Approximation of eigenvalues of Sturm-Liouville differential equations by the sinc-collocation method
by Mary Katherine Jarratt

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics
Montana State University

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A collocation method with sinc function basis elements is developed to approximate the eigenvalues of regular and singular Sturm-Liouville boundary value problems. The method generates a matrix system Ax = &lamba;Bx with A and B both symmetric and positive definite matrices. In the case of singular problems, the matrix norm of B-1 is unbounded; however, the error in the eigenvalue approximation is shown to converge at a fate of exp(-a√N) for a positive and 2N + 1 basis elements for both regular and singular problems. This is done by circumventing the standard error analysis which typically involves ||B-1||.

Test examples of the Sturm-Liouville differential equations given on finite and infinite intervals are included to illustrate the accuracy and implementation of the method. Several of the test examples are used to compare the performance of the sinc-collocation method to other numerical methods.

APPROXIMATION OF EIGENVALUES OF STURM-LIOUVILLE DIFFERENTIAL EQUATIONS BY THE SINC-COLLOCATION METHOD

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A thesis submitted in partial fulfillment
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of

Doctor of Philosophy

in

Mathematics

MONTANA STATE UNIVERSITY
Bozeman, Montana

May, 1987

APPROVAL

of a thesis submitted by

Mary Katherine Jarratt

This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

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ACKNOWLEDGMENTS

I would like to thank Dr. Ken Bowers for the help, support, and encouragement that he gave me as my advisor and friend as I worked on my degree. My thanks also go to Dr. John Lund who introduced me to sinc functions and continually encouraged me during these past five years.

My appreciation is extended to Dr. Norm Eggert for his help and comments as my thesis progressed, and to Dr. Richard Gillette for his careful reading of my thesis.

I thank Dr. Frank Stenger for his interest in me, his encouragement in my work, and his helpful discussions with me about sinc functions.

I also want to thank Rene' Tritz for the excellent typing job she did.

And finally, though not least of all, I want to thank my family for their support and belief in me that this thesis and degree would become reality.

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ABSTRACT

A collocation method with sinc function basis elements is developed to approximate the eigenvalues of regular and singular Sturm-Liouville boundary value problems. The method generates a matrix system $Ax = \lambda Bx$ with A and B both symmetric and positive definite matrices. In the case of singular problems, the matrix norm of B^{-1} is unbounded; however, the error in the eigenvalue approximation is shown to converge at a rate of $\exp(-a\sqrt{N})$ for a positive and 2N + 1 basis elements for both regular and singular problems. This is done by circumventing the standard error analysis which typically involves $\|B^{-1}\|$.

Test examples of the Sturm-Liouville differential equations given on finite and infinite intervals are included to illustrate the accuracy and implementation of the method. Several of the test examples are used to compare the performance of the sinc-collocation method to other numerical methods.

CHAPTER 1

INTRODUCTION

The Sturm-Liouville equation is a second order linear differential equation of the form

$$(1.1) \qquad \frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + (\lambda\rho(x) - g(x))u = 0$$

where λ is a parameter and p, ρ, and g are real with p and ρ strictly positive.

The Sturm-Liouville equation, along with appropriate boundary conditions, describes many physical phenomena typically connected with vibration problems in continuum mechanics, that is, boundary value problems corresponding to simple harmonic standing waves. Several phenomena described in some detail in Birkoff and Rota [1] are the vibrating string where the eigenvalues are proportional to the squared frequency, the longitudinal vibrations of the elastic bar, and the vibrating membrane.

The Sturm-Liouville equation can be written in several forms. The form given in (1.1) is called the self-adjoint form for real λ. Assuming g and ρ are continuous and p is continuously differentiable, solutions to (1.1) exist.

The Liouville normal form of the Sturm-Liouville equation will be the form used throughout Chapters 2-5.

Any differential equation (1.1) with p, $\rho \in C^2$ and $g \in C$ can be rewritten in normal form

$$(1.2) \qquad \frac{d^2\omega}{dt^2} + [\lambda - \hat{g}(t)]\omega = 0$$

where $\hat{g} = g/\rho + (p\rho)^{-1/4}\frac{d^2}{dt^2}[(p\rho)^{1/4}]$. This is accomplished by the change of variable

$$u = \omega(t)/(p(x)\rho(x))^{1/4} \quad \text{and} \quad t = \int \frac{\rho(x)}{p(x)}^{1/2} dx.$$

This transformation is discussed in Birkoff and Rota [1].

The Sturm-Liouville equation in standard form is given by

$$(1.3) \qquad y'' + P(x)y' + Q(x)y = \lambda R(x)y \ .$$

By the change of variable

$$y(x) = u(x)v(x) \quad \text{for} \quad v(x) = \exp\left(-\tfrac{1}{2}\int P dx\right),$$

(1.3) can be rewritten in normal form

$$(1.4) \qquad u'' + q(x)u = \lambda r(x)u$$

where $q(x) = Q(x) - \frac{1}{4}(P(x))^2 - \frac{1}{2}P'(x)$. Simmons [2] discusses at length the theory dealing with (1.3) and (1.4). Each of the forms (1.2), (1.3), and (1.4) are equivalent up to transformations and play different roles in the numerical approximation of their eigenvalues. This will be discussed in more detail in Chapter 5.

The Sturm-Liouville equation (1.1) can be posed on a finite, semi-infinite, or infinite interval. In the case that (1.1) is on a closed, finite interval [a,b], p and ρ are strictly positive on [a,b], and p, ρ, and g are bounded in the interval, (1.1) is called a regular Sturm-Liouville equation. A regular Sturm-Liouville system is defined to be a regular Sturm-Liouville equation on the finite interval [a,b] with two separated endpoint conditions

$$\alpha u(a) + \alpha' u'(a) = 0$$

and

$$\beta u(b) + \beta' u'(b) = 0$$

for α, α', β, and β' real constants. The following theorem, stated with proof in Birkoff and Rota [1], gives information concerning the eigenfunctions of a regular Sturm-Liouville system.

Theorem 1.5: Eigenfunctions of a regular Sturm-Liouville system having different eigenvalues are orthogonal with respect to the weight ρ; that is, if u, λ and v, μ are eigenpairs and $\lambda \neq \mu$,

$$\int_a^b \rho(x)u(x)v(x)\,dx = 0 .$$

The transformation of (1.1) to normal form (1.2) takes a regular Sturm-Liouville system to a regular Sturm-Liouville system, does not change the eigenvalues, and takes functions orthogonal with respect to the weight ρ to functions orthogonal with respect to the weight 1.

