The effects of thermal diffusion and optical absorption on laser generated ultrasonic waves in a solid

by Amitava Roy

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Mechanical Engineering
Montana State University

Abstract:
In this thesis the effects of absorptivity and diffusivity on the displacement of the back surface of a specimen, when its front surface is irradiated with a high energy laser whose temporal pulse shape is represented by a Dirac-delta function, are analyzed for the one-dimensional case. When the specimen is irradiated with laser energy thermoelastic waves are generated inside the specimen. These waves travel from the front to the back surface of the specimen and cause displacement of that surface. Governing differential equations and corresponding boundary conditions approximating this phenomenon are set up. These equations are then solved by Laplace transform method to obtain expressions for temperature distribution inside the specimen and displacment of its back surface.

A sharp spike in displacement-time graph is observed which agrees with the experimentally obtained data. It is also observed that at low absorptivity diffusivity has negligible effect on the displacement-time characteristic. But in the case of high absorptivity, displacement increases significantly with increase in absorptivity. Again for very small diffusivity, peak displacement decreases with increase in absorptivity while for other values of diffusivity peak displacement increases with increase in absorptivity. Also peak displacement does not increase much with increase in diffusivity when the diffusivity is already high.

All of these observations can be explained by considering the heat distribution inside the specimen as a series of discrete point heat sources. The resultant displacement in this case is then the summation of all the displacements due to each individual point heat source.

Using this solution as a Green's function, the author can obtain displacement of the back surface of the specimen for any arbitrary laser pulse shape. Again this formulation helps to determine the relative effects of absorptivity and diffusivity on the displacement characteristic.

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Amitava Roy

This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

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W. E. Larsen — Head, Major Department

Approved for the College of Graduate Studies

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TABLE OF CONTENTS

TABLE OF CONTENTS–Continued

LIST OF TABLES

LIST OF FIGURES

LIST OF FIGURES–Continued

ABSTRACT

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CHAPTER 1

INTRODUCTION

A laser is a special type of high power light source. The main properties of interest that are different in laser radiation as compared to radiation from conventional light sources are the intensity, direction, monochromaticity, coherence, brightness and high power availability. With a simple system it is easily possible to generate short duration pulses of red and infrared laser light with powers of the order of millions of watts. Several billions to trillions of watts have been obtained in a pulse in a more sophisticated system. Moreover, directionality and focusing properties of laser beams make it possible to deliver high irradiation to a small spot.

When a laser beam strikes the surface of a material, some of the beam's energy is absorbed and some energy is reflected from the surface. If absorption of laser energy is high, intense heating is produced which may lead to melting or vaporization of the target. Whenever the surface of a body is subjected to rapid transient heating and the temperature is not sufficiently high to cause melting or vaporization of the material, thermoelastic waves are produced inside the specimen. These waves propagate through the body and can be detected by a suitable device within the body. Cracks or voids in the material can alter the form and characteristics of the travelling wave if their dimensions are large compared to the wavelength of the travelling wave. By detecting and analyzing the altered wave pattern, the location, size and shape of the irregularity in the material can be determined. Laser pulses of very short duration and low intensity can produce thermoelastic waves of sufficiently short wavelength to detect fine micro-cracks, previously undetected by ultrasonic testing methods.

Mechanism of Wave Generation

Light is absorbed in opaque materials by interaction with electrons. A quantum of light energy is absorbed by an electron which is raised to a high energy state. The excited electron will collide with neighboring electrons and other particles in its vicinity and give up energy. In this process energy transfer occurs from one particle to another in the material. This is the same collision process which governs the conduction heat transfer phenomenon in a material. When a laser beam strikes the surface of an opaque material, it is absorbed in the same fashion as light. In a good conductor the mean free time between collisions for electrons is of the order of 10^{-13} to 10^{-14} seconds [1]. Thus during the time in which a laser pulse strikes the surface (in this case 10^{-9}–10^{-7} seconds), an excited electron will make many collisions with other electrons and lattice particles. Therefore it can be said that when a laser beam is absorbed, optical energy is instantaneously converted to heat energy within a volume in which the laser energy is absorbed. The production and distribution of heat energy is so rapid, compared to the laser pulse duration, that the laws of conservation of energy can be applied in the small volume in which energy is absorbed. Therefore it can be said that the concept of temperature and other usual equations of heat flow are all valid in this case and the use of the continuum mechanics approach is justified. When the laser pulse duration is very short (in the order of pico-seconds) then there is no time for an excited electron to distribute its energy amongst its neighboring particles through the collision process. Hence it cannot be said that heat conduction has taken place during that time interval. Thus if the time of interest is very small, the continuum approach cannot be employed and this case will require a different treatment. In this case the laser-pulse duration and the time of interest are long enough to allow application of the usual laws of heat conduction.

