Mono-cable logging system with intermediate tension control : computer simulation

by Albert Vincent Turk

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Engineering Mechanics
Montana State University

Abstract:
A cooperative research project with the USDA-Forest Service has shown the need for a mono-cable logging system with intermediate tension control as an alternative to road construction in the logging area. Intermediate control sheaves will be required to maintain optimum tension throughout long distances and large elevation changes. Preliminary designs have shown the need to model the system to demonstrate the system's feasability.

A series of FORTRAN programs were written to simulate this system, building first upon single span solutions, with the eventual program including the capability of simulating the changes in tension of a system of up to 100 connected spans with as many as 200 loads traversing the system. The individual sheaves along the system path can be modeled as either control or idler sheaves. The solution of the necessary equations required the use of iterative solutions.

Comparison was made of the Newton-Raphson method and Brown's method for the solution of a set of two non-linear equations with two unknowns.

The model satisfies the simulation needs of the system. It allows the monitoring of tension, differential tension across sheaves, and changes in cable length required within each span. The program showed the feasibility of the system and gives direction for future modifications of the model to more closely simulate the actual situation.

The program will provide a solid foundation for current and future work in this area and shows a valid procedure for future modeling.

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ABSTRACT

A cooperative research project with the USDA-Forest Service has shown the need for a mono-cable logging system with intermediate tension control as an alternative to road construction in the logging area. Intermediate control sheaves will be required to maintain optimum tension throughout long distances and large elevation changes. Preliminary designs have shown the need to model the system to demonstrate the system's feasability.

A series of FORTRAN programs were written to simulate this system, building first upon single span solutions, with the eventual program including the capability of simulating the changes in tension of a system of up to 100 connected spans with as many as 200 loads traversing the system. The individual sheaves along the system path can be modeled as either control or idler sheaves. The solution of the necessary equations required the use of iterative solutions. Comparison was made of the Newton-Raphson method and Brown's method for the solution of a set of two non-linear equations with two unknowns.

The model satisfies the simulation needs of the system. It allows the monitoring of tension, differential tension across sheaves, and changes in cable length required within each span. The program showed the feasibility of the system and gives direction for future modifications of the model to more closely simulate the actual situation.

The program will provide a solid foundation for current and future work in this area and shows a valid procedure for future modeling.

CHAPTER 1

INTRODUCTION

As an alternative to building logging roads in rugged and ecologically sensitive areas, the U. S. Forest Service is exploring other methods of timber harvest. A monocable system with intermediate tension control is one such option.[1] The system would be used to transport timber over unrestricted distances from the forest to landing sites, using standing timber or temporary towers, braced with guy lines to support both idler and control sheaves. The supports are arranged in a zig-zag pattern to facilitate passage of suspended loads. This also allows a greater tractive force to be applied by the control sheaves. The addition of braking/pulling sheaves at intermediate points allows an increase of the possible elevational change using a given cable size. As one span is given more cable to relieve tension neighboring spans have to give up cable and are subject to tension increases. Coordination of the cable speed adjustment provided by these sheaves is thus necessary to prevent a combination of adjustments causing tensions outside the operating range and a resultant system failure. Mechanisms to measure tension on either side of a sheave have

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[1]William E. Larsen, Michael J. Gonsior, W. Robert Taylor, "USFS - MSU Cooperative Research Project", attachment 2. USDA Forest Service Intermountain Research Station and Montana State University College of Engineering, 1985.

been designed as well as a control system for hydraulically powered sheaves.[2] This thesis explains the derivation of the equations of cable tension and load position, and discusses the merits of the iterative methods of solution that are required. The system of combined spans is then assembled with a scheme for coordinated response using characteristics of the proposed control sheaves. Various geographical layouts and system response times may then be simulated and the modifications to layout and system components tested to determine system response as a function of time.

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[2]Ibid., attachment 1.

