Comparison of numerical approximation methods for the solution of first order differential equations
by Leon J D Rouge

A THESIS Submitted to the Graduate Faculty In partial fulfillment of the requirements for the degree of Master of Science In Applied Mathematics
Montana State University

Abstract:
Since the solution of an nth order differential equation can be reduced to the solution of a system of first order differential equations, we shall concern ourselves only with the solution of the latter. In our discussion we shall consider the basic conditions needed to assure a solution by Picard’s Method, a short description of the method, determination of the error inherent in the method, extensions of the method within the region of convergence, procedures to minimize errors, and an illustration of its application.

Parallelling Picard's Method, we shall analyze the method of Taylor's series. In similar manner the difference methods are presented, pointing out in particular that, although these methods are more accurate than the analytic methods such as Picard's and Taylor's, they are step-by-step numerical approximations and, unless the results are fitted into an expression such as Newton's formula, they cannot produce a solution in analytic form.

An analytic discussion follows, outlining recommended procedures to be followed in the solution of first order differential equations, emphasizing the need for careful preliminary analysis of error terms, spacings, and the inherent characteristics of each method in order to lead most efficiently to desired results.

As an illustration of the problems encountered and the accuracy obtained by each method, an example is presented and results compared.

COMPARISON OF NUMERICAL APPROXIMATION
METHODS FOR THE SOLUTION OF
FIRST ORDER DIFFERENTIAL EQUATIONS

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I. INTRODUCTION

Given an nth order differential equation of the form

$$y^{(n)}(x) = f(x,y,y', \ldots y^{(n-1)})$$

subject to initial conditions $y(x_o) = y_o$, $y'(x_o) = y'_o, \ldots y^{(n-1)}(x_o) = y_o^{(n-1)}$. By introduction of parameters $y_1, y_2, \ldots y_{n-1}$, the equation can be reduced to the system

$$y' = y_1,\ y_1' = y_2, \ldots y'_{n-1} = f(x,y,y_1,\ldots y_{n-1})$$

each of which is a differential equation of first order.

For example, the second order differential equation

$$y'' = f(x,y,y')$$

with initial conditions $y(x_o) = y_o$, $y'(x_o) = y'_o$, by the substitution $y'(x) = p(x)$, reduces to the system

$$p'(x) = f(x,y,p)$$

$$y'(x) = p(x)$$

Thus, our basic problem is the solution of first order differential equations of the form

$$y'(x) = f(x,y)$$

with initial condition $y(x_o) = y_o$. In our discussion we shall consider the basic conditions needed to assure a solution by each method presented, a short description of each method, an analysis of the accuracy obtained, and an analytic comparison of the various methods. Finally, as an illustration of the application of each method, a problem is presented.

II. PICARD'S METHOD

1. Existence of solution. The existence and uniqueness of a solution $y(x)$ of the differential equation

$$y' = f(x,y) \tag{1}$$

with initial condition $y(x_0) = y_0$ are guaranteed if the following three basic assumptions are satisfied:

(a) $f(x,y)$ is singlevalued in a region R of the xy-plane which includes the point (x_0,y_0).

(b) $f(x,y)$ is continuous in R, hence

$$|f(x,y)| \leq M \tag{2}$$

for all (x,y) in R.

(c) $f(x,y)$ satisfies a Lipschitz condition for any two points (x,y_1) and (x,y_2) in R:

$$|f(x,y_2) - f(x,y_1)| < K \tag{3}$$

where K is a constant dependent on $f(x,y)$ but independent of (x,y_1) and (x,y_2).

We define the sequence of functions:

$$y_0(x) = y_0 \tag{4}$$
$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(x, y_n(x))\,dx \qquad n = 1, 2, 3, \ldots$$

We will show that for x within a certain interval this sequence has a limit as $n \to \infty$, which satisfies (1).

From (4) it follows:

(5) $$|y_1(x) - y_0| = \left|\int_{x_0}^{x} f(x,y_0)\,dx\right| \leq \int_{x_0}^{x} |f(x,y_0)|\,dx \leq \int_{x_0}^{x} M\,dx \leq M|x-x_0|$$

Hence if we choose a constant α small enough, then

(6) $$|x-x_0| < \alpha, \quad |y_1(x) - y_0| < M\alpha$$

defines a new region R* about (x_0, y_0) which will be contained in R.

By induction we then obtain from (4)

(7) $$|y_n(x) - y_0| \leq \int_{x_0}^{x} |f(x, y_{n-1}(x))|\,dx \leq M|x-x_0| < \beta$$

so that $y_n(x)$ will remain in R* for $|x - x_0| < \alpha$.

To prove $\lim_{n\to\infty} y_n(x) = y(x)$ for all x in $|x - x_0| < \alpha$, observe that

(8) $$|y_2(x) - y_1(x)| \leq \int_{x_0}^{x} |f(x,y_1(x)) - f(x,y_0)|\,dx \leq \int_{x_0}^{x} K|y_1(x) - y_0|\,dx$$
$$\leq \int_{x_0}^{x} KM|x-x_0|\,dx = KM\frac{|x-x_0|^2}{2!}$$

Similarly

(9) $$|y_n(x) - y_{n-1}(x)| \leq K^{n-1}M\frac{|x-x_0|^n}{n!}$$

Since:

(10) $$|y_n(x) - y_0| \leq |y_n(x) - y_{n-1}(x)| + |y_{n-1}(x) - y_{n-2}(x)| + \cdots + |y_1(x) - y_0|$$

we evidently have

(11) $$|y_n(x) - y_0| \leq \sum_{k=1}^{n} MK^{k-1}\frac{|x-x_0|^k}{k!}$$