There are Sturm-Liouville systems of interest that are not regular. That is, the function h (h = p, ρ, and/or g) is singular at one or both endpoints of the interval on which the problem is posed or one or both endpoints is infinite. If either or both of the preceeding situations occur, (1.1) is called a singular Sturm-Liouville equation. Comparable to Theorem 1.5 is

Theorem 1.6: Eigenfunctions u and v, with different eigenvalues, of a singular Sturm-Liouville system are orthogonal with respect to the weight ρ whenever

$$\lim_{\alpha \to a^+,\ \beta \to b^-} p(x)[u(x)v'(x) - v(x)u'(x)] \Big|_{x=\alpha}^{\beta} = 0$$

where b can be infinite (Birkoff [1]).

Many differential equations that arise in physical problems have singular points, that is, points at which P and/or Q are not analytic in (1.3). As Simmons [2] describes, choosing the most physically appropriate solution is many times determined by the behavior of the solution at a singular point.

The two types of singular points are regular and irregular. A singular point x_0 is called regular, if $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ are analytic at x_0. If these two conditions are not satisfied, x_0 is called an irregular singular point. In terms of the normal form

(1.4), z_0 is a regular singular point if q has at worst a pole of order two at z_0.

Several numerical approaches are used to solve for the eigenvalues of Sturm-Liouville systems. They include finite differences [3], [4], [5], [6], [7], finite elements [5], [6], [8], [9], [10], [11], Galerkin [4], [5], [8], [9], [10], [11], the tau method [4], [12], [13], and collocation [4], [12], [13].

The finite difference method is straightforward for problems posed on a finite interval. It yields a matrix system $A\vec{x} = \lambda B\vec{x}$ which is symmetric [3].

The Rayleigh-Ritz approximation method described in Examples 5.1, 5.2, and 5.3 is a Galerkin method using polynomials or piecewise polynomials as basis functions [8], [10]. The tau method, using Chebyshev basis functions [4], is demonstrated in Example 5.4.

The sinc-collocation method defines an approximate solution to the Sturm-Liouville system at a finite set of nodes using sinc functions as basis functions. Before the complete description of the method is given in Chapter 4, Chapters 2 and 3 discuss some preliminary ideas. Chapter 2 introduces some general mathematical notions which form part of the basis for the analysis and implementation of the sinc-collocation method. Basic information on the sinc function and various properties of it are found in Chapter 3. Chapter 5 gives specific examples of Sturm-Liouville systems using the sinc-collocation method and compares these results with results of other methods mentioned earlier.

CHAPTER 2

PRELIMINARY RESULTS

For the formulation and implementation of the sinc-collocation method, several general mathematical notions need to be considered. This chapter gives a general discussion of the WKB approximation to solutions of particular differential equations, indicial equations related to the solution of differential equations, and the method of solving a generalized eigenvalue problem. Two theorems concerning eigenvalues will be included for completeness. Concluding the chapter is a general discussion of the Rayleigh-Ritz method of solving boundary value problems.

The following discussion of the WKB approximation is due to Olver [14]. Suppose $\omega(x)$ is a function satisfying

$$(2.1) \qquad \frac{d^2\omega}{dx^2} = f(x)\omega(x)$$

for x real or complex and f some given positive function. To approximate the solution ω of (2.1), let $\eta(x)$ be three times differentiable and

$$W(x) = \{\eta'(x)\}^{\frac{1}{2}}\omega(x) .$$

A new differential equation in W is

(2.2) $$\frac{d^2W}{d\eta^2} = \{(\dot{x})^2 f(x) + (\dot{x})^{1/2} \frac{d^2}{d\eta^2}((\dot{x})^{-1/2})\}W$$

where dots represent differentiation with respect to η. This is called the Liouville transformation. Without loss of generality, $\eta(x)$ can be chosen so that $(\dot{x})^2 f(x) = 1$; hence,

$$\eta(x) = \int (f(x))^{1/2}\, dx \quad .$$

Given that f is twice differentiable, (2.2) can be written

(2.3) $$\frac{d^2W}{d\eta^2} = (1 + \phi(x))W$$

where

$$\phi = -f^{-3/4} \frac{d^2}{dx^2}(f^{-1/4}) \quad .$$

If ϕ is neglected, (2.3) is a separable differential equation and the two independent solutions are $e^{\pm\eta}$. Noting that

$$\eta'(x) = (f(x))^{1/2}$$

and using the original variables, an approximation to ω of (2.1) is given by

(2.4) $$\omega \simeq Af^{-1/4}\exp\left(\int f^{1/2}dx\right) + Bf^{-1/4}\exp\left(-\int f^{1/2}dx\right)$$

where A and B are arbitrary constants. This approximation is called the WKB (in recognition of the works of Wentzel, Kramers, and Brillouin) approximation or Liouville-Green approximation. The functions

$$f^{-1/4}\exp\left(\int f^{1/2}dx\right)$$

and

$$f^{-1/4}\exp\left(-\int f^{1/2}dx\right)$$

are called LG (Liouville-Green) functions.

The accuracy of the approximation, of course, depends on the magnitude of ϕ. One obvious case where the approximation fails is when zeros of f lie in the interval over which the differential equation is posed.

This approximation will be used in the example section in Chapter 5 when considering the asymptotic behavior of the solution of a differential equation at infinity for the problems posed on the infinite interval $(0,\infty)$.

The indicial equation of a differential equation, also used in Chapter 5, will give, from the method of Frobenius, a series solution at singular points. The singular points of interest are $x = 0$ in the examples posed on $(0,\infty)$ and $x = a$ and $x = b$ on the interval (a,b).

Simmons [2] has the following discussion. Given the differential equation

(2.5) $$-u''(x) + (g(x) - \lambda\rho(x))u(x) = 0 ,$$

the method of Frobenius gives, for a singular point at $x = a$,

(2.6) $$u(x) = (x - a)^s \sum_{j=0}^{\infty} \gamma_j(x - a)^j .$$

An assumption made is that $\gamma_0 \neq 0$ since another power of $x - a$ could have been incorporated in $(x - a)^s$ if $\gamma_0 = 0$.

