Abstract:
A one-dimensional theoretical model for the laser generated displacement waves in a solid strip with a transparent liquid lying on its front surface is presented. For mathematical simplicity, the liquid medium is treated as a half-space and it is assumed that no vaporization of the liquid or the solid occurs at the interface. Spatial distribution of the laser energy deposited in the solid is assumed to decay exponentially with depth. Regarding the temporal distribution of the laser energy as a delta function in time, a Green’s function in time for the displacement waves in the solid is obtained. Assuming uncoupled thermoelasticity theory, the Laplace transform technique is used to solve the one-dimensional heat conduction and wave equations. The solution for the wave equation is specialized to obtain the displacement at the back surface of the solid. The waveform obtained at the back surface is compared with that obtained for a solid strip in the absence of the liquid. It is shown that the peak displacements obtained in both the cases are the same. The higher postpeak displacements obtained in the presence of the liquid medium are attributed to the surface constraint imposed by the liquid medium on the solid medium. It is also shown that the thermal diffusion of heat into the liquid from the interface has negligible effects on the waveform generated at the back surface of the solid.
ONE-DIMENSIONAL MODEL FOR LASER GENERATED ELASTIC WAVES IN A SOLID LAYER THE ILLUMINATED SURFACE OF WHICH IS CONSTRAINED BY A TRANSPARENT LIQUID MEDIUM

by

Harischandra Prasad Cherukuri

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Mechanical Engineering

MONTANA STATE UNIVERSITY
Bozeman, Montana

July 1989
APPROVAL

of a thesis submitted by

Harischandra Prasad Cherukuri

This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

Date:

July 7, 1989

Chairperson, Graduate Committee

Approved for the Major Department

Date:

July 12, 1989

Head, Major Department

Approved for the College of Graduate Studies

Date:

July 12, 1989

Graduate Dean
STATEMENT OF PERMISSION TO USE

In presenting this thesis in partial fulfillment of the requirements for a master's degree at Montana State University, I agree that the Library shall make it available to borrowers under rules of the Library. Brief quotations from this thesis are allowable without special permission, provided that accurate acknowledgment of source is made.

Permission for extensive quotation from or reproduction of this thesis may be granted by my major professor, or in his absence, by the Dean of Libraries when, in the opinion of either, the proposed use of the material is for scholarly purposes. Any copying or use of the material in this thesis for financial gain shall not be allowed without my written permission.

Signature  [Signature]

Date  [Date]
ACKNOWLEDGEMENTS

This work was supported by the Department of Interior's Bureau of Mines under Contract No. J0134035 through Department of Energy Contract No. DE-AC07-76ID01570. The author would like to thank Idaho National Engineering Laboratory and in particular Dr. K.L. Telschow for his assistance in this project. The author would also like to express his appreciation to Prof. R.J. Conant for his active participation in this project and for the many ways in which he guided me.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>APPROVAL</td>
<td>ii</td>
</tr>
<tr>
<td>STATEMENT OF PERMISSION TO USE</td>
<td>iii</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>iv</td>
</tr>
<tr>
<td>TABLE OF CONTENTS</td>
<td>v</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>vii</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>viii</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>ix</td>
</tr>
<tr>
<td>1. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>Laser Induced Stress Waves in Solids</td>
<td>3</td>
</tr>
<tr>
<td>Previous Work</td>
<td>4</td>
</tr>
<tr>
<td>Statement of The Problem</td>
<td>7</td>
</tr>
<tr>
<td>2. FORMULATION</td>
<td>10</td>
</tr>
<tr>
<td>Approach</td>
<td>10</td>
</tr>
<tr>
<td>Basic Equations of Linear Thermoelasticity</td>
<td>11</td>
</tr>
<tr>
<td>for An Isotropic Homogeneous Body</td>
<td>13</td>
</tr>
<tr>
<td>Assumptions</td>
<td>16</td>
</tr>
<tr>
<td>Governing Equations</td>
<td></td>
</tr>
<tr>
<td>3. TEMPERATURE SOLUTION</td>
<td>20</td>
</tr>
<tr>
<td>Method of Solution</td>
<td>20</td>
</tr>
<tr>
<td>Temperature Solution</td>
<td>21</td>
</tr>
</tbody>
</table>
### TABLE OF CONTENTS—Continued

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4. DISPLACEMENT SOLUTION</td>
<td>26</td>
</tr>
<tr>
<td>Displacement Solution</td>
<td>26</td>
</tr>
<tr>
<td>5. OBSERVATIONS AND RESULTS</td>
<td>43</td>
</tr>
<tr>
<td>Waveform Generated at the Back Surface of the Solid</td>
<td>43</td>
</tr>
<tr>
<td>Physical Interpretation</td>
<td>48</td>
</tr>
<tr>
<td>6. CONCLUSION</td>
<td>57</td>
</tr>
<tr>
<td>REFERENCES CITED</td>
<td>59</td>
</tr>
<tr>
<td>APPENDICES</td>
<td>62</td>
</tr>
<tr>
<td>Appendix A: Program to calculate the temperature distribution in the solid medium</td>
<td>63</td>
</tr>
<tr>
<td>Appendix B: Program to calculate the waveform generated at the back surface of the solid medium</td>
<td>68</td>
</tr>
</tbody>
</table>
LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Physical properties of copper and water</td>
<td>44</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Pulse echo normal incidence method for detecting flaws</td>
<td>2</td>
</tr>
<tr>
<td>2. Schematic representation of the solid-liquid system under normal laser pulse incidence</td>
<td>9</td>
</tr>
<tr>
<td>3. Temperature distribution in the copper specimen as a function of depth for different times</td>
<td>45</td>
</tr>
<tr>
<td>4. Waveform generated at the back surface of the copper specimen as a function of time with the water layer present</td>
<td>46</td>
</tr>
<tr>
<td>5. Waveform generated at the back surface of the copper specimen as a function of time, in the absence of the water layer</td>
<td>47</td>
</tr>
<tr>
<td>6. Effect of increasing bulk modulus on the waveform generated at the back surface of the copper specimen</td>
<td>52</td>
</tr>
<tr>
<td>7. Effect of diffusion of heat into the water layer from the interface, on the waveform generated at the back surface of the copper specimen</td>
<td>53</td>
</tr>
<tr>
<td>8. Effect of diffusion of heat into the water layer, on the temperature distribution in the copper specimen</td>
<td>54</td>
</tr>
<tr>
<td>9. Temperature distribution in the copper specimen as a function of depth for different times</td>
<td>55</td>
</tr>
<tr>
<td>10. Waveform generated at the back surface of the copper specimen as a function of time</td>
<td>56</td>
</tr>
<tr>
<td>11. Format of the data input files for temperature calculation</td>
<td>64</td>
</tr>
<tr>
<td>12. Program to calculate the temperature distribution in the solid medium</td>
<td>65</td>
</tr>
<tr>
<td>13. Format of the data input files for displacement calculation</td>
<td>69</td>
</tr>
<tr>
<td>14. Program to calculate the waveform generated at the back surface of the solid medium</td>
<td>70</td>
</tr>
</tbody>
</table>
A one-dimensional theoretical model for the laser generated displacement waves in a solid strip with a transparent liquid lying on its front surface is presented. For mathematical simplicity, the liquid medium is treated as a half-space and it is assumed that no vaporization of the liquid or the solid occurs at the interface. Spatial distribution of the laser energy deposited in the solid is assumed to decay exponentially with depth. Regarding the temporal distribution of the laser energy as a delta function in time, a Green’s function in time for the displacement waves in the solid is obtained. Assuming uncoupled thermoelasticity theory, the Laplace transform technique is used to solve the one-dimensional heat conduction and wave equations. The solution for the wave equation is specialized to obtain the displacement at the back surface of the solid. The waveform obtained at the back surface is compared with that obtained for a solid strip in the absence of the liquid. It is shown that the peak displacements obtained in both the cases are the same. The higher postpeak displacements obtained in the presence of the liquid medium are attributed to the surface constraint imposed by the liquid medium on the solid medium. It is also shown that the thermal diffusion of heat into the liquid from the interface has negligible effects on the waveform generated at the back surface of the solid.
CHAPTER I

INTRODUCTION

Ultrasonic techniques are finding increased use and importance in the field of nondestructive testing of materials. Before the advent of these techniques, methods like striking a specimen and listening for the characteristic "ring" were used for detecting flaws in the materials. With these methods only the gross defects could be detected since the wavelength of the audible sound waves is large compared to the size of the defects. However, with the development of the reliable methods for generating and detecting ultrasonic waves, even the presence of small defects in the components under inspection can be found since the wavelength of ultrasonic waves is very small compared to the size of the defect.

For example, in the pulse echo normal incidence method used for detecting flaws, cracks, etc., pulse sending and pulse detecting transducers are both placed on one side of the specimen being tested (see Figure 1). If no flaws are present in the specimen, the displacement detected will be as shown in Figure 1.a, where each spike is detected by the transducer after times \( nt_0 \), \( n = 1, 2, \ldots \) etc., with \( t_0 = 2L/v \), \( L \) the length of the specimen and \( v \) the velocity of the ultrasonic pulse in the specimen. On the other hand, if there is a flaw in the specimen such as the one (highly exaggerated) shown in Figure 1.b, there will be a short spike detected in between the two major spikes.
One of the widely used methods for generating ultrasonic waves for nondestructive testing purposes is to make use of the inverse piezoelectric effect. However, this method has the disadvantage in that the piezoelectric transducer has to be coupled to the workpiece. This coupling, depending on its strength and nature, has the effect of influencing the ultrasonic wave apart from causing wear at the interface due to abrasion.

Efforts have therefore been made to utilize other physical effects to generate ultrasonic waves which make mechanical contact with the specimen unnecessary. One of these effects is rapid heating. By heating a body suddenly, mechanical stresses can be produced in the material due to the thermal expansion of the
material. These stresses then initiate elastic waves. If the heating is for very short duration and is intense, very high frequencies and shock waves can be produced.

There are many ways of delivering the required thermal energy to the body. Of these, particularly suitable is the laser induced heating. The suitability of lasers for this purpose comes from the intensity and the directionality properties of the laser beams. These two properties allow high power densities (irradiances) to be delivered in very short times to the absorbing material although the total power carried by the beam may be small. Other advantages include the high reproducibility of laser generated ultrasound [1] and higher frequencies than can be obtained with piezoelectric devices. Also, resonance phenomena which occur with piezoelectric devices do not occur in laser induced ultrasonic waves.

In the present work, the author is concerned with the ultrasonic wave generation due to laser induced heating.

Laser Induced Stress Waves in Solids

There are basically two mechanisms by which thermal waves are produced in solids due to laser interaction.

When a short duration laser beam of power density sufficiently high to produce stress waves but small enough not to cause any phase change is absorbed by a target material (here, a solid), high thermal energies are deposited for a very short interval of time causing high spatial thermal gradients. This induces localized rapid thermal expansion in the medium, which is resisted by the surrounding medium. This resistance may be so high that the material is subjected to rapidly varying strain field, which results in the generation of thermal stress waves.

On the other hand, if a high intensity laser beam strikes the target material, some of the material may get evaporated. The velocity with which the ejected
material leaves the target surface can be so high that, by the momentum conservation principle, the target is imparted an impulse. This results in the transmission of a shock wave through the target.

Now if the surface of the target is coated with a thin film of liquid or a transparent solid, due to the surface constraint imposed by the film it has been found that the resulting acoustic transmission through the solid is stronger. In the presence of a liquid film, if no phase changes occur in either of the two media and if it is assumed that the laser beam energy is deposited completely in the solid, there will be thermoelastic waves generated in the solid due to internal heat source and in the liquid due to surface heating from the solid-liquid interface.

In the next section some of the previous work that was done to understand the above mentioned phenomena is reviewed.

**Previous Work**

Perhaps the first person to suggest the possibility of generating elastic waves due to absorption of radiation from the high powered light sources was White [2]. He assumed that the laser energy deposition in the solid is equivalent to surface heating of the solid and analyzed the one-dimensional elastic wave generation in isotropic elastic bodies under several transient surface heating conditions such as uniform and non-uniform heating of the surface of a semi-infinite body, and uniform heating of the end of a long rod. The case of input heat flux varying harmonically with time was studied in detail. It was shown that the dependence of the stress wave amplitude on frequency and the properties of the body heated is governed by the kind of constraint applied at the surface and by the distance through which the heating takes place. It was also shown that the stress amplitude for a constrained surface is much greater than that for a free surface.
Unlike White, Ready [3] considered the optical absorption effects by assuming that the energy absorbed by the body decreases exponentially across the thickness of the body. However, he determined only the temperature distribution and did not analyze the deformation or the stress field. A general solution for the temperature distribution in the body due to a laser pulse of arbitrary temporal shape was obtained using Duhamel's integral method. This solution was then used to show that for an ordinary laser pulse of a sequence of microsecond duration spikes, the surface temperature follows the laser spikes and that the properties of the body such as thermal conductivity are important for the temperature distribution in the body. In the same work, the effects of vaporization of the material from the surface of the body due to ordinary laser pulses and Q-switched laser pulses were also investigated experimentally.

Since Ready, there have been many investigations, both analytical and theoretical, into the laser generated elastic waves in both solids and liquids.

Of these, as far as the laser generated stress waves in solids are concerned, particularly worthy of mentioning is the work by the group of Scruby, Dewhurst, Hutchins and Palmer ([1], [4] and [5]). They studied experimentally the waveforms produced due to the pulsed laser irradiation over a wide range of material conditions such as the presence and the absence of surface plasmas, free metal surfaces and metal surfaces modified with transparent liquid or solid coatings. Based on the experiments, neglecting thermal diffusion and treating the thermal source as constrained to the surface, three-dimensional axisymmetric models for these different conditions were obtained. The model for the first case predicted an inward displacement at the back surface, while the models for the remaining two cases predicted an outward displacement. Although these results were in agreement with those obtained experimentally, the model developed for the first case
does not predict an initial outward displacement spike (which is observed in the experimental results) as thermal diffusion was neglected. Doyle [6] took the effects of thermal diffusion into consideration and obtained a model which predicts the spike. Brechtel [7] investigated the conditions under which diffusion effects can be neglected and showed that if the depth of thermal diffusion for the laser pulse duration is large compared to the optical attenuation depth, the surface and volume generated models of laser induced heating yield the same results. On the other hand, if the thermal diffusion depth for the laser pulse duration is comparable or less than the attenuation depth, the results obtained from the two models may differ significantly.