CHAPTER 2

DERIVATION OF SINGLE SPAN EQUATIONS

The Classic Catenary

The free-body diagram of a segment of cable is shown with spatial coordinates x and z in Figure 1.[1] The cable weight per unit length is w (considered constant), and the tension T is a function of position. For a segment of finite length δs, spanning finite spatial dimensions δx and δz, the equation of horizontal equilibrium is

$$[T(dx/ds) + (d(T(dx/ds))/ds)\delta s] - T(dx/ds) = 0 \tag{2.1}$$

$$\text{or } d(T(dx/ds))/ds = 0. \tag{2.2}$$

Thus the change in T(dx/ds) with respect to the variable s (span position) is zero. Integrating with respect to s

$$\int [d(T(dx/ds))/ds]ds = T(dx/ds) + C = 0. \tag{2.3}$$

This includes C as a constant of integration. Letting dx/ds represent the cosine of the cable slope angle Θ, where H is the horizontal component of cable tension (constant throughout the span) the equation becomes

$$T(dx/ds) = T\cos\Theta = H \quad \text{or} \quad T = H(ds/dx) = H\sec\Theta. \tag{2.4}$$

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[1]H. Max Irvine, Cable Structures, (Cambridge: MIT Press, 1981), 5.

The equation of vertical equilibrium is

(2.5) $$[T(dz/ds) + (d(T(dz/ds))/ds)\delta s + w\delta s] - T(dz/ds) = 0$$

(2.6) $$\text{or } d(T(dz/ds))/ds = -w.$$

Thus the change in $T(dz/ds)$ is not zero with respect to the variable s. Integrating with respect to s

(2.7) $$\int [d(T(dz/ds))/ds]ds = \int -wds.$$

Letting dz/ds represent the sine of the angle Θ

(2.8) $$T(dz/ds) = T \sin \Theta = -ws + C$$

An alternate form of equation 2.6 can be found in combination with a manipulation of equation 2.4.[1]

(2.9) $$d((H(ds/dx))(dz/ds))/ds = H[d(dz/dx)/ds] = -w$$

(2.10) $$H[(d(dz/dx)/dx)(dx/ds) + (d(dz/dx)/dz)(dz/ds)] = -w$$

(2.11) $$H[(d^2z/dx^2)(dx/ds) + (d(dz/dz)/dx)(dz/ds)] = -w$$

(2.12) $$H(d^2z/dx^2)(dx/ds) = -w$$

(2.13) $$H(d^2z/dx^2) = -w(ds/dx)$$

In the case of $-w(ds/dx)$ (the load intensity per unit span) equal to a constant K_1, 1.13 can be integrated twice to give a parabolic profile.

(2.14) $$z = \tfrac{1}{2}K_1x^2 + K_2x + K_3.$$

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[1] Ibid., 4-6.

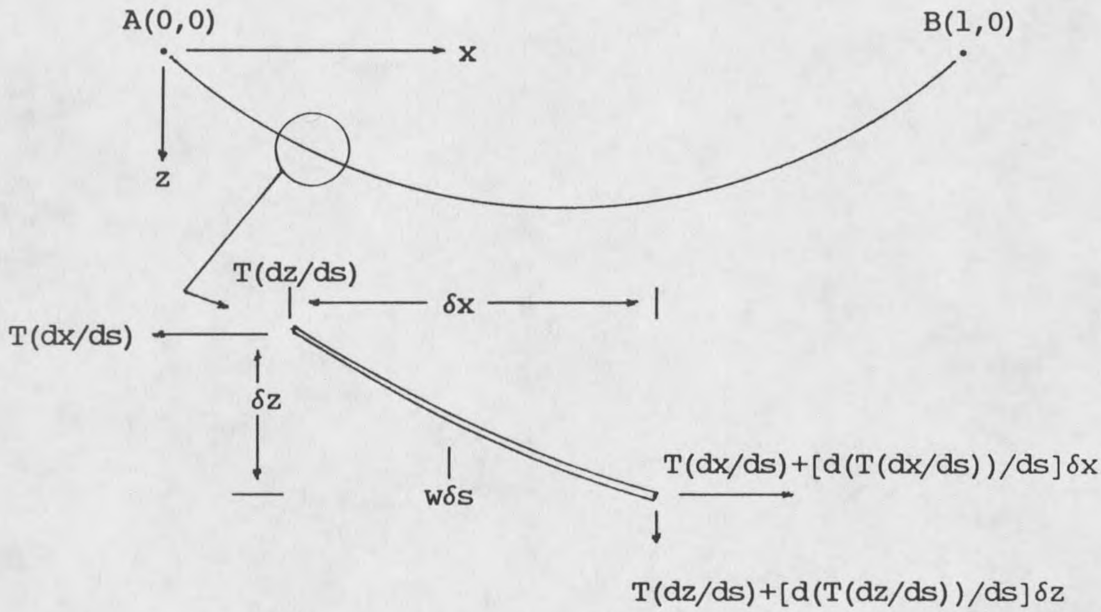

Figure 1. Free-body diagram of a cable segment.

