Application of boundary integral equation method to two-dimensional frictional contact problems

by Bhushan Wasudeo Dandekar

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mechanical Engineering
Montana State University

Abstract:
In this thesis, the boundary integral equation technique is used to analyze the two-dimensional elastic contact problems with Coulomb friction. In this method, since only the boundary needs to be modeled, which contains the unknown contact zone, the data preparation time is considerably reduced. Also, since both tractions and displacements are retained as unknowns, the contact stresses are calculated directly.

Navier's equations for the bodies in contact are transformed into simultaneous linear equations using the standard BIE technique. Each body is divided into a potential contact zone and a non-contact zone. The equations obtained from the contact conditions are written explicitly such that a blocked coefficient matrix is obtained. An incremental-iterative technique is applied to the potential contact zone equations to find the correct contact area and the associated surface tractions.

Results for several frictionless and frictional problems are presented. In the case of frictionless problems the results are found to agree well with the analytical results. The frictional problems include an unloading problem and an advancing contact problem, for which no analytical solutions exist.

In most cases, equal size linear elements are found to give best results. The arrangement of stick-slip zones is affected by the node locations and large number of elements may be required to produce satisfactory results. It is shown that in the presence of friction, the problem needs to be solved by applying the load in a large number of load increments or by computing the load increment sizes such that each node pair just barely comes in contact and then dividing these load increments into smaller ones.

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APPROVAL

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Bhushan Wasudeo Dandekar

This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citation, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

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ABSTRACT

In this thesis, the boundary integral equation technique is used to analyze the two-dimensional elastic contact problems with Coulomb friction. In this method, since only the boundary needs to be modeled, which contains the unknown contact zone, the data preparation time is considerably reduced. Also, since both tractions and displacements are retained as unknowns, the contact stresses are calculated directly.

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In most cases, equal size linear elements are found to give best results. The arrangement of stick-slip zones is affected by the node locations and large number of elements may be required to produce satisfactory results. It is shown that in the presence of friction, the problem needs to be solved by applying the load in a large number of load increments or by computing the load increment sizes such that each node pair just barely comes in contact and then dividing these load increments into smaller ones.

CHAPTER 1

INTRODUCTION

Many problems in mechanical system design involve parts that come in contact with each other. Bolted joints, bearings, gears and rings of a chain are some examples where the external forces are transmitted through the contact of two or more deformable bodies. When these parts come in contact under the action of external forces, a region of contact is formed and a pressure and shear stress distribution develops in that region. Normally, the pressure distribution that exists in the contact region is neglected and the transmission of forces is simplified into resultant forces which are important in the analysis of the structure. This simplification is justified by SAINT VENANT'S principle which roughly states that the pressure distribution in the contact region has a negligible effect at points far from the contact region.

In many cases, the load carrying capacity of the structure largely depends on the stress distribution in the contact area through which load transfer takes place. If the contact area is small, local regions of high stress may be formed, increasing the risk of crack initiation and propagation.

Wear of the mating parts along the contact surface is also an important factor in the study of contact mechanics. Due to surface roughness of the solids in contact, initial contact occurs only at the asperities. The load is therefore supported by the asperities which, even under small loads, deform plastically. These then break under an imposed tangential traction, resulting in material loss (Sarkar, 1976). A potential wear situation also exists whenever there is relative motion between two solids in contact. Sliding motion can cause adhesive and abrasive wear, while cyclic loading can cause fretting and fatigue wear thereby increasing the risk of fatigue failure. These friction and wear processes in materials depend on the nature of the true contact area (Sarkar,1976).

Development of different techniques to analyze contact problems with friction is thus important in order to increase understanding of the influence of different parameters that affect the nature of the contact area, and to enable the study of associated deformation and stress patterns.

The general problem of elastic bodies in contact remains one of the most difficult problems in solid mechanics. As the two deformable bodies come in contact under the action of external loads, the contact area changes progressively. In most cases the extent of contact is a priori unknown. This presents a geometric non-linearity even if the material is assumed to be linearly elastic in behavior.

When frictional forces are present between two contacting surfaces, the problem becomes even more difficult. In the presence of frictional forces the contact area may exhibit a region of adhesion and a region of slip which are unknown and often of a more complicated form than the contact area itself (Turner,1979). Frictional forces are inherently non-conservative in nature and dissipate energy over any load cycle applied to a system. Thus, when friction is considered, the stress distribution in the two deformable bodies in contact may depend on the entire history of loading.

The presence of frictional forces and geometric non-linearity has limited the analytical study of contact problems. Analytical solutions to frictionless problems are restricted to simple geometries while solutions to contact problems with friction are very rare. In fact, the existence of solutions to problems in which Coulomb's law of friction holds still remains to be proven (Campos et.al., 1982).