Until now, all the work considered above dealt with neglecting the effects of stress or displacement fields upon the temperature field. Strikwerda and Scott [8], using coupled thermoelastic theory, obtained a one-dimensional theoretical model for the temperature and the displacement fields produced due to short duration laser pulses. They used perturbation methods to arrive at the solutions. Explicit results for a Gaussian pulse shape and an exponential decay pulse shape were presented. Results for the temperature field were compared with those given in [3] and found to be compatible.

The work of Scruby et. al. ([1],[4] and [5]) also showed that the modification of the surface (i.e. coated targets) leads to an enhancement in the generation of longitudinal waveforms at the backside. Their work considers both the cases where there is a phase change of either the target material or the coating material and where there is no phase change at all.

However, before Scruby et al., there were many others who investigated the possibility of strengthening the acoustic wave generation by coating the laser irradiated targets.
Anderholm [9] developed a technique in which the material is confined between two transparent materials. When a laser pulse of short duration hits the target, a confined, high energy plasma is produced which then generates high amplitude stress waves into the transparent materials. The technique of O'keefe and Skeen [10] involves partially vaporizing the thin coating on the target material through conduction from the laser heated surface. The subsequent expansion of the gas and the plasma confinement at the interface causes stress waves to be generated in the target material.

In the last two cases, to increase the coupling of the laser energy to the target, the laser irradiations were generated in the vacuum. Fox [11] showed experimentally that the stress enhancements can be achieved without plasma confinement and in atmospheric air by coating the target surface with optically dense thin films such as of oil or water.

However, to the author's knowledge, theoretical work concerned with the effects of laser irradiation of coated targets was done only by Scruby et al. Their work does not consider the diffusion effects. As a first step toward understanding the effects of diffusion on this problem (assuming there is no phase change), it is the purpose of the present work to consider the one-dimensional aspects of the problem taking diffusion into account. The actual problem that will be dealt with is presented next.

**Statement of The Problem**

In this thesis, the author will be concerned with the case where the laser induced heating causes temperature changes and therefore displacement changes only along the thickness. It is to be noted that although this is far from the reality, it is certainly possible to realize approximate homogeneous behavior in
the transverse directions with regard to temperatures and displacements when the transverse dimensions of the laser beam are large compared to the depth to which heat is propagated during the laser pulse time (page 70, [12]). Therefore it can be assumed that the laser pulse incidence is normal to the liquid-solid interface.

The main aim is to develop a one-dimensional theoretical model for the prediction of laser induced displacement waves in a solid medium of finite thickness with a liquid layer resting on the front side of it (Figure 2).

The model to be developed should fulfill the following requirements:

1. It should be valid for a laser pulse of arbitrary temporal shape.
2. It should explain why higher displacements are observed in the presence of a liquid medium than in the absence of the liquid medium.
3. Theoretical displacements obtained from this model for the back surface of the solid should match the experimentally obtained results.

It should be noted that displacement of interest is that at the back surface of the solid, since it is usually at the back surface that the laser generated ultrasonic waves are measured. (In non-destructive testing, such a method where the ultrasonic wave is generated at one surface of the test specimen and is detected at the other surface is called the through transmission normal incidence method. This can be visualized from Figure 1 by placing a displacement detecting transducer at the back surface instead of at the front surface.) This also means that time of interest is only until the second arrival of the wave at the back surface, since the waves reflected from flaws present in the solid take times ranging from about one to three times the time that a longitudinal wave generated at the interface of the solid and the liquid, takes to reach the back surface of the solid.
Figure 2. Schematic representation of the solid-liquid system under normal laser pulse incidence.

To avoid the interference of the reflected waves from the upper surface of the liquid with the waves in the solid, it is also assumed that the liquid layer is thick enough so that there is no reflected wave from the upper liquid surface for the times of interest.
CHAPTER 2

FORMULATION

In this chapter, the approach taken to solve the problem proposed in the last chapter is presented, and then the equations governing the stress waves generated in the solid due to laser heating and the acoustic waves generated in the liquid due to heating from the interface are developed.

Approach

To obtain the displacement and the temperature fields in both the liquid and the solid media, uncoupled thermoelastic theory is assumed i.e., the temperature changes resulting from the changes in the deformation field are neglected. Optical absorption effects are taken into account by assuming that the laser deposition of energy into the solid through the liquid medium drops exponentially along the thickness of the solid medium as assumed in [3].

In order to meet the requirement (see Chapter 1): that the obtained solutions be valid for a laser pulse of arbitrary temporal shape, first Green's functions for the temperature and displacement fields are obtained by replacing the function representing the laser pulse temporal shape by the Dirac delta function in time. Once the Green's functions are obtained, the thermal and displacement fields at any time due to a desired temporal shape of the laser pulse can be obtained by employing the superposition principle (integration).

In the next section some of the basic equations of linear thermoelasticity are reviewed.
Basic Equations of Linear Thermoelasticity for An Isotropic Homogeneous Body

When a body is subjected to temperature variations it is known that the body undergoes deformations. In the presence of external constraints or if the temperature is not uniform, these variations give rise to stresses. Conversely, when a body is subjected to deformations, a rise or a drop in the temperature of the body may occur. Therefore, the problem of determining the displacement in a body becomes a coupled one since to know the displacement field, one needs to know the temperature distribution in the body and to know the temperature distribution one needs to know the displacement field in the body.

However, if it is assumed that the effect of the deformation field on the temperature field is negligible, the temperature field can be determined independent of the deformation field and then using the known thermal distribution, the displacement field can be obtained.

For the present problem, it is assumed that this indeed is the case. This seems reasonable since the thermoelastic coupling coefficient for most of the materials is small compared to unity. Intuitively, it can be argued that since the exciting source for the present case is thermal and since the mechanical changes are due to this, the variations in the temperature field due to the deformation changes are small. Further discussion on the conditions under which uncoupling between the temperature and the deformation fields is valid can be found in [13].

Thus for an uncoupled problem, the basic governing equations for the temperature field and the deformation field for an isotropic, linear thermoelastic solid with no body forces are given by [14]:

\[ T_{1,ii} - \frac{1}{k_1} \frac{\partial T_1}{\partial t} + \frac{h_1}{c_1} = 0 \]  
(2.1)

\[ (\lambda + \mu)u_{1i,jj} + \mu u_{ij,ji} - \gamma (3\lambda + 2\mu) T_{1,i} = \rho_1 \ddot{u}_{1i} \]  
(2.2)
where the standard indicial notation is used. In the above equations $T_1(x, y, z, t)$ and $u_1(x, y, z, t)$ are respectively the temperature and displacement fields in the solid, $h_1$ is the thermal source per unit volume (can be a function of space and time), $k_1$, $c_1$ and $\rho_1$ are, respectively, the thermal diffusion, thermal conductivity and density of the medium.

For an isotropic, linear thermoelastic liquid with no body forces, the conduction equation remains the same as Equation (2.1). However, the wave equation becomes [15]:

$$B u_{2,i,i} - B \beta T_{2,i} = \rho_2 \ddot{u}_2$$

where $u_2(x, y, z, t)$ is the displacement field in the liquid and $B$ is the bulk modulus, $\rho_2$ is the density, $\beta$ is the coefficient of volume thermal expansion.

The stress-strain relations for an isotropic, homogeneous, linearly thermoelastic solid are given by the Duhamel-Neumann constitutive equations [3]:

$$\tau_{ij} = 2\mu u_{1i,j} + \lambda u_{kk} \delta_{ij} + \beta T_1 \delta_{ij}$$

and for the liquid in terms of pressure [15]:

$$p = -B (u_{2i,i} - \beta T_2)$$

These are the basic equations for this work. In the next two sections, when the problem is formulated mathematically, these equations are used for describing the temperature and displacement distributions in the liquid and the solid media.

Often when modeling a physical problem, certain assumptions idealizing the physical phenomenon are necessary. Here also, to make the mathematics tractable, some assumptions are made and are presented next.
To analyze the deformation fields in the solid and the liquid media due to absorption of laser energy by the solid, the following assumptions (apart from the uncoupling condition) are made. These make the mathematics manageable and yet yield a physically possible model.

Assumption 1: Linear elastic, isotropic, homogeneous thermal and mechanical behavior is assumed for both the media. With this assumption, the equations presented in the previous chapter can be used to describe the displacement and temperature fields in the two media.

Assumption 2: Thermal and mechanical properties of the two media are assumed to be independent of the temperature. For many solids and liquids, the variations of the properties are relatively small over a fairly wide range of temperatures. Therefore this assumption seems reasonable.

Assumption 3: The ordinary thermodynamical concept of temperature is valid for the times of interest and therefore the thermal energy distribution in the two media can be analyzed by using the macroscopic laws of heat conduction. The explanation for this assumption follows the one given by Ready [3]. The transfer of heat in conducting media is governed by the collision processes between electrons, and between electrons and the lattice phonons. The mean free time between collisions for electrons is of the order $10^{-14}$ to $10^{-13}$ seconds and since times of interest are of the order $10^{-8}$ to $10^{-7}$ seconds, many collisions will have occurred and the energy absorbed by one electron will be distributed and passed onto the lattice. Therefore, the light energy can be assumed to be transferred instantaneously into heat energy which in turn implies that temperature in the ordinary sense is a valid concept.
Assumption 4: Heat transfer to the liquid medium occurs only through conduction from the interface and there is no loss of energy to the liquid from the laser pulse as it passes through the liquid towards the solid. This implies that heating of liquid occurs only through surface heating from the interface and that there are no thermal sources in the liquid.

Assumption 5: No phase change takes place in either of the two media. This means that the total absorbed energy by the solid medium should be small enough to prevent any phase change in either medium but yet high enough to generate waves.

Assumption 6: Both the liquid and the solid media can be modelled as half-spaces when solving for the temperature distribution. As was mentioned in the first chapter, times of interest are about three times the time a longitudinal wave originated at the solid-liquid interface takes to travel across to the back surface of the solid. Since elastic waves travel at much greater speeds than the thermal waves do, the back surface of the solid remains unaffected by the thermal gradients generated in its interior for the times of interest. That this can be assumed is seen from the following argument.

The rapidity with which a material accepts and conducts thermal energy is determined by the thermal diffusivity of the material. The higher the diffusivity, higher is the depth of penetration of heat. The time required for the thermal energy to travel a specified penetration depth $x$ is given \([16]\) approximately by $x^2/4k$, where $k$ is the diffusivity. Thus for metal specimens of thicknesses in the order of 0.1 cm to 1 cm, the time constant is of the order of milliseconds. On the other hand, the time that an elastic wave takes to travel across these thicknesses is of the order of microseconds. Therefore, times of interest are in the range of a few microseconds which implies that for the temperature calculations, the solid medium can safely be regarded as a half-space.
For the same reason the liquid medium is also assumed to be a half-space for temperature calculations (in fact, for the liquid medium, as $k$ is small, thermal time constant will be higher).

**Assumption 7:** For the calculation of displacement field, we treat the solid as a finite medium and the liquid medium as a half-space. The reason for treating the solid as a finite medium is obvious from the fact that ultimately it is the displacement at the back surface of the solid that is actually required.

It was mentioned in the first chapter that the thickness of the liquid medium is such that the waves generated at the interface and travelling in the liquid do not have the time to get reflected at the upper surface and to interfere with the waves generated in the solid. It is for this reason that the liquid medium is assumed to be of infinite thickness.

**Assumption 8:** Energy lost by the solid through reradiation and convection is negligible. Since the time intervals of interest and the laser pulse duration are short as mentioned in the first chapter, this is a reasonable assumption. Even for laser energies high enough to cause phase change, the losses due to reradiation are only about 1% for short times. For further discussion on this, the reader is referred to Chapter 4 in [12].

**Assumption 9:** The deposition of laser energy in the solid is assumed to be distributed according to $f(t)Q_0 \alpha \exp(-\alpha z)$ where $f(t)$ is a function in time describing the temporal shape of the laser pulse, $Q_0$ is the laser energy incident per unit area, $\alpha$ is the optical absorption coefficient of the solid and $z$ is spatial coordinate along the thickness of the two media. The spatial distribution $\exp(-\alpha z)$ of laser energy corresponds to the exponential decay of the electromagnetic radiation in solids. By allowing $f(t)$ to be arbitrary, the displacement field due to a laser pulse of arbitrary temporal shape can be calculated. This is achieved by using the
Green's function method to be discussed later. This assumption also allows the laser interaction with the solid medium to be modelled as a thermal source in the solid.

**Governing Equations**

As mentioned before, if the coupling between the temperature field and the displacement field is neglected, the diffusion equation can be solved first, without the knowledge of the displacement field. The displacement equation can then be solved for the displacement field using the known temperature distribution. Also since the proposed problem is one-dimensional, temperature and displacement fields are functions only of time and the coordinate describing the thickness of the two media (i.e., \( z \) in Figure 2).

To find the temperature distribution, treating the two media as two half-spaces (Assumption 6) in contact with each other, let the coordinate system be chosen with the positive \( z \) direction along the thickness of the solid with \( z = 0 \) describing the interface. Thus \( z \) is positive in the solid and is negative in the liquid medium.

Let \( T_1(z, t) \) be the transient temperature distribution in the solid and \( T_2(z, t) \) be that in the liquid. Let \( \rho_1, k_1 \) and \( c_1 \) respectively be the density, diffusivity and conductivity of the solid medium. Let \( \rho_2, k_2 \) and \( c_2 \) be those of the liquid medium.