Computing u" from (2.6), substituting that and (2.6) into (2.5), combining corresponding powers of $(x - a)$, and equating those to zero yields a system of equations, the first of which is

$$\gamma_0[s(s - 1) - (g_a - \lambda\rho_a)] = 0 \tag{2.7}$$

where

$$\begin{cases} g_a = \lim\limits_{x \to a}(x - a)^2 g(x) \\ \\ \rho_a = \lim\limits_{x \to a}(x - a)^2 \rho(x) \ . \end{cases} \tag{2.8}$$

The indicial equation is the bracketed piece of (2.7), that is,

$$s(s - 1) - (g_a - \lambda\rho_a) = 0 \ . \tag{2.9}$$

Once s has been calculated, (2.6) is the series solution of u(x) near the singularity $x = a$.

Solving for the eigenvalues of Sturm-Liouville differential equations by the sinc-collocation method will be done by solving a generalized eigenvalue problem, that is,

$$A\vec{x} = \lambda B\vec{x} \tag{2.10}$$

where λ and $\vec{x} \neq \vec{0}$ are the eigenpair of A and B which are real, square matrices. This means that there exists a nonzero vector $\vec{x}$ if and only if there exists a number λ satisfying

(2.11) $\det[A - \lambda B] = 0$.

Hohn discusses this problem in [15] and included here are some ideas important to the development of this thesis.

Consider the case where A is a symmetric matrix and B is both symmetric and positive definite (these assumptions hold for all Sturm-Liouville problems in normal form for which the sinc-collocation method is applied). In this case, the generalized eigenvalue problem is equivalent to a regular eigenvalue problem. That is, there is a matrix W so that when

$$\det[W - \lambda I] = 0$$

is solved for λ, the same λ's will solve (2.10). These eigenvalues will all be real, since W is a symmetric matrix.

The details of the preceeding paragraph follow. Because B is symmetric, there exists an orthogonal matrix U such that

$$U^T B U = D(\mu)$$

where $D(\mu)$ is a diagonal matrix with diagonal elements μ_i, the eigenvalues of B. Since B is positive definite, $\mu_i > 0$ for all i. Put $R = D(\mu^{-½})$. Then

$$\begin{aligned} R^T(U^T B U)R &= R^T D(\mu) R \\ &= I . \end{aligned}$$

Set S = UR. S is nonsingular since

$$\det S = \det(UR)$$
$$= \det U \cdot \det R$$
$$= \det U \left(\prod_{i=1}^{n} \mu_i^{-1/2} \right) \neq 0 \ .$$

This is nonzero because U is orthogonal ($U^T U = I$ implies $\det U^T \cdot \det U = 1$) and $\mu_i > 0$ for all i. Also

$$S^T B S = R^T (U^T B U) R$$
$$= I \ .$$

Because S is nonsingular, solving the generalized eigenvalue problem $\det[A - \lambda B] = 0$ is equivalent to $\det S^T [A - \lambda B] S = 0$. Then

$$\det[S^T A S - \lambda S^T B S] = \det[W - \lambda I] = 0 \quad (2.12)$$

which is a regular eigenvalue problem with $W = S^T AS$.

To see that the eigenvalues of (2.12) are real, note that

$$(S^T A S)^T = S^T A^T (S^T)^T$$
$$= S^T A S$$

by the symmetry of A; hence, W is symmetric. Since $S^T AS$ is symmetric there exists an orthogonal matrix Q so that

$$Q^T (S^T A S) Q = D(\lambda)$$

where λ is in the spectrum of $S^T AS$, that is, $\lambda \in \sigma(S^T AS)$ and, equivalently, $\lambda \in \sigma(A)$.

$$(SQ)^T A (SQ) = D(\lambda)$$

and

$$(SQ)^T B (SQ) = Q^T (S^T B S) Q$$
$$= Q^T (I) Q = I \ .$$

Hence, there exists a matrix $V = SQ$ that simultaneously diagonalizes A and B relative to conjugacy. That is,

$$V^T A V = D(\lambda) \tag{2.13}$$

with the eigenvalues of A making up the matrix $D(\lambda)$ and

$$V^T B V = I \ . \tag{2.14}$$

Finally, two theorems will be stated here, without proof, that will be needed later in the paper. They can be found in Parlett [16]. The Monotonicity Theorem gives bounds on the eigenvalues of the sum of two matrices.

Theorem 2.15: Monotonicity Theorem.

Let $W = A + Y$ where W, A, and Y are all square, real, symmetric matrices. Let $\omega_i \in \sigma(W)$, $\alpha_i \in \sigma(A)$, and $y_i \in \sigma(Y)$ so that the eigenvalues are ordered, that is, for $i < j$, $\beta_i \le \beta_j$ for $\beta = \omega, \alpha$, and y. Then

$$\alpha_1 + y_j \le \omega_j \le y_j + \alpha_n$$

and

$$y_1 + \alpha_j \le \omega_j \le \alpha_j + y_n \ .$$

The Courant-Fischer Max-Min Theorem gives the eigenvalues of the matrix A via the Rayleigh quotient.

Theorem 2.16: Courant-Fischer Max-Min Theorem

Let A be a symmetric, n x n matrix and $\alpha_i \in \sigma(A)$ such that for $i < j$, $\alpha_i \le \alpha_j$. Then for $\vec{x} \ne \vec{0}$

$$\alpha_n = \max_{\vec{x} \in R^n} \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}} ,$$

$$\alpha_1 = \min_{\vec{x} \in R^n} \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}} ,$$

and

$$\alpha_k = \max_{R^{k-1}} \min_{\vec{x} \in R^{k-1}} \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}}$$

where R^{k-1} ranges over all (k - 1)-dimensional subspaces of R^n.

Many boundary value problems that describe physical phenomena satisfy a variational property. That is, a differential equation Au = f with specified boundary conditions can be solved by minimizing a functional of the form

$$F(u) = (Au,u) - 2(u,f)$$

where $(\cdot,\cdot)$ denotes an inner product. The variational property is fundamental to the Rayleigh-Ritz method of solving boundary value problems.

To carry out the Rayleigh-Ritz method, it is necessary to find the stationary values, or critical points, of the functional R[v]. The Rayleigh quotient is given by

$$(2.17) \qquad R[v] = \frac{a[v]}{b[v]} = \frac{\int_a^b \{(v')^2 + gv^2\}dx}{\lambda \int_a^b \rho v^2 dx}$$

for a differential equation of the form

$$-v'' + gv = \lambda \rho v$$

with appropriate boundary conditions. The stationary values occur where the gradient of R vanishes. A suitable set of basis functions is chosen to approximate v in (2.17) and the result forms the approximate to the eigenvalue λ.