It is apparent that the only way of overcoming the above mentioned difficulties encountered in the study of contact problems is through the use of numerical techniques, and much of the current research in contact mechanics is directed towards developing such techniques (Torstenfelt, 1983, Cheng and Kikuchi, 1985, Bathe and Chaudhary, 1985).

In developing these numerical techniques the following factors are usually taken into account:

1. Contact problems may involve complex geometries and different friction laws.
2. Primary interest lies in obtaining the unknown contact area and the contact stresses.
3. Incremental-iterative solution techniques are essential to find the contact area and the contact stresses in the presence of frictional forces.

The finite element method has been used with considerable success to solve contact problems. The major drawback of the finite element method is its large data preparation time. Since both contacting bodies must be modeled entirely, the data preparation time and the execution time can be high; this is especially true when only the contact area and the contact stresses are of importance. In many cases the contact region is unknown and the model may have to be modified several times before reliable results are obtained. This means that substantial modeling experience is necessary to produce acceptable results.

Another numerical technique which has been developed to solve contact problems is the integral equation method. This method imposes certain restrictions on the shape of the body and is not as general as the finite element method. These two methods are discussed in detail in the following chapter.

In recent years, the boundary element method has emerged as a strong alternative to the finite element method. This method is well suited for the analysis of contact problems, and offers several advantages over current numerical methods. Since the governing differential equations are transformed into integral equations defined on the boundary, only the boundary need be modeled. By employing the boundary element method, the unknowns in the potential contact zone can be obtained. Once the equations are solved on the boundary, stresses and displacements inside the domain can be obtained by using the computed boundary tractions and displacements. Thus, the unnecessary computation time required in solving the equations over the entire domain is saved. Furthermore, reduction in the dimensionality of the problem makes the modeling simple and reduces data preparation time. This offers a significant advantage over the finite element method when solutions to contact problems are sought.

Approach

In this dissertation the boundary element analysis of two-dimensional contact problems is considered. The problems involve the contact of homogeneous isotropic linearly elastic bodies for which Coulomb's law of friction holds. In contact problems the solution of Navier's equations coupled with contact conditions is required. Navier's equations are transformed into integral equations defined on the boundary and then the standard collocation method is employed to obtain a system of simultaneous linear equations. These equations, along with the contact conditions provide sufficient equations to solve for unknown displacements and tractions in the contact zone. Each body is divided into a potential contact zone and a non-contact zone. For the potential contact zone, the equations obtained from the contact conditions are written explicitly so that the equations associated with the potential contact zone can be separated from those outside of the contact zone. After separating these equations, an incremental-iterative technique is applied to the potential contact zone equations to find the correct contact area and the associated surface tractions and displacements. The boundary unknowns outside of the contact zone can then be found by using the computed surface tractions and displacements. A Fortran computer code is developed to obtain numerical solutions for two-dimensional contact problems.

Solutions to several problems with and without friction are presented. For frictionless problems the results are compared with analytical solutions and in the case of frictional problems the results are compared with analytical solutions and other numerical methods whenever possible.

CHAPTER 2

LITERATURE REVIEW

Analytical Treatment of Contact Problems

The classical problem of frictionless contact between two elastic spheres was solved by Hertz in the 1880's. Since then problems concerning elastic bodies in contact have received considerable attention. Excellent accounts of the analytical techniques used in the treatment of contact problems, as well as the solutions to many interesting problems can be found in the works of Gladwell (1980) and de Pater and Kalker (1975).

The complexity of contact problems with friction has restricted most of the analytical treatment to frictionless problems. The work done by Spence (1973,1975,1986) accounts for most of the analytical solutions to frictional contact problems and provides valuable information on the role of friction in elastic contact problems. Spence (1973) shows that when a monotonically increasing load is applied to a rigid plane indentor in contact with an elastic half space, the region of contact divides into an inner adhesive region and an outer annulus of inward slip. The radius of adhesion is a function of material constants and the coefficient of friction, and is evaluated as an eigenvalue of a Fredholm integral equation. In a recent paper Spence (1986) considers the problem of a rigid plane indentor on an elastic half space subjected to a monotonically increasing normal force P, and a shear force Q. The ratio Q/mP, where m is the friction coefficient, is constant and is less than 1 during deformation. A solution exhibiting an unsymmetrical region of adhesion surrounded by regions of slip in opposite directions is obtained. The results include the effect of the shear force Q on the extent of the adhesion region.

Motivated by Spence's earlier work, Prasad and Dasgupta (1975), and Chiu and Prasad (1976) study the plane strain compression of an elastic rectangle by rigid rough planes and a pair of rigid rough punches, respectively. Numerical results consist of shearing traction, normal pressure and the zone of adhesion for various values of Poisson's ratio and the friction coefficient. The zone of adhesion in both problems is found to be independent of the magnitude of loading, but depends on the friction coefficient, Poisson's ratio and the geometrical parameters.