Within the scope of the uncoupled theory of thermoelasticity, then, \( T_1(z, t) \) has to satisfy the following one-dimensional form of the heat conduction equation (2.1):

\[
T_{1,zz} + \frac{Q_0}{c_1} \exp(-\alpha z) f(t) = \frac{1}{k_1} T_{1,tt}, \quad 0 \leq z \leq \infty, \quad t > 0
\]

(2.6)

where the interaction of the laser pulse with the solid is modelled as a thermal source in the solid (Assumption 9).
Assumption 4 implies that there is no internal heat source in the liquid medium. Therefore, $T_2(z, t)$ has to satisfy the following one-dimensional form of the heat conduction equation (2.1) with no thermal sources:

$$T_{2,zz} = \frac{1}{k_2} T_{2,t} , \quad -\infty \leq z \leq 0 , \ t > 0 . \quad (2.7)$$

The boundary conditions for the above equations are obtained as follows:

At the interface $z = 0$, temperature and heat continuities require that

$$T_1(0, t) = T_2(0, t) \quad (2.8)$$

and

$$c_1 T_1, (0, t) = c_2 T_2, (0, t) . \quad (2.9)$$

The other two conditions come from Assumption (6) according to which, for the times of interest, the back surface of the solid and the front surface of the liquid are not affected by thermal gradients in the interior of the solid-liquid system. Therefore

$$\lim_{z \to -\infty} T_1(z, t) \text{ is bounded} \quad (2.10)$$

and

$$\lim_{z \to -\infty} T_2(z, t) \text{ is bounded} \quad (2.11)$$

and the initial conditions are

$$T_1(z, 0) = 0 \quad (2.12)$$

and

$$T_2(z, 0) = 0 . \quad (2.13)$$

The associated one-dimensional displacement equation for the solid medium obtained from Equation (2.2) is

$$u_{1,zz} = \delta_{1,z} + \frac{1}{C_{L_1}^2} u_{1,tt} , \quad 0 \leq z \leq b , \ t > 0 \quad (2.14)$$
where $u_1(z, t)$ is the displacement field in the solid medium, $C_{L_1}$ is the longitudinal wave speed in the solid medium,

$$C_{L_1} = \left(\frac{\lambda + 2\mu}{\rho_1}\right)^{1/2}, \quad \text{(2.15)}$$

$$\delta = \left(\frac{3\lambda + 2\mu}{\lambda + 2\mu}\right) \gamma, \quad \text{(2.16)}$$

and $\gamma$ is the coefficient of linear thermal expansion. The one-dimensional wave equation for the liquid obtained from Equation (2.3) is

$$u_{2,zz} = \beta T_{2,z} + \frac{1}{C_{L_2}^2} u_{2,tt}, \quad -\infty < z < 0, \ t > 0 \quad \text{(2.17)}$$

where $u_2(z, t)$ is the displacement in the liquid medium, $C_{L_2}$ is the acoustic wave speed in the liquid medium, and

$$C_{L_2} = \left(\frac{\beta}{\rho_2}\right)^{1/2}. \quad \text{(2.18)}$$

Imposing the displacement and stress continuities respectively at the interface, the conditions

$$u_1(0, t) = u_2(0, t) \quad \text{(2.19)}$$

and

$$\sigma_{1z}^1(0, t) = \sigma_{2z}^2(0, t) \quad \text{(2.20)}$$

are obtained.

Requiring the back surface of the solid medium to be traction free,

$$\sigma_{zz}^1(b, t) = 0. \quad \text{(2.21)}$$

Requiring that displacement be bounded as $z$ approaches $-\infty$ gives

$$\lim_{z \to -\infty} u_2(z, t) \text{ is bounded}. \quad \text{(2.22)}$$
The initial conditions are

\[ u_1(z,0) = u_{1,t}(z,0) = 0 \quad (2.23) \]

and

\[ u_2(z,0) = u_{2,t}(z,0) = 0 . \quad (2.24) \]

Equations (2.20) and (2.21) can be expressed in terms of displacements using the proper constitutive equations. From the one-dimensional form of the Duhamel-Neumann constitutive equations for isotropic materials (Equation (2.4)):

\[ \sigma_{zz}^1 = (\lambda + 2\mu)u_{1,z} - \gamma(3\lambda + 2\mu)T_1(z,t) . \quad (2.25) \]

For liquid media, stress in terms of displacement is given by (Equation (2.5)):

\[ \sigma_{zz}^2 = B(u_{2,z} - \beta T_2(z,t)) \quad (2.26) \]

since stress is the negative of pressure.

Equations (2.20) and (2.21) with the help of Equations (2.25) and (2.26) become

\[ u_{1,z}(0,t) - (\delta - \varsigma\beta)T_1(0,t) = \Omega u_{2,z}(0,t) \quad (2.27) \]

and

\[ u_{1,z}(b,t) = \delta T_1(b,t) \quad (2.28) \]

where

\[ \varsigma = \frac{B}{(\lambda + 2\mu)} . \quad (2.29) \]

Making use of the conditions (2.8) through (2.13), the equations (2.6) and (2.7) are solved for the temperature distribution in the two media, in the next chapter. In Chapter 4, Equations (2.14) and (2.17) are solved for the displacement field using the boundary conditions (2.19), (2.22), (2.27) and (2.28) and the initial conditions (2.23) and (2.24).
CHAPTER 3
TEMPERATURE SOLUTION

Method of Solution

The temperature distribution in the two media, for a laser pulse of arbitrary temporal shape, is obtained in two steps. First, the function \( f(t) \) which represents the laser pulse shape in time is replaced by a Dirac delta function \( \delta(t) \) and the two equations (2.6) and (2.7) are solved for \( T_1(z,t) \) and \( T_2(z,t) \). The solutions (Green’s functions in time) obtained give temperature response of the two media at position \( z \) and and time \( t \) due to a thermal source acting instantaneously at time \( t = 0 \). Replacing \( t \) in the solutions by \( t - \tau \), then gives the temperature response due to a thermal source acting instantaneously at \( t = \tau \). This follows from the translation property of the Green’s functions (see [17]).

In the second step, the temperature response at any time \( t \) due to a laser pulse of arbitrary temporal shape \( f(t) \) is obtained by integrating the above Green’s functions multiplied by \( f(\tau) \) from \( 0 \) to \( t \).

Since the Laplace transform method is used in this work for solving differential equations, it is to be noted that only the transformed temperature solutions are needed for solving the wave equations. However, for completeness and for later analysis, the complete solution is carried out.
Temperature Solution

Temperature solutions are obtained by using the Laplace transform technique in the time variable \( t \). The Laplace transform pairs are defined by

\[
\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t) \exp(-st)\,dt
\]

and

\[
\mathcal{L}^{-1}\{F(s)\} = f(t) = \int_{a-i\infty}^{a+i\infty} F(s) \exp(st)\,dt
\]

where \( a \) is chosen so that all the singularities of \( F(s) \) are to the left of \( \text{Re}(s) = a \) in the complex plane.

With this definition, the Laplace transforms of Equations (2.6) and (2.7) with \( \delta(t) \) replacing \( f(t) \) become

\[
\mathcal{T}_{1,zz}(z,s) + \frac{Q_0 \alpha}{c_1} \exp(-\alpha z) = \frac{s}{k_1} \mathcal{T}_1(z,s) \quad 0 \leq z \leq \infty
\]

and

\[
\mathcal{T}_{2,zz}(z,s) = \frac{s}{k_2} \mathcal{T}_2(z,s) \quad -\infty \leq z \leq 0
\]

where \( \mathcal{T}_i(z,t), i = 1, 2, \) represents the Green's function for temperature distribution in the \( i^{th} \) medium due to an instantaneous heat source applied at time equals zero i.e., the Green's functions are the solutions of Equations (2.6) and (2.7) when the function in time in these equations is replaced by a Dirac delta function at time equals zero. \( \mathcal{T}_i(z,s), i = 1, 2, \) represents the corresponding Green's function in the \( i^{th} \) medium in the transformed space.

The boundary conditions (2.8) through (2.11) in the transformed space become

\[
\mathcal{T}_1(0,s) = \mathcal{T}_2(0,s)
\]

\[
c_1 \mathcal{T}_{1,z}(0,s) = c_2 \mathcal{T}_{2,z}(0,s)
\]
Equations (3.1) and (3.2) are ordinary differential equations and their solutions are given by

\[ T_1(z, s) = A_1(s) \exp \left( \sqrt{\frac{s}{k_1}} z \right) + B_1(s) \exp \left( -\sqrt{\frac{s}{k_1}} z \right) + T_{1p}(z, s), \quad 0 \leq z \leq \infty \]  

and

\[ T_2(z, s) = A_2(s) \exp \left( \sqrt{\frac{s}{k_2}} z \right) + B_2(s) \exp \left( -\sqrt{\frac{s}{k_2}} z \right), \quad -\infty \leq z \leq 0 \]  

where \( A_1(s), A_2(s), B_1(s) \) and \( B_2(s) \) are to be determined from the boundary conditions. The last term on the right-hand side of the first equation above is the particular solution of the equation (3.1). Using undetermined coefficients, this is obtained as

\[ T_{1p}(z, s) = \frac{Q_0 \alpha k_1}{c_1} \frac{\exp(-az)}{(s - \alpha^2 k_1)} \]  

The boundary conditions (3.5) and (3.6) together with Equations (3.7), (3.8) and (3.9) require that \( A_1(s) \) and \( B_2(s) \) be zero. The conditions (3.3) and (3.4) then give

\[ B_1(s) = \frac{PT_0}{\sqrt{s} (\alpha^2 k_1 - s)} \left( \alpha + \frac{c_2}{c_1} \sqrt{\frac{s}{k_2}} \right) \]  

and

\[ A_1(s) = B_1(s) + \frac{T_0}{s - \alpha^2 k_1} \]  

where

\[ P = \frac{c_1}{\sqrt{k_1}} + \frac{c_2}{\sqrt{k_2}} \]
and

\[ T_0 = \frac{Q_0 \alpha k_1}{c_1} \]  \hspace{1cm} (3.13)

Thus,

\[ \bar{T}_1(z, s) = B_1(s) \exp \left( -\sqrt{\frac{s}{k_1}} z \right) + \bar{T}_{1p}(z, s), \hspace{0.5cm} 0 \leq z \leq \infty \]  \hspace{1cm} (3.14)

and

\[ \bar{T}_2(z, s) = A_2(s) \exp \left( \sqrt{\frac{s}{k_2}} z \right), \hspace{0.5cm} -\infty \leq z \leq 0 \]  \hspace{1cm} (3.15)

where \( B_1(s) \) and \( A_2(s) \) are given by (3.11) and (3.12). Using partial fractions,

\[ \frac{1}{s - \alpha^2 k_1} = \frac{1}{2\alpha \sqrt{k_1}} \left( \frac{1}{\sqrt{s - \alpha \sqrt{k_1}}} - \frac{1}{\sqrt{s + \alpha \sqrt{k_1}}} \right) \]  \hspace{1cm} (3.16)

Substituting this in Equations (3.14) and (3.15) and using the following inverse Laplace transform formulae [18],

\[ \mathcal{L}^{-1} \left\{ \frac{\exp(-q\sqrt{s})}{p + \sqrt{s}} \right\} = \frac{1}{\sqrt{\pi t}} \exp \left( -\frac{q^2}{4t} \right) \]

\[ - p \exp(pq + p^2t) \text{erfc} \left( p\sqrt{t} + \frac{q}{2\sqrt{t}} \right), \quad q \geq 0 \]  \hspace{1cm} (3.17)

\[ \mathcal{L}^{-1} \left\{ \frac{\exp(-q\sqrt{s})}{\sqrt{s(p + \sqrt{s})}} \right\} = \exp(pq + q^2t) \text{erfc} \left( p\sqrt{t} + \frac{q}{2\sqrt{t}} \right), \quad q \geq 0 \]  \hspace{1cm} (3.18)

and

\[ \mathcal{L}^{-1} \left\{ \frac{1}{s - p} \right\} = \exp(pt) \]  \hspace{1cm} (3.19)

the Green’s functions in time for the temperature are obtained from Equations (3.14) and (3.15) as

\[ T_1(z, t) = T_0 \exp \left( -\frac{z^2}{4k_1 t} \right) \left[ \exp(x_1^2) \text{erfc} x_1 + (1 - 2P_1 n) \exp(x_2^2) \text{erfc} x_2 \right] H(t) \]  \hspace{1cm} (3.20)

and

\[ T_2(z, t) = T_0 \exp \left( -\frac{z^2}{4k_2 t} \right) \exp(x_3^2) \text{erfc}(x_3) H(t) \]  \hspace{1cm} (3.21)
where $H(t)$ is the Heaviside’s function and

$$x_1 = \left( \alpha \sqrt{k_1} t - \frac{z}{\sqrt{4k_1} t} \right),$$  
(3.22)

$$x_2 = \left( \alpha \sqrt{k_1} t + \frac{z}{\sqrt{4k_1} t} \right),$$  
(3.23)

$$x_3 = \left( \alpha \sqrt{k_1} t - \frac{z}{\sqrt{4k_2} t} \right),$$  
(3.24)

$$P_1 = \frac{P}{\sqrt{k_1}},$$  
(3.25)

and

$$n = \frac{c_2}{c_1} \sqrt{\frac{k_1}{k_2}}.$$  
(3.26)

Equations (3.20) and (3.21) give the thermal response of the two media due to an instantaneous application of thermal energy in the solid medium at time equal to zero. The thermal response due to an instantaneous heat source applied at time $\tau_0$, i.e. when $\delta(t)$ is replaced by $\delta(t - \tau_0)$, is then obtained from the translation property of the Green’s functions, by shifting $t$ in these equations by $\tau_0$.

Thus, the temperature distributions in the two media due to a thermal source applied instantaneously at $\tau_0$ are given by $T_1(z, t - \tau_0)$ and $T_2(z, t - \tau_0)$.

Now, the temperature distribution $\theta_i(t)$, $i = 1, 2$, in the two media due to a laser pulse of temporal shape $f(t)$ is obtained from the superposition principle of Green’s functions as follows:

$$\theta_1(z, t) = \int_0^\infty T_1(z, t - \tau_0) f(\tau_0) d\tau_0$$  
(3.27)

and

$$\theta_2(z, t) = \int_0^\infty T_2(z, t - \tau_0) f(\tau_0) d\tau_0.$$  
(3.28)
Since the two Green's functions are multiplied by $H(t - \tau_0)$, the upper limit in the above integrals actually reduces to $t$, which physically means that the temperature distribution at a time $t$ is only due the superposition of all the thermal sources that acted only before and at $t$. Any future thermal sources do not have any influence on the current temperature.
CHAPTER 4

DISPLACEMENT SOLUTION

Similar to the temperature solution, displacement solution is also obtained in two steps. In the first step, displacement due to a sudden application of laser energy at time equals zero (with the spatial distribution being an exponential decay along the thickness as given in Assumption 9, Chapter 2) is obtained. This gives a Green's function in time for the displacement. The deformation response due to an instantaneous laser source acting only at \( t = \tau \) is then obtained by using the translation property of Green's functions.

In the second step using the superposition principle of the Green's functions, displacement fields in the two media due to a laser pulse of arbitrary temporal shape are obtained.

Displacement Solution

In the first step, Green's functions in time for the displacement fields in the two media are obtained by substituting for temperature terms in the wave equations (2.14) and (2.17) and in the boundary conditions (2.27) and (2.28), the Green's functions for temperature obtained in the last chapter.

Laplace transform technique is again used to obtain the required solutions. The Laplace transforms of Equations (2.14) and (2.17) (after making the above mentioned replacements) with the help of the initial conditions (2.23) and (2.24), the following differential equations are obtained:

\[
\frac{d^2 \bar{w}_1}{dz^2} (z, s) = \delta \frac{dT_1}{dz} (z, s) + \frac{s^2}{C_L^2} \bar{w}_1 (z, s)
\]  

(4.1)
and
\[
\frac{d^2 \bar{w}_2}{dz^2} (z, s) = \beta \frac{dT_2}{dz} (z, s) + \frac{s^2}{C_{L_2}^2} \bar{w}_2 (z, s) \tag{4.2}
\]
where \(w_i(z, t), i = 1, 2\) represent the Green's functions in time for the displacements and \(\bar{w}_i(z, s), i = 1, 2\) represent their Laplace transforms in the same order.