The values of the constants can be determined using the given boundary conditions. When the quantity $-w(ds/dx)$ is not constant, equation 2.13 can be solved using the geometric constraint relating ds to dx and dz.

(2.15) $$dx^2 + dz^2 = ds^2$$

(2.16) $$ds/dx = (1 + (dz/dx)^2)^{\frac{1}{2}}$$

Equation 2.13 then becomes

(2.17) $$H(d^2z/dx^2) = -w(1 + (dz/dx)^2)^{\frac{1}{2}}$$

(2.18) $$\text{or } d^2z/dx^2 = -(w/H)(1 + (dz/dx)^2)^{\frac{1}{2}}.$$

With cable ends at coordinates (0,0) and (0,£) equation 2.17 is satisfied by[1]

(2.19) $$z = (H/w)\{\cosh(w£/2H) - \cosh([w/H][(£/2) - x])\}$$

From the relationship developed in equation 2.15 cable length s between the origin and any x coordinate can be found.

(2.20) $$s = \int_0^x \{1 + (dz/dx)^2\}^{\frac{1}{2}}\, dx$$

(2.21) $$s = (H/w)\{\sinh(w£/2H) - \sinh([w/H][(£/2) - x])\}$$

Given a cable of length L_0 spanning the gap of horizontal dimension £ the horizontal component of tension may be found by solving equation 2.22 for H (assuming w is known and constant).

(2.22) $$\sinh(w£/2H) - (wL_0/2H) = 0$$

The tension at any x coordinate is given by

(2.23) $$T = T(x) = H \cosh\{[w/H][(£/2) - x]\}.$$

A different development of the equations begins with equations 2.2 and 2.6 expressed in trigonometric form.[2]

(2.24) $$d(T \cos \Theta)/ds = 0$$

(2.25) $$d(T \sin \Theta)/ds = w(s)$$

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[1]Ibid., 6.

[2]Irving H. Shames, Engineering Mechanics: Statics and Dynamics, (Englewood Cliffs: Prentice Hall, Inc. 1980), 240.

Here w is not constant but considered to be a function of position s along the span. Integrating both equations with respect to s gives

(2.26) $$T \cos \Theta = H \quad \text{or} \quad T = H / \cos \Theta.$$

(2.27) $$T \sin \Theta = \int w(s)ds + C_1'$$

(2.28) $$T \sin \Theta = (H / \cos \Theta)\sin \Theta = H \tan \Theta = H(dz/dx)$$

(2.29) $$dz/dx = (1/H)\int w(s)ds + C_1$$

Equation 2.29 cannot be integrated directly as shown as the right side involves a function of s, but may be integrated with respect to s by eliminating dz using the geometric relationship of equation 2.15.

(2.30) $$(dz/dx)^2 + (dx/dx)^2 = (ds/dx)^2$$

(2.31) $$dz/dx = ((ds/dx)^2 - 1)^{\frac{1}{2}}$$

Inserting the representation of dz/dx into equation 2.29 and separating variables

(2.32) $$((ds/dx)^2 - 1)^{\frac{1}{2}} = (1/H)\int w(s)ds + C_1.$$

(2.33) $$ds/dx = \{1 + [(1/H)\int w(s)ds + C_1]^2\}^{\frac{1}{2}}$$

(2.34) $$x = \int \{1 + [(1/H)\int w(s)ds + C_1]^2\}^{-\frac{1}{2}} ds + C_2$$

Equation 2.34 may be integrated when the function w(s) is known and the constants of integration may be evaluated from boundary conditions. Function w(s) is a constant in the case of an inextensible cable of constant mass per unit length. The profile of an inextensible cable loaded by its own weight is known as a catenary.

The Elastic Catenary

Irvine's Development

An actual cable will stretch when under tension in accordance to Hooke's Law

(2.35) $$\sigma = E\epsilon$$

where E is the cable elastic modulus, σ is the stress, and ϵ is the strain. With a cable suspended between points A (0,0) and B (l,h) the unstrained profile is a function of cable position s, as shown in Figure 2. Allowing the cable to deform under load to a strained profile the location of a given point s is now defined by the Lagrangian coordinate p.[1] The weight of the entire cable of unstrained length L_o between points A and B is W.

