Selected topics in theoretical mathematics
by Deborah Gibson McAtee

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics
Montana State University

Abstract:
Mathematics is a large and varied field, going far beyond what most people learned to call math in school; in fact "math" and "mathematics" are as dissimilar as "spelling" and "creative writing" are in the study of language. What most people call math—the manipulation of numbers according to set, memorized rules—mathematicians tend to refer to as accounting; real mathematics has to do with concepts and ideas rather than numbers. At its most advanced level, mathematics is as much an art as a science, more like writing a novel than balancing a checkbook. At their best, mathematicians are, in a very real sense, writing a novel in a foreign language, a novel filled with all the sense of excitement and discovery of hidden secrets of a great mystery story; the difference is that there is no way to flip to the back of the book for a peek at the ending.

And yet, in spite of all the excitement, people associate words like "boring", "dry", and "mechanical" with mathematics because they have never seen any of the beauty that the topic can display. They are led to believe that they can't do mathematics because they had trouble with algebra or the multiplication tables— as if all great writers are perfect spellers. This thesis is an attempt to introduce people to some of the concepts of mathematics in a non-technical, and hopefully entertaining, way.

I have chosen some of the topics that I find the most interesting in classical mathematics (Cardinal and Ordinal Numbers, Geometry), as well as two new and flourishing fields (Fractals, Fuzzy Logic); I have also addressed the question "What is a Mathematical Proof?" In all of these essays, I have tried to present the ideas involved without getting side-tracked by the mechanics and technical details, and to show that mathematics really does involve ideas and concepts in much the same way that philosophy or theoretical physics does. This thesis is not intended to be a complete or technical dissertation, but rather an introduction to mathematics for the intelligent adult who is curious about what mathematicians do.

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APPROVAL

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Deborah Gibson McAtee

This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

5/23/88
Date

Chairperson, Graduate Committee

Approved for the Major Department

5/23/88
Date

Head, Major Department

Approved for the College of Graduate Studies

5/26/88
Date

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ABSTRACT

Mathematics is a large and varied field, going far beyond what most people learned to call math in school; in fact "math" and "mathematics" are as dissimilar as "spelling" and "creative writing" are in the study of language. What most people call math—the manipulation of numbers according to set, memorized rules—mathematicians tend to refer to as accounting; real mathematics has to do with concepts and ideas rather than numbers. At its most advanced level, mathematics is as much an art as a science, more like writing a novel than balancing a checkbook. At their best, mathematicians are, in a very real sense, writing a novel in a foreign language, a novel filled with all the sense of excitement and discovery of hidden secrets of a great mystery story; the difference is that there is no way to flip to the back of the book for a peek at the ending.

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INTRODUCTION

Mathematics is a large and varied field, going far beyond what most people learned to call math in school; in fact, "math" and "mathematics" are as dissimilar as "spelling" and "creative writing" are in the study of language. What most people call math—the manipulation of numbers according to set, memorized rules—mathematicians tend to refer to as accounting; real mathematics has to do with concepts and ideas rather than numbers. At its most advanced level, mathematics is as much an art as a science, more like writing a novel than balancing a checkbook. At their best, mathematicians are, in a very real sense, writing a novel in a foreign language, a novel filled with all the sense of excitement and discovery of hidden secrets of a great story; the difference is that there is no way to flip to the back of the book for a peek at the ending.

And yet, in spite of all the excitement, people associate words like "boring", "dry", and "mechanical" with mathematics because they have never seen any of the beauty that the topic can display. They are led to believe that they can't do mathematics because they had trouble with algebra or the multiplication tables—as if all great writers are perfect spellers. Most people know much more mathematics than they realize and lack only the language in which to express it. For instance, everyone understands the basic idea behind different metrics: there are different ways to measure distance. Asked the distance to a favorite restaurant in town, almost everyone replies "Twelve blocks over and four up" instead of giving the distance "as the crow flies"; anyone who answers that way is using what mathematicians call the taxi–cab metric, while the distance as the crow flies is called the Euclidean metric. This is an example of an important topic in theoretical mathematics being used naturally in daily life, and it shows that no one is too stupid to understand at least some of the concepts involved.

Mathematics is rightly placed in the humanities, concerned as it is with ideas and concepts, but it also bridges the gap between the humanities and the sciences in its applications. Mathematics is at once brilliantly abstract and fundamentally applied; all sciences build on mathematics, but mathematics also soars above the applications to the high reaches inhabited by philosophy and art. Mathematics is much more than algebra or calculus; what follows is merely an introduction to some of the ideas to be found in the field.

