One-dimensional model for laser generated elastic waves in a solid layer the illuminated surface of which is constrained by a transparent liquid medium

by Harischandra Prasad Cherukuri

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Mechanical Engineering
Montana State University

Abstract:
A one-dimensional theoretical model for the laser generated displacement waves in a solid strip with a transparent liquid lying on its front surface is presented. For mathematical simplicity, the liquid medium is treated as a half-space and it is assumed that no vaporization of the liquid or the solid occurs at the interface. Spatial distribution of the laser energy deposited in the solid is assumed to decay exponentially with depth. Regarding the temporal distribution of the laser energy as a delta function in time, a Green's function in time for the displacement waves in the solid is obtained. Assuming uncoupled thermoelasticity theory, the Laplace transform technique is used to solve the one-dimensional heat conduction and wave equations. The solution for the wave equation is specialized to obtain the displacement at the back surface of the solid. The waveform obtained at the back surface is compared with that obtained for a solid strip in the absence of the liquid. It is shown that the peak displacements obtained in both the cases are the same. The higher postpeak displacements obtained in the presence of the liquid medium are attributed to the surface constraint imposed by the liquid medium on the solid medium. It is also shown that the thermal diffusion of heat into the liquid from the interface has negligible effects on the waveform generated at the back surface of the solid.

ONE-DIMENSIONAL MODEL FOR LASER GENERATED ELASTIC WAVES IN A SOLID LAYER THE ILLUMINATED SURFACE OF WHICH IS CONSTRAINED BY A TRANSPARENT LIQUID MEDIUM

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APPROVAL

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Harischandra Prasad Cherukuri

This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency, and is ready for submission to the College of Graduate Studies.

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ABSTRACT

A one-dimensional theoretical model for the laser generated displacement waves in a solid strip with a transparent liquid lying on its front surface is presented. For mathematical simplicity, the liquid medium is treated as a half-space and it is assumed that no vaporization of the liquid or the solid occurs at the interface. Spatial distribution of the laser energy deposited in the solid is assumed to decay exponentially with depth. Regarding the temporal distribution of the laser energy as a delta function in time, a Green's function in time for the displacement waves in the solid is obtained. Assuming uncoupled thermoelasticity theory, the Laplace transform technique is used to solve the one-dimensional heat conduction and wave equations. The solution for the wave equation is specialized to obtain the displacement at the back surface of the solid. The waveform obtained at the back surface is compared with that obtained for a solid strip in the absence of the liquid. It is shown that the peak displacements obtained in both the cases are the same. The higher postpeak displacements obtained in the presence of the liquid medium are attributed to the surface constraint imposed by the liquid medium on the solid medium. It is also shown that the thermal diffusion of heat into the liquid from the interface has negligible effects on the waveform generated at the back surface of the solid.

CHAPTER 1

INTRODUCTION

Ultrasonic techniques are finding increased use and importance in the field of nondestructive testing of materials. Before the advent of these techniques, methods like striking a specimen and listening for the characteristic "ring" were used for detecting flaws in the materials. With these methods only the gross defects could be detected since the wavelength of the audible sound waves is large compared to the size of the defects. However, with the development of the reliable methods for generating and detecting ultrasonic waves, even the presence of small defects in the components under inspection can be found since the wavelength of ultrasonic waves is very small compared to the size of the defect.

For example, in the pulse echo normal incidence method used for detecting flaws, cracks, etc., pulse sending and pulse detecting transducers are both placed on one side of the specimen being tested (see Figure 1). If no flaws are present in the specimen, the displacement detected will be as shown in Figure 1.a, where each spike is detected by the transducer after times nt_0, $n = 1, 2, \ldots$ etc., with $t_0 = 2L/v$, L the length of the specimen and v the velocity of the ultrasonic pulse in the specimen. On the other hand, if there is a flaw in the specimen such as the one (highly exaggerated) shown in Figure 1.b, there will be a short spike detected in between the two major spikes.

