=',E15.6,/
91: 927X,'CL=',E15.6,'1')
92: WRITE(108, 900) FF
93: 900 FOR'AT(///20X,'THE FORCING FREQ. IS ',E15.6,1X,'CPS',/11)
94: C NEW CALL SUBROUTINE MATINIT (MATRIX INITIALIZE)
95: C ITS ONLY ARGUMENT IS THE NUMBER OF BEAMS, N
96: CALL MATINIT(N)
97: C NEW PUT BOUNDARY CONDITIONS IN THE MATRIX LABELED ARRAY
98: C THE SUBROUTINE MATINIT USES ROWS 9=16,25=32,41=48,..,N-8THRU N
99: C DO NOT USE THESE ROWS FOR BOUNDARY CONDITION EQUATIONS
100: C SEE SUBROUTINE BC FOR ITS NOTATION
101: CALL BC(0,0,1,1,1,MEW(I),PHI(I),CD(I))
CALL BC(1,0,1,3,1,MEW(1),PHI(1),CSL(1))
CALL BC(0,1,1,5,1,MEW(1),PHI(1),CD(1))
CALL BC(2,1,1,7,1,MEW(1),PHI(1),CM(1))
CALL BC(3,1,1,39,1,MEW(1),PHI(1),CS(1))
CALL BC(0,1,-1,5,2,MEW(2),PHI(2),CD(2))
CALL BC(0,0,1,17,2,MEW(2),PHI(2),CSL(2))
CALL BC(1,0,1,19,2,MEW(2),PHI(2),CSL(2))
CALL BC(2,1,1,21,2,MEW(2),PHI(2),CM(2))
CALL BC(0,1,1,37,2,MEW(2),PHI(2),CD(2))
CALL BC(3,1,1,39,2,MEW(2),PHI(2),CS(2))
CALL BC(1,1,1,21,3,MEW(3),PHI(3),CSL(3))
CALL BC(2,1,-1,23,3,MEW(3),PHI(3),CM(3))
CALL BC(0,0,1,33,3,MEW(3),PHI(3),CD(3))
CALL BC(1,0,1,35,3,MEW(3),PHI(3),CSL(3))
CALL BC(0,1,-1,37,3,MEW(3),PHI(3),CD(3))
CALL BC(3,1,1,39,3,MEW(3),PHI(3),CS(3))
CALL SC(I,0,1,3,1,MEW(1),PHI(1),CSL(1))
CALL SC(I,1,0,1,5,1,MEW(1),PHI(1),CD(1))
CALL SC(I,2,1,1,7,1,MEW(1),PHI(1),CM(1))
CALL SC(I,3,1,1,39,1,MEW(1),PHI(1),CS(1))
CALL SC(I,0,1,-1,5,2,MEW(2),PHI(2),CD(2))
CALL SC(I,0,0,1,17,2,MEW(2),PHI(2),CSL(2))
CALL SC(I,1,0,1,19,2,MEW(2),PHI(2),CSL(2))
CALL SC(I,2,1,1,21,2,MEW(2),PHI(2),CM(2))
CALL SC(I,0,1,1,37,2,MEW(2),PHI(2),CD(2))
CALL SC(I,3,1,1,39,2,MEW(2),PHI(2),CS(2))
CALL SC(I,1,1,1,21,3,MEW(3),PHI(3),CSL(3))
CALL SC(I,2,1,-1,23,3,MEW(3),PHI(3),CM(3))
CALL SC(I,0,0,1,33,3,MEW(3),PHI(3),CD(3))
CALL SC(I,1,0,1,35,3,MEW(3),PHI(3),CSL(3))
CALL SC(I,0,1,-1,37,3,MEW(3),PHI(3),CD(3))
CALL SC(I,3,1,1,39,3,MEW(3),PHI(3),CS(3))
NOW BUILD THE 16XN+1 COLUMN OF ARRAY
F(1,49)=CD(1)*ZPC(1)
F(2,49)=CD(1)*ZPS(1)
F(5,49)=CD(1)*ZPC(1)+CD(2)*ZPC(2)
F(6,49)=CD(1)*ZPS(1)+CD(2)*ZPS(2)
F(17,49)=CD(2)*ZPC(2)
F(18,49)=CD(2)*ZPS(2)
F(33,49)=CD(3)*ZPC(3)
F(34,49)=CD(3)*ZPS(3)
F(37,49)=CD(2)*ZPC(2)+CD(3)*ZPC(3)
F(38,49)=CD(2)*ZPS(2)+CD(3)*ZPS(3)
SUBROUTINE MAWRITE MAY BE CALLED AT THIS POINT
CALL IT WILL DO IS WRITE THE MATRIX ARRAY
NOW CALL THE SUBROUTINE RESULTS(N)
ITS ONLY ARGUMENT IS THE NUMBER OF BEAMS (N)
RESULTS CALCULATES AND WRITES EVERYTHING FOR THE END OF BEAM 1
CALL RESULTS(N)
SUBROUTINE BC(N1,L,J1,IR,NB,MEW PHI,C)