The boundary conditions (2.19), (2.22), (2.27) and (2.28), after replacing the temperatures in (2.27) and (2.28) by their Green's functions, upon taking Laplace transforms become:

\[
\bar{w}_1 (0, s) = \bar{w}_2 (0, s), \tag{4.3}
\]

\[
\lim_{z \to -\infty} \bar{w}_2 (z, s) \text{ is bounded}, \tag{4.4}
\]

\[
\bar{w}_1, z (0, s) - (\delta - \sigma \beta) \bar{T}_1 (0, 2) = \sigma \bar{w}_2, z (0, s), \tag{4.5}
\]

and

\[
\bar{w}_1, z (b, s) = \delta \bar{T}_1 (b, s) = 0. \tag{4.6}
\]

The complete solutions of (4.1) and (4.2) are given by:

\[
\bar{w}_1 (z, s) = d_1 (s) \cosh(sz/C_{L_1}) + d_2 (s) \sinh(sz/C_{L_1}) + \bar{w}_{1p} (z, s) \tag{4.7}
\]

and

\[
\bar{w}_2 (z, s) = d_3 (s) \cosh(sz/C_{L_2}) + d_4 (s) \sinh(sz/C_{L_2}) + \bar{w}_{2p} (z, s) \tag{4.8}
\]

where \(d_i(s), i = 1, 2, 3, 4\) are to be determined from the boundary conditions (4.3) through (4.6) and \(\bar{w}_{1p} (z, s)\) and \(\bar{w}_{2p} (z, s)\) are the particular solutions of (4.1) and (4.2) respectively, found by the method of undetermined coefficients. These are

\[
\bar{w}_{1p} (z, s) = -B_1 (s) \frac{s}{s/k_1 - s^2/C_{L_1}^2} \exp \left( -\sqrt{\frac{s}{k_1}} z \right) + \frac{C_{L_1}^2 \delta T_0 \alpha \exp(\alpha z)}{(s - \alpha k_1)(s^2 - \alpha^2 C_{L_1}^2)} \tag{4.9}
\]

and

\[
\bar{w}_{2p} (z, s) = \frac{\beta A_2 (s) \sqrt{s/k_2}}{s/k_2 - s^2/C_{L_2}^2} \exp \left( \sqrt{\frac{s}{k_2}} z \right) \tag{4.10}
\]
where $A_1(s)$ and $B_2(s)$ are as given by Equations (3.10) and (3.11).

Upon substituting (4.9) and (4.10) into the boundary conditions (4.3), (4.5) and (4.6), the following equations are obtained after some algebraic manipulations:

\[
d_1(s) - d_2(s) = \bar{w}_{2p}(0, s) - \bar{w}_{1p}(0, s) = M_1(s) \quad , \tag{4.11}
\]

\[
\left( \frac{s}{C_{L_1}} \right) d_2(s) - \zeta \left( \frac{s}{C_{L_2}} \right) d_4 = \zeta \bar{w}_{2p,z}(0, s) + (\delta - \zeta \beta) \bar{T}_1(0, s) - \bar{w}_{1p}(0, s) = M_2(s) , \tag{4.12}
\]

\[
\left( \frac{s}{C_{L_1}} \right) d_1(s) \sinh \left( \frac{sb}{C_{L_1}} \right) + \left( \frac{s}{C_{L_1}} \right) d_2(s) \cosh \left( \frac{sb}{C_{L_1}} \right) = \delta \bar{T}_1(b, s) - \bar{w}_{1p,z}(b, s) = M_3(s) \tag{4.13}
\]

and

\[
d_3(s) = d_4(s) \quad . \tag{4.14}
\]

Equations (4.11), (4.12) and (4.13) are three simultaneous equations in three unknowns $d_1$, $d_2$ and $d_3$. Solving these,

\[
d_1(s) = C_{L_1} \Omega \left[ \left( \frac{\zeta}{C_{L_2}} M_1(s) - \frac{M_2(s)}{s} \right) \cosh \frac{sb}{C_{L_1}} + \frac{M_3(s)}{s} \right] \quad , \tag{4.15}
\]

\[
d_2(s) = C_{L_1} \Omega \left[ \left( - \frac{\zeta}{C_{L_2}} M_1(s) + \frac{M_2(s)}{s} \right) \sinh \frac{sb}{C_{L_1}} + \frac{s}{C_{L_1}} \frac{M_3(s)}{s} \right] \quad , \tag{4.16}
\]

\[
d_3(s) = C_{L_1} \Omega \left[ - \frac{1}{C_{L_1}} \sinh \frac{sb}{C_{L_1}} + \cosh \frac{sb}{C_{L_1}} \left( \frac{M_2(s)}{s} \right) + \frac{M_3(s)}{s} \right] \tag{4.17}
\]

and

\[
d_4(s) = d_3(s) \quad . \tag{4.18}
\]

where

\[
\Omega = \frac{1}{\left( \frac{\zeta}{C_{L_2}} \cosh \frac{sb}{C_{L_1}} + \sinh \frac{sb}{C_{L_1}} \right)} . \tag{4.19}
\]
Substituting (4.15) and (4.16) into (4.7), and (4.17) and (4.18) into (4.8), the Green's functions in the transformed space are obtained as

\[
\bar{w}_1(z, s) = \Omega \left[ R_1(s) \cosh \frac{s(b - z)}{C_{L_1}} + R_2(s) \cosh \frac{sz}{C_{L_1}} + \frac{z}{C_{L_2}} \sinh \left( \frac{sz}{C_{L_1}} \right) \right] \bar{w}_{1p}(z, s)
\]

(4.20)

and

\[
\bar{w}_2(z, s) = \frac{\Omega}{2} \left[ \{ R_4(s) + R_3(s) \} \exp \left( \frac{sb}{C_{L_1}} \right) + \{ R_4(s) - R_3(s) \} \exp \left( -\frac{sb}{C_{L_1}} \right) + 2R_2(s) \right] \exp \left( \frac{sz}{C_{L_2}} \right) \bar{w}_{2p}(z, s)
\]

(4.21)

where

\[
R_1(s) = \left[ \frac{z}{C_{L_2}} \left( M_1(s) - M_2(s) \right) \right] C_{L_1}, \quad \text{(4.22)}
\]

\[
R_2(s) = \left( M_3(s) C_{L_1} \right) / s, \quad \text{(4.23)}
\]

\[
R_3(s) = -M_1(s), \quad \text{(4.24)}
\]

and

\[
R_4(s) = \left( \frac{M_2(s) C_{L_1}}{s} \right). \quad \text{(4.25)}
\]

The functions (including \( \Omega \)) multiplying \( R_i(s), i = 1, 2, 3, 4 \) are either exponentials or functions of exponentials. By the second shifting theorem of Laplace transforms,

\[
\mathcal{L}^{-1} \{ \exp(-as)F(t) \} = F(t - a)H(t - a) \quad .
\]

Therefore the only contribution of these functions is towards Heaviside's functions and a shift in the time variable. However these functions bring out clearly the wave nature of the problem, so the exact contributions of these to the displacement solution will be considered first. From Equation (4.19),

\[
\Omega = \frac{1}{\frac{z}{C_{L_2}} \cosh \frac{sb}{C_{L_1}} + \sinh \frac{sb}{C_{L_1}}} \quad . \quad \text{(4.26)}
\]
the wave nature of the problem, so the exact contributions of these to the dis-
placement solution will be considered first. From Equation (4.19),

$$\Omega = \frac{1}{\xi \frac{C_{L_1}}{C_{L_2}} \cosh \frac{sb}{C_{L_1}} + \sinh \frac{sb}{C_{L_1}}}$$

(4.26)

This can be rewritten as

$$\Omega = \frac{\exp(-sb/C_{L_1})}{B \left(1 + \frac{A}{B} \exp(-2sb/C_{L_1})\right)}$$

(4.27)

where

$$A = \xi \frac{C_{L_1}}{C_{L_2}} - 1$$

(4.28)

and

$$B = \xi \frac{C_{L_1}}{C_{L_2}} + 1$$

(4.29)

Noticing that $A/B$ is less than one, the terms inside the brackets in the de-
nominator of the above expression for $\Omega$ can be expanded using binomial theorem
as follows:

$$\begin{align*}
\Omega &= \frac{2 \exp(-sb/C_{L_1})}{B} \left[1 - \left(\frac{A}{B}\right) \exp\left(-\frac{2sb}{C_{L_1}}\right)\right] \\
&\quad + \left(\frac{A}{B}\right)^2 \exp\left(-\frac{4sb}{C_{L_1}}\right) - \cdots
\end{align*}$$

(4.30)

With this, substituting $\Omega$ in the expressions for the displacement of fields,

$$\bar{\omega}_1(s, z) = \frac{R_1(s)}{B} \{H_{11}(z, s) + H_{12}(z, s)\}$$

(4.31)

$$+ R_2(s) \{H_{21}(z, s) + H_{22}(z, s)\} + \bar{\omega}_{1p}(z, s)$$

and

$$\bar{\omega}_2(s, z) = R_3(s)H_{31}(z, s) + \{R_4(s) + R_3(s)\}H_{32}(z, s)$$

(4.32)

$$+ \{R_4(s) - R_3(s)\}H_{33}(z, s) + \bar{\omega}_{2p}(z, s)$$

where

$$H_{11}(z, s) = Q \exp(-sz/C_{L_1})$$

(4.33)
The physical interpretation that can be obtained from the Heaviside functions will now be discussed.

\[ H_{12}(z, s) = Q \exp \left( -s \left( \frac{b}{C_{L_1}} + \frac{b-z}{C_{L_1}} \right) \right) \]  

\[ H_{21}(z, s) = Q \exp \left( -s \left( \frac{b-z}{C_{L_1}} \right) \right) \]  

\[ H_{22}(z, s) = \frac{Q}{B} \exp \left( -s \left( \frac{b+z}{C_{L_1}} \right) \right) \]  

\[ H_{31}(z, s) = \frac{Q}{B} \exp \left( \frac{sz}{C_{L_2}} \right) \]  

\[ H_{32}(z, s) = \frac{Q}{B} \exp \left( -s \left( \frac{2b}{C_{L_1}} - \frac{z}{C_{L_2}} \right) \right) \]  

\[ H_{33}(z, s) = 2Q \exp \left( -s \left( \frac{b}{C_{L_1}} - \frac{z}{C_{L_2}} \right) \right) \]  

and

\[ Q = \left[ 1 - \frac{A}{B} \exp \left( -\frac{2sb}{C_{L_1}} \right) + \frac{A^2}{B^2} \exp \left( -\frac{4sb}{C_{L_1}} \right) - \cdots \right] \]  

The physical interpretation that can be obtained from the Heaviside functions will now be discussed.

\( H_{11}(z, s) \) represents waves travelling to the right from \( z = 0 \) to \( z = b \) starting at times \( t = 0 \), \( 2b/C_{L_1}, \ 4b/C_{L_1} \), etc. \( H_{22}(z, s) \) represents waves travelling to the right from \( z = 0 \) to \( z = b \) starting at times \( t = b/C_{L_1}, \ 3b/C_{L_1} \), etc. \( H_{12}(z, s) \) represents waves travelling to the left from \( z = b \) to \( z = 0 \) starting at times \( t = b/C_{L_1}, \ 3b/C_{L_1} \), etc. \( H_{21}(z, s) \) represents waves travelling to the left from \( z = b \) to \( z = 0 \) at times \( t = 0 \), \( 2b/C_{L_1}, \ 4b/C_{L_1} \), etc.

\( H_{31}(z, s) \) represents waves travelling from \( z = 0 \) to the left in the liquid medium starting from \( t = 0 \), \( 2b/C_{L_1}, \ 4b/C_{L_1} \), etc. \( H_{32}(z, s) \) and \( H_{33}(z, s) \) also represent the waves travelling from \( z = 0 \) to the left in the liquid medium but starting respectively at times \( t = 2b/C_{L_1}, \ 4b/C_{L_1} \), etc. and at \( t = b/C_{L_1}, \ 3b/C_{L_1} \), etc.
With this interpretation, the $R_i(s)$ terms in Equations (4.31) and (4.32) will now be inverted.

Substituting Equations (4.11) to (4.13) in Equations (4.22) to (4.25) and then making use of Equations (4.9), (4.10), (3.10) and (3.11), the following are obtained:

$$R_1(s) = -\frac{\zeta \beta C_{L_1}}{\sqrt{k_1}} \frac{P T_0}{s} \left( \frac{1}{\sqrt{s + \frac{c_{L_1}^2}{k_1}}} \right) \left( \frac{1}{\sqrt{s + \frac{c_{L_1}^2}{k_2}}} \right)$$

$$- \frac{T_0 \delta C_{L_1}}{\sqrt{k_1}} \frac{1}{\sqrt{s(\sqrt{s + \alpha \sqrt{k_1})}}} \left( \frac{1}{s - \frac{c_{L_1}^2}{k_1}} \right)$$

$$+ \frac{T_0 \delta C_{L_1}^3}{k_1} \left( \frac{1}{s^2 - \alpha^2 C_{L_1}^2} \left( \frac{1}{s - \frac{c_{L_1}^2}{k_1}} \right) \right)$$

$$(4.40)$$

$$= \frac{\delta T_0 \alpha \zeta C_{L_1}^3}{C_{L_1}} \left\{ \frac{P}{\sqrt{k_1}} \left( \frac{1}{\frac{c_{L_1}^2}{k_1} - s} \right) \left( 1 + \frac{C_{L_1}^2}{\alpha C_1 \sqrt{k_2}} \right) \right\}$$

$$+ \frac{1}{s^2 - \alpha^2 C_{L_1}^2}$$

$$R_2(s) = \frac{C_{L_1} \delta T_0 P}{s - \frac{c_{L_1}^2}{k_1}} \left( \alpha + \frac{C_{L_2}}{C_1 \sqrt{\frac{s}{k_2}}} \right) \exp \left( -\frac{\sqrt{\frac{s}{k_1}} b}{\sqrt{s(\alpha^2 k_1 - s)}} \right)$$

$$+ \frac{C_{L_1} \delta T_0 \exp(-\alpha b)}{(k_1 \alpha^2 - s)} \frac{s}{(C_{L_1}^2 \alpha^2 - s^2)}$$