There are several ways by which thermoelastic waves can be generated in a solid with the help of laser, utilizing different laser systems and/or modification of the specimen surface on which the laser beam is incident. The simplest among them is the laser irradiation of a clean specimen surface without any chemical coating on it by a laser pulse which strikes the metal surface normally. The intensity of the laser pulse must not be high enough to cause melting or damage of the surface. The amount of energy absorbed depends on the material's optical absorption coefficient. The reciprocal of the optical absorption coefficient is the depth at which the laser intensity drops to $1/e$ of its original intensity where e denotes the exponential term. For metals optical absorption coefficient varies between 10^5/cm. to 10^6/cm., which is very large. Thus for a metal specimen the absorption of laser energy as it travels through the specimen is extremely high. Therefore in each consecutive layer of the specimen, less and less energy is available for absorption and this results in the production of a steep spatial temperature gradient within the solid. This temperature gradient, in turn, produces a strain field (thermoelastic effect). Again the rise of temperature is extremely rapid, as the laser pulse has a steep temporal gradient and it is of very short duration, which causes the total laser energy to be absorbed in a very short period of time. The steep temporal gradient of the temperature produces a rapidly changing strain field. Due to the rapid change in strain with respect to time, an elastic wave is generated. Thus when there is no melting and/or vaporization of the material due to absorption of laser energy, the wave generation in the solid is principally associated with thermoelastic effects.

In this thesis the effect of absorptivity and diffusivity of a material on the temperature distribution in the specimen and displacement of its back surface, caused by the generated thermoelastic waves, are investigated.

Literature Review

The generation of acoustic pulses in a solid by laser irradiation of its surface was first suggested by White [2], in 1963. White showed experimentally that high frequency elastic waves are produced in the solid by pulses of electromagnetic energy and light, upon their absorption at the surfaces of elastic solids and fluids. In another paper [3] White has analyzed the process of elastic wave production and its propagation through a solid which is subjected to transient heating with a laser, with particular emphasis on the case of the input flux varying harmonically with respect to time. He related the elastic wave amplitude to the characteristics of the heat input flux and the thermal and elastic properties of the body. In the process he showed the proportionality of the stress wave amplitude and absorbed power density. In this paper he assumed that all the heat is absorbed at the surface of the body.

Carome, Clark and Moller [4] have described the exposure of a liquid with high optical absorptivity, to a Q-spoiled ruby laser and the resulting development of stress waves. After them Penner and Sharma [5] have investigated theoretically the thermal stress development in partially transparent rods of infinite length for a one-dimensional geometry prior to ablation and thermal equilibrium. For this they assumed a particular simplified temperature profile along the material depth. J.F. Ready [1] described methods by which one can calculate the temperature distribution for any arbitrary laser pulse shape when the laser energy absorbed drops off exponentially along the specimen depth.

Much later, in 1980, Scruby et. al. [6] performed quantitative experimental measurements in the generation of elastic waves by laser radiation. They found that the thermoelastic source generated both longitudinal (L) and shear (S) waves, but the latter predominates at the epicenter. They recorded the displacement of the surface opposite to the surface on which the laser strikes and observed a sharp spike is present in the displacement, signalling the arrival of the first longitudinal wave. Subsequently, the displacement became negative, i.e. in the direction opposite to the laser propagation.

Telschow and Conant [7] have observed that the spike in the displacement can be explained through the use of one-dimensional models that account for optical penetration and thermal diffusion into the material. They considered a single point source buried at the depth "H" below the surface and they showed that a positive precursor signal is produced. Next, optical penetration is taken into account by distributing point sources with an exponentially decaying magnitude with depth into the material. This model also produced a precursor signal whose shape reflects the temperature profile with depth. Finally the effects of thermal diffusion on the precursor signal were considered. In this case they assumed that all the laser energy is absorbed at the surface.

Objective

In this thesis an idealization is made about how the laser-pulse is absorbed by a specimen along its thickness when one surface of it is irradiated by a high energy laser-pulse so that the energy intensity absorbed at any particular depth of the specimen is known. It is assumed that laser energy absorbed in the material decays exponentially along the depth of the specimen. Based on this assumption, a closed form solution of temperature distribution along the specimen depth is obtained. Then, the displacement of one surface of the specimen is calculated for this temperature distribution and for the one-dimensional-case when both diffusivity and absorptivity in the material are present. After that the influence of optical absorption and diffusivity of the material on the temperature distribution in the material and more importantly, on the elastic waveform generated, is determined.

Experimental Set Up

A laser beam, properly aligned so that it strikes the material surface normally, is incident on one surface of the specimen (Figure 1). The laser pulse is absorbed all along the material depth as it travels across the specimen and causes the temperature to rise throughout the specimen depth, though temperature falls off rapidly from the front to the back surface of the specimen, as noted previously and as shown in Figure 2a. Due to the temperature rise, elastic waves are generated at each point P along the depth of the specimen (Figure 2b). The energy carried by each of the waves generated throughout the specimen will decrease at a high spatial rate from the front to the back surface due to less temperature rise from one surface to the other. Each of these waves generated at point P travels toward both surfaces of the specimen as shown in Figure 2b and causes displacement of the surfaces when it arrives. After being reflected at the two surfaces the waves travel in the opposite directions. The displacement of one surface at any particular instant would be due to the sum of the effects of all the waves, both original and reflected, reaching that surface at that instant. Also due to diffusion, the temperature profile through the thickness of the material will change with time. This will modify the different characteristics of the waves generated at each point of the specimen as time proceeds. This phenomenon also affects the displacement of any surface with time. A transducer placed at one surface of the specimen (Figure 1) measures the movement of that surface. The transducer is maintained axial with respect to the laser beam, i.e. at the epicenter with respect to the acoustic source. Instead of the transducer, any other device (e.g. laser beam) may also be used for measuring the displacement of that surface.

