Derivation of the equations of motion of a dissipative hydrodynamic system by a guage transformation of the lagrangian density
by George Hiroshi Saito

A thesis submitted to the Graduate Faculty in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE in Physics
Montana State University

Abstract:
This investigation is concerned with the derivation of the equations of motion of a real fluid from a Lagrangian density function. Dissipation of energy by viscous forces is introduced into the Lagrangian density function hy a hydrodynamic gauge transformation of the second kind analogous to the transformation used in electrodynamic systems. The equations of motion for the viscous fluid are analyzed, and are found consistent with the Navier-Stokes equation when the fluid flow is taken to he irrotational, the ratio of the fluid velocity to the velocity of sound is small, and the terms quadratic in the viscous coefficient are negligible.

The effects of the proposed transformation on the hydrodynamic field are analyzed through construction of the various densities- associated with the field. Employing the conservation of momentum, the Navier-Stokes equation for a viscous fluid is obtained. From this result, it is concluded that the validity of the hydrodynamic gauge transformation used to introduce viscosity into the equations of motion of a real fluid is established.

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ABSTRACT

This investigation is concerned with the derivation of the equations of motion of a real fluid from a Lagrangian density function. Dissipation of energy by viscous forces is introduced into the Lagrangian density function by a hydrodynamic gauge transformation of the second kind analogous to the transformation used in electrodynamic systems. The equations of motion for the viscous fluid are analyzed, and are found consistent with the Navier-Stokes equation when the fluid flow is taken to be irrotational, the ratio of the fluid velocity to the velocity of sound is small, and the terms quadratic in the viscous coefficient are negligible.

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I

INTRODUCTION

The purpose of the study undertaken in this thesis is to construct a Lagrangian density function which provides the equations of motion of a dissipative, classical hydrodynamic system. Equations of motion obtained from the Lagrangian density function by means of the variational principle must be consistent with the Navier-Stokes equation of hydrodynamics. This is the criteria which will be used to determine the correct form of the Lagrangian density function for a real fluid.

For the treatment of a conservative hydrodynamic system, the variational principle has been used with success. However, for a nonconservative system in which there is a dissipation of energy, the treatment of the system is not as well defined. The difficulty arises in the formation of an appropriate Lagrangian density function for the system. Presently, there is no accepted, standard technique for the treatment of a general dissipative energy system by a variational principle. In a real fluid, it is known that the viscous stresses result in mechanical energy being dissipated from the system. This dissipative process takes place irreversibly as a continual degradation of system energy.

In this thesis, the attempt is made to introduce viscosity into the equations of motion of an irrotational fluid by considering the analogy between dissipative processes in hydrodynamics and electrodynamics and by employing a gauge transformation. This transformation, following electrodynamics, is called a gauge transformation of the second kind. It is shown that the transformation does not change the wave functions of a quantum hydrodynamic representation of the system.

To avoid confusion, a discussion of the established gauge transformation of hydrodynamics is included which is denoted as a gauge transformation of the first kind, in keeping with the electrodynamic nomenclature. In the following section, the discussion begins with a review of variational techniques.

II

FORMULATION OF THE VARIATIONAL PRINCIPLE

Introduction

Variational methods coupled with a canonical formalism are very versatile and have enjoyed widespread application. In addition, a classical canonical formalism for a system is a necessary prerequisite for the subsequent quantization of the system, Wentzel[1], Schiff[2], and Ito[3].

For this study, the techniques of the variational principle will be applied to attempt a classical canonical formalism of a dissipative hydrodynamic system. However, before developing the field equations from a generalized Lagrangian density, the methods used in classical mechanics will be described briefly, see for example Goldstein[4], Landau and Lifshitz[5], and Corben and Stehle[6].

Discrete Mechanical System

The equations of motion of classical particle mechanics are obtained from the Principle of Least Action or Hamilton's Principle which states that an action integral of the form

$$I = \int_{t_1}^{t_2} L \, dt \quad , \qquad (1)$$

has an extremum for the path of the motion in the time interval from t_1 to t_2, and the integrand function is defined to be the Lagrangian for the system.

The condition for the integral to be stationary is given by

$$\delta \int_{t_1}^{t_2} L\left(q, \frac{\partial q}{\partial t}, t\right) dt = 0 \quad ,$$

where it has been assumed that the Lagrangian is a function only of the generalized coordinate q and the first time derivative of the generalized coordinate. Carrying out the variation, the equation of motion for the system is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \quad ,$$

with

$$\dot{q} = \frac{\partial q}{\partial t} \quad .$$

For a system with n degrees of freedom, as given by the number of generalized coordinates, the Lagrangian must be varied independently with respect to each of the generalized coordinates. For n generalized coordinates, the equations of motion for a discrete, conservative mechanical system are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad ,$$

$$i = 1, 2, \cdots, n \quad .$$

The above set of second order equations represents the Euler-Lagrange differential equations for the system.

When the potential energy is a function of the generalized coordinates and the kinetic energy is quadratic in the generalized velocity coordinates, the Lagrangian can be expressed as

$$L = T - V \quad ,$$

where T is the kinetic energy,

and V is the potential energy.

When expressed in generalized coordinates, the Lagrangian formulation is invariant with respect to the coordinate system[4].

Hamilton's principle may be extended to nonconservative systems[4, 5] and is given in the form

$$\delta \int_{t_1}^{t_2} (T + W)\, dt = 0 \quad ,$$

where T is the kinetic energy,

and W is the virtual work.

In this form, it is required that the first variation of the integral of the kinetic energy and the virtual work done by the forces on the system during the displacement must vanish. Other forms of the equations of motion for nonconservative and nonholonomic mechanical systems are well established in the literature[4, 5, 6].

Continuous Mechanical System

The treatment of the transition from a discrete mechanical system to a continuous mechanical system is given by Goldstein[4].

For this type of system, the action intergral to be varied has the form

$$I = \int \mathcal{L} \; dx_1 \, dx_2 \, dx_3 \, dx_4 \quad , \qquad (2)$$

where x_i represents x, y, z, and t.

The integrand function in equation (2) is often called the differential Lagrangian function or the Lagrangian density function since the Lagrangian can be expressed in the form

$$L = \int \mathcal{L} \; dx_1 \, dx_2 \, dx_3 \quad .$$

The significant difference between the Lagrangian density function and the integrand function of equation (1) is the role of the independent variables. For a discrete system, the independent variables are the generalized coordinates. In the transition to a continuous mechanical system, properties of the system become described by independent field variables with the coordinates of configuration space serving, to use an analogy, as a sort of book-keeping system to keep track of the independent variables. The coordinates no longer have the significance that they previously did in the description of a discrete, mechanical system.

The variational formalism may also be used to obtain the description of fields, Wentzel[1], and Landau and Lifshitz[7]. A classical field is specified by one or more independent functions of