CHANGE THE COMMON CARD TO THE APPROPRIATE FORM. SEE MAINLINE

COMMON/ARRAY/F(48,43)

THIS SUBROUTINE CALCULATES BOUNDARY CONDITIONS BY

CALCULATING 16 COLUMNS AND 1 ROW FOR EACH CALL.

ARRANGE THE MATRIX ARRAY SO THAT BEAM 1 EQUATIONS ARE IN

COLUMNS 1-8 FOR THE ZC FUNCTION & COLUMNS 9-16 FOR

THE ZS FUNCTION. THEN BEAM 2 EQUATIONS GO INTO THE

NEXT 16 COLUMNS (17-32). THEN THE ZC FUNCTIONS FOR

BEAM 2 GO INTO COLUMNS 17-24 & THE ZS FUNCTION IN

COLUMNS 25-32....ETC THRU BEAM N

THE ARGUMENTS OF THE SUBROUTINE ARE:

N1....ORDER OF DERIVATIVE, IS EITHER 0,1,2,3
L....POINT ON BEAM AT WHICH EVALUATION OCCURS; 0 OR 1
J1....SIGN OF BOUNDARY CONDITION; 1 OR -1
IR....ROW IN WHICH BOUNDARY CONDITION IS TO BE PLACED
NB....BEAM NUMBER; 1,2,3,...,N
MEW....JUST PUT IN MEW(NB); WHERE NB IS BEAM NUMBER
PHI....JUST PUT IN PHI(NB); WHERE NB IS BEAM NUMBER
C....CONSTANT WHICH MAKES THE BOUNDARY CONDITION

DIMENSIONAL

IF N1=0, USE CD(NB)
IF N1=1, USE CSL(NB)
IF N1=2, USE CM(NB)
IF N1=3, USE CS(NB)

REAL MEW

IF(L.EQ.0)Z=0.0
IF(L.EQ.1)Z=1.0
A=MEW*CBS(PHI)
B=MEW*SIN(PHI)
S13=CH(A*Z)*COS(B*Z)*J1
S14=CH(A*Z)*SIN(B*Z)*J1
S23 = S^u(A*Z)*C8S(B*Z)*J1
S24 = S^u(A*Z)*S^n(B*Z)*J1
T13 = C^(h)(B*Z)*C8S(A*Z)*J1
T14 = C^(h)(B*Z)*S^n(A*Z)*J1
T23 = S^u(B*Z)*C8S(A*Z)*J1
T24 = S^u(B*Z)*S^n(A*Z)*J1
IC = B*16-15
IR = IR+1
ICS = IC+8
IF(1,EG,0)G8 T6 40
G9 T6 (15,20,30),N1
10 A1 = S23*A-S14*B
A2 = S14*A+S23*B
A3 = S13*A-S24*B
A4 = S24*A+S13*B
A5 = T23*3-T14*A
A6 = T14*B+T23*A
A7 = T13*3-T24*A
A8 = T24*3+T13*A
S5 T6 50
20 A = MEW*2*C8S(2*PHI)
B = MEW*2*SIN(2*PHI)
A1 = S13*A+S24*B
A2 = S24*A+S13*B
A3 = S23*A-S14*B
A4 = S14*A+S23*B
A5 = T24*B-T13*A
A6 = T13*3-T24*A
A7 = T14*B+T23*A
A8 = T23*3+T14*A
S9 T6 50
30 A = MEW*3*C8S(3*PHI)
B = MEW*3*SIN(3*PHI)
A1 = S23*A-S14*B
\begin{verbatim}
68:  A2=S14*A+S23*B
69:  A3=S13*A-S24*B
70:  A4=S24*A+S13*B
71:  A5=T14*A-T23*B
72:  A6=-T23*A-T14*B
73:  A7=T24*A-T13*B
74:  A8=-T13*A-T24*B
75:  G9 T0 5C
76:  40 A1=S13
77:  A2=S24
78:  A3=S23
79:  A4=S14
80:  A5=T13
81:  A6=T24
82:  A7=T23
83:  A8=T14
84:  50 F(I,R,I,C)=F(I,R,S,I,C)=A1*C
85:  F(I,R,I,C+1)=F(I,R,S,I,C+1)=A2*C
87:  F(I,R,I,C+3)=F(I,R,S,I,C+3)=A4*C
88:  F(I,R,I,C+4)=F(I,R,S,I,C+4)=A5*C
89:  F(I,R,I,C+5)=F(I,R,S,I,C+5)=A6*C
91:  F(I,R,I,C+7)=F(I,R,S,I,C+7)=A8*C
92:  RETURN
93:  END
\end{verbatim}
SUBROUTINE ZF(N,M,A1,Z)