FRACTALS

Fractals have attracted growing interest in the last couple of years, in part because computer graphics can produce fabulous full–color pictures of them. Fractals seem to be everywhere, from computer magazines to glossy books, from physics journals to economic conferences; the articles are almost always accompanied by illustrations and graphs, and frequently by full–color graphics designed to catch attention as much as to make a point. Fractals have even become an art form, with computers churning out ever more spectacular graphs whose appeal is aesthetic rather than scientific. But there is more to fractals than just pretty pictures.

Fractals are objects that have fractional dimensions, possess an attribute called self–similarity, and that are typically created by means of iteration. Fractals were so–named because of their fractional dimension, so let's look at this definition first.

Euclidean geometry teaches us that a point has zero dimension, a line has one dimension (length), a plane or flat surface has two (length and width), and space has three dimensions (length, width, and height). This seems simple enough—everything can easily be classified into one category or another—until we try to determine the dimension of a ball of string. From a great distance, the ball appears to be a point, and so has zero dimension; from closer, it appears as a ball and has three dimensions; closer yet, and the ball is seen to be made up of one long line of string and so must have one dimension; still closer, and the string can be seen to have width and height as well as length, and we're back to three dimensions.

So is the ball of string zero, one, or three dimensional? Obviously, it depends on how closely we look.

Now consider this line:

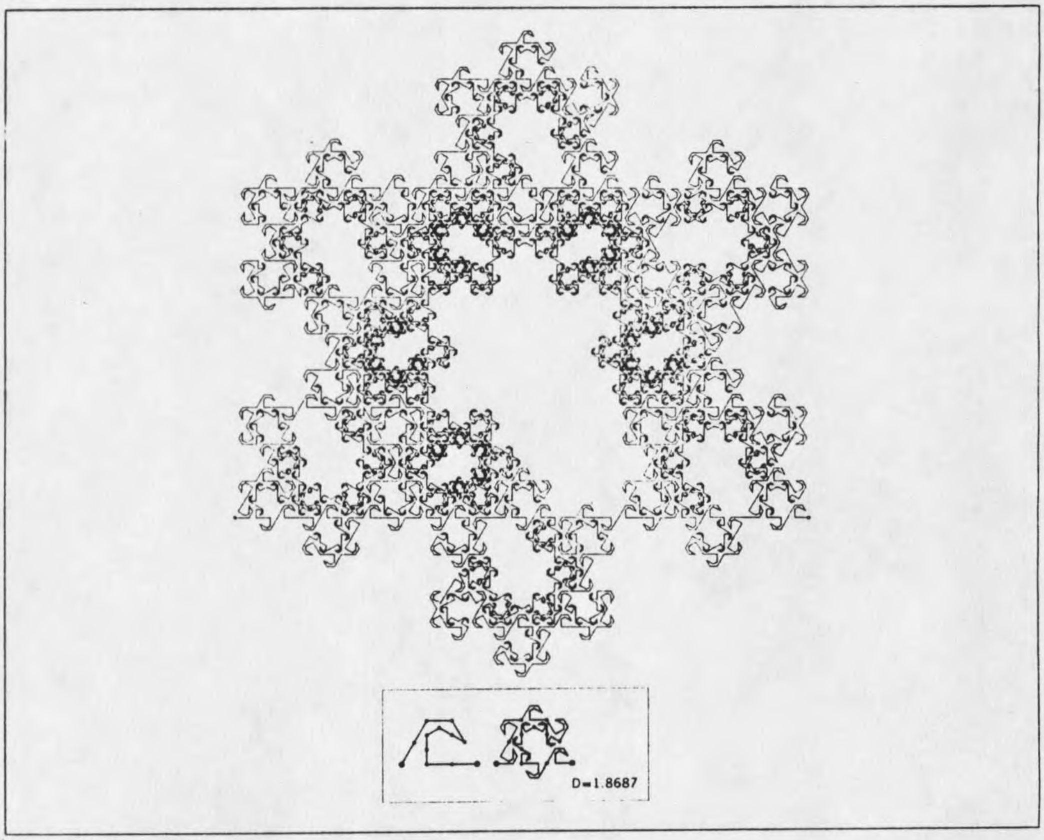

Figure 1. 1.87-dimensional line

The line itself is obviously one–dimensional, and lies in a two–dimensional plane. But in fractal dimensions, the line's dimension is between 1 and 2 (in fact, it is 1.8687) because, in Peter Sorenson's words, "...what follows will be an intuitive explanation rather than a technical one—the more complicated the wiggles get, the more the line's single dimension approaches the second dimension, until it could become infinitely wiggly, infinitely long, fill the plane, and be thoroughly two–dimensional." In other words, the more the line fills the plane, the closer it gets to having the same dimension as the plane. The dimension is more than one but not yet two, so the line must have a dimension of one and a fraction—hence the name fractals.