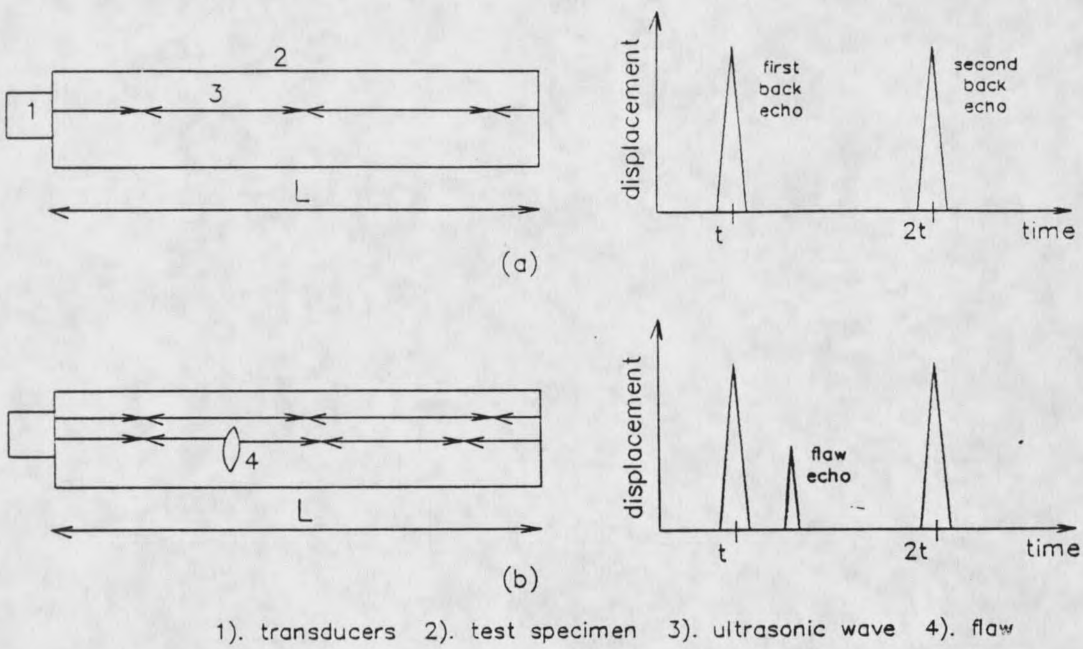

Figure 1. Pulse echo normal incidence method for detecting flaws.

One of the widely used methods for generating ultrasonic waves for nondestructive testing purposes is to make use of the inverse piezoelectric effect. However, this method has the disadvantage in that the piezoelectric transducer has to be coupled to the workpiece. This coupling, depending on its strength and nature, has the effect of influencing the ultrasonic wave apart from causing wear at the interface due to abrasion.

Efforts have therefore been made to utilize other physical effects to generate ultrasonic waves which make mechanical contact with the specimen unnecessary. One of these effects is rapid heating. By heating a body suddenly, mechanical stresses can be produced in the material due to the thermal expansion of the material. These stresses then initiate elastic waves. If the heating is for very short duration and is intense, very high frequencies and shock waves can be produced.

There are many ways of delivering the required thermal energy to the body. Of these, particularly suitable is the laser induced heating. The suitability of lasers for this purpose comes from the intensity and the directionality properties of the laser beams. These two properties allow high power densities (irradiances) to be delivered in very short times to the absorbing material although the total power carried by the beam may be small. Other advantages include the high reproducibility of laser generated ultrasound [1] and higher frequencies than can be obtained with piezoelectric devices. Also, resonance phenomena which occur with piezoelectric devices do not occur in laser induced ultrasonic waves.

In the present work, the author is concerned with the ultrasonic wave generation due to laser induced heating.

Laser Induced Stress Waves in Solids

There are basically two mechanisms by which thermal waves are produced in solids due to laser interaction.

When a short duration laser beam of power density sufficiently high to produce stress waves but small enough not to cause any phase change is absorbed by a target material (here, a solid), high thermal energies are deposited for a very short interval of time causing high spatial thermal gradients. This induces localized rapid thermal expansion in the medium, which is resisted by the surrounding medium. This resistance may be so high that the material is subjected to rapidly varying strain field, which results in the generation of thermal stress waves.

On the other hand, if a high intensity laser beam strikes the target material, some of the material may get evaporated. The velocity with which the ejected material leaves the target surface can be so high that, by the momentum conservation principle, the target is imparted an impulse. This results in the transmission of a shock wave through the target.