(2.36) $$W = wL_o$$

At A define the vertical and horizontal components of tension as V and H respectively. Allowing s to equal zero at A and equal L_o at B the components of tension at a point s along the cable are defined as

(2.37) $$T \sin \Theta = T(dz/dp) = V - W(s/L_o)$$

(2.38) $$\text{and } T \cos \Theta = T(dx/dp) = H.$$

Θ is now defined by the deformed position of the cable. Equation 2.35 may now be written in terms of the deformation. Allowing A_o to equal

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[1]Irvine, Cable Structures, 16.

the unstrained cross sectional area, L to equal the strained length of the cable, and E to equal the cable elastic modulus

(2.39) $$T/A_o = E((dp/ds) - 1),$$

(2.40) $$\text{or } T = EA_o((dp/ds) - 1).$$

The end conditions of s = p = 0 at (0,0) and $s = L_o$, p = L at (£,h) allow solution for x, z, and T as functions of s. The geometric constraint to satisfy becomes

(2.41) $$(dx/dp)^2 + (dz/dp)^2 = 1.$$

(2.42) $$(H/T)^2 + ((V - Ws/L_o)/T)^2 = 1$$

This yields the desired expression for tension as a function of s.

(2.43) $$(H^2 + (V - Ws/L_o)^2)^{\frac{1}{2}} = T(s)$$

Working from equations 2.38 and 2.40

(2.44) $$dx/dp = H/T(s).$$

(2.45) $$dp/ds = (T(s)/EA_o) + 1$$

(2.46) $$dx/ds = (dx/dp)(dp/ds) = (H/EA_o) + H/T(s)$$

(2.47) $$x = H \int_0^s (1/EA_o) + (H^2 + (V - Ws/L_o)^2)^{-\frac{1}{2}}] \; ds$$

When incorporating the end conditions at (0,0) the resultant form is

(2.48) $$x(s) = Hs/EA_o + (HL_o/W)[\sinh^{-1}(V/H) - \sinh^{-1}\{(V - Ws/L_o)/H\}].$$

Using equations 2.37 and 2.40

(2.49) $$dz/dp = (V - Ws/L_o)/T(s).$$

(2.50) $$dz/ds = (dz/dp)(dp/ds) = \{(V - Ws/L_o)/T(s)\}\{(T(s)/EA_o) + 1\}$$

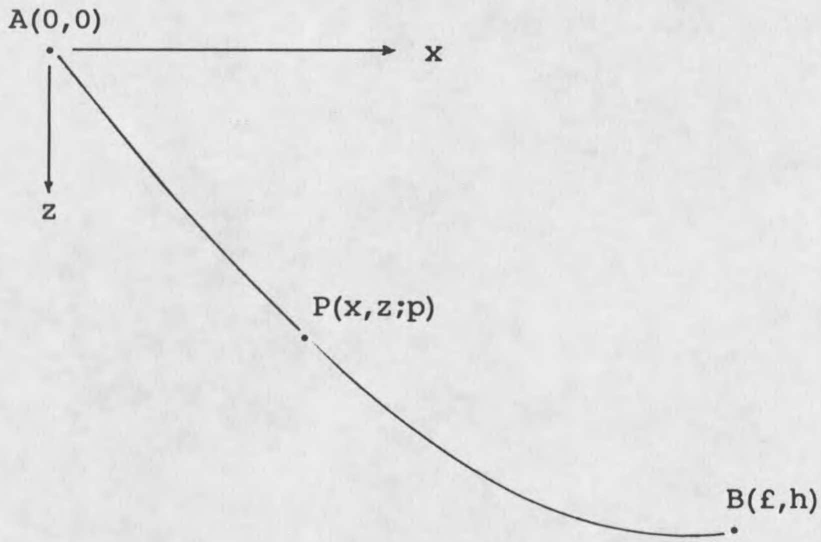

Figure 2. Coordinates for the elastic catenary.

$$z = \int_0^s [\{(V - Ws/L_o)/EA_o\} + \{(V - Ws/L_o)/T(s)\}]ds \tag{2.51}$$

Integration with respect to s, and use of end conditions at (0,0) yields

$$z(s) = (HL_o/W)[\{1+(V/H)^2\}^{\frac{1}{2}} - \{1+((V - Ws/L_o)/H)^2\}^{\frac{1}{2}}] + (Ws/EA_o)\{(V/W) - (s/2L_o)\}. \tag{2.52}$$

By substituting L_o for s in equations 2.48 and 2.52 values for h and l will be found, provided the correct values of H and V are used in the equations. Setting the equations in the form $f(H,V) = 0$ yields a set of two nonlinear simultaneous equations with two unknowns.

$$x(H,V) - h = 0 \tag{2.53}$$

$$z(H,V) - £ = 0 \tag{2.54}$$

An iterative solution must be pursued for the values of H and V. Since H and V are functions of T and Θ the latter two unknowns may be solved for when that is more convenient. In methods that require the simultaneous solution of equations, T and Θ must be used, as H and V are not independant variables.