The problems discussed above have a stationary contact zone, i.e. the contact surface is known a priori. Frictional contact problems with an unknown contact zone have not yet been solved. In fact, as noted in Chapter 1, the existence of solutions to such problems where Coulomb's friction law is assumed to hold remains to be proven.

Numerical Treatment of Contact Problems

The vast majority of numerical solutions to contact problems have been obtained using finite element methods. Numerous articles have been written on the use of these methods to solve contact problems. Some of these articles, along with integral equation techniques, are reviewed in this section.

Chan and Tuba (1971) write the finite element equations for each body, separating the force vector into externally applied loads and loads in the possible contact zone. The external loads are then applied incrementally and at each increment an iterative scheme is used to determine the contact zone. Plane stress and plane strain problems using Coulomb friction are solved. A similar approach is used by Ohte (1973), and Tsuta and Yamaji (1973).

Francavilla and Zienkiewicz (1975) write the equations for displacements in the possible contact zone in terms of the contact pressure and the flexibility coefficients. Contact is brought about by assuming a portion of one body to be fixed, while a portion of the other body is given rigid movement. An expression is developed relating the pressure in the contact zone to the initial clearance and the rigid displacements. Once a suitable pressure distribution is found by an iterative scheme, the displacements in the assumed contact zone can be determined. A major advantage of this method is that only a relatively small system of equations is used during the iteration process. Their work is extended by Sachdeva, et.al. (1981) to include force boundary conditions and by Sachdeva and Ramakrishna (1981) to include Coulomb friction. The latter paper presents several examples of frictional contact for problems with proportional loading.

Schafer (1975) introduces special bond elements in the contact zone. With the assumption that a relation between the shear stress and the frictional deformation is known, the stiffness matrix for a bond element is developed by employing the principle of virtual work. The bond element stiffness matrix, along with the usual element stiffness matrix are then used to develop a finite element program. Various bond elements which can be used in construction technology are discussed. Stadter and Weiss (1979) construct a special gap element which replaces the open space between the two bodies, in the potential contact zone. The elastic modulus of a gap element is adjusted according to the normal strain in that element. Conditions for separation and contact are developed using normal strain. An iterative technique based on a stress invariance principle is developed to solve frictionless contact problems.

Gaertner (1977) addresses the plane frictional contact problem by utilizing triangular elements in which the six nodal unknowns are the normal and tangential components of displacement, rotation, tangential strain, and the normal and the tangential stress components. An incremental iterative procedure is used to solve the problem. Results for the frictionless case include Hertz's problem and contact between a connecting rod and a shaft.

Tseng and Olson (1981) use the mixed finite element method, in which both nodal displacements and stresses are retained as unknowns. Thus, contact conditions can be incorporated directly. An incremental loading procedure is used. The numerical results show rather poor agreement with the theoretical results.

Okamoto and Nakazawa (1979) construct contact elements for three dimensional frictional contact problems in which mixed contact conditions are considered. The use of contact elements allows the equilibrium equations and the geometrical boundary conditions to be treated as additional equations independent of the stiffness equations. As a result, only a part of the system of equations related to the contact zone is required to be solved at each load increment. The size of the load increments is calculated such that the contact status of only one node pair changes in each increment. Torstenfelt (1983, 1984) uses a similar approach for automatic load incrementation. His work concentrates on developing a method that can be implemented in any general purpose finite element computer program and does not use any special elements in the contact zone. Numerical results consist of the problem of an indentation of an elastic half space by a rigid circular punch. Both loading and unloading phases are considered. Results are in good agreement with the closed form solution for the loading phase and with the numerical results obtained by Turner (1979) for the unloading phase.

Campos, et. al. (1982) develop variational principles for the analysis of contact problems with Coulomb friction, in which the normal stress distribution in the contact zone is prescribed. First, a finite element approximation of the problem without friction is obtained. The normal contact pressure determined from this solution is then used as a first approximation for the corresponding problem with friction but with prescribed normal pressure. Pires and Oden (1983) extend this work to develop variational principles for incremental analysis of quasi-static contact problems with friction. Non-classical friction laws developed by Oden and Pires (1983) are used. Non-proportional oscillating loads are considered in analyzing the indentation of an elastic body by a rigid punch. Using variational principles, Turner (1979) has solved the problem of contact between a rigid circular cylinder and an isotropic homogeneous linearly elastic half space for the cases of frictionless contact, adhesive contact, and frictional contact for which analytical solutions are available. Turner also considers a problem of frictional unloading for which no analytical solutions are available.