Now if the surface of the target is coated with a thin film of liquid or a transparent solid, due to the surface constraint imposed by the film it has been found that the resulting acoustic transmission through the solid is stronger. In the presence of a liquid film, if no phase changes occur in either of the two media and if it is assumed that the laser beam energy is deposited completely in the solid, there will be thermoelastic waves generated in the solid due to internal heat source and in the liquid due to surface heating from the solid-liquid interface.

In the next section some of the previous work that was done to understand the above mentioned phenomena is reviewed.

Previous Work

Perhaps the first person to suggest the possibility of generating elastic waves due to absorption of radiation from the high powered light sources was White [2]. He assumed that the laser energy deposition in the solid is equivalent to surface heating of the solid and analyzed the one-dimensional elastic wave generation in isotropic elastic bodies under several transient surface heating conditions such as uniform and non-uniform heating of the surface of a semi-infinite body, and uniform heating of the end of a long rod. The case of input heat flux varying harmonically with time was studied in detail. It was shown that the dependence of the stress wave amplitude on frequency and the properties of the body heated is governed by the kind of constraint applied at the surface and by the distance through which the heating takes place. It was also shown that the stress amplitude for a constrained surface is much greater than that for a free surface.

Unlike White, Ready [3] considered the optical absorption effects by assuming that the energy absorbed by the body decreases exponentially across the thickness of the body. However, he determined only the temperature distribution and did not analyze the deformation or the stress field. A general solution for the temperature distribution in the body due to a laser pulse of arbitrary temporal shape was obtained using Duhamel's integral method. This solution was then used to show that for an ordinary laser pulse of a sequence of microsecond duration spikes, the surface temperature follows the laser spikes and that the properties of the body such as thermal conductivity are important for the temperature distribution in the body. In the same work, the effects of vaporization of the material from the surface of the body due to ordinary laser pulses and Q-switched laser pulses were also investigated experimentally.

Since Ready, there have been many investigations, both analytical and theoretical, into the laser generated elastic waves in both solids and liquids.

Of these, as far as the laser generated stress waves in solids are concerned, particularly worthy of mentioning is the work by the group of Scruby, Dewhurst, Hutchins and Palmer ([1], [4] and [5]). They studied experimentally the waveforms produced due to the pulsed laser irradiation over a wide range of material conditions such as the presence and the absence of surface plasmas, free metal surfaces and metal surfaces modified with transparent liquid or solid coatings. Based on the experiments, neglecting thermal diffusion and treating the thermal source as constrained to the surface, three-dimensional axisymmetric models for these different conditions were obtained. The model for the first case predicted an inward displacement at the back surface, while the models for the remaining two cases predicted an outward displacement. Although these results were in agreement with those obtained experimentally, the model developed for the first case does not predict an initial outward displacement spike (which is observed in the experimental results) as thermal diffusion was neglected. Doyle [6] took the effects of thermal diffusion into consideration and obtained a model which predicts the spike. Brechtel [7] investigated the conditions under which diffusion effects can be neglected and showed that if the depth of thermal diffusion for the laser pulse duration is large compared to the optical attenuation depth, the surface and volume generated models of laser induced heating yield the same results. On the other hand, if the thermal diffusion depth for the laser pulse duration is comparable or less than the attenuation depth, the results obtained from the two models may differ significantly.

Until now, all the work considered above dealt with neglecting the effects of stress or displacement fields upon the temperature field. Strikwerda and Scott [8], using coupled thermoelastic theory, obtained a one-dimensional theoretical model for the temperature and the displacement fields produced due to short duration laser pulses. They used perturbation methods to arrive at the solutions. Explicit results for a Gaussian pulse shape and an exponential decay pulse shape were presented. Results for the temperature field were compared with those given in [3] and found to be compatible.

The work of Scruby et. al. ([1],[4] and [5]) also showed that the modification of the surface (i.e. coated targets) leads to an enhancement in the generation of longitudinal waveforms at the backside. Their work considers both the cases where there is a phase change of either the target material or the coating material and where there is no phase change at all.

However, before Scruby et al., there were many others who investigated the possibility of strengthening the acoustic wave generation by coating the laser irradiated targets.

Anderholm [9] developed a technique in which the material is confined between two transparent materials. When a laser pulse of short duration hits the target, a confined, high energy plasma is produced which then generates high amplitude stress waves into the transparent materials. The technique of O'keefe and Skeen [10] involves partially vaporizing the thin coating on the target material through conduction from the laser heated surface. The subsequent expansion of the gas and the plasma confinement at the interface causes stress waves to be generated in the target material.