Gonsior's Development

From the starting point of the cable segment shown in Figure 3, the equations of equilibrium may be stated as[1]

(2.55) $$T \cos \Theta - (T + dT) \cos (\Theta + d\Theta) = 0$$

(2.56) and, $$T \sin \Theta - (T + dT) \sin (\Theta + d\Theta) - w ds = 0.$$

The weight per unit length of the deformed cable (w) can be expressed in terms of the strain (ϵ) and unstrained weight per unit length ($\tilde{w}$).

(2.55) $$w = \tilde{w} / (1 + \epsilon)$$

With the unstrained cross sectional area expressed as A_o the strain may be expressed as

(2.56) $$\epsilon = T/EA_o.$$

(2.57) $$w = \tilde{w} / (1 + T/EA_o)$$

Letting $1/EA_o = K$ equation 57 becomes

(2.58) $$w = \tilde{w} / (1 + TK).$$

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[1]William E. Larsen, Michael J. Gonsior, W. Robert Taylor, "USFS - MSU Cooperative Research Project" A4.4.

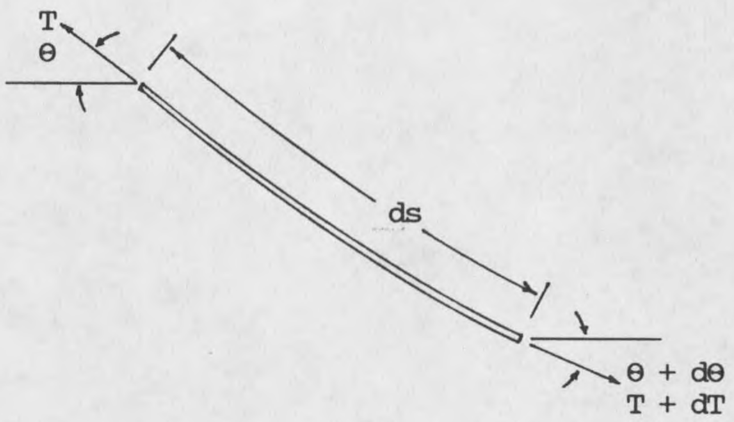

Figure 3. Free-body diagram of cable segment.

By expanding equations 2.55 and 2.56 using the identities for sine and cosine of a sum the equations may be restated as in equation 2.59.

(2.59) $$T \cos \Theta - (dT + T)(\cos \Theta \cos d\Theta - \sin \Theta \sin d\Theta) = 0$$

(2.60) $$T \sin \Theta - (dT + T)(\sin \Theta \cos d\Theta + \cos \Theta \sin d\Theta) = wds$$

As an angle α goes to zero the cosine of α approaches 1 and the sine of α becomes α. Thus by letting sin $d\Theta$ equal $d\Theta$, and cos $d\Theta$ equal 1

(2.61) $$T \cos \Theta - (dT + T)(\cos \Theta - d\Theta \sin \Theta) = 0$$

(2.62) $$\text{or} \quad dTd\Theta \sin \Theta - dT \cos \Theta + Td\Theta \sin \Theta = 0,$$

(2.63) $$T \sin \Theta - (dT + T)(\sin \Theta + d\Theta \cos \Theta) = wds$$

(2.64) $$\text{or} \quad dT \sin \Theta + dTd\Theta \cos \Theta + Td\Theta \cos \Theta + wds = 0.$$

Ignoring terms involving products of differential quantities

(2.65) $$Td\Theta \cos \Theta + dT \sin \Theta = wds = (\tilde{w} / (1 + TK))ds,$$

(2.66) $$\text{and} \quad dT \cos \Theta = Td\Theta \sin \Theta.$$

(2.67) $$dT/T = \tan\Theta \, d\Theta$$

(2.68) $$\int dT/T = \int \tan\Theta \, d\Theta$$

(2.69) $$\ln(T) = \ln(\sec\Theta) + C$$

(2.70) $$\ln(T) - \ln(\sec\Theta) = \ln(T/\sec\Theta) = \ln(T\cos\Theta) = C$$

(2.71) $$T\cos\Theta = e^C = C' = H \quad \text{or} \quad T = H\sec\Theta$$

(2.72) $$dT/d\Theta = H(\tan\Theta\sec\Theta) \quad \text{or} \quad dT = H\tan\Theta\sec\Theta\, d\Theta$$