Another numerical technique used to solve contact problems is the integral equation approach in which the normal component of the surface displacement of each body is written as the product of the normal pressure and an influence function, integrated over the entire contact zone. These integrals are then used in a kinematic constraint equation, resulting in a singular Fredholm integral equation of the first kind. Discretization of the integral equation is accomplished by dividing the contact region into a number of rectangular cells over which contact pressure is assumed to be uniform. The resulting system of linear algebraic equations is then solved numerically. For details of this technique the reader is referred to the work of Singh and Paul (1974), Paul and Hashemi (1980, 1981).

The main advantages of the integral equation approach lies in the simplicity of the resulting equations, and the relatively small amount of computational effort required to obtain a numerical solution. Only the potential contact zone is modeled so that the data preparation time is minimal. However, the influence functions are generally based on the Boussinesq function for the half space, so only the problems where geometries are reasonably approximated by a half space can be solved. In conformal contact the Boussinesq function cannot be assumed to be valid. Some alternate means of developing the influence functions, such as the empirical approach used by Paul and Hashemi (1980), must be found to solve the vast majority of problems. In these methods the shape of the bodies away from the contact region is disregarded, restricting their use in the analysis and design of machine elements where deformation and stress patterns of the entire body are often necessary. The integral equation approach is thus a special technique used for solving contact problems and does not have the generality offered by finite element and boundary element methods.

To date, the finite element method remains the most widely employed means of solving contact problems. Complex geometries can be handled routinely, and determination of the complete stress distribution in the body can be readily accomplished once the displacements in the contact zone are known. In recent years general purpose, efficient algorithms for stress analysis using finite element methods have been developed which can be extended to analyze contact problems with relative ease. However, since elements have to be defined over the entire body, modeling complex geometries can result in significant data preparation time. High execution time is inevitable since it is necessary to construct the complete stiffness matrix for both bodies. In the stiffness formulation of the finite element method the displacements are the primary unknowns. In the analysis of contact problems the contact forces are of primary interest, especially when friction is present. Except in the mixed formulation of Tseng and Olson, equilibrium conditions in the contact zone cannot be used directly in the formulation. If the flexibility formulation is used, then a matrix inversion step is required for each body. Introduction of special gap or contact elements makes it possible to retain both displacements and contact forces as primary unknowns but the total number of unknowns which necessarily have to be involved in the iterative process increases. In some of these formulations the system matrix is not symmetric (Okamoto and Nakazawa 1979), which nullifies one of the benefits of using finite element method.

In lieu of the above discussion the boundary integral equation method seems a natural choice for solving elastic contact problems. It retains the generality of the finite element method and offers the simplicity of the integral equation approach in modeling contact problems.

Boundary Integral Equation Method

To solve boundary value problems in classical elastostatics, the boundary integral equation (BIE) method uses the singular solution of the Navier equations. This solution along with the reciprocal work theorem yield an integral equation which provides a relation between boundary displacements and corresponding boundary tractions. The equation can be used to generate a set of simultaneous integral equations from suitable boundary data. In all but the simplest of problems, the integral equation cannot be solved analytically. However, suitable numerical techniques have been developed for solving these equations.

The original development of the BIE method is due to Jawson and Ponter (1963), who apply the method to elastic torsion problems. Their work is extended by Jawson (1963) and Symm (1963) to the solution of problems in potential theory.

Rizzo (1967) develops the BIE formulation for two dimensional problems in linear elasticity. Cruse (1969) extends it to three dimensional problems and also presents a numerical solution technique analogous to the finite element method. Computation of internal displacements and stresses is also included.

Lachat (1975) and Lachat and Watson (1975, 1976) develop a general numerical technique for two and three dimensional elastostatics problems. They introduce several improvements that provide a structure to the BIE method as a numerical technique, and aid in improving the accuracy and efficiency of the BIE method. They represent both geometry and boundary data parametrically in which the variation of these quantities is expressed in terms of the nodal values and shape functions of intrinsic coordinates. This representation allows use of Gaussian quadrature for efficient numerical integration. The order of the quadrature formula to be used is determined for each segment based on calculated error bounds for the formulae and the rapidity of the variation of the integrand within the segment. Examples are presented using linear, quadratic and cubic variation for the boundary data.

The boundary integral equation method has also been extended to solve inelastic problems. Solutions techniques for applying the BIE method to problems with constitutive equations representing various material responses are now being developed. For current developments and research trends in the BIE method the reader is referred to Brebbia et. al. (1984).

The BIE method offers several important advantages over the finite element method and the integral equation method for the numerical solution of contact problems. As mentioned previously, the data preparation time is reduced considerably relative to the finite element methods, since only the boundary needs to be modeled. Input data can be easily changed and less experience is required by the analyst to produce acceptable results.

Since both tractions and displacements are retained as unknowns, the contact equations can be written explicitly and hence, different friction laws can be incorporated easily. Tractions and displacements are computed with the same accuracy and surface tractions in the contact zone, often the most important part of the solutions, are calculated directly.