In the last two cases, to increase the coupling of the laser energy to the target, the laser irradiations were generated in the vacuum. Fox [11] showed experimentally that the stress enhancements can be achieved without plasma confinement and in atmospheric air by coating the target surface with optically dense thin films such as of oil or water.

However, to the author's knowledge, theoretical work concerned with the effects of laser irradiation of coated targets was done only by Scruby et al. Their work does not consider the diffusion effects. As a first step toward understanding the effects of diffusion on this problem (assuming there is no phase change), it is the purpose of the present work to consider the one-dimensional aspects of the problem taking diffusion into account. The actual problem that will be dealt with is presented next.

Statement of The Problem

In this thesis, the author will be concerned with the case where the laser induced heating causes temperature changes and therefore displacement changes only along the thickness. It is to be noted that although this is far from the reality, it is certainly possible to realize approximate homogeneous behavior in the transverse directions with regard to temperatures and displacements when the transverse dimensions of the laser beam are large compared to the depth to which heat is propagated during the laser pulse time (page 70, [12]). Therefore it can be assumed that the laser pulse incidence is normal to the liquid-solid interface.

The main aim is to develop a one-dimensional theoretical model for the prediction of laser induced displacement waves in a solid medium of finite thickness with a liquid layer resting on the front side of it (Figure 2).

The model to be developed should fulfill the following requirements:

1. It should be valid for a laser pulse of arbitrary temporal shape.
2. It should explain why higher displacements are observed in the presence of a liquid medium than in the absence of the liquid medium.
3. Theoretical displacements obtained from this model for the back surface of the solid should match the experimentally obtained results.

It should be noted that displacement of interest is that at the back surface of the solid, since it is usually at the back surface that the laser generated ultrasonic waves are measured. (In non-destructive testing, such a method where the ultrasonic wave is generated at one surface of the test specimen and is detected at the other surface is called the through transmission normal incidence method. This can be visualized from Figure 1 by placing a displacement detecting transducer at the back surface instead of at the front surface.) This also means that time of interest is only until the second arrival of the wave at the back surface, since the waves reflected from flaws present in the solid take times ranging from about one to three times the time that a longitudinal wave generated at the interface of the solid and the liquid, takes to reach the back surface of the solid.