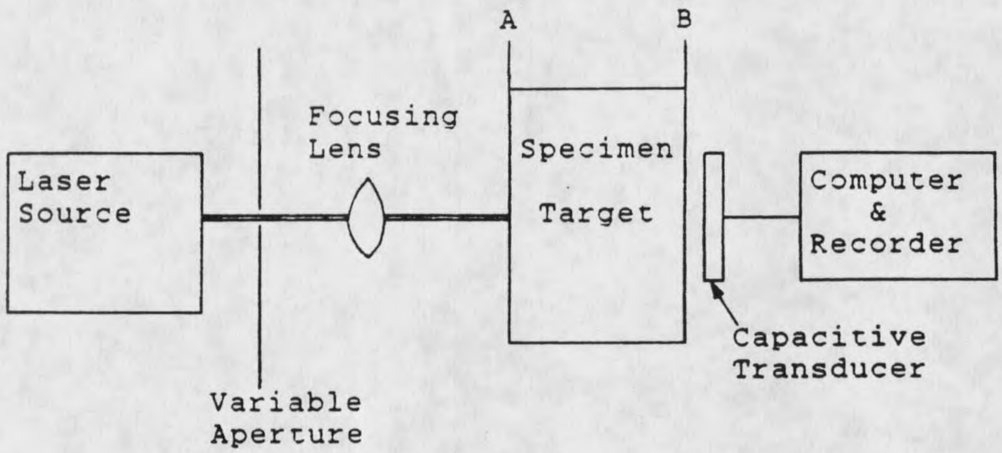

Figure 1. Schematic of the apparatus.

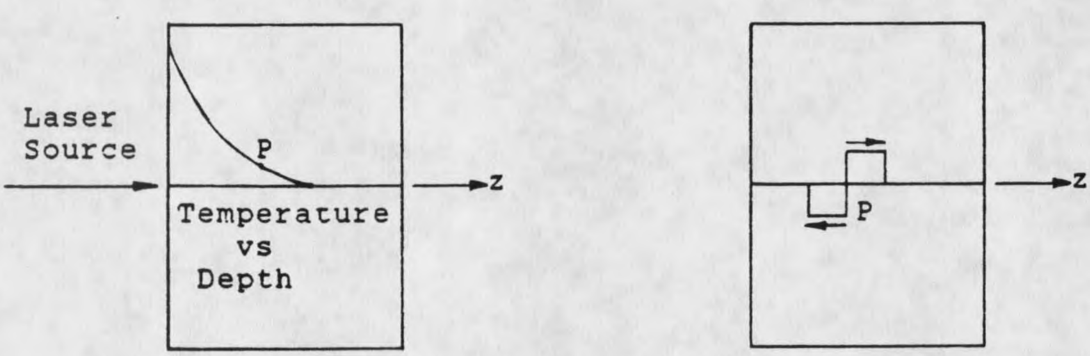

Figure 2. (a) Temperature distribution and (b) wave generation along the depth of the specimen due to sudden temperature rise at point P.

Many experiments like the one described above have been reported previously [1,6]. Typically in most of the cases Q-switched laser pulses were used. Peak pulses were of the order of 10^7–10^8 watts and power densities were less than 10^7 watts/cm^2 to avoid melting or damage of the specimen surface. The total energy carried by each pulse was about 60 mj. The duration of laser pulses were several tens of nano-seconds. The resulting characteristic displacement obtained in many of these experiments is shown in Figure 3, which is taken from [6].

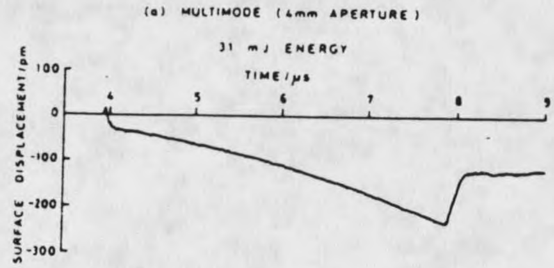

Figure 3. Experimentally obtained displacement variation with time.

CHAPTER 2

FORMULATION OF THE PROBLEM

In this chapter several assumptions regarding heat conduction and wave propagation through the specimen, appropriate to the experimental set up, are made and the governing differential equations together with the corresponding boundary conditions, representing an idealization of the real physical phenomenon, are established. In formulating the problem it must be noted that interest focuses on knowing the temperature distribution within the material and the displacement of one surface of the specimen (shown in Figure 1 as surface B) for only a very short period of time, i.e. the time required for the generated elastic wave to travel not more than three times the depth of the specimen. This is because after this time, there would be repetition of result as the original waves would keep bouncing back and forth from both the surfaces and as time progresses, the new waves produced would have negligibly small amplitude to contribute to the net displacement of the specimen surface, due to a slow drop in temperature with respect to time throughout the specimen after some time. For example, longitudinal wave speed through copper is $4.66 \times 10^{+5}$ cm/sec. In this case, specimen depth is approximately 2.5 cm. so that the author is interested in the time it takes for the wave to travel 7.5 cm., which can be calculated to be $1.6 \times 10^{+4}$ ns. The assumptions made in formulating the problem are as follows:

i) The laser-pulse strikes over a large area of the specimen compared to the specimen depth through which heat flow occurs in this short time of observation, which is of the order of 10^{+4} ns. The depth of penetration of heat through a material in time t is given approximately by the equation:

$$D = \sqrt{4kt} \tag{2.1}$$

where D is the depth through which heat flows, k is thermal diffusivity, and t is time of interest. For example, in copper (which has a diffusivity of 1.1234 $cm^2/sec.$), heat penetrates up to a depth of .0085 cm. in the time of interest which was calculated before to be $1.6 \times 10^{+4}$ ns. for copper. From reference [1] it can be said that for a Q-switched laser pulse a minimum spot diameter of 0.05 cm. is expected, so that the area affected would be much wider than deep. Therefore heat flow through the target can be treated as one-dimensional, from which it is concluded that there is no variation of temperature in the direction perpendicular to laser axis.

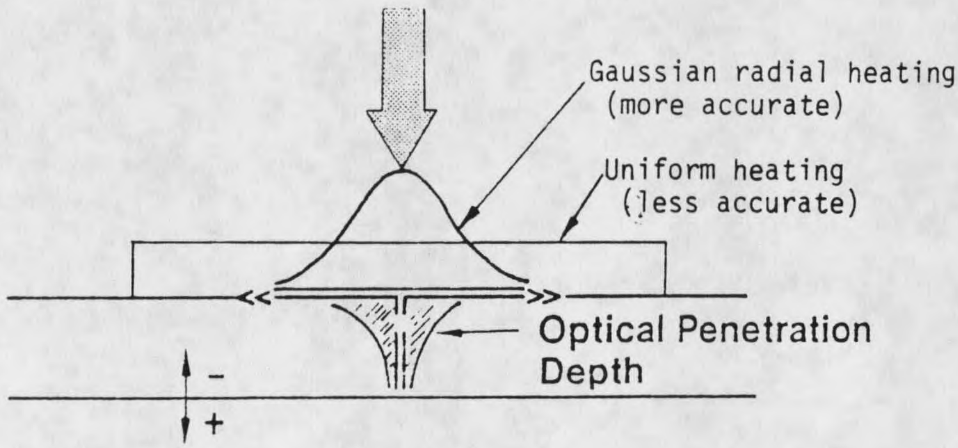

Figure 4. Assumed uniform and non-uniform spatial distribution of laser pulse.

ii) The laser is assumed to be of uniform power density in the transverse direction. This assumption is not good because an actual laser shows considerable spatial non-uniformity (Figure 4). One can idealize the variation of laser intensity in the transverse direction as a Gaussian profile. However this assumption will make the problem two-dimensional. For this case the one-dimensional model would give results of the same order.

Using these two assumptions and the fact that the generated elastic wave is plane, one non-vanishing displacement component in the direction of laser propagation is left. Therefore it can be said that the problem is one-dimensional. Hence, all components of stress and strain tensor, apart from the normal component acting on the plane, perpendicular to laser direction, vanish.

iii) The material is isotropic and homogeneous.

iv) In order to remain in the realm of linear-thermoelasticity, it is assumed that the increment of the material temperature as compared to the reference temperature of the material is small [8, page 99].

v) The material properties do not change with temperature. While this is not strictly true, most changes in the properties of metal tend to be small over fairly wide temperature ranges. For a more complete treatment the temperature variation of the thermal properties must be taken into account.

vi) The thermoelastic coupling between the heat-conduction and the displacement equation for thermoelasticity is ignored by suppressing the term of mechanical origin in the heat-conduction equation. This can be done a for short time of observation [8, page 138], as is the case here.

With these assumptions the governing equations of heat-conduction and displacement can be written as follows:

$$\frac{\partial^2 \theta}{\partial z^2} - \frac{1}{k}\frac{\partial \theta}{\partial t} = -\frac{A}{K} \tag{2.2}$$

and

$$C_L^2 \frac{\partial^2 u}{\partial z^2} - \frac{E\alpha}{(1-2\nu)\rho} \frac{\partial \theta}{\partial z} = \frac{\partial^2 u}{\partial t^2} \tag{2.3}$$

where θ is temperature difference with the reference temperature, z is the coordinate representing depth of the specimen (see Figure 2a), t is time, k is thermal diffusivity of the material, K is thermal conductivity of the material, A is heat generated per unit time per unit volume of the specimen (it is a function of both time and depth of the specimen), C_L is longitudinal wave speed through the material, u is non-vanishing displacement component in the z direction, E is Young's modulus of the material, α is the coefficient of linear thermal expansion of the material, ν is Poisson's ratio, and ρ is mass density.

In order to specify the form of A in Equation (2.2) it is assumed [9, page 72] that: i) the amount of laser energy absorbed by the material (i.e. converted into heat) decreases exponentially along the depth of the specimen; ii) the duration of the laser-pulse shape is infinitely small compared to the time of observation i.e. the time taken by the generated thermoelastic wave to traverse three times the depth of the specimen. Thus the temporal pulse shape of the laser can be represented by a Dirac-delta function. Therefore

$$A(z,t) \approx b\delta(t)\exp(-bz) \tag{2.4}$$

where b is the absorption coefficient of the material and $\delta(t)$ is the Dirac-delta function. The factor b comes in the exponential argument because it determines the amount of heat absorption by the specimen along its depth as laser energy passes through it. After integrating the expression for A, with respect to time and depth, between the limits 0 and ∞, the total energy absorbed by the specimen should be obtained. In this case, the total energy absorbed is assumed to be unity. Therefore to get unity after integration of the expression for A, the expression must be multiplied by the factor b. Thus it can be said that in this case b acts as a normalization factor.