Combining equations 2.65, 2.71, and 2.72

(2.73) $$Hd\Theta + H\tan^2\Theta\, d\Theta = (\tilde{w}/(1 + HK\sec\Theta))ds.$$

(2.74) $$(1 + \tan^2\Theta)Hd\Theta(1 + (HK\sec\Theta))/\tilde{w} = ds$$

(2.75) $$(\sec^2\Theta)(1+HK\sec\Theta)(H/\tilde{w})d\Theta = (\sec^2\Theta + HK\sec^3\Theta)(H/\tilde{w})d\Theta = ds$$

Integrating both sides of equation 2.75 between proper limits,

(2.76) $$s - s_o = (H/\tilde{w})\{\tan\Theta - \tan\Theta_o + \tfrac{1}{2}KH[\ln(\sec\Theta + \tan\Theta) + \tan\Theta\sec\Theta - \ln(\sec\Theta_o + \tan\Theta_o) - \tan\Theta_o\sec\Theta_o]\}.$$

The quantity $s - s_o$ is the distance of strained cable between the points (x,z) and (x_o,z_o) where the tensions are T and T_o, with cable slopes Θ and Θ_o respectively. The distance $(x - x_o)$ was previously defined as $\pounds$, between points $(0,0)$ and $(\pounds,h)$. Since H is constant throughout the span

(2.77) $$T\cos\Theta = T_o\cos\Theta_o = H.$$

A typical span is illustrated in Figure 4. The equation of vertical equilibrium may be written as

(2.78) $$T\sin\Theta - T_o\sin\Theta_o - W_c = 0.$$

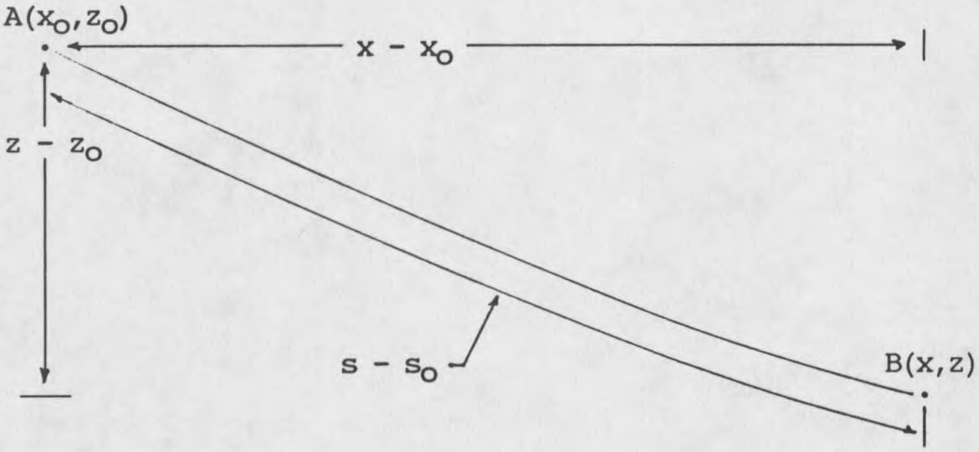

Figure 4. Typical span.

If W_C is the weight of the cable in the span, the unstrained length may be found to be

(2.79) $$L_O = W_C/\tilde{w} = (T \sin \Theta - T_O \sin \Theta_O)/w.$$

By combining equations 2.76, 2.77, and 2.79 total cable stretch may be found.

(2.80) $$L - L_O = (KH^2/2\tilde{w})[\ln(\sec \Theta + \tan \Theta) + \tan \Theta \sec \Theta - \ln(\sec \Theta_O + \tan \Theta_O) - \tan \Theta_O \sec \Theta_O]$$

In order to find equations for x and z as functions of Θ the chain rule is used.