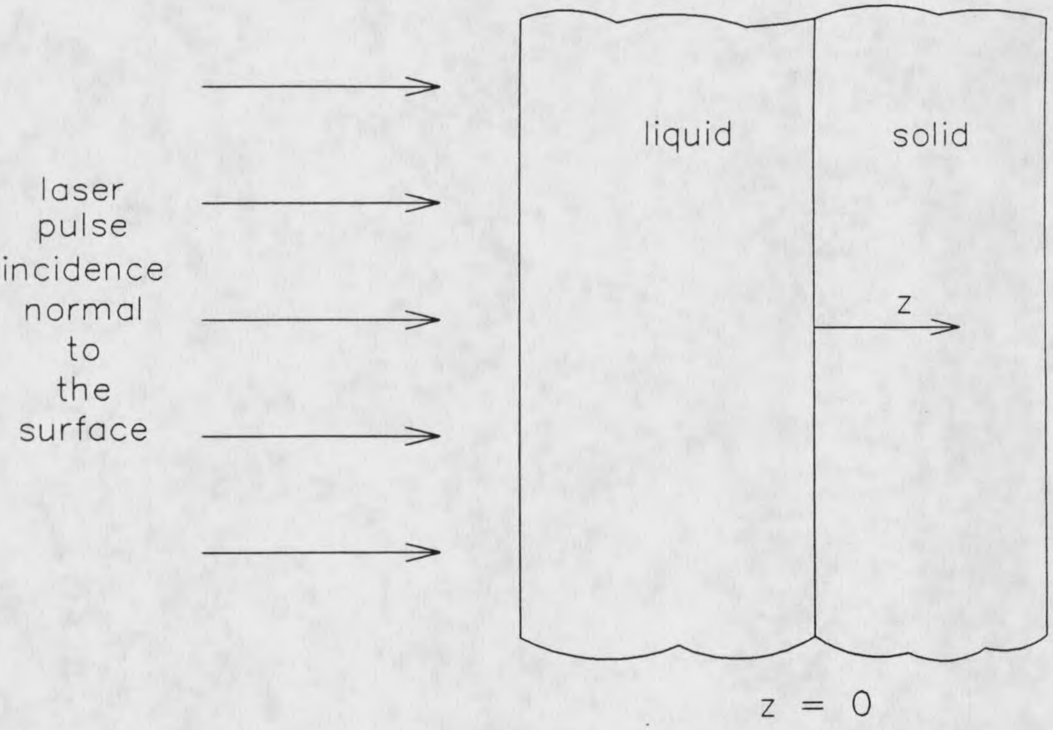

Figure 2. Schematic representation of the solid-liquid system under normal laser pulse incidence.

To avoid the interference of the reflected waves from the upper surface of the liquid with the waves in the solid, it is also assumed that the liquid layer is thick enough so that there is no reflected wave from the upper liquid surface for the times of interest.

CHAPTER 2

FORMULATION

In this chapter, the approach taken to solve the problem proposed in the last chapter is presented, and then the equations governing the stress waves generated in the solid due to laser heating and the acoustic waves generated in the liquid due to heating from the interface are developed.

Approach

To obtain the displacement and the temperature fields in both the liquid and the solid media, uncoupled thermoelastic theory is assumed i.e., the temperature changes resulting from the changes in the deformation field are neglected. Optical absorption effects are taken into account by assuming that the laser deposition of energy into the solid through the liquid medium drops exponentially along the thickness of the solid medium as assumed in [3].

In order to meet the requirement (see Chapter 1) that the obtained solutions be valid for a laser pulse of arbitrary temporal shape, first Green's functions for the temperature and displacement fields are obtained by replacing the function representing the laser pulse temporal shape by the Dirac delta function in time. Once the Green's functions are obtained, the thermal and displacement fields at any time due to a desired temporal shape of the laser pulse can be obtained by employing the superposition principle (integration).

In the next section some of the basic equations of linear thermoelasticity are reviewed.

Basic Equations of Linear Thermoelasticity for An Isotropic Homogeneous Body

When a body is subjected to temperature variations it is known that the body undergoes deformations. In the presence of external constraints or if the temperature is not uniform, these variations give rise to stresses. Conversely, when a body is subjected to deformations, a rise or a drop in the temperature of the body may occur. Therefore, the problem of determining the displacement in a body becomes a coupled one since to know the displacement field, one needs to know the temperature distribution in the body and to know the temperature distribution one needs to know the displacement field in the body.

However, if it is assumed that the effect of the deformation field on the temperature field is negligible, the temperature field can be determined independent of the deformation field and then using the known thermal distribution, the displacement field can be obtained.

For the present problem, it is assumed that this indeed is the case. This seems reasonable since the thermoelastic coupling coefficient for most of the materials is small compared to unity. Intuitively, it can be argued that since the exciting source for the present case is thermal and since the mechanical changes are due to this, the variations in the temperature field due to the deformation changes are small. Further discussion on the conditions under which uncoupling between the temperature and the deformation fields is valid can be found in [13].

Thus for an uncoupled problem, the basic governing equations for the temperature field and the deformation field for an isotropic, linear thermoelastic solid with no body forces are given by [14]:

$$T_{1,ii} - \frac{1}{k_1}\frac{\partial T_1}{\partial t} + \frac{h_1}{c_1} = 0 \tag{2.1}$$

$$(\lambda + \mu)u_{1i,jj} + \mu u_{1j,ji} - \gamma(3\lambda + 2\mu)T_{1,i} = \rho_1 \ddot{u}_{1i} \tag{2.2}$$

where the standard indicial notation is used. In the above equations $T_1(x,y,z,t)$ and $u_1(x,y,z,t)$ are respectively the temperature and displacement fields in the solid, h_1 is the thermal source per unit volume (can be a function of space and time), k_1, c_1 and ρ_1 are, respectively, the thermal diffusion, thermal conductivity and density of the medium.