WRITE BY DANIEL F. PRILL SUMMER 1970

CALCULATES THE Z-FUNCTIONS

REAL M,N,

DIMENSION F1(I,2),M(2,2),N(2,2),G1(2,1),F2(1,2),G2(2,1),A1(8)

1,D(2,2),Z1(1,2),Z2(1,1)

F1(1,1)=A1(1)
F1(1,2)=A1(2)
G1(1,1)=A1(3)
G1(2,1)=A1(4)
F2(1,1)=A1(5)
F2(1,2)=A1(6)
G2(1,1)=A1(7)
G2(2,1)=A1(8)

CALL MAMUL(F1,M,Z1,1,2,2)
CALL MAMUL(Z1,G1,Z2,1,2,1)
Z=Z+Z2(1,1)
CALL MAMUL(F2,N,Z1,1,2,2)
CALL MAMUL(Z1,G2,Z2,1,2,1)
Z=Z+Z2(1,1)
RETURN
END
SUBROUTINE RESULTS(NBS)

WRITTEN BY DANIEL F. PRILL     SUMMER 1970

CALCULATES ALL RESULTS FOR THE ENDS OF THE BEAMS

NEED TO CHANGE THE COMMON CARD. SEE MAINLINE

COMMON/ARRAY/F(48,49)
COMMON/PARTIC/APC(3),ZPS(3),&CONST/H(48),PARAM/A(3),B(3),MEW(3),

PHI(3)

DIMENSION M(2,2),N(2,2),M1(2,2),N1(2,2),X(201),A1(8),D1(2,2),

S(4),S1(4)

REAL M,N,M1,N1

DO 5 J=1,201

X(J)=(J-1)/200.

CONTINUE

I=NBS*16

I=I+1

CALL SBLTN(F,H,I-I)

11=1

12=2

13=3

14=4

15=5

16=6

17=7

18=8

19=9

20=10

21=11

22=12

23=13

24=14

25=15

26=16

DO 100 I=1,NBS
IF (1 > 3) G0 TO 10

11 = 1 + 16
12 = 12 + 16
13 = 13 + 16
14 = 14 + 16
15 = 15 + 16
16 = 16 + 16
17 = 17 + 16
18 = 18 + 16
19 = 19 + 16
20 = 20 + 16
21 = 21 + 16
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66 = 66 + 16
67 = 67 + 16