$$R_3(s) = \frac{-B_1 \delta \sqrt{\frac{s}{k_1}}}{\frac{s}{k_1} - \frac{c_{L_1}^2}{k_1}} + \frac{T_0 \alpha \delta C_{L_1}^2}{(s - k_1 \alpha^2)(s^2 - \alpha^2 C_{L_1}^2)}$$

$$- \frac{\beta A_2 \sqrt{s/k_2}}{\frac{s}{k_2} - \frac{c_{L_2}^2}{k_2}}$$

$$(4.42)$$

and
\[
R_4(s) = \frac{\delta \beta C_{L_1} A_2(s)}{s - \frac{\sigma_{L_2}^2}{k_1}} + \delta C_{L_1} \frac{B_1(s)}{s - \frac{\sigma_{L_1}^2}{k_1}} + \frac{(\delta T_0 C_{L_1})s}{(s^2 - \alpha^2 C_{L_1}^2)(s - k_1 \alpha^2)}
\]

where \(B_1(s)\) and \(A_2(s)\) are given by Equations (3.10) and (3.11). These, after some algebraic work, become

\[
B_1(s) = \frac{T_0 P}{\sqrt{s}} \left( \alpha + \frac{c_2}{c_1} \sqrt{\frac{s}{k_2}} \right) \frac{1}{\alpha^2 k_1 - s}
\]

and

\[
A_2(s) = \frac{P_1 T_0}{\sqrt{s}(\sqrt{s} + \alpha \sqrt{k_1})}
\]

where \(P_1\) is given by Equation (3.25). For inverting \(R_1(s)\), it is first noted that, using partial fractions,

\[
\frac{1}{(\sqrt{s} + \alpha \sqrt{k_1}) (\sqrt{s} + \frac{\sigma_{L_2}^2}{\sqrt{k_2}})} = \left( \frac{1}{\sqrt{s} + \alpha \sqrt{k_1}} - \frac{1}{\sqrt{s} + \frac{\sigma_{L_2}^2}{\sqrt{k_2}}} \right) \frac{\sqrt{k_2}}{C_{L_2} - \alpha \sqrt{k_1} k_2}
\]

\[
\frac{1}{\sqrt{s}(\alpha \sqrt{k_1} + \sqrt{s})} = \left( \frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s} + \alpha \sqrt{k_1}} \right) \frac{1}{\alpha \sqrt{k_1}}
\]

\[
\frac{1}{(s^2 - \alpha^2 C_{L_1}^2) (s - \frac{\sigma_{L_1}^2}{k_1})} = \frac{k_1}{2\alpha C_{L_1}^2 (\alpha k_1 + C_{L_1})} \frac{1}{(s + \alpha C_{L_1})}
\]

\[
+ \frac{k_1}{2\alpha C_{L_1}^2 (\alpha k_1 - C_{L_1})} \frac{1}{s - \alpha C_{L_1}}
\]

\[
+ \frac{k_1^2}{C_{L_1}^2 (C_{L_1}^2 - \alpha^2 k_1^2)} \frac{1}{s - C_{L_1}^2 / k_1}
\]

\[
\]
\[
\frac{1}{s \left( s - \frac{C_{L_1}^2}{k_1} \right) (s - k_1 \alpha^2)} = \frac{k_1^2}{C_{L_1}^2 \left( C_{L_1}^2 - k_1^2 \alpha^2 \right)} \frac{1}{s - \frac{C_{L_1}^2}{k_1}} + \frac{1}{\alpha^2 \left( C_{L_1}^2 - k_1^2 \alpha^2 \right)} \frac{1}{s - \alpha^2 \alpha} \tag{4.49}
\]

\[
- \frac{1}{\alpha^2} \frac{1}{s},
\]

\[
\frac{1}{(s - k_1 \alpha^2) \left( \frac{C_{L_1}^2}{k_1} - s \right)} = \left( \frac{1}{s - k_1 \alpha^2} - \frac{1}{s - \frac{C_{L_1}^2}{k_1}} \right) \frac{k_1}{k_1^2 \alpha^2 - C_{L_1}^2} \tag{4.50}
\]

and

\[
\frac{1}{(s^2 - \alpha^2 C_{L_1}^2) (s - k_1 \alpha^2)} = \frac{1}{2 \alpha^2 C_{L_1} (C_{L_1} + k_1 \alpha)} \frac{1}{s + \alpha C_{L_1}} + \frac{1}{2 \alpha^2 C_{L_1} (C_{L_1} - k_1 \alpha)} \frac{1}{s - \alpha C_{L_1}} + \frac{1}{\alpha^2 (k_1^2 \alpha^2 - C_{L_1}^2)} \frac{1}{s - k_1 \alpha^2} \tag{4.51}
\]

Substituting these in the expression for \( R_1(s) \) and using the following inverse Laplace transform formula [18],

\[
\mathcal{L}^{-1} \left\{ \frac{1}{s \pm p} \right\} = \exp(\pm pt), \tag{4.52}
\]

\[
\mathcal{L}^{-1} \left\{ \frac{1}{s(\sqrt{s} + p)} \right\} = \frac{1}{p} \left( 1 - \exp(p^2 t) \text{erf} c(p\sqrt{t}) \right), \quad p \geq 0 \tag{4.53}
\]

\[
\mathcal{L}^{-1} \left\{ \frac{1}{(\sqrt{s} + p)(s - q^2)} \right\} = \frac{\exp(q^2 t) \{ p - q \text{erf} c(q\sqrt{t}) \} - p \exp(p^2 t) \text{erf} c(p\sqrt{t})}{p^2 - q^2}, \tag{4.54}
\]
the inverse of $R_1(s)$ is obtained as

$$
R_1(t) = \exp(\alpha^2 k_1 t) \text{erfc} \alpha \sqrt{k_1 t} \left( \frac{M_{11}}{\alpha} - \frac{M_{12}}{\alpha} + \frac{M_{13}}{\alpha} \frac{C_{L_1}}{C_{L_2}} P_1 n \right)
$$

$$
- \exp(C_{L_1}^2 t/k_1) \text{erfc} \left( \frac{t}{k_1} \right) \left( M_{11} + M_{13} P_1 \frac{C_{L_1}}{C_{L_2}} \right) \frac{k_1}{C_{L_1}}
$$

$$
+ \exp(C_{L_2}^2 t/k_2) \text{erfc} \left( \frac{t}{k_2} \right) \left( M_{11} \frac{\sqrt{k_1 k_2}}{C_{L_2}} \right)
$$

$$
- M_{11} \frac{C_{L_1}}{\alpha^2 P_1 k_1} \sinh(\alpha C_{L_1} t) - M_{13} \frac{k_1}{C_{L_2}} \cosh(\alpha C_{L_1} t)
$$

$$
+ \exp(C_{L_2}^2 t/k_1) M_{11} \left( \frac{n}{\alpha} + \frac{k_1}{C_{L_1}} \right) B
$$

$$
+ M_{12} \left( \frac{1}{\alpha} - \frac{\sqrt{k_1 k_2}}{C_{L_2}} \right) + M_{13} \left( \frac{C_{L_1}^2 - k_1^2 \alpha^2}{C_{L_1} C_{L_2} \alpha} \right) P_1
$$

where

$$
M_{11} = \frac{T_0 P \delta C_{L_1} \alpha \sqrt{k_1}}{C_{L_1}^2 - k_1^2 \alpha^2},
$$

$$
M_{12} = \frac{\zeta \beta C_{L_1} P T_0}{C_{L_2} - \alpha \sqrt{k_1 k_2}} \frac{\sqrt{k_2}}{k_1}
$$

and

$$
M_{13} = \frac{\delta T_0 \zeta C_{L_1}^2}{C_{L_1}^2 - k_1^2 \alpha^2}
$$

It can be seen that $M_{11}$, $M_{12}$ and $M_{13}$, $P_1$ (Equation (3.25)) and $n$ (Equation (3.26)) are non-dimensional constants.

In inverting $R_2(t)$, the following partial fractions in addition to Equations (4.50) and (4.51) are used:

$$
\frac{1}{s - \alpha^2 k_1} = \left( \frac{1}{\sqrt{s} - \alpha \sqrt{k_1}} - \frac{1}{\sqrt{s} + \alpha \sqrt{k_1}} \right) \frac{1}{2\alpha \sqrt{k_1}}
$$

and

$$
\frac{1}{s - \frac{C_{L_1}^2}{k_1}} = \left( \frac{1}{\sqrt{s} - \frac{C_{L_1}}{\sqrt{k_1}}} - \frac{1}{\sqrt{s} + \frac{C_{L_1}}{\sqrt{k_1}}} \right) \frac{\sqrt{k_1}}{2C_{L_1}}
$$
With the help of the following inverse Laplace transforms [18],

\[
\mathcal{L}^{-1} \left\{ \frac{\exp(-q\sqrt{s})}{p + \sqrt{s}} \right\} = \frac{1}{\sqrt{\pi t}} \exp \left( \frac{-q^2}{4t} \right) 
\]

\[ - p \exp(pq + p^2 t) \text{erfc} \left( p\sqrt{t} + \frac{q}{2\sqrt{t}} \right), \quad q \geq 0 \]  

(4.61)

and

\[
\mathcal{L}^{-1} \left\{ \frac{\exp(-q\sqrt{s})}{\sqrt{s}(p + \sqrt{s})} \right\} = \exp(pq + p^2 t) \text{erfc} \left( p\sqrt{t} + \frac{q}{2\sqrt{t}} \right), \quad q \geq 0 , \quad (4.62)
\]

the inverse \( R_2(t) \) of \( R_2(s) \) becomes

\[
R_2(t) = \frac{M_{11}}{2} \exp \left( -\frac{b^2}{4k_1 t} \right) \left[ \exp(x_{11}^2) \text{erfc}(x_{11}) \left\{ -\frac{n}{2} - \frac{k_1}{C_{L_1}} \right\} 
\]

\[ + \exp(x_{12}^2) \text{erfc}(x_{12}) \left\{ -\frac{n}{2} + \frac{k_1}{C_{L_1}} \right\} \]

\[ + \exp(y_{11}^2) \text{erfc}(y_{11}) \left\{ \frac{n}{2} + \frac{1}{2} \right\} \]

\[ + \exp(y_{12}^2) \text{erfc}(y_{12}) \left\{ \frac{n}{2} - \frac{1}{2} \right\} \]

(4.63)

\[ - \frac{M_{11}\sqrt{k_1}}{P\alpha} \exp(\alpha^2 k_1 t - ab) \]

\[ + \frac{C_{L_1} \sinh C_{L_1} \alpha t + k_1 \alpha \cosh C_{L_1} \alpha t}{P\alpha^2 \sqrt{k_1}} M_{11} \exp(-\alpha b) \]

where

\[ n = \frac{c_2}{c_1} \sqrt{\frac{k_1}{k_2}} , \quad (4.64) \]

\[ x_{11} = \frac{b}{\sqrt{4k_1 t}} - C_{L_1} \sqrt{\frac{t}{k_1}} , \quad (4.65) \]

\[ x_{12} = \frac{b}{\sqrt{4k_1 t}} + C_{L_1} \sqrt{\frac{t}{k_1}} , \quad (4.66) \]

\[ y_{11} = \frac{b}{\sqrt{4k_1 t}} - \alpha \sqrt{k_1 t} , \quad (4.67) \]

and

\[ y_{12} = \frac{b}{\sqrt{4k_1 t}} + \alpha \sqrt{k_1 t} . \quad (4.68) \]
With the help of the partial fractions (4.49), (4.50) and (4.51) and the inverse Laplace transforms given by (4.53) and (4.54), in addition to

\[
\frac{1}{s \left(s - \frac{c_1^2}{k_2}\right)} = \frac{k_2}{C_{L_2}^2} \left(\frac{1}{s - \frac{c_1^2}{k_2}} - \frac{1}{s}\right),
\]

(4.69)

the inverse Laplace transform of \( R_3(s) \) is obtained as

\[
R_3(t) = -M_{31} \left[ \frac{C_{L_1}^2 - k_1^2 \alpha^2}{\alpha C_{L_1}^2} + \frac{k_1}{C_{L_1}} \left( \frac{\alpha k_1}{C_{L_1}} + n \right) \exp(C_{L_1}^2 t/k_1) \right]
\]

\[
+ \frac{1}{\alpha} \exp(\alpha^2 k_1 t)erf(c(\alpha \sqrt{k_1} t))
\]

\[
- n \frac{k_1}{C_{L_1}} \exp(C_{L_1}^2 t/k_1)erfc \left( C_{L_1} \sqrt{\frac{t}{k_1}} \right)
\]

\[
- \sqrt{k_1} \alpha C_{L_1} \cosh(C_{L_1} \alpha t) + k_1 \alpha \sinh(C_{L_1} \alpha t) \right] \right]
\]

(4.70)

\[
M_{31} = \frac{\delta T_0 P C_{L_1}^2}{\sqrt{k_1} \left(C_{L_1}^2 - \alpha^2 k_1^2\right)},
\]

(4.71)

\[
M_{32} = \frac{\beta PT_0}{\sqrt{k_1}},
\]

(4.72)
and \( n \) is as given by (4.64). Finally, substituting Equation (4.50) and

\[
\frac{s}{(s^2 - \alpha^2 C_{L_1}^2)(s - k_1 \alpha^2)} = \frac{-1}{2\alpha(C_{L_1} + k_1 \alpha)} \frac{1}{s + \alpha C_{L_1}} + \frac{1}{2\alpha(C_{L_1} - k_1 \alpha)} \frac{1}{s - \alpha k_1} + \frac{k_1}{k_1^2 \alpha^2 - C_{L_1}^2} \frac{1}{s - \alpha^2 k_1}
\]

(4.73)
in the expression for \( R_4(s) \) and using Equations (4.52), (4.54) and the following inverse Laplace transform formula [18],

\[
\mathcal{L}^{-1} \left\{ \frac{q^2 - p^2}{\sqrt{s}(s - p^2)(\sqrt{s} + q)} \right\} = \exp(p^2 t) \left[ \frac{q}{p} \text{erfc} p\sqrt{t} - 1 \right]
\]

\[
+ \exp(q^2 t) \text{erfc} q\sqrt{t}
\]