Equivalently, by the variational property, the differential equation can be solved. Minimize the Galerkin form

$$(2.18) \qquad \int_a^b [(v')^2 + gv^2]dx = \lambda \int_a^b \rho v^2 dx$$

with respect to a set of basis functions where an approximation to v is given by

$$(2.19) \qquad \tau(x) = \sum_{i=1}^{n} c_i \tau_i(x)$$

with $\{\tau_i\}_{i=1}^n$ the set of basis functions.

Both approaches, for the same choice of basis functions, yield the same matrix system $A\vec{x} = \lambda B\vec{x}$. They also involve integrations that usually require a numerical quadrature. When the differential equation is posed on an infinite interval the problem in general requires the truncation of the interval. For example, in the minimization of the Galerkin form for a problem on the interval $(0,\infty)$, the elements a_{ij} of A are

$$(2.20) \qquad a_{ij} = \int_0^K [c_i \tau_i'(x) \tau_j'(x) + g(x) c_i \tau_i(x) \tau_j(x)]dx$$

and the elements b_{ij} of B are

$$(2.21) \qquad b_{ij} = \int_0^K \rho(x) c_i \tau_i(x) \tau_j(x) dx$$

for fixed $K \in (0,\infty)$. For each particular $g(x)$ and $\rho(x)$ the truncation point K needs to be determined, and the numerical integrations need to be carried out. In the case where $g(x)$ and/or $\rho(x)$ are singular, a_{ij} and/or b_{ij} are singular integrals. This requires more computational work in constructing the entries of the matrices A and B.

CHAPTER 3

SINC-COLLOCATION METHOD

The sinc-collocation method applied to the calculation of eigenvalues of Sturm-Liouville problems depends on several aspects of the theory of sinc functions [17]. The development of the method begins with the sinc approximating formulas -- an interpolation formula and a quadrature rule for the entire real line. From these the method of conformal mappings allows a Sturm-Liouville problem on an arbitrary interval to be solved by the same basic method. From this development come the matrices of the sinc-collocation method.

The sinc function is defined by

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \quad .$$

If f is defined over the whole real line, then for $h > 0$,

$$C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh)\,\mathrm{sinc}\left(\frac{x - kh}{h}\right) \tag{3.1}$$

is called the Whittaker cardinal expansion for f whenever the series converges. The properties of this expansion have been studied extensively by Stenger [17]. To use (3.1) as an approximation on $\mathbb{R}^1$, define

$$C_{M,N}(f,h)(x) = \sum_{k=-M}^{N} f(kh)\operatorname{sinc}\left(\frac{x - kh}{h}\right) .$$

The following definition and theorem are found in [17].

Definition 3.2: For $d > 0$, let $B(S)$ be the family of functions f that are analytic in the infinite strip $S = \{z \in C \mid |\mathrm{Im} z| < d\}$ and satisfy

$$\int_{-d}^{d} |f(t + is)|\, ds \to 0 \quad \text{as} \quad t \to \pm\infty$$

and

$$N(f) = \lim_{s \to d^-} \int_{-\infty}^{\infty} (|f(t + is)|^2 + |f(t - is)|^2)dt < \infty \quad .$$

Theorem 3.3: Assume $f \in B(S)$. If $h > 0$ and $\pi d/h > 1$, then

$$\left\|\frac{d^2 f}{dt^2} - \frac{d^2}{dt^2} C(f,h)\right\|_2 \le \left(\frac{\pi}{h}\right)^2 \frac{N(f)}{\sinh(\pi d/h)} \quad .$$

Since $$\frac{d^2}{dt^2}(C_{M,N}(f,h)(t)) = \sum_{k=-M}^{N} f(kh) \frac{d^2}{dt^2} \operatorname{sinc}\left(\frac{t - kh}{h}\right)$$

and if there exist positive constants C, α, and β so that

$$(3.4) \qquad |f(t)| \le C \begin{cases} e^{\alpha t} , & t \in (-\infty,0] \\ e^{-\beta t} , & t \in (0,\infty) \end{cases}$$

then

(3.5)

$$\left\|\frac{d^2f}{dt^2} - \frac{d^2}{dt^2}\, C_{M,N}(f,h)\right\|_2 \le \left(\frac{\pi}{h}\right)^2 \left\{\frac{N(f)}{\sinh(\pi d/h)} + \frac{C}{5h}\left(\frac{e^{-\alpha h}}{\alpha} + \frac{e^{-\beta h}}{\beta}\right)\right\}.$$

For the selections $h = \left(\frac{\pi d}{\alpha M}\right)^{1/2}$ and $N = \left[\left[(\alpha/\beta)M\right]\right]$,

$$(3.6) \qquad \left\|\frac{d^2f}{dt^2} - \frac{d^2}{dt^2}\, C_{M,N}(f,h)\right\|_2 \le KM^{5/4} \exp(-(\pi d\alpha M)^{1/2})$$

where K depends on f and d.

Error bound theorems involving $C_{M,N}$ and $\frac{d}{dt}C_{M,N}$ also appear in [17]. However, for the method of this paper, those bounds are not necessary.

Many important examples are not posed on the whole real line. The following definition will allow a problem posed on an arbitrary interval in the complex plane to be mapped to the infinite strip S of Definition 3.2, thus allowing Theorem 3.3 to be used.

Definition 3.7: Let D be a simply-connected domain in the complex plane with boundary points a and b. For a given $d > 0$, let $\phi(z)$ be a conformal mapping of D onto S such that $\phi(a) = -\infty$ and $\phi(b) = \infty$. Define
$\Gamma = \{\psi(\eta) : \eta \in R,\ -\infty < \eta < \infty\}$ where $\psi = \phi^{-1}$,
$z_k = \psi(kh)$ for $k = 0, \pm 1, \pm 2, \ldots,$ and $h > 0$. See Figure 1.