Now the initial and boundary conditions are set, after making appropriate assumptions, of Equations (2.2) and (2.3).

Initial condition of heat conduction equation:

There is only one initial condition for Equation (2.2). It is assumed that initially the specimen is at the reference temperature. Therefore the initial condition of the heat-conduction equation is given by

$$\theta(z,0) = 0 \quad . \tag{2.5}$$

Boundary conditions of heat conduction equation:

In order to formulate the two boundary conditions of Equation (2.2), the physics of the problem must be considered. For short laser pulses heat is confined to a small area and heat losses from the area are generally small compared to the radiation flux incident on the area. While laser flux densities of interest are of the order of 10^{+6} watts/cm^2 or greater, even at elevated temperature, thermal radiation amounts to be of the order of 10^{+3} watts/cm^2 for solid materials. Moreover, the author is interested in a small interval of time. Therefore during that time the total loss of heat from the material surface will be negligibly small. In that case the external surface at which the laser strikes the specimen can be treated as insulated and

$$\frac{\partial \theta}{\partial z}_{(z=0)} = 0 \quad . \tag{2.6}$$

For a majority of materials, thermal diffusivity is quite low and the rate at which heat is propagated through the solid is considerably less than the rate at which the elastic wave travels. For example, in the case of copper, as shown before, during the time the elastic wave travels three times the depth of the specimen (which is calculated to be $1.6 \times 10^{+4}$ ns.), heat penetrates up to a depth of .0085 cm., which is negligibly small compared to the 2.5 cm. depth of the specimen. Thus in the time of observation, the surface which is opposite to the one on which the laser beam is incident will have no information about the heat that propagates through it. Therefore the specimen can be considered a half-space with respect to heat conduction. The limitation of this assumption is the neglect of heat conduction from the region which is very near to the transducer. But in case of metal, absorptivity being very high, almost all the laser energy is absorbed very near to the surface on which the laser beam is incident. As such the assumption will introduce negligible error. Moreover, as the depth in this half-space goes to infinity, the temperature should be bounded, i.e.

$$\lim_{z \to \infty} \theta(z,t) \text{ is bounded} \quad . \tag{2.7}$$

In solving the wave equation, the finite depth of the specimen is taken into consideration.

<u>Initial conditions of wave equation</u>:

As the wave equation (2.3) is second order in time, there will be two initial conditions. It is assumed that there is no displacement of the specimen initially. This gives

$$u(z,0) = 0 \quad . \tag{2.8}$$

The material is initially at rest, that is, the velocity of any material particle is zero at the initial instant. Therefore

$$\frac{\partial u}{\partial t}_{(t=0)} = 0 \quad . \tag{2.9}$$

<u>Boundary conditions of wave equation:</u>

The boundary conditions for the wave equation are derived from the fact that two boundaries are stress free. Hence

$$\sigma_{zz}(0,t) = 0 \tag{2.10}$$

and

$$\sigma_{zz}(D,t) = 0 \tag{2.11}$$

where D is depth of the specimen and σ_{zz} is non-vanishing stress-component.

Also, from the Duhamel-Neumann equation for isotropic material and one-dimensional case,

$$\sigma_{zz} = (2\mu + \lambda)\epsilon_{zz} - \frac{E\alpha}{(1-2\nu)}\,\theta \tag{2.12}$$

where

$$\mu = \frac{E}{2(1+\nu)} \quad ,$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad ,$$

ϵ_{zz} is non-vanishing strain component and μ and λ are the well known Lame constants. Applying Equation (2.12) to the first boundary condition (Equation (2.10)),

$$\begin{aligned} \epsilon_{zz(z=0)} &= \frac{E\alpha}{(1-2\nu)(2\mu+\lambda)}\,\theta_{(z=0)} \\ &= \frac{\alpha(1+\nu)}{(1-\nu)}\,\theta_{(z=0)} \quad . \end{aligned} \tag{2.13}$$

Therefore, since

$$\epsilon_{zz} = \frac{\partial u}{\partial z} \tag{2.14}$$

the result is

$$\frac{\partial u}{\partial z}_{(z=0)} = \frac{\alpha(1+\nu)}{(1-\nu)}\,\theta_{(z=0)} \quad . \tag{2.15}$$

Similarly, from the second boundary condition (Equation (2.11)),

$$\frac{\partial u}{\partial z}_{(z=D)} = \frac{\alpha(1+\nu)}{(1-\nu)}\, \theta_{(z=D)} \quad . \tag{2.16}$$

This completes the formulation of the problem.

CHAPTER 3

SOLUTION OF THE EQUATIONS

In Chapter 2 the problem was formulated. Now both the heat conduction and wave-propagation equations (Equations (2.2) and (2.3) respectively) can be solved. The Laplace transform method is used in solving these equations.