The BIE method does not impose any restrictions on the shape of the boundary, and therefore complex geometries can be handled routinely. The half-space restrictions on the shape of the body imposed by the integral equation method do not apply.

CHAPTER 3

FORMULATION OF BOUNDARY INTEGRAL EQUATION

In this chapter the basic equations of boundary integral equations (BIE) are developed within the context of the linear theory of elasticity. The development of BIE is now well documented; for details the reader is referred to Brebbia et.al. (1984).

In what follows, all work is referred to a Cartesian coordinate system. The notation used is the usual Cartesian tensor notation, with implied summation on repeated indices, and partial differentiation denoted by comma-index.

The analysis is restricted to the classical elastostatics problems, which is based on the following assumptions:

1. Linear stress-strain relations.
2. Small deformation theory holds, i.e. equilibrium equations can be referred to the undeformed configuration.

In small deformation theory the strain tensor components ε_{IJ} are expressed in terms of the displacement vector components u_I as

$$\epsilon_{IJ} = 1/2(\partial u_I/\partial X_J + \partial u_J/\partial X_I) \qquad (3.1)$$

Consider an elastic body occupying a domain Ω bounded by a surface Γ. If σ_{IJ} are the stress tensor components and B_I

are the body force vector components then the equilibrium equations are

$$\partial\sigma_{IJ}/\partial X_J + B_I = 0 \qquad \text{in } \Omega. \qquad (3.2)$$

Moment equilibrium can be used to show that the stress tensor is symmetric, i.e.,

$$\sigma_{IJ} = \sigma_{JI} . \qquad (3.3)$$

For a homogeneous isotropic elastic medium the stress-strain relations are

$$\sigma_{IJ} = 2G\epsilon_{IJ} + 2G\nu/(1-2\nu)\epsilon_{KK}\delta_{IJ} . \qquad (3.4)$$

G and ν designate the shear modulus and the Poisson's ratio, respectively. δ_{IJ} is the Kronecker delta whose properties are

$$\delta_{IJ} = 0 \qquad I \neq J$$
$$= 1 \qquad I = J. \qquad (3.5)$$

Using Eq. (3.4) in Eq. (3.2), and using Eq. (3.1) in the result, the usual Navier's equations are obtained

$$Gu_{I,JJ} + G/(1-2\nu)u_{J,JI} + B_I = 0 \qquad \text{in } \Omega. \qquad (3.6)$$

A particular solution to these partial differential equations must satisfy, in addition, the boundary conditions for displacements on the surface Γ_u and tractions on Γ_t, respectively given as

$$u_I(x) = q_I(x) \qquad x \in \Gamma_u ,$$

and

$$t_I(x) = \sigma_{IJ}n_J = p_I(x) \qquad x \in \Gamma_t . \qquad (3.7)$$

where $\underline{n}$ is the unit outward normal to the surface Γ and $\underline{t}$ is the traction vector at a point $x \in \Gamma$. $q_I(x)$ are the prescribed displacements on the surface Γ_u, and $p_I(x)$ are the prescribed tractions on Γ_t. In a well posed problem at any given point on the boundary either the displacements $u_I(x)$ or the tractions $t_I(x)$ are known. The problem is to find a solution to Eq. (3.6) subject to the boundary conditions (3.7).

Plane State of Strain or Stress

In solving two-dimensional problems in elasticity, if the X_3 component of displacement vector $\underline{u}$ is constant and if the displacements u_1, u_2 are functions of X_1 and X_2 only, the body is said to be in plane strain state parallel to the X_1, X_2-plane. In plane strain the following conditions hold:

$$\partial u_1/\partial X_3 = \partial u_2/\partial X_3 = 0 \; ; \; u_3 = \text{constant}. \qquad (3.8)$$

The Navier's equations for plane strain (I, J =1,2) are given by Eq. (3.6).

If the stress components in the X_3 direction vanish everywhere,

$$\sigma_{31} = \sigma_{32} = \sigma_{33} = 0, \qquad (3.9)$$

the state of stress is said to be plane stress parallel to the X_1, X_2-plane. Substituting Eq. (3.4) and (3.9) in Eq. (3.2) the Navier's equations for plane stress are obtained (Fung, 1965),

$$Gu_{I,JJ} + G(1+\nu)/(1-\nu)\, u_{J,JI} + B_I = 0 \text{ in } \Omega. \qquad (3.10)$$

Eq. (3.6) is identical to Eq. (3.10), if ν in Eq. (3.6) is replaced by $\nu/(1+\nu)$. Hence, a plane strain formulation may be used to solve plane stress problems if ν is replaced by $\nu/(1+\nu)$.