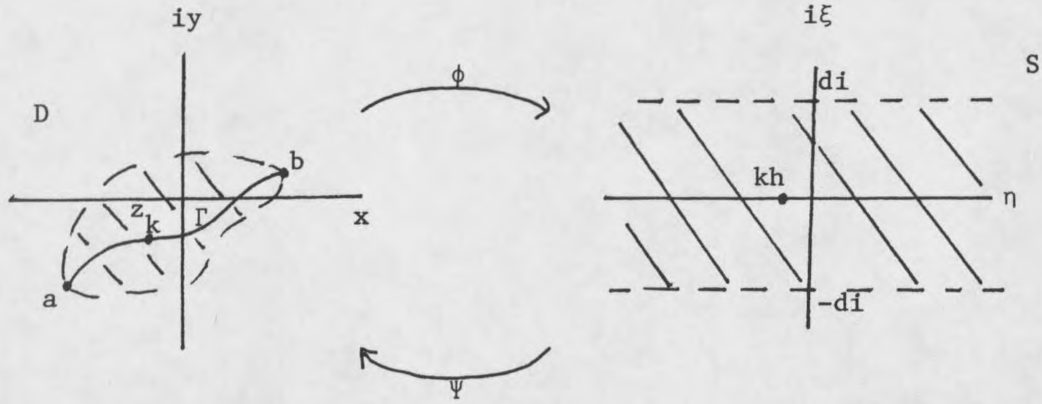

Figure 1: The Domain D and Strip S

The particular intervals of interest here are subsets of the real line. For the finite interval $(a,b) \subset \mathbb{R}$ an appropriate mapping ϕ is

$$\phi_1(x) = \ln\left(\frac{x - a}{b - x}\right) .$$

For the semi-infinite interval two maps will be considered. They are

$$\phi_2(x) = \ln(x)$$

and

$$\phi_3(x) = \ln\big(\sinh(x)\big) .$$

Figure 2 shows the domains mapped by these to the infinite strip.

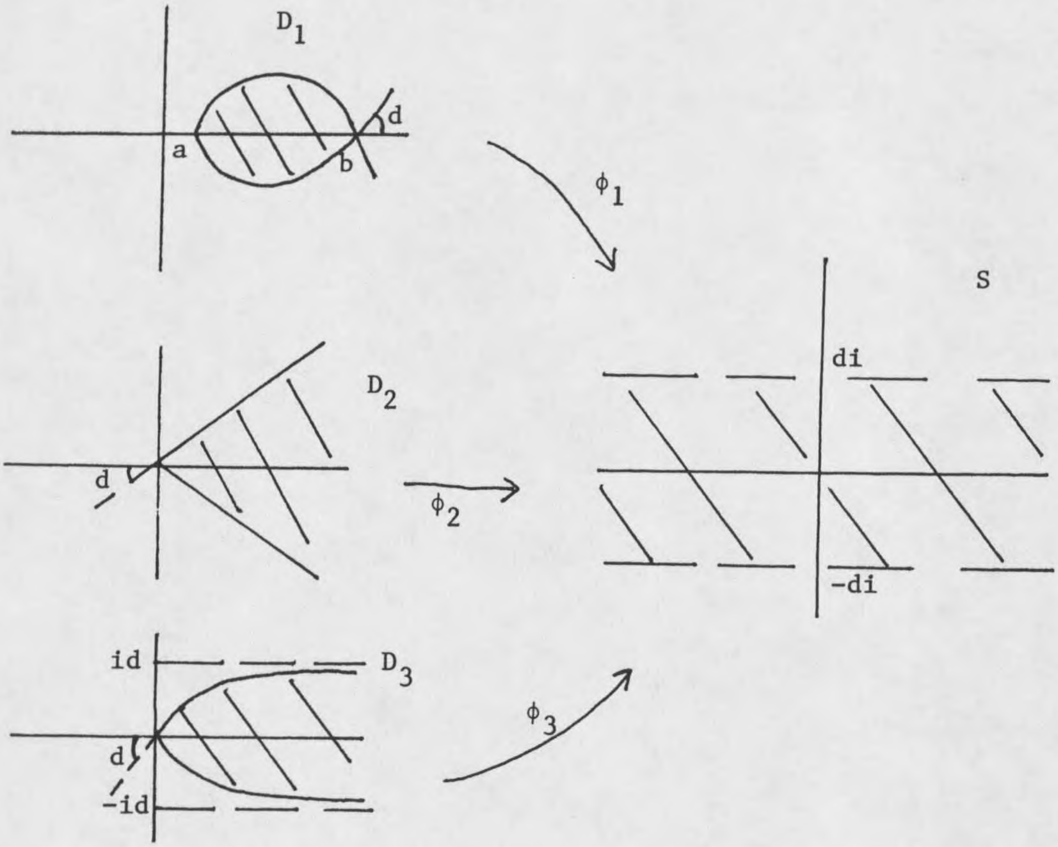

Figure 2: The Mappings ϕ_i and Their Domains

The procedure for choosing ϕ_2 or ϕ_3 is somewhat problem dependent and will be discussed in greater detail in Chapter 5.

It will be convenient to note here that for ϕ_1

$$z_k = \phi_1^{-1}(kh) = \frac{be^{kh} + a}{e^{kh} + 1} ,$$

for ϕ_2

$$z_k = \phi_2^{-1}(kh) = e^{kh} ,$$

and for ϕ_3

$$z_k = \phi_3^{-1}(kh) = \ln\left[e^{kh} + \sqrt{e^{2kh} + 1}\right] .$$

Definition 3.8: Let d, ϕ, and ψ be as in Definition 3.7. Let B(D) denote the family of functions f that are analytic in D, a simply-connected domain in the complex plane, that satisfy

$$\int_{\psi(\eta+L)} |f(z)dz| \to 0 \quad \text{as} \quad \eta \to \pm\infty$$

where $L = \{i\xi \mid |\xi| < d\}$ and

(3.9) $$N(f) \equiv \liminf_{\substack{C\to\partial D \\ C\subset D}} \int_C |f(z)dz| < \infty .$$

The following theorem [17] gives a quadrature rule with error for functions which are elements of B(D).

Theorem 3.10: For $f \in B(D)$

(3.11) $$\int_\Gamma f(z)dz - h \sum_{p=-\infty}^{\infty} \frac{f(z_p)}{\phi'(z_p)} = (i/2)\int_{\partial D} \frac{f(z)K(\phi,h)(z)}{\sin(\pi\phi(z)/h)}\, dz$$

where $K(\phi,h)(z) = \exp[(i\pi\phi(z))\mathrm{sgnIm}(\phi(z))/h]$. Note that

(3.12) $$|K(\phi,h)(z)| = \exp(-\pi d/h) \text{ for } z \in \partial D.$$

The quadrature rule given in Theorem 3.10 will be used in approximating inner products defined by

$$(r,s) = \int_\Gamma r(z)s(z)w(z)dz$$

where w(z) is a weight function and Γ is the curve given in Definition 3.7. The two specific inner products of interest here are $(u'', S_k \circ \phi)$ and $(gu, S_k \circ \phi)$ where

$$S_k \circ \phi(z) = \text{sinc}\left[\frac{\phi(z) - kh}{h}\right],$$

with $\phi(z)$ being the conformal map in Definition 3.7 and g a function defined on Γ. Then, by two integrations by parts,