Solution of Heat-Conduction Equation

Taking Laplace transform of Equation (2.2) and applying the initial condition (Equation (2.5)),

$$\frac{\partial^2 \bar{\theta}}{\partial z^2} - \frac{s}{k}\,\bar{\theta} = -\frac{b}{K}\,\exp(-bz) \tag{3.1}$$

where

$$\bar{\theta} = \bar{\theta}(z,s) = \int_0^\infty \theta(z,t)e^{-st}\,dt \quad .$$

The complete solution of Equation (3.1) is given by

$$\bar{\theta}(z,s) = c_1(s)\exp\left(\sqrt{\frac{s}{k}}\;z\right) + c_2(s)\exp\left(-\sqrt{\frac{s}{k}}\;z\right) - \frac{b\exp(-bz)}{\rho c(kb^2 - s)} \quad . \tag{3.2}$$

Applying the boundary condition given by Equation (2.7) to Equation (3.2),

$$c_1(s) = 0 \quad . \tag{3.3}$$

Applying the boundary condition given by Equation (2.6) to Equation (3.2),

$$c_2(s) = \frac{b^2\sqrt{k}}{\rho c(kb^2 - s)\sqrt{s}} \quad . \tag{3.4}$$

Therefore the solution of Equation (3.1) is given by

$$\bar{\theta}(z,s) = \frac{b}{\rho c(kb^2 - s)}\left\{\frac{\sqrt{k}\exp\left(-\sqrt{\frac{s}{k}}\;z\right)}{\sqrt{s}} - \exp(-bz)\right\} \tag{3.5}$$

and after differentiation,

$$\frac{\partial\bar{\theta}(z,s)}{\partial z} = -\frac{b^2}{\rho c(kb^2 - s)}\left\{\exp\left(-\sqrt{\frac{s}{k}}\,z\right) - \exp(-bz)\right\} \quad . \tag{3.6}$$

<u>Inverse Laplace Transform</u>:

In order to take the inverse Laplace transform of Equation (3.5) it is rearranged, using partial fractions as follows:

$$\bar{\theta}(z,s) = -\frac{b}{2\rho c}\left\{\frac{\exp\left(-\sqrt{\frac{s}{k}}\,z\right)}{\sqrt{s}(\sqrt{s} - \sqrt{k}\,b)} - \frac{\exp\left(-\sqrt{\frac{s}{k}}\,z\right)}{\sqrt{s}(\sqrt{s} + \sqrt{k}\,b)} + \frac{b\exp(-bz)}{\rho c(s - kb^2)}\right\} \quad . \tag{3.7}$$

The inverses of these transforms can be found in [10]. After taking inverse Laplace transform of Equation (3.7) and after simplifying,

$$\begin{aligned}\theta(z,t) = \frac{b}{2\rho c}\exp(b^2 kt)\Bigg\{&\exp(bz)erfc\left(b\sqrt{kt} + \frac{z}{2\sqrt{kt}}\right)\\ &+ \exp(-bz)erfc\left(b\sqrt{kt} - \frac{z}{2\sqrt{kt}}\right)\Bigg\} \quad .\end{aligned} \tag{3.8}$$

This is the *complete solution* of the heat-conduction equation and it gives the distribution of temperature along the depth of the specimen for a laser-pulse which has the form of Dirac-delta function with respect to time. This solution is valid for any material with both thermal-conductivity and diffusivity, i.e. this solution is of the most general type as far as material properties are concerned. Now this solution can be used as a Green's function to calculate the temperature distribution due to a laser of any arbitrary temporal pulse shape using the following expression:

$$T(z,t) = \int_0^\infty f(\tau)\theta(z,t-\tau)d\tau$$

where $f(t)$ is the temporal pulse shape of the laser and $\theta(z,t-\tau)$ is obtained from Equation (3.8).

Solution of Wave Equation

Taking the Laplace transform of the differential equation (2.3) and applying both the initial conditions (Equations (2.8) and (2.9)), after simplification using Equation (3.6),

$$\frac{\partial^2 \bar{u}(z,s)}{\partial z^2} - \frac{s^2}{C_L^2}\,\bar{u}(z,s) = - \frac{E\alpha b^2}{\rho^2 C C_L^2(1-2\nu)}\left\{\frac{\exp\left(-\sqrt{\frac{s}{k}}\,z\right)}{(kb^2-s)} - \frac{\exp(-bz)}{(kb^2-s)}\right\} \quad . \tag{3.9}$$

Again

$$C_L^2 = \frac{\lambda+2\mu}{\rho} = \frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)} \quad .$$

Substituting this value of C_L in Equation (3.9),

$$\frac{\partial^2 \bar{u}(z,s)}{\partial z^2} - \frac{s^2}{C_L^2}\,\bar{u}(z,s) = - \frac{\alpha b^2(1+\nu)}{\rho c(1-\nu)}\left\{\frac{\exp\left(-\sqrt{\frac{s}{k}}\,z\right)}{(kb^2-s)} - \frac{\exp(-bz)}{(kb^2-s)}\right\} \quad . \tag{3.10}$$