(2.81) $$dx/d\Theta = (dx/ds)(ds/d\Theta) = (ds/d\Theta)\cos \Theta$$

(2.82) $$dz/d\Theta = (dz/ds)(ds/d\Theta) = (ds/d\Theta)\sin \Theta$$

From equation 2.76, when differentiated with respect to Θ

(2.83) $$ds/d\theta = (H/\tilde{w})(\sec^2\theta + KH \sec^3\theta).$$

(2.84) $$dx/d\theta = H(\sec\theta + KH \sec^2\theta)$$

(2.85) $$dz/d\theta = H(\tan\theta \sec\theta + KH \tan\theta \sec^2\theta)$$

Separating variables and integrating both sides

(2.86) $$(x - x_o) = (H/\tilde{w})\int_{\theta_o}^{\theta}(\sec\theta + KH \sec^2\theta)d\theta.$$

(2.87) $$(x - x_o) = (H/\tilde{w})[\ln((\sec\theta + \tan\theta)/(\sec\theta_o + \tan\theta_o)) + KH(\tan\theta - \tan\theta_o)]$$

(2.88) $$(z - z_o) = (H/\tilde{w})\int_{\theta_o}^{\theta}(\tan\theta \sec\theta + KH \tan\theta \sec^2\theta)d\theta$$

(2.89) $$(z - z_o) = (H/\tilde{w})[(\sec\theta - \sec\theta_o) + \tfrac{1}{2}KH(\tan^2\theta - \tan^2\theta_o)]$$

Thus we again have two simultaneous equations from which two unknown values may be found. Further manipulation yields equations in the form

(2.90) $$\theta_o = \sin^{-1}[(1 - f)/(1 + f)],$$

(2.91) $$\theta = \sin^{-1}[(g - 1)/(g + 1)],$$

(2.92) $$f = [\cos^2\theta/(1 + \sin\theta)^2]e^{\alpha},$$

(2.93) $$g = [(1 + \sin\theta_o)^2/\cos^2\theta_o]e^{\alpha},$$

(2.94) $$\alpha = 2[(\tilde{w}/H)(x - x_o) - KH(\tan\theta - \tan\theta_o)].$$

This will allow for an iterative solution for one value of θ, given H, $(x - x_o)$, and the other value of θ.[1] It can be shown that Irvine's and Gonsior's derivations are totally equivalent.

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[1]Ibid., A4.8

Response to a Point Load

The slope of the cable section at a point and the magnitude of the components of cable tension at the point are directly related. It has been shown that the horizontal component of tension is constant, thus it is the change in the vertical component which is responsible for the change in cable slope. The vertical component of tension changes with distance s in a continuous manner. When a point load is applied to the cable there is a discontinuity in the cable slope at this point. A cable with a point load of weight M, located a distance of unstrained cable s_1 from the origin (0,0) will have slope Θ with tan Θ equal to V/H at the initial point, tan Θ_1 equal to $((V - (Ws_1/L_o))/H)$ immediately preceding the load, tan Θ_2 equal to $((V - M - (Ws_1/L_o))/H)$ immediately following the load, and tan Θ_o equal to $((V - M - W)/H)$ at the terminal point (ℓ,h), as in Figure 5. The equations of position x and z must satisfy the geometric constraint that they be equal on each side of the discontinuity. This is done using proper values for the constants of integration. The equations for T, x, and z take two forms, one valid on $0 \le s \le s_1$, and the other valid on $s_1 \le s \le L_o$. Letting Θ_s be the slope at the point s, for $0 \le s \le s_1$ (i. e., preceding the point load)[1]

$$T(s) = \{H^2 + (V - Ws/L_o)^2\}^{\frac{1}{2}} \tag{2.95}$$

$$\begin{aligned} x(s) &= Hs/EA_o + (HL_o/W)[\sinh^{-1}(V/H) - \sinh^{-1}((V - Ws/L_o)/H)] \\ &= T\cos\Theta\{s/EA_o + (L_o/W)[\sinh^{-1}(\tan\Theta) - \sinh^{-1}(\tan\Theta_s)]\} \end{aligned} \tag{2.96}$$

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[1]Irvine, Cable Structures, 20-21.

$$(2.97)\quad z(s) = (HL_o/W)[\{1 + (V/H)^2\}^{\frac{1}{2}} - \{1 + ((V - Ws/L_o)/H)^2\}^{\frac{1}{2}}] + (Ws/EA_o)[(V/W) - (s/2L_o)]$$

$$= ((T\cos\theta)L_o/W)[\sec\theta - \sec\theta_s] + (Ws/EA_o)[(\tan\theta) - (s/2L_o)].$$