$$(u'', S_k \circ \phi) = \int_\Gamma u''(z) S_k \circ \phi(z) w(z) dz$$

$$= u'(z)[S_k \circ \phi(z)w(z)]\Big|_a^b - \int_\Gamma u'(z)[S_k \circ \phi(z)w(z)]'dz$$

$$= -u(z)[S_k \circ \phi(z)w(z)]'\Big|_a^b + u'(z)[S_k \circ \phi(z)w(z)]\Big|_a^b$$

$$+ \int_\Gamma u(z)[S_k \circ \phi(z)w(z)]''dz$$

$$\equiv B_T + \int_\Gamma u(z)[S_k \circ \phi(z)w(z)]''dz .$$

Note that if

$$u^{(1-j)}(z)[S_k \circ \phi(z)w(z)]^{(j)}\Big|_a^b = 0 \quad \text{for } j = 0,1,$$

then $B_T = 0$ and using (3.11)

$$(u'', S_k \circ \phi) = \int_\Gamma u(z)[S_k \circ \phi(z)w(z)]''dz$$

$$= h \sum_{p=-\infty}^{\infty} \frac{u(z_p)[S_k \circ \phi(z)w(z)]''\big|_{z=z_p}}{\phi'(z_p)} + E_1(u)$$

where

$$(3.13)\qquad E_1 = \left(i/2\right)\int_{\partial D} \frac{K(\phi,h)(z)u(z)[S_k \circ \phi(z)w(z)]''}{\sin(\pi\phi(z)/h)}\,dz\ .$$

The weight $w(z)$ must be chosen to force $B_T = 0$ and to bound $|E_1|$. Consider then the term $[S_k \circ \phi(z)w(z)]''$ in E_1.

$$\begin{aligned}[S_k \circ \phi(z)w(z)]'' &= [S_k \circ \phi(z)w'(z) + w(z)S_k' \circ \phi(z)\phi'(z)]' \\ &= S_k'' \circ \phi(z)(\phi'(z))^2 w(z) + S_k \circ \phi(z)w''(z) \\ &\quad + S_k' \circ \phi(z)(\phi'(z)w'(z) + (w(z)\phi'(z))')\ .\end{aligned}$$

For $w(z) = (\phi'(z))^{-1/2}$, the coefficient of $S_k' \circ \phi(z)$ is zero. As will be shown later, this choice of $w(z)$ will satisfy all the necessary properties, including $B_T = 0$. It also simplifies the inner product

$$\begin{aligned}(u'',S_k \circ \phi) &= h\sum_{p=-\infty}^{\infty} \frac{u(z_p)\left[S_k \circ \phi(z)(\phi'(z))^{-1/2}\right]''\Big|_{z=z_p}}{\phi'(z_p)} + E_1(u) \\ &= h\sum_{p=-\infty}^{\infty} u(z_p)\Big\{S_k'' \circ \phi(z_p)(\phi'(z_p))^{1/2} \\ &\quad + S_k \circ \phi(z_p)\frac{1}{\phi'(z_p)}\left((\phi'(z))^{-1/2}\right)''\Big|_{z=z_p}\Big\} + E_1(u) \\ &= \frac{1}{h}\sum_{p=-\infty}^{\infty} u(z_p)\delta_{kp}^{(2)}(\phi'(z_p))^{1/2} \\ (3.14)\qquad &\quad + hu(z_k)\frac{1}{\phi'(z_k)}\left((\phi'(z))^{-1/2}\right)''\Big|_{z=z_k} + E_1(u)\end{aligned}$$

where

$$(3.15)\qquad h^2 \frac{d^2}{d\phi^2}(S_k \circ \phi(z))\Big|_{z=z_p} = \delta_{kp}^{(2)} = \begin{cases} -\pi^2/3 & , \ p = k \\ \dfrac{-2(-1)^{p-k}}{(p-k)^2} & , \ p \neq k \end{cases}$$

and

$$(3.16)\qquad S_k \circ \phi(z_p) = \delta_{kp}^{(0)} = \begin{cases} 1 \ , & p = k \\ 0 \ , & p \neq k \end{cases}$$

by Stenger [17].

To bound the modulus of E_1, the following inequalities are needed:

$$\tfrac{1}{2} \left\| \frac{\frac{d^2}{d\phi^2}(S_k \circ \phi(z))}{\sin(\pi\phi(z)/h)} \right\|_{z\in\partial D} \leq \frac{\pi d + h \tanh\left(\frac{\pi d}{h}\right)}{\pi d^3 \tanh\left(\frac{\pi d}{h}\right)} + \frac{\pi}{2hd}$$

$$\equiv C_2(h,d)$$

and

$$(3.17)\qquad \tfrac{1}{2} \left\| \frac{S_k \circ \phi(z)}{\sin(\pi\phi(z)/h)} \right\|_{z\in\partial D} \leq \frac{h}{2\pi d} \equiv C_0(h,d)$$

also from Stenger [17]. Now consider

$|E_1(u)| =$

$$\frac{i}{2} \int_{\partial D} \frac{K(\phi,h)(z)u(z)\left\{S_k'' \circ \phi(z)(\phi'(z))^{3/2} + \dfrac{S_k \circ \phi(z)}{((\phi'(z))^{\frac{1}{2}})''}\right\}}{\sin(\pi\phi(z)/h)} dz$$

$$\leq e^{-\pi d/h}\left\{C_2(h,d)N(u(\phi')^{3/2}) + C_0(h,d)N\left(u\left((\phi')^{-\frac{1}{2}}\right)''\right)\right\}$$

where $N(f)$ is as in (3.9) and the bound on $K(\phi,h)$ is as in (3.12).

For the other inner product of interest, with $w(z) = (\phi'(z))^{-1/2}$,

$$(gu, S_k \circ \phi) = h \sum_{p=-\infty}^{\infty} \frac{g(z_p)u(z_p)S_k \circ \phi(z_p)}{(\phi'(z_p))^{3/2}} + E_2(gu)$$

(3.18)

$$= h \frac{g(z_k)u(z_k)}{(\phi'(z_k))^{3/2}} + E_2(gu)$$

by (3.11) and (3.16) where

(3.19)
$$E_2(gu) = \frac{i}{2} \int_{\partial D} \frac{g(z)u(z)K(\phi,h)(z)S_k \circ \phi(z)(\phi'(z))^{-1/2}}{\sin(\pi\phi(z)/h)} dz .$$

Bounding the modulus of E_2 gives

(3.20) $$|E_2| \le e^{-\pi d/h} C_0(h,d)N(gu(\phi')^{-1/2})$$

by (3.9), (3.12), and (3.17).