(4.74)
the inverse of \( R_4(s) \) is obtained as

\[
R_4(t) = \exp(\alpha^2 k_1 t) \text{erfc}(\alpha \sqrt{k_1 t}) \left( -\frac{M_{11}}{\alpha} - \frac{M_{41}}{\alpha} \right)
\]

\[
+ \exp(C_{L_2}^2 t/k_2) \text{erfc} \left( C_{L_2} \sqrt{\frac{t}{k_2}} \right) \left( M_{41} \frac{\sqrt{k_1 k_2}}{C_{L_2}} \right)
\]

\[
+ \exp(C_{L_2}^2 t/k_2) \frac{M_{41}}{\alpha} \left( \frac{C_{L_2} - \alpha\sqrt{k_1 k_2}}{C_{L_2}} \right)
\]

\[
+ \exp(C_{L_1}^2 t/k_1) \text{erfc} C_{L_1} \sqrt{\frac{t}{k_1}} M_{11} \left( \frac{k_1}{C_{L_1}} \right)
\]

\[
- \exp(C_{L_1}^2 t/k_1) \frac{M_{11}}{\alpha} \left( \frac{k_1}{C_{L_1}} + n \right)
\]

\[
+ \frac{M_{11}}{\alpha} \frac{1}{P \alpha \sqrt{k_1}} \left( C_{L_1} \sinh(\alpha C_{L_1} t) + k_1 \alpha \cosh(\alpha C_{L_1} t) \right)
\]

(4.75)

The inverse Laplace transforms of the particular solutions (4.9) and (4.10) will now be obtained. Substituting the partial fractions given by (4.49) to (4.51) in (4.9) and using the inverse Laplace transforms given by (4.61) and (4.62), the.
The inverse of \( w_{1p}(z,s) \) is obtained as

\[
 w_{1p}(z,t) = -\frac{M_{11}}{2} \left[ \exp \left( -\frac{z^2}{4k_1t} \right) \exp(x_{21}^2)\text{erfc}(x_{21}) \left( \frac{k_1}{C_{L_1}} + \frac{n}{\alpha} \right) 
+ \exp(x_{22}^2)\text{erfc}(x_{22}) \left( \frac{k_1}{C_{L_1}} - \frac{n}{\alpha} \right) 
- \exp(y_{21}^2)\text{erfc}(y_{21})(1 + n) \frac{C_{L_1}}{\alpha^2 k_1} 
- \exp(y_{22}^2)\text{erfc}(y_{22})(1 - n) \frac{C_{L_1}}{\alpha^2 k_1} \right] 
- \frac{PT_0 \delta}{\alpha \sqrt{k_1}} \text{erfc} \left( \frac{z}{\sqrt{4k_1t}} \right) 
- M_{11} \frac{C_{L_1}}{P \alpha^2 \sqrt{k_1}} \exp(-\alpha z + \alpha^2 k_1 t) 
+ \frac{M_{11}}{P \alpha^2 \sqrt{k_1}} (C_{L_1} \cosh C_{L_1} \alpha t + ak_1 \sinh C_{L_1} \alpha t) \exp(-\alpha z) 
\]

where

\[
x_{21} = \frac{z}{\sqrt{4k_1t}} - C_{L_1} \sqrt{\frac{t}{k_1}}, \tag{4.77}
\]

\[
x_{22} = \frac{z}{\sqrt{4k_1t}} + C_{L_1} \sqrt{\frac{t}{k_1}}, \tag{4.78}
\]

\[
y_{21} = \frac{z}{\sqrt{4k_1t}} - \alpha \sqrt{k_1 t} \tag{4.79}
\]

and

\[
y_{22} = \frac{z}{\sqrt{4k_1t}} + \alpha \sqrt{k_1 t} \tag{4.80}
\]

To invert \( w_{2p}(z,s) \), it is first noted that

\[
\left( \frac{1}{s - \frac{C_{L_1}^2}{k_2}} \right) (\sqrt{s} + \alpha \sqrt{k_1}) = \frac{k_2}{2C_{L_2}(C_{L_2} - \alpha \sqrt{k_1} k_2)} \frac{1}{\sqrt{s} + \frac{C_{L_2}^2}{\sqrt{k_2}}} 
+ \frac{k_2}{2C_{L_2}(C_{L_2} + \alpha \sqrt{k_1} k_2)} \frac{1}{\sqrt{s} - \frac{C_{L_2}^2}{\sqrt{k_2}}} 
+ \frac{k_2}{\alpha^2 k_1 k_2 - C_{L_2}^2} \frac{1}{\sqrt{s} + \alpha \sqrt{k_1}} \tag{4.81}
\]
Substituting this in the expression for $w_{2p}(z, s)$ and making use of the inverse Laplace transform formula given by Equation (4.61) and the equation following [18]:

\[ \mathcal{L}^{-1} \left\{ \frac{\exp(-q\sqrt{s})}{s(p + \sqrt{s})} \right\} = \text{erfc} \frac{1}{2\sqrt{t}} - \exp(pq + p^2t) \text{erfc} \left( p\sqrt{t} + \frac{q}{2\sqrt{t}} \right), \quad q \geq 0, \tag{4.82} \]

the inverse Laplace transform of $w_{2p}(z, s)$ (noting that $z \leq 0$) is obtained as

\[
w_{2p}(z, t) = \beta PT_0 \left[ \frac{1}{\alpha} \sqrt{\frac{k_2}{k_1}} \text{erfc} \left( -\frac{z}{4\sqrt{k_2}t} \right) + \exp \left( -\frac{z^2}{4k_2t} \right) \left\{ -\frac{k_2}{2(C_{L_2} + \alpha\sqrt{k_1k_2})} \exp(x_{31}^2) \text{erfc} x_{31} + \frac{k_2}{2(C_{L_2} - \alpha\sqrt{k_1k_2})} \exp(x_{32}^2) \text{erfc} x_{32} - \frac{C_{L_2}^2}{C_{L_2}^2 - \alpha^2 k_1 k_2} \sqrt{\frac{k_2}{k_1}} \frac{1}{\alpha} \exp(x_{33}^2) \text{erfc} x_{33} \right\} \right] \tag{4.83}\]

where

\[
x_{31} = -C_{L_2} \sqrt{\frac{t}{k_2}} - \frac{z}{\sqrt{4k_2}t} \tag{4.84}\]

\[
x_{32} = C_{L_1} \sqrt{\frac{t}{k_2}} - \frac{z}{\sqrt{4k_2}t} \tag{4.85}\]

and

\[
x_{33} = \alpha \sqrt{k_1} t - \frac{z}{\sqrt{4k_2}t} \tag{4.86}\]

Using Equations (4.26), (4.31) and (4.32) along with the expressions just obtained for $R_i(t), i = 1, 2, 3, 4$, the final displacement solutions for the times of interest
become

\[
\begin{align*}
\bar{w}_1(z,t) &= \frac{1}{B} \left[ R_1 \left( t - \frac{z}{C_{L_1}} \right) H \left( t - \frac{z}{C_{L_1}} \right) \right. \\
&\quad - \frac{A}{B} R_1 \left( t - \frac{2b}{C_{L_1}} - \frac{z}{C_{L_1}} \right) H \left( t - \frac{2b}{C_{L_1}} - \frac{z}{C_{L_1}} \right) \\
&\quad + R_1 \left( t - \frac{b}{C_{L_1}} - \frac{b-z}{C_{L_1}} \right) H \left( t - \frac{b}{C_{L_1}} - \frac{b-z}{C_{L_1}} \right) \\
&\quad + \left[ R_2 \left( t - \frac{b-z}{C_{L_1}} \right) H \left( t - \frac{b-z}{C_{L_1}} \right) \\
&\quad - \frac{A}{B} R_2 \left( t - \frac{2b}{C_{L_1}} - \frac{b-z}{C_{L_1}} \right) H \left( t - \frac{2b}{C_{L_1}} - \frac{b-z}{C_{L_1}} \right) \\
&\quad - \frac{A}{B} R_2 \left( t - \frac{b}{C_{L_1}} - \frac{z}{C_{L_1}} \right) H \left( t - \frac{b}{C_{L_1}} - \frac{z}{C_{L_1}} \right) \\
&\quad + w_{1p}(z,t), \quad 0 \leq t \leq \frac{3b}{C_{L_1}}
\end{align*}
\]

and

\[
\begin{align*}
\bar{w}_2(z,t) &= \frac{1}{B} \left[ R_3 \left( t + \frac{z}{C_{L_2}} \right) \\
&\quad + R_4 \left( t + \frac{z}{C_{L_2}} \right) \right] H \left( t + \frac{z}{C_{L_2}} \right) \\
&\quad - \frac{A}{B} \left[ R_3 \left( t - \frac{2b}{C_{L_1}} + \frac{z}{C_{L_2}} \right) \\
&\quad + R_4 \left( t - \frac{2b}{C_{L_1}} + \frac{z}{C_{L_2}} \right) \right] H \left( t - \frac{2b}{C_{L_1}} + \frac{z}{C_{L_2}} \right) \\
&\quad + \left\{ R_4 \left( t - \frac{2b}{C_{L_1}} + \frac{z}{C_{L_2}} \right) \\
&\quad - R_3 \left( t - \frac{2b}{C_{L_1}} + \frac{z}{C_{L_2}} \right) \right\} H \left( t - \frac{2b}{C_{L_1}} + \frac{z}{C_{L_2}} \right) \\
&\quad + 2R_2 \left( t - \frac{b}{C_{L_1}} + \frac{z}{C_{L_2}} \right) H \left( t - \frac{b}{C_{L_1}} + \frac{z}{C_{L_2}} \right) \\
&\quad + w_{2p}(z,t), \quad 0 \leq t \leq \frac{3b}{C_{L_1}}
\end{align*}
\]
Equations (4.87) and (4.88) are the required Green’s functions in time for the times of interest.

Since it is the displacement at the back surface of the solid medium that is actually needed, specializing (4.87) for \( z = b \),

\[
    w_1(b, t) = \frac{1}{B} \left[ 2R_1 \left( t - \frac{b}{C_{L_1}} \right) H \left( t - \frac{b}{C_{L_1}} \right) \right. \\
    + R_2(t)H(t) - \frac{2A}{B} R_2 \left( t - \frac{2b}{C_{L_1}} \right) H \left( t - \frac{2b}{C_{L_1}} \right) \\
    + w_{1p}(z, t), \quad 0 \leq t < \frac{3b}{C_{L_1}}.
\]

The second term in (4.87) vanishes for \( z = b \) since times of interest are only until \( t = 3b/C_{L_1} \). The waveform \( u_1(b, t) \) generated at \( z = b \) due to a pulse of arbitrary temporal shape \( f(t) \), obtained by the superposition principle of the Green’s functions, is then given by

\[
    u_1(b, t) = \int_0^t w_1(b, t - \tau_0) f(\tau_0) d\tau_0. \quad (4.90)
\]
CHAPTER 5
OBSERVATIONS AND RESULTS

In this chapter, using the temperature solution obtained in Chapter 3 and the displacement solution obtained in the last chapter, the displacement waveform, as a function of time, generated at the back surface of the solid is analysed. Analysis is confined to only the Green's functions in time for temperature and displacement in the solid medium. Waveform generated due to an arbitrary pulse shape can be obtained by the superposition of the Green's functions in time for displacements as mentioned before in Chapter 4 and is not considered here. The solid medium is taken as made up of copper and the liquid medium is taken to be water. A physical interpretation of the waveform generated is made. Also, a discussion on the influence of various parameters involved in the generation of the waveform at the back surface of the solid is presented.

Waveform Generated at the Back Surface of the Solid

In order to analyze the results obtained in Chapter 3 and Chapter 4, copper-water system is assumed. Thickness of the solid layer is taken to be 25 mm. Using the values shown in the table for the material properties, the temperature solution given by Equation (3.20) and the waveform generated at the back surface of the solid as given by Equation (4.89) are plotted in Figure 3 and Figure 4 respectively (in evaluating the error functions use is made of the FORTRAN routine given in [19]). In Figure 3 the non-dimensional temperature is obtained by dividing the actual temperature with $T_0$ given by Equation (3.13). The significance of $T_0$ is
that it represents the initial interface temperature. In Figure 4, the non-dimensional displacement is obtained by dividing the actual displacement with $\delta T_0/\alpha$ where $\delta$ is given by Equation (2.16). The non-dimensional time is obtained by dividing the actual time with $b/C_L$ i.e. with the time that a wave generated at the interface takes to reach the back surface of the solid.

<table>
<thead>
<tr>
<th>Property</th>
<th>Copper</th>
<th>Water</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thermal Diffusivity</td>
<td>$1.1234 \text{ cm}^2/\text{s}$</td>
<td>$0.0013084 \text{ cm}^2/\text{s}$</td>
</tr>
<tr>
<td>Thermal Conductivity</td>
<td>$3.86 \text{ N/cm}^{-\circ C}$</td>
<td>$0.00552 \text{ W/cm}^{-\circ C}$</td>
</tr>
<tr>
<td>Lame Constant $\lambda$</td>
<td>$9.5 \times 10^6 \text{ N/cm}^2$</td>
<td></td>
</tr>
<tr>
<td>Lame Constant $\mu$</td>
<td>$4.5 \times 10^6 \text{ N/cm}^2$</td>
<td></td>
</tr>
<tr>
<td>Bulk Modulus</td>
<td></td>
<td>$2.18 \times 10^6 \text{ N/cm}^2$</td>
</tr>
<tr>
<td>Longitudinal Velocity</td>
<td>$4.66 \times 10^5 \text{ cm/s}$</td>
<td>$1.48 \times 10^5 \text{ cm/s}$</td>
</tr>
<tr>
<td>Absorption Coefficient</td>
<td>$10^5 - 10^6/\text{cm}$</td>
<td></td>
</tr>
<tr>
<td>Linear Thermal Expansion Coefficient</td>
<td>$0.17 \times 10^{-4} /\circ C$</td>
<td>$0.18 \times 10^{-5} /\circ C$</td>
</tr>
</tbody>
</table>

Table 1. Physical properties of copper and water.

In Figure 5, the waveform generated at the back surface of the solid, in the absence of the liquid medium, is plotted as a function of time. Comparing this with Figure 4, it can be seen that the two curves are identical until the peak is reached and once the peak is reached, the displacements produced in the presence of the liquid are higher than those produced in the absence of the liquid.
optical absorption coefficient = 1000000/cm.

Figure 3. Temperature distribution in the copper specimen as a function of depth for different times.
optical absorption coefficient = 1000000/cm.

Figure 4. Waveform generated at the back surface of the copper specimen as a function of time with the water layer present.
optical absorption coefficient = 1000000/cm.