WRITE (10, 60) I
60 FORMAT(////40X,'BEAM'1X,I3,1X,'C9NSTANTS ARE;////)  
61: J=16*I  
62: J1=J-15  
63: WRITE(108,70)(H(J2),J2=J1,J)  
64: 70 FORMAT(8E15.6)  
65: WRITE(109,90)I  
66: 80 FORMAT('1',5X,'THE DEFLECTION AND DERIVATIVE NUMBERS 1 THRU 4 !  
67: 1 FOR BEAM'1X,I3,1X,'ARE PRINTED BELOW'1/5X,'EVERYTHING IS IN !  
68: 2 IN N0DIMENSIONAL FORM'1/5X,'NUMBERS START AT N0N0DIMENSIONAL !  
69: 3 LENGTH 0F ZEROS'///)  
70: IF(I*GT*1)G0 T0 30  
71: C1=A(I)  
72: C2=B(I)  
73: A1(1)=CH(C1)  
74: A1(2)=SH(C1)  
75: A1(3)=COS(C2)  
76: A1(4)=SIN(C2)  
77: A1(5)=CH(C2)  
78: A1(6)=SH(C2)  
79: A1(7)=COS(C1)  
80: A1(8)=SIN(C1)  
81: YMAX=Q.Q  
82: CALL ZFJN(M,N,A1,'ZC)  
83: ZC=ZC-ZFC(I)  
84: CALL ZFJN(M1,N1,A1,ZS)  
85: ZS=ZS-ZPS(I)  
86: T  =  3.14159*(J-I)/ISO,  
87: Y0=ZC*COS(T)+ZS*SIN(T)  
88: IF(AES(Y0).GT.YMAX)YMAX=ABS(Y0)  
89: T= T+1  
90: CONTINUE  
91: T=T+1  
92: CONTINUE  
93: D5 20 J=1,181  
94: T=3*(J-I)/180,  
95: Y0=ZC*COS(T)+ZS*SIN(T)  
96: IF(AES(Y0).GT.YMAX)YMAX=ABS(Y0)  
97: 20 CONTINUE  
98: T=T+1  
99: D5 100 J=1,201  
100: C1=A(I)*X(J)
C2 = P(I) * X(J)
A1(1) = CH(C1)
A1(2) = SH(C1)
A1(3) = COS(C2)
A1(4) = SIN(C2)
A1(5) = CH(C2)
A1(6) = SH(C2)
A1(7) = COS(C1)
A1(8) = SIN(C1)

CALL ZFUN(M, N, A1, ZC)
ZC = ZC - ZPC(I)

CALL ZFUN(M1, N1, A1, ZS)
ZS = ZS - ZPS(I)

DB 40 J1 = 1, 4

CALL RECUR(J1, MEW(I), PHI(I), M, N, D1, D2)

CALL ZFUN(D1, D2, A1, ZJC)

S(J1) = ZJC

CALL RECUR(J1, MEW(I), PHI(I), M1, N1, D1, D2)

CALL ZFUN(D1, D2, A1, ZJC)

S1(J1) = ZJC

YO = ZC * COS(T) + ZS * SIN(T)
Y1 = S(1) * COS(T) + S(1) * SIN(T)
Y2 = S(2) * COS(T) + S(1) * SIN(T)
Y3 = S(3) * COS(T) + S(1) * SIN(T)
Y4 = S(4) * COS(T) + S(1) * SIN(T)

WRITE(108, 400) Y0, Y1, Y2, Y3, Y4

WRITE(108, 90) T1

SUBROUTINE MATWRITE(NBS)
C WRITTEN BY DANIEL F. PRILL  SUMMER 1970
C WRITES THE MATRIX, ARRAY
C NEED TO CHANGE THE COMMON CARD
COMMON/ARRAY/F(48,49)
K=NBS*16
K1=K+1
WRITE(108,10)K,K1
10 FORMAT('11/,///,20X,'THE MATRIX TO BE SOLVED FOLLOWS:/',
120X,'THE MATRIX IS OF SIZE: I3,I3,I3,I3, I3,'/) DO 20 I=1,K
20 WRITE(108,30)I
30 FORMAT('//,40X,'REW: I3//') WRITE(1C8,40)(F(I,J),J=1,K) CONTINUE
40 FORMAT('8E15.6') 20 CONTINUE
WRITE(1C8,50)K1
50 FORMAT('//,40X,'THE',I3,I3,I3,'TH COLUMN IS:/'
52 WRITE(1C8,40)(F(J,K1),J=1,K) WRITE(1C8,60)
60 FORMAT('1') RETURN
END
SUBROUTINE RECUR(N1,MEW,PHI,M,N,D1,D2)