Truncating the sum in (3.14) leads to the matrices of the method. Bounding the error, denoted here by T, caused by this truncation will be discussed in Chapter 4. Letting $p = -M, \ldots, N$ (3.14) becomes

$$(u'', S_k \circ \phi) = \frac{1}{h} \sum_{p=-M}^{N} u(z_p)\delta_{kp}^{(2)}(\phi'(z_p))^{1/2}$$

$$+ h\, u(z_k) \frac{1}{\phi'(z_k)} \left((\phi'(z))^{-1/2}\right)'' \Big|_{z = z_k}$$

$$+ E_1(u) + T.$$

Letting k run from -M to N and ignoring the errors, the inner product becomes, in matrix form,

$$\left\{\frac{1}{h} I^{(2)} D(\sqrt{\phi'}) + hD\left(\frac{1}{\phi'}\left((\phi')^{-1/2}\right)''\right)\right\}\vec{u}$$

where D(r) denotes a diagonal matrix of dimension m x m where m = M + N + 1 with the (i,i)-th element given by $r(z_i)$. The matrix $\vec{u}$ is of dimension m x 1 with ith element $u(z_i)$. Finally, the matrix $I^{(2)}$ has the same dimension as the diagonal matrix with the (k,p)-th element given by $\delta_{kp}^{(2)}$ in (3.15). Explicitly this is

$$I^{(2)} = \begin{bmatrix} \pi^2/3 & \frac{-2}{1^2} & \frac{2}{2^2} & \cdots & (-1)^{m-1}\frac{2}{(m-1)^2} \\ \frac{-2}{1^2} & \pi^2/3 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \frac{-2}{1^2} \\ (-1)^{m-1}\frac{2}{(m-1)^2} & \cdots & & \frac{-2}{1^2} & \pi^2/3 \end{bmatrix}.$$

This matrix is symmetric, negative definite, with all its eigenvalues lying in the open interval $(-\pi^2, 0)$ [17]. It is also a Toeplitz matrix, that is, a matrix with all diagonals constant.

The matrix describing $(gu, S_k \circ \phi)$ in (3.18) is a diagonal matrix of dimension m x m given by $hD\left(\frac{gu}{(\phi')^{3/2}}\right)$.

Notice that with (3.16) no truncation error is incurred in this inner product.

CHAPTER 4

ERROR OF SINC-COLLOCATION METHOD

Most numerical methods used to approximate the eigenvalues of Sturm-Liouville problems lead to a discrete system $A\vec{u} = \lambda B\vec{u}$. The same is true for the sinc-collocation method. This section carries out the convergence analysis for the sinc-collocation method of the eigenvalue approximation. A standard proof of convergence for other numerical methods, in particular, finite differences [3], involves $\|B^{-1}\|$ and the norm of the local truncation error. The method often cannot be applied when the original problem is singular and $\|B^{-1}\|$ is not uniformly bounded. The analysis for the sinc-collocation method is valid whether the Sturm-Liouville problem is regular or singular.

The form of the differential equation considered here is the normal form given by

$$(4.1) \qquad \begin{cases} -u''(x) + g(x)u(x) = \lambda\rho(x)u(x) \\ u(a) = u(b) = 0 \end{cases}$$

It will be convenient to transform (4.1) to a differential equation on the whole real line by using the change of variable

$$\omega(t) = (u\sqrt{\phi'}) \circ \psi(t)$$

where ϕ and ψ are as described in Definition 3.7. To simplify the transformation several steps will be used.

Let

$$(4.2) \qquad u(x) = \omega(x)(\phi'(x))^{-1/2} .$$

Then

$$u'(x) = \omega'(x)(\phi'(x))^{-1/2} + \omega(x)((\phi'(x))^{-1/2})'$$

and

$$u''(x) = \omega''(x)(\phi'(x))^{-1/2} + 2\omega'(x)\left((\phi'(x))^{-1/2}\right)' + \omega(x)\left((\phi'(x))^{-1/2}\right)'' .$$

Replacing the above statements in (4.1) yields

$$(4.3) \qquad -\omega''(x) - 2\omega'(x)(\phi'(x))^{1/2}\left((\phi'(x))^{-1/2}\right)' + \omega(x)\left\{ g(x) - (\phi'(x))^{1/2}\left((\phi'(x))^{-1/2}\right)''\right\} = \lambda\rho(x)\omega(x) .$$

Given $\omega \in B(S)$ in Definition 3.2, let ω satisfy the bounds in (3.4). Using the change of variable (4.2) in the form $\omega(x) = u(x)\sqrt{\phi'(x)}$, (3.4) takes the form

$$(4.4) \qquad |\omega(x)| = |u(x)\sqrt{\phi'(x)}| \le C \begin{cases} e^{-\alpha|\phi(x)|} & x \in \Gamma_a \\ e^{-\beta|\phi(x)|} & x \in \Gamma_b \end{cases}$$

where $\Gamma_a = \{\psi(t): \ t \in (-\infty, 0]\}$

and $\Gamma_b = \{\psi(t): \ t \in (0, \infty)\}$.

Hence the boundary conditions in (4.1) can be seen to be

$$(4.5) \qquad \lim_{t\to\pm\infty} \omega(t) = 0 .$$

Let $x = \psi(t)$ which implies $t = \phi(x)$. Then

$$\frac{d\omega}{dx} = \frac{d\omega}{dt}\frac{dt}{dx} = \omega'(t)\phi'(x)$$

and

$$\frac{d^2\omega}{dx^2} = \frac{d}{dx}\left(\omega'(t)\phi'(x)\right)$$

$$= \omega'(t)\phi''(x) + \omega''(t)(\phi'(x))^2 \ .$$

Now (4.3) becomes

$$(4.6) \qquad -\omega''(t) - \omega'(t)\left\{\frac{\phi''(x)}{(\phi'(x))^2} + 2(\phi'(x))^{-\frac{1}{2}}\left((\phi'(x))^{-\frac{1}{2}}\right)'\right\}$$

$$+ \omega(t)\left\{\frac{g(x)}{(\phi'(x))^2} - (\phi'(x))^{-3/2}\left((\phi'(x))^{-\frac{1}{2}}\right)''\right\}$$

$$= \lambda \frac{\rho(\psi(t))}{(\phi'(x))^2}\omega(t) \ .$$

A short computation shows that the coefficient of $\omega'(t)$ is zero. Using the fact that

$$\phi'(x) = \frac{1}{\psi'(\phi(x))} = \frac{1}{\psi'(t)}$$

in (4.6) yields

$$(4.7) \qquad L\omega = -\omega''(t) + \omega(t)\gamma_g(t) = \lambda\rho(\psi(t))(\psi'(t))^2\omega(t)$$

for

$$(4.8) \qquad \gamma_g(t) = g(\psi(t))(\psi'(t))^2 - (\psi'(t))^{3/2}\left(\sqrt{\psi'(t)}\right)'' \ .$$