For the point s located beyond the point load, $s_1 \leq s \leq L_o$

$$(2.98)\quad x(s) = Hs/EA_o\ (HL_o/W)[\sinh^{-1}(V/H) - \sinh^{-1}((V-M-(Ws/L_o))/H) + \sinh^{-1}((V-M-(Ws_1/L_o))/H) - \sinh^{-1}((V-(Ws_1/L_o))/H)]$$

$$= \{s/EA_o + (L_o/W)[\sinh^{-1}(\tan\theta) - \sinh^{-1}(\tan\theta_s) + \sinh^{-1}(\tan\theta_2) - \sinh^{-1}(\tan\theta_1)]\}T\cos\theta,$$

$$(2.99)\quad z(s) = (HL_o/W)[\{1 + (V/H)^2\}^{\frac{1}{2}} - \{1 + ((V-M-(Ws/L_o))/H)^2\}^{\frac{1}{2}} + \{1 + ((V-M-(Ws_1/L_o))/H)^2\}^{\frac{1}{2}} - \{1 + ((V-(Ws_1/L_o))/H)^2\}^{\frac{1}{2}} + (MW/(HEA_oL_o))\{s_1 - s\}] + (Ws/EA_o)[(V/W) - (s/2L_o)]$$

$$= ((T\cos\theta)L_o/W)[\sec\theta - \sec\theta_s + \sec\theta_2 - \sec\theta_1] + (M[s_1 - s] + Ws[\tan\theta - (s/2L_o)])/EA_o,$$

$$(2.100)\quad T(s) = \{H^2 + ((V - M - (Ws/L_o))/H)^2\}^{\frac{1}{2}}.$$

To find H and V (or T and θ) for a particular span substitute $s = L_o$ in equations 2.99 and 2.100, with proper root values found when $x(H,V) = 1$ and $z(H,V) = h$, as in equations 2.53 and 2.54.

Multiple Point Loads on a Span

Sinclair has developed a dimensionless formulation for multiply loaded spans, taking advantage of the recursion becoming apparent in the single point load formulation, and simplifying the notation.[1] The dimensionless terms used in the formulas are given as

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[1]Irvine, Cable Structures, 22–24.

(2.101) $\Omega = x/L_o$ horizontal coordinate of strained profile

(2.102) $\mu = z/L_o$ vertical coordinate of strained profile

(2.103) $\sigma = s/L_o$ Lagrangian coordinate of unstrained profile

(2.104) $\sigma_n = s_n/L_o$ Lagrangian load coordinate $(n = 1, 2, \ldots, N)$

(2.105) $\delta = h/L_o$ relative vertical displacement of end points

(2.106) $\tau = l/L_o$ cable aspect ratio

(2.107) $\Theta = T/W$ cable tension

(2.108) $\epsilon = H/W$ horizontal reaction at all points

(2.109) $\alpha = V/W$ vertical reaction at initial point

(2.110) $\beta = W/EA_o$ flexibility factor

(2.111) $\phi_n = M_n/W$ applied concentrated vertical loads

(2.112) $$\Phi_n = \sum_{j=-1}^{n} \phi_j$$

We define $\sigma_0 = 0$, $\phi_{-1} = 0$, and $\phi_0 = 0$. The equation of vertical equilibrium at a point on the strained profile is

(2.113) $$T(dz/dp) = V - \sum_{i=0}^{n} M_i - Ws/L_o$$

The equations for horizontal and vertical position and tension are

(2.114) $$\Theta(\sigma) = \{\epsilon^2 + (\alpha - \Phi_n - \sigma)^2\}^{\frac{1}{2}}$$

(2.115) $$\Omega(\sigma) = \epsilon[\beta\sigma + \sinh^{-1}(\alpha/\epsilon) - \sinh^{-1}((\alpha - \Phi_n - \sigma)/\epsilon) + \sum_{i=0}^{n}\{\sinh^{-1}((\alpha - \Phi_i - \sigma_i)/\epsilon) - \sinh^{-1}((\alpha - \Phi_{i-1} - \sigma_i)/\epsilon)\}]$$

(2.116) $$\mu(\sigma) = \beta\sigma(\alpha - \sigma/2) + \{\epsilon^2 + \alpha^2\}^{\frac{1}{2}} - \{\epsilon^2 + (\alpha - \Phi_n - \sigma)^2\}^{\frac{1}{2}} + \sum_{i=0}^{n} [\beta\phi_i(\sigma_i - \sigma) + \{\epsilon^2 + (\alpha - \Phi_i - \sigma_i)^2\}^{\frac{1}{2}} - \{\epsilon^2 + (\alpha - \Phi_{i-1} - \sigma_i)^2\}^{\frac{1}{2}}].$$