For an isotropic, linear thermoelastic liquid with no body forces, the conduction equation remains the same as Equation (2.1). However, the wave equation becomes [15]:

$$\mathcal{B}u_{2j,ji} - \mathcal{B}\beta T_{2,i} = \rho_2 \ddot{u}_2 \tag{2.3}$$

where $u_2(x,y,z,t)$ is the displacement field in the liquid and $\mathcal{B}$ is the bulk modulus, ρ_2 is the density, β is the coefficient of volume thermal expansion.

The stress-strain relations for an isotropic, homogeneous, linearly thermoelastic solid are given by the Duhamel-Neumann constitutive equations [3]:

$$\tau_{ij} = 2\mu u_{1i,j} + \lambda u_{kk}\delta_{ij} + \beta T_1 \delta_{ij} \tag{2.4}$$

and for the liquid in terms of pressure [15]:

$$\mathcal{P} = -\mathcal{B}(u_{2i,i} - \beta T_2) \quad . \tag{2.5}$$

These are the basic equations for this work. In the next two sections, when the problem is formulated mathematically, these equations are used for describing the temperature and displacement distributions in the liquid and the solid media.

Often when modeling a physical problem, certain assumptions idealizing the physical phenomenon are necessary. Here also, to make the mathematics tractable, some assumptions are made and are presented next.

Assumptions

To analyze the deformation fields in the solid and the liquid media due to absorption of laser energy by the solid, the following assumptions (apart from the uncoupling condition) are made. These make the mathematics manageable and yet yield a physically possible model.

Assumption 1: Linear elastic, isotropic, homogeneous thermal and mechanical behavior is assumed for both the media. With this assumption, the equations presented in the previous chapter can be used to describe the displacement and temperature fields in the two media.

Assumption 2: Thermal and mechanical properties of the two media are assumed to be independent of the temperature. For many solids and liquids, the variations of the properties are relatively small over a fairly wide range of temperatures. Therefore this assumption seems reasonable.

Assumption 3: The ordinary thermodynamical concept of temperature is valid for the times of interest and therefore the thermal energy distribution in the two media can be analyzed by using the macroscopic laws of heat conduction. The explanation for this assumption follows the one given by Ready [3]. The transfer of heat in conducting media is governed by the collision processes between electrons, and between electrons and the lattice phonons. The mean free time between collisions for electrons is of the order 10^{-14} to 10^{-13} seconds and since times of interest are of the order 10^{-8} to 10^{-7} seconds, many collisions will have occurred and the energy absorbed by one electron will be distributed and passed onto the lattice. Therefore, the light energy can be assumed to be transferred instantaneously into heat energy which in turn implies that temperature in the ordinary sense is a valid concept.

Assumption 4: Heat transfer to the liquid medium occurs only through conduction from the interface and there is no loss of energy to the liquid from the laser pulse as it passes through the liquid towards the solid. This implies that heating of liquid occurs only through surface heating from the interface and that there are no thermal sources in the liquid.

Assumption 5: No phase change takes place in either of the two media. This means that the total absorbed energy by the solid medium should be small enough to prevent any phase change in either medium but yet high enough to generate waves.

Assumption 6: Both the liquid and the solid media can be modelled as half-spaces when solving for the temperature distribution. As was mentioned in the first chapter, times of interest are about three times the time a longitudinal wave originated at the solid-liquid interface takes to travel across to the back surface of the solid. Since elastic waves travel at much greater speeds than the thermal waves do, the back surface of the solid remains unaffected by the thermal gradients generated in its interior for the times of interest. That this can be assumed is seen from the following argument.

The rapidity with which a material accepts and conducts thermal energy is determined by the thermal diffusivity of the material. The higher the diffusivity, higher is the depth of penetration of heat. The time required for the thermal energy to travel a specified penetration depth x is given [16] approximately by $x^2/4k$, where k is the diffusivity. Thus for metal specimens of thicknesses in the order of 0.1 cm to 1 cm, the time constant is of the order of milliseconds. On the other hand, the time that an elastic wave takes to travel across these thicknesses is of the order of microseconds. Therefore, times of interest are in the range of a few microseconds which implies that for the temperature calculations, the solid medium can safely be regarded as a half-space.

For the same reason the liquid medium is also assumed to be a half-space for temperature calculations (in fact, for the liquid medium, as k is small, thermal time constant will be higher).