Somigliana's Identity

Boundary integral equations for elasticity can be developed using the method of weighted residuals (Brebbia, et.al., 1984). These methods usually bear no physical relation to the problem, therefore the equations presented here are derived via Somigliana's identity, which can be deduced from Betti's reciprocal work theorem. This theorem states (Sokolnikoff, 1956):

If an elastic body is subjected to two systems of body and surface forces, then the work that would be done by the first system t_I, B_I in acting through the displacements u'_I due to the second system of forces is equal to the work that would be done by the second system t'_I, B'_I in acting through the displacements u_I due to the first system of forces.

The theorem can be expressed mathematically as

$$\int_\Omega B_I u'_I \, d\Omega + \int_\Gamma t_I u'_I \, d\Gamma = \int_\Omega B'_I u_I \, d\Omega + \int_\Gamma t'_I u_I \, d\Gamma \, . \qquad (3.11)$$

where Ω is the region bounded by Γ.

Let the body force components B'_I be the unit loads applied at a source point $\underline{\xi} \in \Omega$ in each of the coordinate directions given by unit vectors e_I. Then the body force distribution can be expressed as

$$B'_I = \Delta(\underline{\xi} - \underline{x})e_I . \qquad (3.12)$$

where $\Delta(\underline{\xi} - \underline{x})$ is the Dirac delta function, $\underline{x} \in \Omega$ is an observation point, and the underscore indicates a vector. The Dirac delta function has the following properties:

$$\Delta(\underline{\xi} - \underline{x}) = 0 \qquad \underline{x} \neq \underline{\xi} ,$$

$$\int_\Omega g(\underline{x})\, \Delta(\underline{\xi} - \underline{x}) = g(\underline{\xi}) \qquad \underline{\xi} \in \Omega$$
$$= 0 \qquad \underline{\xi} \notin \Omega . \qquad (3.13)$$

Therefore the first integral on the right hand side in Eq. (3.11) then becomes

$$\int_\Omega B'_I u_I \, d\Omega = \int_\Omega e_I \, u_I(\underline{x})\Delta(\underline{\xi} - \underline{x}) \, d\Omega = u_I(\underline{\xi})e_I . \qquad (3.14)$$

If a unit load is applied successively in each of the coordinate directions, then the displacements u'_J and the tractions t'_J can be represented in the following form:

$$u'_J = U_{IJ}(\underline{\xi}, \underline{x})\, e_I ,$$
$$t'_J = T_{IJ}(\underline{\xi}, \underline{x})\, e_I . \qquad (3.15)$$

where $U_{IJ}(\underline{\xi},\underline{x})$ and $T_{IJ}(\underline{\xi},\underline{x})$ represent the displacements and tractions, respectively in the J direction at $\underline{x}$ corresponding to an unit point force acting in the I direction at $\underline{\xi}$. Substituting Eq. (3.14) and (3.15) into Eq. (3.11), the displacement components u_I at $\underline{\xi}$ can be written in the following form

$$u_I(\underline{\xi}) = \int_\Gamma U_{IJ}(\underline{\xi}, \underline{x})\, t_J(\underline{x})\, d\Gamma(\underline{x}) - \int_\Gamma T_{IJ}(\underline{\xi}, \underline{x})\, u_J(\underline{x})\, d\Gamma(\underline{x})$$
$$+ \int_\Omega U_{IJ}(\underline{\xi}, \underline{x})\, B_J(\underline{x})\, d\Omega(\underline{x}) . \qquad (3.16)$$

The above equation is Somigliana's identity for displacements. This identity is the starting point for the BIE method. The terms $U_{IJ}(\underline{\xi},\underline{x})$ are the fundamental solutions of Navier equations, i.e., they satisfy

$$Gu_{I,JJ} + G/(1-2\upsilon)u_{J,JI} + \Delta(\underline{\xi} - \underline{x})\, e_I = 0 . \qquad (3.17)$$

Fundamental Solutions

These fundamental solutions, or free-space Green's functions, are necessary for the boundary integral equation formulation. They are defined over the entire region Ω, except at the source point $\underline{\xi} \in \Omega$, but are not required to satisfy boundary conditions on Γ. If the fundamental solution is forced to satisfy the boundary conditions then the Green's function for the specific problem is obtained.