Fractals can also have dimensions between zero and one or between two and three. Points that almost cover a line, such as something called the Cantor set, are an example of the former, and the radiator of your car provides an analogy for the latter. The sheet metal out of which your radiator was built was once a flat, two–dimensional surface; however, the manufacturing process folds it in such a way that it starts to fill the three–dimensional space in which it sits. So the radiator can be seen as having a dimension between the two of the plane and the three of the space.

Besides having fractional dimensions, most fractals have a characteristic known as self–similarity. Perhaps the best example from outside mathematics is Jonathan Swift's verse:

So, Nat'ralists observe, a Flea
Hath smaller Fleas that on him prey;
And these have smaller Fleas to bite 'em;
And so proceed ad infinitum.

According to this verse, no matter how closely you look at the flea, you will see the same thing: fleas biting fleas. This idea, that the same thing appears at all scales, is called self–similarity. This similarity at different scales can be exact, with a precise duplication at each scale, as in Figure 2, or it can be a more approximate similarity. This latter type is usually called statistical or stochastic similarity, and it means that views at different scales look "almost" alike, as in Figure 3; "'stochastic' is a learnedly elegant way of saying 'random'," as Paul Halmos commented.

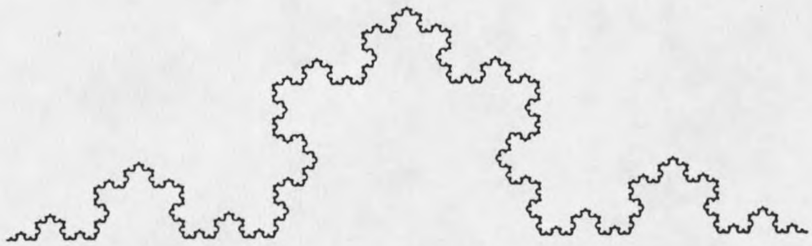

Figure 2. Koch snowflake

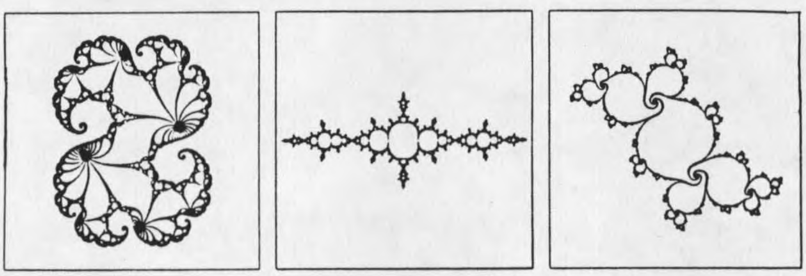

Figure 3. Julia sets

In Figure 2, it is easy to explain how to get the figure: Take a line and replace the middle third with an equilateral triangle;

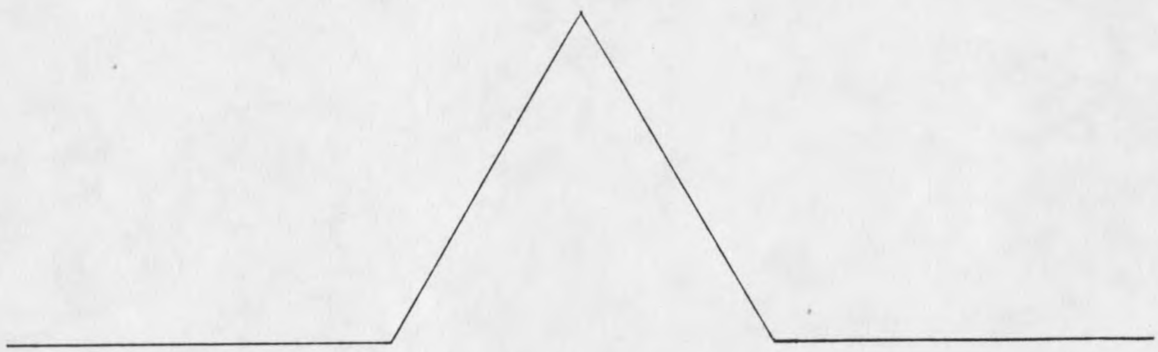

Figure 4. Koch snowflake

now take each straight line segment, remove the middle third, and replace it with a triangle;