Figure 5. Waveform generated at the back surface of the copper specimen as a function of time, in the absence of the water layer.
Physical Interpretation

In order to explain the physical behavior behind this, first consider the case of a solid layer where the two surfaces are stress-free. Unless otherwise mentioned the following discussion is restricted only until \( r = 3 \). The thermal energy deposited by the laser beam in the solid can be viewed as a continuous structure of point sources of varying intensity. Each of the point sources emits waves travelling to the left (i.e., toward \( z = 0 \)) and to the right (i.e., toward \( z = b \)) toward the two free surfaces \([20]\). When these hit the free surfaces, they get reflected and the reflected waves possess the same sign of displacement as the incident waves do. Since in this case, the thermal energy deposited in the solid decays exponentially with depth, positive displacement waves travelling to the right toward \( z = b \) and negative displacement waves travelling to the left toward \( z = 0 \) are generated initially. The positive displacement waves travelling to the right upon reflection at \( z = b \), produce double the displacement that they alone can produce. And the negative displacement waves upon reflection at \( z = 0 \), produce negative displacement reflected waves travelling toward \( z = b \). These reflected waves upon reaching \( z = b \), produce twice the displacement that they alone can produce. These arrive at and after \( r = 1 \). The effect of these negative displacement waves is to reduce the positive displacement induced by the positive displacement waves. This results in a drop in the displacement at the back surface after \( r = 1 \). Now, since the negative and the positive displacement waves produced from a point source have equal displacements in magnitude, it is expected that the waveform be symmetrical about \( r = 1 \). This will be the case if the thermal diffusion effects are neglected. However, in this case, since thermal diffusion is present, there will be new waves
generated at later times which skew the shape of the waveform. The curve in Figure 5 represents this behavior.

Now consider the case of a solid layer, where its front surface is rigidly fixed and the back surface is free. In this case, since the back surface is free, the positive displacement waves behave the same way as they did in the previous case. However, the negative displacement waves travelling to the left upon reaching \( z = 0 \), get reflected as positive displacement waves as the front surface is fixed. These waves when they reach the back surface add to the positive displacement already produced by the waves arrived earlier. Thus the displacements produced in this case after \( \tau = 1 \) show an increasing trend [21].

The foregoing discussion then suggests that the higher displacements predicted by the curve in Figure 4 can possibly be due to the partial surface constraint imposed by the liquid layer on the front surface of the solid. That this is the reason can be seen from the following discussion.

Assume that a liquid layer is present on the solid layer. Again, the positive displacement waves generated in the solid behave the same way as they did in the previous cases until \( \tau = 1 \). The negative displacement waves however, upon reaching \( z = 0 \) apart from getting reflected partially, also get transmitted. If the surface constraint imposed by the liquid is sufficiently low i.e. if the bulk modulus of the liquid is sufficiently low, then the reflected waves although reduced in magnitude still remain negative. These upon reaching \( z = b \), reduce the displacement produced earlier by the positive displacement waves. However, this reduction is not as high as before and therefore higher displacements with a decreasing trend are produced. Now if the bulk modulus is sufficiently high, the reflected waves may become positive which then upon reaching \( z = b \) increase the displacement.
Thus after \( r = 1 \), an increasing trend of the displacement in the waveform will be seen.

Figure 6 illustrates this behavior, where curves are drawn for increasing bulk modulus. It can be seen that the higher the bulk modulus, higher is the displacement produced at the back surface. The curves showing a monotonic increase in the displacement are similar to those obtained by Conant [21] for a fixed front surface.

Now look at the effect of surface heating of the liquid layer on the displacement at the back surface of the solid layer. In Figure 7, waveform produced at the back surface of the solid in the presence of diffusion of heat into the liquid and in the absence of diffusion of heat into the liquid are plotted. As the figure shows, the two curves lie on top of each other. This is expected since the thermal diffusivity of the liquid medium (here, water) is very low. This is also supported by Figure 8 where temperature distribution in the solid medium in the presence and in the absence of the liquid medium is presented. The difference caused by the liquid medium can be seen to be negligible.

Using the values in Table 1 for \( \alpha = 10^9 / \text{cm.} \), temperature distribution and the corresponding waveform generated at the back surface are plotted in Figure 9 and Figure 10 respectively. Comparing Figure 9 with Figure 3, it is clear that the temperature redistribution process is much slower in this case than in the case of \( \alpha = 10^6 / \text{cm.} \). The reason for this can be attributed to the extremely high thermal gradients produced in the latter case. In the former case, the thermal gradients are low enough so that the subsequent wave generation at the interior points of the solid layer can be neglected. It can also be observed in Figure 10 that the peak produced is smaller than that in Figure 4 and the waveform is wider. The physical basis for this comes again from the fact that for large \( \alpha \),
more thermal energy is deposited in a smaller volume which results in a much more rapid thermal expansion than for the case of low $\alpha$. For further details on the effects of the optical absorption on the waveform generated, the reader is referred to [22].

Thus it can be concluded that the higher displacements generated at the back surface are due only to the surface constraint imposed by the liquid medium at the front surface of the solid and that higher is the constraint (i.e., higher is the bulk modulus), higher will be the displacement generated at the back surface of the solid.
optical absorption coefficient = 1000000/cm.

Figure 6. Effect of increasing bulk modulus on the waveform generated at the back surface of the copper specimen
optical absorption coefficient = 1000000/cm.

Figure 7. Effect of diffusion of heat into the water layer from the interface, on the waveform generated at the back surface of the copper specimen.
optical absorption coefficient = 1000000/cm.

--- water layer present
--- water layer absent

time = 0.002ns

Figure 8. Effect of diffusion of heat into the water layer, on the temperature distribution in the copper specimen
optical absorption coefficient = 1000/cm.

Figure 9. Temperature distribution in the copper specimen as a function of depth for different times.
optical absorption coefficient = 1000/cm.

Figure 10. Waveform generated at the back surface of the copper specimen as a function of time.
CHAPTER 6

CONCLUSION

In the first chapter, it was mentioned that the model for the prediction of the waveform generated at the back surface of the solid should fulfill two requirements. First, it should be able to explain either physically or mathematically, the reason for obtaining stronger waveform (experimentally observed) at the back surface of the solid in the presence of an optically transparent liquid medium. In the last chapter, it was concluded that the stronger waveform is due to the surface constraint imposed by the liquid layer on the front surface of the solid. Higher the bulk modulus of the liquid, higher is the postpeak displacement produced at the back surface of the solid.

The second requirement to be satisfied is that the waveform predicted by the theoretical model should match the experimentally observed waveform. Experimental results show that in the presence of a liquid layer on the front surface of a solid specimen, a stronger waveform with a higher peak than that can be generated in the absence of the liquid layer results. The waveform predicted by the model developed in this work is identical to the waveform obtained in the absence of the liquid layer until at least the peak is reached. The author feels that this incompatibility between experimental and theoretical results can be due to two reasons. First, the theoretical model developed in this work doesn't take into account the possibility of the liquid medium getting vaporized by absorption of heat from the interface. In Chapter 2, it was assumed that the laser power is such
as not to cause any vaporization in either of the two media. However, if the laser power is increased to a level sufficient to cause vaporization of the liquid medium alone, then it is possible that, due to the recoil imparted to the solid medium by the vaporizing liquid particles a higher peak may be obtained. The second reason can be the assumption that the laser beam diameter is large enough so that the transverse thermal distribution in the solid specimen can be taken to be uniform. Thus to obtain theoretical results compatible with the experimental results, further investigation needs to be done taking into account the vaporization of the liquid medium or by resorting to a two-dimensional analysis or both.

However, the model developed has its advantages too. If the laser power is low enough not to cause any phase changes in either of the two media and the laser beam diameter is large enough, then the model can be used to predict the waveform at the back surface of the solid. Using the Green’s function in time for the displacement field in the solid, displacement as a function of time due to a laser pulse of arbitrary temporal shape can be obtained and the resulting waveform will have higher amplitude than can be obtained in the absence of the liquid medium. The higher amplitudes can effectively be used in determining the microstructure of the solid medium [23].
REFERENCES CITED
REFERENCES CITED


APPENDIX A

PROGRAM TO CALCULATE THE TEMPERATURE DISTRIBUTION IN THE SOLID MEDIUM
Figure 11. Format of the data input files for temperature calculation.

File: Input.tern
Line 1: !SOLID \( c_1 \quad \kappa_1 \quad \alpha \)
Line 2: \( c_2 \quad \kappa_2 \)

File: Control.tern
Line 1: \( N \quad \text{EPSN} \quad \text{AMAXTIME} \quad \text{NTOUS} \quad \text{TOUSTEP} \quad \text{TEPSN} \)
Line 2: \( \text{ZSTEP1} \quad \text{ZSTEP2} \quad \text{ZSTEP} \)

Explanation of the terms used above

!SOLID is a flag used for the calculation of the temperature distribution in the solid medium in the absence of the liquid medium.

\( N \) = maximum number of iterations

\( \text{EPSN} \) = minimum value of the temperature desired

\( \text{AMAXTIME} \) = maximum time

\( \text{NTOUS} \) = number of time steps

\( \text{TOUSTEP} \) = time step size

\( \text{TEPSN} \) = control parameter to take into account the round-off error in the computer (should be less than \( \text{TOUSTEP} \))

\( \text{ZSTEP1} \) = step size in \( z \)

\( \text{ZSTEP2} \) = step size in \( z \)

\( \text{ZSTEP} \) = when \( z \) equals this, step size is changed from \( \text{ZSTEP1} \) to \( \text{ZSTEP2} \)
Figure 12. Program to calculate the temperature distribution in the solid medium.

```
PROGRAM TEMDLT
* PROGRAM TO CALCULATE THE TEMPERATURE PROFILE
* IN THE SOLID MEDIUM WHEN THE THERMAL ENERGY IS
* DEPOSITED SUDDENLY
* VARIABLE SPECIFICATION
IMPLICIT REAL*8(A-H,K,O-Z)
INTEGER COUNT1
PARAMETER (NMAX=1000)
DIMENSION PZ1(NMAX),T1(NMAX),T1NOLIQ(NMAX),G1(NMAX)
EXTERNAL ERFUNC

COMMON/BLOK1/ALPHA,K1,QO
COMMON/BLOK2/TOU,SQRTPI
COMMON/BLOK3/PARAM,PARAM1,PARAM2
COMMON/BLOK4/C,D,IGAFLG

* OPEN THE FILES FOR INPUT AND OUTPUT
OPEN(UNIT=11,FILE='INPUT.TEM',STATUS='OLD*)
OPEN(UNIT=12,FILE='CONTROL.TEM',STATUS='OLD*')
OPEN(UNIT=16,FILE='SOL.TEM',STATUS='NEW*')
OPEN(UNIT=17,FILE='SOL.GRD',STATUS='NEW*')
OPEN(UNIT=18,FILE='SOL.NOL',STATUS='NEW*')
OPEN(UNIT=19,FILE='GRD.MAT',STATUS='NEW*')

* READ THE INPUT DATA FROM THE FILES
READ(12,*)N,EPSN5AMAXTIME,NTOUS,TOUSTEP,TEPSN
READ(12,*)ZSTEP1,ZSTEP2,ZSTEP
READ(11,*)ISOLID,C1,AK1,ALPHA
READ(11,*)C2,AK2

PI=22.D0/7.D0
P1=(C1/DSQRT(AK1))/(C1/DSQRT(AK1)+C2/DSQRT(AK2))
AN=(C2/C1)*DSQRT(AK1/AK2)
TOU=0.D0

WRITE(16,1110)NTOUS
WRITE(17,1110)NTOUS
WRITE(18,1110)NTOUS
DO 999 I=1,N
   IFLAG=0
   Z=0.D0
   COUNT1=1
   A=ALPHA*DSQRT(AK1*TOU)
999 CONTINUE
```
888   PZ1(COUNT1)=Z
     IF(TOU.EQ.0.D0)THEN
       T1(COUNT1)=DEXP(-ALPHA*Z)
       G1(COUNT1)=-DEXP(-ALPHA*Z)
       T1NOLIQ(COUNT1)=T1(COUNT1)
     ELSE
       B=Z/(2.DO*DSQRT(AK1*TOU))
       DUM1=ERFUNC(A,B,-1)
       DUM2=ERFUNC(A,B,1)
       DUM3=DEXP(-Z*Z/(4.DO*AK1*TOU))
       DUM4=1.DO/DSQRT(AK1*TOU*ALPHA*ALPHA*PI)
       IF(!SOLID.LT.0)THEN
         T1(COUNT1)=(DUM1+(1.DO-2.DO*P1*AN)*DUM2)*0.5D0
         G1(COUNT1)=(-DUM1+(1.DO-2.DO*P1*AN)*DUM2
                    +2.DO*P1*AN*DUM3*DUM4)*0.5D0
       ELSE
         T1(COUNT1)=(DUM1+(1.DO-2.DO*P1*AN)*DUM2)*0.5D0
         G1(COUNT1)=(-DUM1+(1.DO-2.DO*P1*AN)*DUM2
                    +2.DO*P1*AN*DUM3*DUM4)*0.5D0
       T1NOLIQ(COUNT1)=(DUM1+DUM2)*0.5D0
     ENDIF
     ENDIF
     IF(T1(COUNT1).LE.EPSN)THEN
       WRITE(16,1110)COUNT1
       WRITE(17,1110)COUNT1
       WRITE(18,1110)COUNT1
       DO 99 J=I,J,COUNT1
         WRITE(16,1111)PZ1(J),T1(J)
         WRITE(17,1111)PZ1(J),G1(J)
         WRITE(19,1112)PZ1(J),G1(J)
       IF(ISOLID.GE.0)THEN
         WRITE(18,1111)PZ1(J),T1NOLIQ(J)
       ENDIF
         99 CONTINUE
         GO TO 990
     ELSE
       IF(Z.LT.ZSTEP)THEN
         Z=Z+ZSTEP1
       ELSE
         Z=Z+ZSTEP2
       ENDIF
       COUNT1 = COUNT1+1
       GO TO 888
     ENDIF

IF(DABS(TOU-AMAXTIME).LE.TEPSN)THEN
   GO TO 2222
ELSE
   TOU=TOU+TOUSTEP
ENDIF

CONTINUE

FORMAT(13,2X,I3)
FORMAT(10.7,F10.7,E25.17)
FORMAT(10.7,2X,F10.7,E25.17)

CONTINUE

END

REAL*8 FUNCTION ERFUNC(X,Y,IFLAG)

IMPLICIT REAL*8 (A-H,O-Z)

DSQRTPI = DSQRT(22.D0/7.D0)

IF(IFLAG.GE.0.D0)THEN
   ARG = X+Y
ELSE
   ARG = X-Y
ENDIF

Z = DABS(ARG)
T = l.D0/(l.D0+0.5D0*Z)

DERFCC = (-Z*Z-1.26551223DO+T*(1.00002368DO+T*
          (.37409196D0+T*.09678418D0+T*(-.18628806D0
          +T*.37886807D0+T*(-1.13520398D0+T*(1.48851587D0
          +T*(-.82215223D0+T*.17087277D0))))))
IF(Z.LE.7.0.D0)THEN
   IF (IFLAG.NE.2) THEN
      ERFUNC = DEXP(-Y*Y+ARG*ARG+DERFCC)*T
   ELSE
      ERFUNC = DEXP(ARG*ARG+DERFCC)*T
   ENDIF
ELSE
   IF(IFLAG.NE.2)THEN
      ERFUNC = DEXP(-Y*Y)/(DSQRTPI*ARG)
   ELSE
      ERFUNC = 1.D0/(SQRTPID*ARG)
   ENDIF
ENDIF

IF(ARG.LT.0.D0)THEN
   ERFUNC = 2.D0*DEXP(-Y*Y+ARG*ARG) -ERFUC
ENDIF

END
APPENDIX B

PROGRAM TO CALCULATE THE WAVEFORM GENERATED AT THE BACK SURFACE OF THE SOLID MEDIUM
Figure 13. Format of the data input files for displacement calculation.