WRITTEN BY DANIEL F. PRILL SUMMER 1970

CALCULATES INTEGRATION CONSTANTS FOR EACH DERIVATIVE

REAL J,K,M,N,MEW

DIMENSION J(2,2),K(2,2),M(2,2),N(2,2),D(2,2),E(2,2),D1(2,2),
D2(2,2),D3(2,2),D4(2,2)

J(1,1)=0.0
J(1,2)=1.0
J(2,1)=1.0
J(2,2)=0.0

K(1,1)=0.0
K(1,2)=-1.0
K(2,1)=1.0
K(2,2)=0.0

CALL MATC(A,MEW**2*COS(2*PHI))
CALL MATC(B,MEW**2*SIN(2*PHI))
CALL MAMUL(J,M,D,2,2,2,2)
CALL MAMUL(M,K,E,2,2,2,2)
CALL MATC(A,D*D3,2,2,2,2)
CALL MATC(B,E*D4,2,2,2,2)
CALL MATC(A,M*E*D3,2,2,2,2)
CALL MATC(B,E*D4,2,2,2,2,2)
CALL MATC(A,D*D3,2,2,2,2)
CALL MATC(A,M*E*D3,2,2,2,2)
CALL MATC(A,D*D3,2,2,2,2)
CALL MATC(A,M*E*D3,2,2,2,2)

END
CALL MATC(B,E,D4,2,2)
CALL MATA(D3,D4,D1,2,2)
A = -A
CALL MAMUL(J,N,D,2,2,2)
CALL MAMUL(D,K,E,2,2,2)
CALL MATC(A,N,D3,2,2)
CALL MATC(B,E,D4,2,2)
CALL MATA(D3,D4,D2,2,2)

G9 TO 5C

30 A = NEW**3*COS(3*PHI)  
B = NEW**3*SIN(3*PHI)  
CALL MAMUL(J,N,D,2,2,2)  
CALL MAMUL(M,K,E,2,2,2)  
CALL MATC(A,D,D3,2,2)  
CALL MATC(B,E,D4,2,2)  
CALL MATA(D3,D4,D1,2,2)
A = -A  
B = -B
CALL MAMUL(J,N,D,2,2,2)  
CALL MAMUL(N,K,E,2,2,2)  
CALL MATC(B,D,D3,2,2)  
CALL MATC(A,E,D4,2,2)  
CALL MATA(D3,D4,D2,2,2)
G9 TO 5D

40 A = NEW**4*COS(4*PHI)  
B = NEW**4*SIN(4*PHI)  
CALL MAMUL(J,M,D,2,2,2)  
CALL MAMUL(D,K,E,2,2,2)  
CALL MATC(A,M,D3,2,2)  
CALL MATC(B,E,D4,2,2)  
CALL MATA(D3,D4,D1,2,2)
B = -B
CALL MAMUL(J,N,D,2,2,2)  
CALL MAMUL(D,K,E,2,2,2)
CALL MATC(A, N, D3, 2, 2)
CALL MATC(B, E, D4, 2, 2)
CALL MATA(D3, D4, D2, 2, 2)
50 RETURN
END
SOLUTION OF SIMULTANEOUS EQUATIONS BY GAUSSIAN ELIMINATION