The solution of Eq. (3.17) for three-dimensional problems was first obtained by Kelvin (Love, 1929) as

$$U_{IJ}(\underline{\xi}, \underline{x}) = (1+\upsilon)/8\pi E(1-\upsilon)r\ [(3-4\upsilon)\delta_{IJ} + r_{,I}r_{,J}] \qquad (3.18)$$

where E is the Young's modulus and its relationship between G and ν is

$$E = 2G(1+\upsilon) . \qquad (3.19)$$

$r = r(\underline{\xi}, \underline{x})$ is the distance between source point and observation point:

$$r_I = X_I(\underline{x}) - X_I(\underline{\xi}) ,$$
$$r = (r_I r_I)^{1/2} . \qquad (3.20)$$

The notation $r_{,I}$ indicates differentiation with respect to X_I

$$r_{,I} = \partial r_I / \partial X_I = r_I / r . \tag{3.21}$$

Eq. (3.18) is the solution for the displacements at any point $\underline{x}$ due to a unit point load at a point $\underline{\xi}$ in an infinite elastic medium; note that the solution is singular at the source point, $\underline{x} = \underline{\xi}$. Using the stress-strain relation Eq. (3.4) and Eq. (3.18) a similar expression for tractions can be obtained, that is:

$$T_{IJ}(\underline{\xi}, \underline{x}) = \{1/8\pi(1-\upsilon)r^2\} [(1-2\upsilon)(r_{,I}n_J - r_{,J}n_I) - n_m r_{,m}\{(1-2\upsilon)\delta_{IJ} + 3r_{,I}r_{,J}\}] . \tag{3.22}$$

Note that $U_{IJ}(\underline{\xi}, \underline{x})$ and $T_{IJ}(\underline{\xi}, \underline{x})$ are symmetrical in $\underline{\xi}$ and $\underline{x}$.

By integrating this solution along an infinite line, the solution for a line load of unit intensity in an infinite elastic medium is obtained (Danson, 1983) and this modified solution can be used for plane strain problems. In integrating Eq. (3.18) the displacements are found to be infinite since the line load extends to infinity. Even though the displacements are infinite the strains will be finite and by finding the strain field, a fundamental solution for the plane strain problem is then obtained. These expressions are

$$U_{IJ}(\underline{\xi}, \underline{x}) = (1+\upsilon)/4\pi E(1-\upsilon)r \; [(3-4\upsilon)\ln(1/r)\delta_{IJ} + r_{,I}r_{,J}]. \tag{3.23}$$

and

$$T_{IJ}(\xi, x) = 1/4\pi(1-\upsilon)r\,[(1-2\upsilon)(r_{,I}n_J - r_{,J}n_I) - n_m r_{,m}\{(1-2\upsilon)\delta_{IJ} + 2r_{,I}r_{,J}\}]\,. \quad (3.24)$$

The plane strain equations are valid for plane stress problems if ν and E are replaced by

$$\upsilon' = \upsilon/(1+\upsilon)\,,$$
$$E' = E(1+2\upsilon)/(1+\upsilon^2)\,. \quad (3.25)$$

Boundary Integral Equation

Eq. (3.16) gives displacements at a point $\underline{\xi}$ in Ω, in terms of surface integrals of boundary displacements and tractions, and a volume integral of the body force distribution. It provides a means of determining displacements on the interior once the boundary tractions and displacements are known. If the source point $\underline{\xi} \in \Omega$ is moved to the boundary, an expression containing only boundary data will result, which can be used to obtain the boundary unknowns.

Care must be taken in evaluating Eq. (3.16) as $\underline{\xi}$ is moved to the boundary since Gauss's divergence theorem, which is used in deriving Betti's theorem (Eq. 3.11), requires all functions to be continuous. Since the kernels $T_{IJ}(\underline{\xi},\underline{x})$ and $U_{IJ}(\underline{\xi},\underline{x})$ are singular at the source point, some material must be added where $\underline{\xi} \in \Omega$ intersects the boundary so that the source point can be enclosed in portion of a small sphere of radius δ (Fig. 1). Eq. (3.16) is evaluated over the boundary $\Gamma-\Gamma^{*}+\Gamma_{\delta}$ and the domain $\Omega' = \Omega + \Omega_{\delta}$, as δ --> 0 :

$$\delta_{IJ}u_J(\xi) = \text{Lim } \delta \to 0 \ \{\int_{\Gamma-\Gamma^{*}+\Gamma\delta} U_{IJ}(\xi,x)t_J d\Gamma(x)$$

$$- \int_{\Gamma-\Gamma^{*}+\Gamma\delta} T_{IJ}(\xi,x)u_J(x)d\Gamma(x) + \int_{\Omega'} U_{IJ}(\xi,x)B_J(x)d\Omega(x)\}. \qquad (3.26)$$

As δ --> 0 , Ω' --> Ω; therefore the integral with body force term B_J does not change. Consider the second integral in the above equation :

$$\text{Lim}\delta\to 0 \int_{\Gamma-\Gamma^{*}+\Gamma\delta} T_{IJ}(\xi,x)\, u_J(x)\, d\Gamma(x) =$$

$$\text{Lim}\delta\to 0 \ [\int_{\Gamma-\Gamma^{*}} T_{IJ}(\xi,x)u_J(x)d\Gamma(x) + \int_{\Gamma\delta} T_{IJ}(\xi,x)u_J(x)d\Gamma(x)]. \qquad (3.27)$$