File: Input.dat
Line 1: $k_1 \ c_1 \ \lambda \ \mu \ C_{L_1} \ \alpha$
Line 2: $k_2 \ c_2 \ B \ \beta \ C_{L_2}$

File: Control.dat
Line 1: TINIT TSTEP1 TSTEP2 TSTEP3 FAC1 FAC2 FAC3
Line 2: AL L

Explanation of the terms used above

TINIT = initial time
TSTEP1 = time step used outside a region containing the waveform peak
TSTEP2 = time step used for times slightly less than the longitudinal wave arrival time
TSTEP3 = time step used for times slightly greater than the longitudinal wave arrival time
FAC1 = this multiplies the longitudinal wave arrival time to determine the maximum time for which displacement is calculated
FAC2 = this determines the time until which TSTEP1 is to be used and from which TSTEP2 is to be used
FAC3 = this determines the time until which TSTEP3 is to be used and from which TSTEP1 is to be used
AL = depth of the solid layer
L = a flag (should always be 1)
Figure 14. Program to calculate the waveform generated at the back surface of the solid medium.

PROGRAM WAVE
* PROGRAM TO CALCULATE THE DISPLACEMENT FIELD
* IN THE SOLID-LIQUID SYSTEM.

* VARIABLES
IMPLICIT REAL*8 (A-H,O-Z)
EXTERNAL ERFUNC
COMMON/SOLPROP/AK1,C1,ALBD1,AMU1,CL1,ALPHA,GAMMA
COMMON/LIQPROP/AK2,C2,LMOD,BETA,CL2
COMMON/CONST/P,AN,AM11,AM12,AM13,DELTA,CHI,A,B,P1
COMMON/GEOM/AL,TSPAN

OPEN(8,FILE='INPUT.DAT',STATUS='OLD')
OPEN(9,FILE='CONTROL.DAT',STATUS='OLD')
OPEN(15,FILE='SOLDIS.MAT',STATUS='NEW')

* READ IN THE MATERIAL PROPERTIES AND CONTROL VARIABLES FROM THE INPUT FILES.
READ(8,*)AK1,C1,ALBD1,AMU1,CL1,ALPHA,GAMMA
READ(8,*)AK2,C2,LMOD,BETA,CL2
READ(9,*)TINIT,TSTEP1,TSTEP2,TSTEP3,FAC1,FAC2,FAC3
READ(9,*)AL,L

* CALCULATE THE VALUES OF THE VARIOUS COMPONENTS
CALL CNSTNT
K = 0
TSPAN = AL/CL1
AMAXT = FAC1*TSPAN
TPREPK = FAC2*TSPAN
TPSTPK = FAC3*TSPAN

* NOW CALCULATE THE DISPLACEMENT IN THE SOLID
T = TINIT
100 IF(T.LE.AMAXT)THEN
   IF((T.GT.TPREPK).AND.(L.EQ.1))THEN
      L = 2
      T = TPREPK
   ENDIF
   CALL DISP(T,SOLDIS)
   TNOND = T/TSPAN
WRITE(15,1001)TNOND,SOLDIS
IF((T.LT.TPREPK).OR.(T.GE.TPSTPK))THEN
  T = T + TSTEP1
ELSE IF(T.LE.TSPAN)THEN
  T = T + TSTEP2
ELSE
  T = T + TSTPE3
ENDIF
K = K+1
GO TO 100
ENDIF
WRITE(15,*K
1001 FORMAT('',2X,E18.10,E25.10)
END

SUBROUTINE CNSTNT
* THIS SUBROUTINE CALCULATES AND RETURNS
* THE CONSTANTS INVOLVED IN THE WAVE EQUATION.
* VARIABLES
IMPLICIT REAL*8 (A-H,O-Z)

C
om m onzsolpropzakij
C
i j
A
l b d ij
A
m u ij
CLI
j
A
lph a
,
+ GAMMA
COMMON/ZLIQPROPZAK2JCI2,BMODJCBETAJCL2
C
om m onzconstzpj
A
n j
A
m iij
A
m m j
A
m isj
D
elt
A
j
CHI
j
A
j
B
j
P
i
* CALCULATE THE CONSTANTS
DUM1 = ALBD1+2.D0*AMU1
DELTA = (S.D0*ALBD1+2.D0*AMU1)*GAMMA/DUM1
CHI = BMOD/DUM1
A = -1.D0+CHI*(CL1/CL2)
B = 1.D0+CHI*(CL1/CL2)
P1 = (C1/DSQRT(AK1))/(C1/DSQRT(AK1)+C2/DSQRT(AK2))
P = P1*DSQRT(AK1)
AN = (C2/C1)*DSQRT(AK1/AK2)
DUM2 = CL1*CL1-ALPHA*ALPHA*AK1*AK1
AM11 = (DELTA*CL1*ALPHA*AK1*(P1)/ DUM2)*
ALPHA/DELTA
+ DUM3 = CL2-ALPHA*AK1R*DSQRT(AK2/AK1)
AM12 = ALPHA*CHI*BETA*CL1*(P1) *DSQRT(AK2/AK1)
+ /(DUM3*DELTA)
AM13 = DELTA*CHI*CL1*CL1*ALPHA/ (DELTA*DUM2)
RETURN
END
Figure 14 (continued)

REAL*8 FUNCTION ERFUNC(X,Y,IFLAG)
IMPLICIT REAL*8 (A-H,O-Z)
DSQRTPI = DSQRT(22.D0/7.D0)
IF(IFLAG.GE.0.D0)THEN
  ARG = X + Y
ELSE
  ARG = X - Y
ENDIF
Z = DABS(ARG)
T = 1.0D0/(1.0D0+0.5D0*Z)
DERFCC = (-Z*Z-1.2655123D0+T*(1.00002368D0+ T*
  +0.37409196D0+T*(0.9678418D0+T*(-0.18628806D0
  +T*(0.27886807D0+T*(-1.13520398D0+ T*(1.48851587D0
  +T*(-0.82215223D0+T*.17087277D0))))))))
IF(Z.LE.700.D0)THEN
  IF(IFLAG.NE.2)THEN
    ERFUNC = DEXP(-X*X+ARG*ARG+DERFCC)*T
  ELSE
    ERFUNC = DEXP(ARG*ARG+DERFCC)*T
  ENDIF
ELSE
  IF(IFLAG.NE.2)THEN
    ERFUNC = DEXP(-X*X)/(DSQRTPI*ARG)
  ELSE
    ERFUNC = l.DO/ (DSQRTPI* ARG)
  ENDIF
ENDIF
IF(ARG.LT.0.D0)THEN
  ERFUNC = 2.D0*DEXP(-X*X+ARG*ARG)-ERFUNC
ENDIF
END

SUBROUTINE R1(T,VALUE)
  SUBROUTINE THAT RETURNS THE VALUE OF THE FUNCTION
  R1(T) IN THE DISPLACEMENT EQUATION.
+
  VARIABLES
IMPLICIT REAL*8 (A-H,O-Z)
+
COMMON/SOLPROP/AK1,C1,ALBD1,AMU1,CL1,ALPHA,GAMMA
COMMON/LIQPROP/AK2,C2,BMOD,BETA,CL2
COMMON/CONST/P,AN,AM11,AM12,AM13,DELTA,CHI,A,B,P1
COMMON/GEOM/AL,TSPAN
+
DUM1 = (AM11-AM12+AM13*(CL1/CL2)*AN*P1)/ALPHA
DUM2 = -(AM11+AM13*P1*AN*(CL1/CL2)) * (AK1/CL1)
SUBROUTINE R2(T, VALUE, FLAG)
* THIS SUBROUTINE EVALUATES AND RETURNS THE VALUE
* OF R2(T)

VARIABLES
IMPLICIT REAL*8 (A-H,O-Z)

COMMON/SOLPROP/AK1,C1,ALBD1,AMU1,CL1,ALPHA,GAMMA
COMMON/LIQPROP/AK2,C2,BMOD,BETA,CL2
COMMON/CONST/P,AN,AM11,AM12,AM13,DELTA,CHI,A,B,P1
COMMON/GEOM/AL,TSPAN

DUM1 = -AN/ALPHA-AK1/CL1
DUM2 = -AN/ALPHA+AK1/CL1
DUM3 = -(AN+1.D0)/ALPHA
DUM4 = (AN-1.D0)/ALPHA
DUM5 = 1.D0/(2.D0*P1*ALPHA)*AM11
DUM6 = CL1/(2.D0*ALPHA*ALPHA*P1*AK1)*AM11

X2 = AL/DSQRT(4.D0*AK1*T)
X3 = CL1*DSQRT(T/AK1)
IF(FLAG.EQ.0.D0)THEN
  IF(T.LE.TSPAN)THEN
    F1 = ERFUNC(X2,X2,-1)
    F5 = (DUM5+DUM6)*DEXP(ALPHA*CL1*T -ALPHA*AL) +
         (DUM5-DUM6)*DEXP(-ALPHA*CL1*T -ALPHA*AL)
  ELSE
    F1 = -ERFUNC(-X2,-X3,-1)
  END IF
ELSE
Figure 14 (continued)

\[ F5 = (DUM5-DUM6) \cdot DEXP(-\alpha \cdot CL1 \cdot T - \alpha \cdot \alpha) \]
ENDIF
ELSE
\[ F1 = \text{ERF FUNC}(X2, X3, -1) \]
\[ F5 = (DUM5+DUM6) \cdot DEXP(\alpha \cdot CL1 \cdot T - \alpha \cdot \alpha) + \]
\[(DUM5-DUM6) \cdot DEXP(-\alpha \cdot CL1 \cdot T - \alpha \cdot \alpha) \]
ENDIF
\[ F2 = \text{ERF FUNC}(X2, X3, 1) \]
\[ X3 = \alpha \cdot \text{D SQ R T}(AK1 \cdot T) \]
\[ F3 = \text{ERF FUNC}(-X2, -X3, -1) \]
\[ F4 = \text{ERF FUNC}(X2, X3, 1) \]
\[ \text{VALUE} = (F1 \cdot DUM1 + F2 \cdot DUM2 + F3 \cdot DUM3 + F4 \cdot DUM4) + \]
\[ \ast \text{AM11} \cdot 0.5D0 + F5 \]
RETURN
END

SUBROUTINE W1P(T, VALUE)
* THIS SUBROUTINE CALCULATES AND RETURNS THE VALUE
* OF THE PARTICULAR SOLUTION OF THE DISPLACEMENT
* EQUATION FOR THE SOLID
*
VARIABLES
IMPLICIT REAL*8 (A-H,O-Z)

COMMON/SOLPROP/AK1,C1,ALBD1,AMU1,CL1,ALPHA,GAMMA
COMMON/LIQPROP/AK2,C2,BMOD,BETA,CL2
COMMON/CONST/P,AN,AM11,AM12,AM13,DELTA,CHI,A,B,P1
COMMON/GEOM/AL,TSPAN

DSQRTPI = DSQRT(22.D0/7.D0)

DUM1 = AK1/CL1+AN/ALPHA
DUM2 = AK1/CL1-AN/ALPHA
DUM3 = (CL1/(ALPHA*ALPHA*AK1))* (1.D0+AN)
DUM4 = -(CL1/(ALPHA*ALPHA*AK1))* (1.D0-AN)
DUM5 = P1
DUM6 = 0.5D0*AM11+CL1/(P1*ALPHA* ALPHA*AK1)
DUM7 = 0.5D0*AM11/(P1*ALPHA)

X2 = AL/DSQRT(4.D0*AK1* T)
X3 = CL1*DSQRT(T/AK1)
IF(T.LE.TSPAN)THEN
\[ F1 = \text{ERF FUNC}(X2, X3, -1) \]
\[ F6 = (DUM6+DUM7) \cdot DEXP(\alpha \cdot CL1 \cdot T - \alpha \cdot \alpha) + \]
\[(DUM6-DUM7) \cdot DEXP(-\alpha \cdot CL1 \cdot T - \alpha \cdot \alpha) \]
ELSE
    F1 = ERFUNC(-X2,-X3,-1)
    F6 = (DUM6-DUM7)*DEXP(-ALPHA*CL1*T -ALPHA*AL)
ENDIF
F2 = ERFUNC(X2,X3,1)
X3 = ALPHA*DSQRT(AK1*T)
F3 = ERFUNC(-X2,-X3,-1)
F4 = ERFUNC(X2,X3,1)
X1 = 0.D0
X3 = 0.D0
F5 = ERFUNC(X2,0.D0,1)
VALUE = -AM11*0.5D0*(DUM1*F1+DUM2*F2 +DUM3*F3
    +DUM4*F4)-F5*DUM5+F6
RETURN
END

SUBROUTINE DISP(T5X)
* SUBROUTINE TO EVALUATE THE DISPLACEMENT IN THE
* SOLID-LIQUID SYSTEM
*
* VARIABLES
IMPLICIT REAL*8 (A-H,O-Z)

COMMON/SOLPROP/AK1,C1,ALBD1,AMU1,CL1,ALPHA,GAMMA
COMMON/LIQPROP/AK2,C2,BMOD,BETA,CL2
COMMON/CONST/P,AN,AM11,AM12,AM13,DELTA,CHI,A,B,P1
COMMON/GEOM/AL,TSPAN

CALL W1P(T,VALP)
CALL R2(T,VAL1,0.D0)
IF(T.GE.TSPAN)THEN
    T1 = T-TSPAN
    CALL R1(T1,VAL2)
ELSE
    VAL2 = 0.D0
ENDIF
IF(T.GE.2.D0*TSPAN)THEN
    T2 = T-2.D0*TSPAN
    CALL R2(T2,VAL3,2.DO)
ELSE
    VAL3 = 0.DO
ENDIF

DUM1 = (2.DO/B)*VAL2+VAL1-(2.DO*A/B)*VAL3
X = (VALP + DUM1)
RETURN
END