N  NUMBER OF SIMULTANEOUS EQUATIONS
L = N - 1
A(I,J)  ELEMENTS OF THE AUGMENTED MATRICES
I  MATRIX ROW NUMBER
J  MATRIX COLUMN NUMBER
JJ TAKES ON VALUES OF THE ROW NUMBERS WHICH ARE POSSIBLE PIVOT ROWS, EVENTUALLY TAKING ON THE VALUE IDENTIFYING THE ROW HAVING THE LARGEST PIVOT ELEMENT
BIG TAKES ON VALUES OF THE ELEMENTS IN THE COLUMN CONTAINING POSSIBLE PIVOT ELEMENTS, EVENTUALLY TAKING ON THE VALUE OF THE ELEMENT USED
TEMP  TEMPORARY NAME USED FOR THE ELEMENTS OF THE ROW SELECTED TO BECOME THE PIVOT ROW, BEFORE THE INTERCHANGE IS MADE
K  INDEX OF A DO LOOP TAKING ON VALUES FROM 1 TO N - 1; IT IDENTIFIES THE COLUMN CONTAINING POSSIBLE PIVOT ELEMENTS
KP1  K + 1
AB  ABSOLUTE VALUE OF A(I,K)
GUO  QUOTIENT A(I,K)/A(K,K)
X(I)  UNKNOWN OF THE SET OF EQUATIONS BEING SOLVED
SUM  SUMMATION OF A(I,J)*X(J) FROM J=I+1 TO N
IN Index of a DO LOOP TAKING ON VALUES FROM 1 TO N - 1
IP1  I + 1
SUBROUTINE SOLTN(A,X,N,M)
DIMENSION A(N,M),X(N)
L = N - 1
DO 12 K = 1,L
JJ = K
BIG = ABS(A(K,K))
KP1 = K + 1
DO 7 I = KP1,N
AB = ABS(A(I,K))
34: IF (BIG-AB) 6,7,7
35: 6 BIG=AB
36: JJ=I
37: 7 CONTINUE
38: IF (JJ-K) 8,10,8
39: 8 DO 9 J=K,M
40: TEMP = A(JJ,J)
41: A(JJ,J)= A(K,J)
42: 9 A(K,J) = TEMP
43: 10 DO 11 I=KP1,N
44: GUBT = A(I,K)/A(K,K)
45: 11 DO 12 J=KP1,M
46: A(I,J)=A(I,J)-GUBT*A(K,J)
47: 12 A(I,K)=0.*
48: X(N)=A(N,M)/A(N,N)
49: 13 DO 14 NN=1,L
50: SUM=0.*
51: 14 I=N+NN
52: 15 IP1=I+1
53: 16 DO 13 J=IP1,N
54: SUM=SUM+A(I,J)*X(J)
55: 17 X(I)=(A(I,M)-SUM)/A(I,I)
56: 18 RETURN
57: END
SUBROUTINE MATINIT(NBS).

WRITTEN BY DANIEL F. PRILL SUMMER 1970

INITIALIZES THE MATRIX LABELED ARRAY

NEED TO CHANGE COMMON CARD. SEE MAINLINE

COMMON/ARRAY/FR(48,49)

M=NBS*16

M1=M+1

DO 10 I=1,M

DO 10 J=1,M1

10 F(I,J)=0.0

DO 100 I=1,NBS

M=I*16

F(M-7,M-14)=F(M-5,M-12)=1.0

F(M-2,M-11)=F(M,M-9)=1.0

F(M-6,M-15)=F(M-4,M-13)=1.0

F(M-3,M-10)=F(M-1,M-8)=1.0

DO 100 J=1,NBS

J1=J-8

J2=M+J1

100 F(J2,J1)=1.0

RETURN

END
SUBROUTINE VAMUL(A,B,C,M,N,MM)

WRITTEN BY DANIEL F. PRILL SUMMER 1970

THIS SUBROUTINE IS FOR THE COMPUTATION OF A MATRIX C FROM C = A * B, WHERE C IS AN M*MM MATRIX, A IS AN M*N, AND B IS AN N*MM.

DIMENSION A(M,N),B(N,MM),C(M,MM)

DO 3 I=1,M
    DO 3 J=1,MM
    C(I,J)=C
    DO 3 K=1,N

3 C(I,J)=C(I,J)+A(I,K)*B(K,J)

RETURN

END
SUBROUTINE MATC(A, C, D, I, J)

WRITTEN BY DANIEL F. PRILL SUMMER 1970

SUBROUTINE MULTIPLIES CONSTANT A TIMES MATRIX C

DIMENSION C(2,2), D(2,2)

DB 10 I=1, I
DB 10 J=1, J

10 D(I, J) = A * C(I, J)

RETURN

END
SUBROUTINE MATA(A,B,C,I,J)

WRITTEN BY DANIEL F. PRILL    SUMMER 1970

THIS SUBROUTINE ADDS MATRIX A TO MATRIX B, C = A + B

DIMENSION A(2,2), B(2,2), C(2,2)

DO 10 M=1, I
    DO 10 N=1, J
  10 C(M,N) = A(M,N) + B(M,N)

RETURN

END
LITERATURE CONSULTED


The forced vibration of viscous damped, N-beam structures