Define an approximation to $\omega(t)$ by

$$(4.9) \qquad C_M(\omega,h)(t) \equiv C_{M,M}(\omega,h)(t) = \sum_{k=-M}^{M} \omega(kh)S_k(t) \ .$$

It is not necessary to define the approximation to be a centered sum, that is, a sum which goes from $-M$ to M. In fact, the computations of Chapter 5 will show that noncentered sums can give the accuracy desired with less summands. It is for simplicity of notation that centered sums are used in the present section.

Substituting (4.9) into (4.7) yields

$$LC_M(\omega,h)(jh) = -\sum_{k=-M}^{M} \frac{1}{h^2}\, \delta_{jk}^{(2)}\omega(kh) + \gamma_g(jh)\omega(jh)$$

$$= \lambda\rho(\psi(jh))(\psi'(jh))^2\omega(jh)$$

for $j = -M, \ldots, M$ and $\delta_{jk}^{(2)}$ as in (3.15). The matrix form for the sinc-collocation method is then given by

$$(4.10)\qquad L\vec{C}_M(\omega,h) = A\vec{\omega} = \lambda B\vec{\omega}$$

where $\quad A = -\dfrac{1}{h^2}\, I^{(2)} + D(\gamma_g)\ ,$

$\quad B = D(\rho(\psi')^2)\ ,$

and $\quad \vec{\omega} = [\omega(-Mh),\ \omega((-M+1)h),\ \ldots,\ \omega(Mh)]^T\ .$

The diagonal matrices and $I^{(2)}$ are as described in Chapter 3, and all are of dimension $2M + 1$ by $2M + 1$.

The equation in (4.10) gives the method of computing the eigenvalues of the Sturm-Liouville problem described by (4.7) and (4.5). Even though the problem has been translated, the eigenvalues computed are the same as in the original problem (4.1).

To develop the error in using (4.10) to approximate the eigenvalues, assume for given ψ and g

$$(4.11)\qquad \gamma_g(t) \geq (\delta(g,\psi,h))^{-1} \equiv \delta^{-1} > 0 \quad \text{for } t \in (-\infty,\infty)\ .$$

The example section, Chapter 5, will say more about this assumption; but, in particular, on a finite interval, (4.11) holds for $g(x) > 0$ on the interval. Using the fact that $-I^{(2)}$ is symmetric, $D(\gamma_g)$ is diagonal, and Theorem 2.15

$$a_m \equiv \min_{a_j \in \sigma(A)} \{a_j\}$$

$$(4.12) \qquad \geq \min_{p_j \in \sigma((-1/h^2) I^{(2)})} \{p_j\} + \min_{d_j \in \sigma(D(\gamma_g))} \{d_j\}$$

$$\geq 0 + \delta^{-1} = \delta^{-1}$$

Assume, also, that λ_0 and ω_0 are an eigenpair to (4.7) with ω_0 normalized by

$$(4.13) \qquad \int_{-\infty}^{\infty} \omega_0^2(t)\rho(\psi(t))(\psi'(t))^2 dt = 1$$

which is equivalent to

$$(4.14) \qquad \int_a^b u_0^2(x)\rho(x)dx = 1$$

for u_0 a solution to (4.1). This is possible since by Sturm-Liouville theory the eigenfunctions are orthogonal with respect to ρ. This normalization of the eigenfunction will be used, for convenience, for the remainder of the section.

Substitute ω_0 into (4.7) and evaluate at $t = jh$ for $j = -M, \ldots, M$, to get

$$L\vec{\omega}_0 = \lambda_0 D(\rho(\psi(jh))(\psi'(jh))^2)\vec{\omega}_0 \ .$$

Subtracting the above equation from (4.10) yields

$$\Delta\vec{\omega}_0 \equiv L\vec{C}_M(\omega_0,h) - L\vec{\omega}_0 = A\vec{\omega}_0 - \lambda_0\mathcal{D}^2\vec{\omega}_0$$
$$= (A - \lambda_0\mathcal{D}^2)\vec{\omega}_0$$

where $\mathcal{D} = D(\sqrt{\rho}\psi')$. By the assumption (4.11), (4.12) guarantees that A is positive definite and by observation $\mathcal{D}^2$ is positive definite. Hence by (2.13) and (2.14) there exist vectors $\vec{z}_i$ and positive eigenvalues $\mu_i \le \mu_j$ for $-M \le i \le j \le M$ so that

$$Z^T A Z = D(\mu) \tag{4.15}$$

and

$$Z^T \mathcal{D}^2 Z = I \tag{4.16}$$

for $Z = [\vec{z}_{-M}\ \vec{z}_{-M+1}\ \cdots\ \vec{z}_M]$,

and

$$A\vec{z}_i = \mu_i \mathcal{D}^2 \vec{z}_i \quad \text{for } i = -M, \ldots, M . \tag{4.17}$$

Since $\{\vec{z}_i\}_{i=-M}^{M}$ are linearly independent, there are constants β_i so that

$$\vec{\omega}_0 = \sum_{i=-M}^{M} \beta_i \vec{z}_i \tag{4.18}$$

which implies

$$\Delta\vec{\omega}_0 = (A - \lambda_0 \mathcal{D}^2) \sum_{j=-M}^{M} \beta_j \vec{z}_j$$

$$= \sum_{j=-M}^{M} \beta_j A\vec{z}_j - \sum_{j=-M}^{M} \lambda_0 \beta_j \mathcal{D}^2 \vec{z}_j$$

$$= \sum_{j=-M}^{M} \beta_j \mu_j \mathcal{D}^2 \vec{z}_j - \sum_{j=-M}^{M} \lambda_0 \beta_j \mathcal{D}^2 \vec{z}_j \quad \text{by (4.17)}$$

$$= \sum_{j=-M}^{M} \beta_j (\mu_j - \lambda_0) \mathcal{D}^2 \vec{z}_j .$$