As δ --> 0, $\Gamma - \Gamma^{*}$-->Γ; therefore only the second integral in Eq. (3.27) needs to be evaluated. The term $u_J(x)$ can be taken outside the integral since $u_J(x)$ --> $u_J(\xi)$ as δ -->0. Thus the integral to be evaluated is

$$I_{IJ} = \text{Lim } \delta \to 0 \int_{\Gamma\delta} T_{IJ}(\xi,x)d\Gamma(x) \, . \qquad (3.28)$$

where the kernel $T_{IJ}(\xi,x)$ is given by the Eq. (3.22). With the use of polar coordinates, this integral can be evaluated in closed form for the two dimensional case (Danson, 1983). Its value in general depends on the shape of the boundary at the point ξ (Fig. 1); for smooth boundary $I_{IJ} = -1/2\ \delta_{IJ}$. For three-dimensional case the form of I_{IJ} is more complex, and is taken in the Cauchy principle value sense. This does not pose any problems, as later it will be shown that I_{IJ} need not be computed explicitly (see Appendix A).

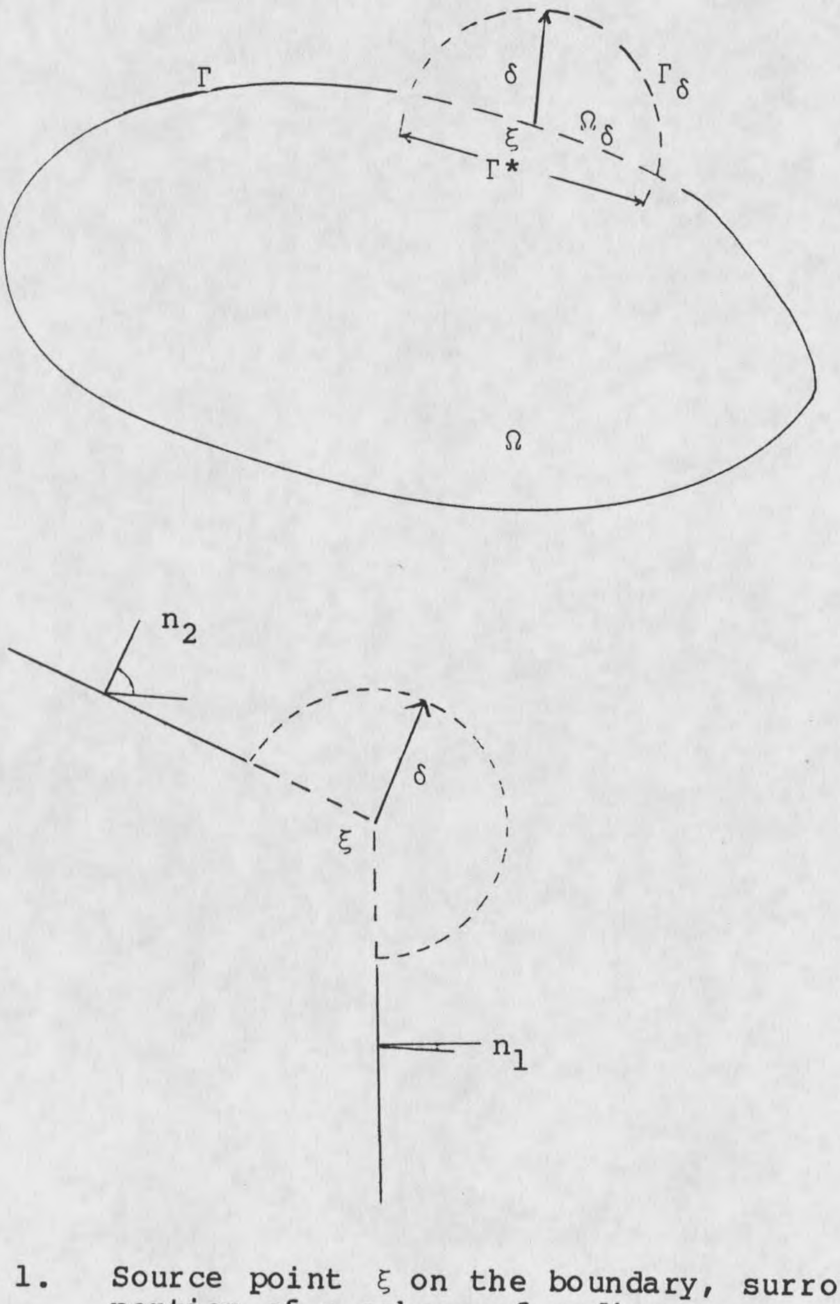

Figure 1. Source point ξ on the boundary, surrounded by a portion of a sphere of radius δ.