The complete solution to Equation (3.10) is given by:

$$\bar{u}(z,s) = A_1(s)\cosh\left(\frac{s}{C_L}\,z\right) + A_2(s)\sinh\left(\frac{s}{C_L}\,z\right) - \frac{\alpha b^2(1+\nu)}{\rho c(1-\nu)}\left\{\frac{\exp\left(-\sqrt{\frac{s}{k}}\,z\right)}{(kb^2-s)\left(\frac{s}{k}-\frac{s^2}{C_L^2}\right)} - \frac{\exp(-bz)}{(kb^2-s)\left(b^2-\frac{s^2}{C_L^2}\right)}\right\} \tag{3.11}$$

where the particular solution is readily obtained using the method of undetermined coefficient. Now the two boundary conditions are applied to determine the two constants $A_1(s)$ and $A_2(s)$ in Equation (3.11). Applying boundary condition given by Equation (2.15),

$$A_2(s) = \frac{\alpha(1+\nu)C_L}{(1-\nu)s}\left[\frac{b^2}{\rho c(kb^2-s)}\left\{\frac{b}{\left(b^2-\frac{s^2}{C_L^2}\right)} - \frac{\sqrt{s}}{\sqrt{k}\left(\frac{s}{k}-\frac{s^2}{C_L^2}\right)}\right\} + \theta(0,s)\right] \quad (3.12)$$

Similarly applying boundary condition given by Equation (2.16),

$$A_1(s) = \frac{\alpha b^2(1+\nu)C_L}{\rho c(1-\nu)s(kb^2-s)}\left\{-\frac{\sqrt{s}\exp\left(-\sqrt{\frac{s}{k}}\,D\right)}{\sqrt{k}\left(\frac{s}{k}-\frac{s^2}{C_L^2}\right)} + \frac{b\exp(-bD)}{\left(b^2-\frac{s^2}{C_L^2}\right)}\right\}\frac{1}{\sinh\left(\frac{s}{C_L}D\right)} - \frac{A_2(s)\cosh\left(\frac{s}{C_L}D\right)}{\sinh\left(\frac{s}{C_L}D\right)} + \frac{C_L\alpha(1+\nu)\theta(D,s)}{(1-\nu)s\sinh\left(\frac{s}{C_L}D\right)} \quad (3.13)$$

Substituting in the two constants, after simplification and rearranging terms Equation (3.11) can be written as follows:

$$u(z,s) = W_1(z,s) + W_2(z,s) + W_3(z,s) + W_4(z,s) + W_5(z,s) + W_6(z,s) \quad (3.14)$$

where

$$W_1(z,s) = -C_L\delta T_0 b\big(U_1(s)\big)\,\frac{\cosh\left(\frac{z}{C_L}s\right)}{\sinh\left(\frac{D}{C_L}s\right)}, \quad (3.15)$$

$$W_2(z,s) = -C_L\delta\big(U_2(s)\big)\,\frac{\cosh\left(\frac{D-z}{C_L}s\right)}{\sinh\left(\frac{D}{C_L}s\right)}, \quad (3.16)$$

$$W_3(z,s) = -C_L\delta T_0 b^2\big(U_3(s)\big)\,\frac{\cosh\left(\frac{D-z}{C_L}s\right)}{\sinh\left(\frac{D}{C_L}s\right)}, \quad (3.17)$$

$$W_4(z,s) = \frac{C_L \delta T_0 b}{\sqrt{k}} \left(U_4(s)\right) \frac{\cosh\left(\frac{D-z}{C_L}\, s\right)}{\sinh\left(\frac{D}{C_L}\, s\right)} \quad , \tag{3.18}$$

$$W_5(z,s) = -\delta T_0 b\big(U_5(z,s)\big) \quad , \tag{3.19}$$

$$W_6(z,s) = C_L \delta\big(U_6(s)\big) \frac{\cosh\left(\frac{z}{C_L}\, s\right)}{\sinh\left(\frac{D}{C_L}\, s\right)} \quad , \tag{3.20}$$

and

$$U_1(s) = \frac{\exp\left(-\frac{D\sqrt{s}}{\sqrt{k}}\right)}{\sqrt{ks}\,(kb^2-s)\left(\frac{s}{k}-\frac{s^2}{C_L^2}\right)} - \frac{b\exp(-bD)}{s(kb^2-s)\left(b^2-\frac{s^2}{C_L^2}\right)} \quad , \tag{3.21}$$

$$U_2(s) = \frac{\theta(0,s)}{s} \quad , \tag{3.22}$$

$$U_3(s) = \frac{1}{s(kb^2-s)\left(b^2-\frac{s^2}{C_L^2}\right)} \quad , \tag{3.23}$$

$$U_4(s) = \frac{1}{\sqrt{s}\,(kb^2-s)\left(\frac{s}{k}-\frac{s^2}{C_L^2}\right)} \quad , \tag{3.24}$$

$$U_5(z,s) = \frac{\exp\left(-\sqrt{\frac{s}{k}}\, z\right)}{(kb^2-s)\left(\frac{s}{k}-\frac{s^2}{C_L^2}\right)} - \frac{\exp(-bz)}{(kb^2-s)\left(b^2-\frac{s^2}{C_L^2}\right)} \quad , \tag{3.25}$$

$$U_6(s) = \frac{\theta(D,s)}{s} \quad , \tag{3.26}$$

$$\delta = \frac{\alpha(1+\nu)}{(1-\nu)} \tag{3.27}$$

and

$$T_0 = \frac{b}{\rho c} \quad . \tag{3.28}$$

Now there is a solution of the wave-equation in the transform space. To get the solution in the z-t space, Equation (3.14) must be inverted.