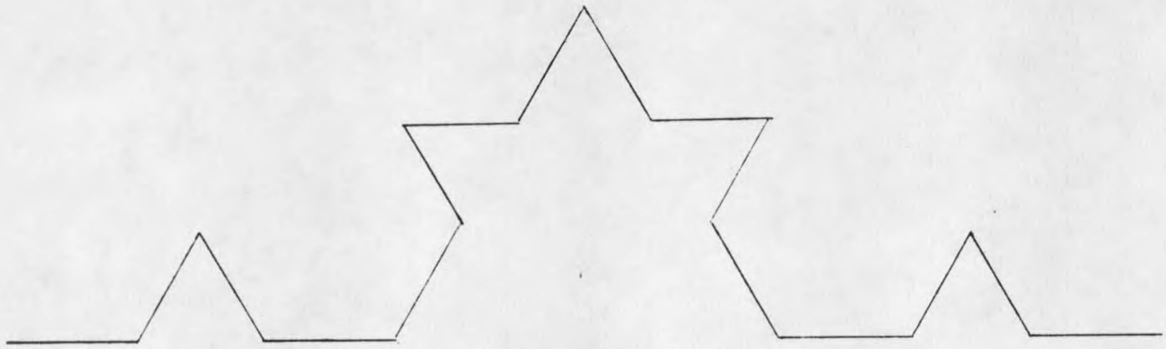

Figure 5. Koch snowflake

do the same thing again;

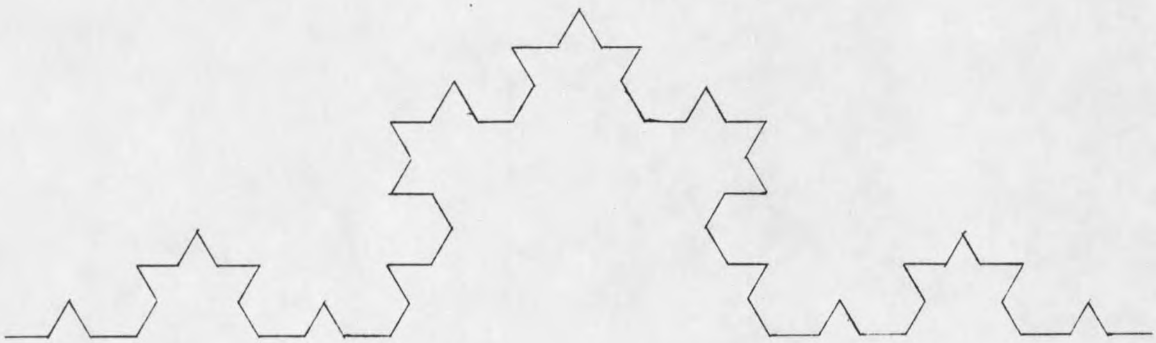

Figure 6. Koch snowflake

and so on. This process can go on as long as you like, limited only by your tolerance level for boredom. It takes a little work to translate these general directions into computer language, but once it's done, everything is known about the figure; there are no surprises. So it's not surprising that mathematicians quickly moved on to more intricate ways of drawing fractals.

The other figures are called Julia sets, after the French mathematician who discovered them, and are some of the more complicated sets in fractal mathematics. In order to have any chance of even realizing its complexity, we have to start with the idea of iteration, or repeated actions. In an interated function, the output of each calculation becomes the input of the next one. For instance, let $f(x) = x^2 + 1$—that is, take a number, square it, and add 1—and let's start with 1 as our first number. Then squaring 1 and adding 1 gives us 2; now take 2 as the number, square it and add 1, and we get 5; now take 5, and ... In mathematical notation, it looks like this:

$$f(1) = 1 + 1 = 2$$

$$f(2) = 4 + 1 = 5$$

$$f(5) = 25 + 1 = 26$$

$$\dots$$

or, on the number line, like this:

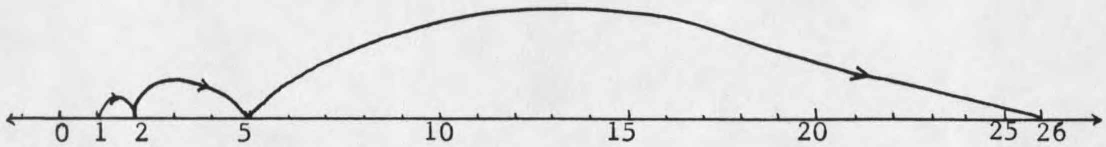

Figure 7. $f(x) = x^2$

with each new result getting farther from 0, or the origin.

While it might seem that new results always move away from the origin, we see something different if we let our function be $f = x^2$ and our first number be .25. Squaring .25 gives us .0625, which is closer to 0. Using this function, $f(x) = x^2$, some iterations will move away from 0 and others will move towards it; if the first number in the iteration is greater than 1, the iterations will move it farther from the origin, while if the first number is less than 1, the iterations will move closer to the origin. If the first number is 1 or -1, the iterations stay the same distance, 1, from the origin, although the iteration starting with -1 will move to 1 after the first iteration.

So far we have been looking at what mathematicians call a map from the line to the line, which just means taking a point on the line, applying a function to it, and looking at where the result ends up on another line. So in the picture below,