Inverse Laplace Transform:

Inversion of the wave equation requires the inversion of the terms $U_1(s)$, $U_2(s)$, $U_3(s)$, $U_4(s)$, $\tilde{U}_5(z,s)$, $U_6(s)$ (Equations (3.21)–(3.26)) together with the appropriate hyperbolic coefficients accompanying them in the expressions for the $W_i(z,s)$, $i = 1-6$ (Equations (3.15)–(3.20)). First consider the effects of the hyperbolic coefficients on the inverse Laplace transform of Equations (3.15) to (3.20). There are two types of hyperbolic terms in the Laplace transforms of the wave equation. From Equations (3.15) and (3.20) the following hyperbolic coefficient is obtained after expansion and using binomial theorem:

$$\frac{\cosh\left(\frac{z}{C_L}\,s\right)}{\sinh\left(\frac{D}{C_L}\,s\right)} = \left\{\exp\left(\frac{z}{C_L}\,s\right) + \exp\left(-\frac{z}{C_L}\,s\right)\right\}\left\{\exp\left(\frac{D}{C_L}\,s\right) - \exp\left(-\frac{D}{C_L}\,s\right)\right\}^{-1}$$

$$= \left\{\exp\left(\frac{z}{C_L}\,s\right) + \exp\left(-\frac{z}{C_L}\,s\right)\right\}\left\{\exp\left(-\frac{D}{C_L}\,s\right) + \exp\left(-\frac{3D}{C_L}\,s\right) + \exp\left(-\frac{5D}{C_L}\,s\right) + \cdots\right\} \tag{3.29}$$

$$= \exp\left(\frac{z-D}{C_L}\,s\right)\left\{1 + \exp\left(-\frac{2D}{C_L}\,s\right) + \exp\left(-\frac{4D}{C_L}\,s\right) + \cdots\right\} + \exp\left(-\frac{z}{C_L}\,s\right)\left\{\exp\left(-\frac{D}{C_L}\,s\right) + \exp\left(-\frac{3D}{C_L}\,s\right) + \exp\left(-\frac{5D}{C_L}\,s\right) + \cdots\right\} .$$

Based on the shifting property of $\exp(-sa)$ the terms inside the first bracket in the above equation represent waves originating at the back surface ($z = D$) at different times and travelling towards the front surface. For example, the first term represents a wave originating at the back surface at time $t = 0$, the second term represents a wave originating at the back surface at time $t = 2D/C_L$ and the third term represents a wave originating at the same surface at time $t = 4D/C_L$. The terms inside the second bracket in Equation (3.29) represent waves originating at the front surface ($z = 0$) at different times and travelling towards the back surface. For example, the first term inside the second bracket represents a wave originating at the front surface at time $t = D/C_L$ and going towards the back surface. Similarly the second and the third term inside the second bracket represent waves originating at the front surface at time $t = 3D/C_L$ and $t = 5D/C_L$ respectively and propagating towards the back surface. Actually the first term inside the second bracket represents the reflection from the front surface of the wave originated at the back surface at time $t = 0$. Similarly, the second term inside the first bracket represents the same wave after it reaches the back surface and is reflected toward the front surface at $t = 2D/C_L$. Therefore it can be said that different waves are reflections of a single original wave at the two surfaces of the specimen. From the hyperbolic term common to Equation (3.16), (3.17) and (3.18),

$$\begin{aligned}\frac{\cosh\left(\frac{D-z}{C_L}\, s\right)}{\sinh\left(\frac{D}{C_L}\, s\right)} &= \exp\left(-\frac{z}{C_L}\, s\right)\left\{1+\exp\left(-\frac{2D}{C_L}\, s\right)\right.\\ &\quad\left.+\exp\left(-\frac{4D}{C_L}\, s\right)+\cdots\right\}\\ &\quad+\exp\left(\frac{z-D}{C_L}\, s\right)\left\{\exp\left(-\frac{D}{C_L}\, s\right)\right.\\ &\quad\left.+\exp\left(-\frac{3D}{C_L}\, s\right)+\exp\left(-\frac{5D}{C_L}\, s\right)+\cdots\right\}\quad.\end{aligned} \tag{3.30}$$

In this case the first, the second and third terms inside the first bracket represent waves originating at the front surface at times $t = 0$, $t = 2D/C_L$, $t = 4D/C_L$ respectively and going towards the back surface. Similarly the first, second and third terms inside the second bracket represent waves originating at the back surface at time $t = D/C_L$, $t = 3D/C_L$ and $t = 5D/C_L$ respectively and propagating towards the front surface. Here also the different waves are reflections of the wave originated at time $t = 0$ at the front surface of the specimen.

Therefore the effect of the hyperbolic terms on the inverse Laplace transform of the $W_i(z, s)$, $i = 1-6$ (Equations (3.15)–(3.20)) is to shift the time in each term of the inverse Laplace transform of the $W_i(z, s)$ by subtracting from it the arguments of the corresponding exponential term which multiply it and then multiplying each of the terms by the Heaviside's function having the shifted time as