Assumption 7: For the calculation of displacement field, we treat the solid as a finite medium and the liquid medium as a half-space. The reason for treating the solid as a finite medium is obvious from the fact that ultimately it is the displacement at the back surface of the solid that is actually required.

It was mentioned in the first chapter that the thickness of the liquid medium is such that the waves generated at the interface and travelling in the liquid do not have the time to get reflected at the upper surface and to interfere with the waves generated in the solid. It is for this reason that the liquid medium is assumed to be of infinite thickness.

Assumption 8: Energy lost by the solid through reradiation and convection is negligible. Since the time intervals of interest and the laser pulse duration are short as mentioned in the first chapter, this is a reasonable assumption. Even for laser energies high enough to cause phase change, the losses due to reradiation are only about 1% for short times. For further discussion on this, the reader is referred to Chapter 4 in [12].

Assumption 9: The deposition of laser energy in the solid is assumed to be distributed according to $f(t)Q_0\alpha\exp(-\alpha z)$ where $f(t)$ is a function in time describing the temporal shape of the laser pulse, Q_0 is the laser energy incident per unit area, α is the optical absorption coefficient of the solid and z is spatial coordinate along the thickness of the two media. The spatial distribution $\exp(-\alpha z)$ of laser energy corresponds to the exponential decay of the electromagnetic radiation in solids. By allowing $f(t)$ to be *arbitrary*, the displacement field due to a laser pulse of arbitrary temporal shape can be calculated. This is achieved by using the

Green's function method to be discussed later. This assumption also allows the laser interaction with the solid medium to be modelled as a thermal source in the solid.

Governing Equations

As mentioned before, if the coupling between the temperature field and the displacement field is neglected, the diffusion equation can be solved first, without the knowledge of the displacement field. The displacement equation can then be solved for the displacement field using the known temperature distribution. Also since the proposed problem is one-dimensional, temperature and displacement fields are functions only of time and the coordinate describing the thickness of the two media (i.e., z in Figure 2).

To find the temperature distribution, treating the two media as two half-spaces (Assumption 6) in contact with each other, let the coordinate system be chosen with the positive z direction along the thickness of the solid with $z = 0$ describing the interface. Thus z is positive in the solid and is negative in the liquid medium.

Let $T_1(z,t)$ be the transient temperature distribution in the solid and $T_2(z,t)$ be that in the liquid. Let ρ_1, k_1 and c_1 respectively be the density, diffusivity and conductivity of the solid medium. Let ρ_2, k_2 and c_2 be those of the liquid medium.

Within the scope of the uncoupled theory of thermoelasticity, then, $T_1(z,t)$ has to satisfy the following one-dimensional form of the heat conduction equation (2.1):

$$T_{1,zz} + \frac{Q_0\,\alpha}{c_1}\,\exp(-\alpha z)f(t) = \frac{1}{k_1}\,T_{1,t} \quad , \quad 0 \le z \le \infty \; , \; t > 0 \qquad (2.6)$$

where the interaction of the laser pulse with the solid is modelled as a thermal source in the solid (Assumption 9).

Assumption 4 implies that there is no internal heat source in the liquid medium. Therefore, $T_2(z,t)$ has to satisfy the following one-dimensional form of the heat conduction equation (2.1) with no thermal sources:

$$T_{2,zz} = \frac{1}{k_2} T_{2,t} \quad , \quad -\infty \leq z \leq 0 \; , \; t > 0 \quad . \tag{2.7}$$

The boundary conditions for the above equations are obtained as follows:

At the interface $z = 0$, temperature and heat continuities require that

$$T_1(0,t) = T_2(0,t) \tag{2.8}$$

and

$$c_1 T_{1,z}(0,t) = c_2 T_{2,z}(0,t) \quad . \tag{2.9}$$

The other two conditions come from Assumption (6) according to which, for the times of interest, the back surface of the solid and the front surface of the liquid are not affected by thermal gradients in the interior of the solid-liquid system. Therefore

$$\lim_{z \to \infty} T_1(z,t) \text{ is bounded} \tag{2.10}$$

and

$$\lim_{z \to -\infty} T_2(z,t) \text{ is bounded} \tag{2.11}$$

and the initial conditions are

$$T_1(z,0) = 0 \tag{2.12}$$

and

$$T_2(z,0) = 0 \quad . \tag{2.13}$$

The associated one-dimensional displacement equation for the solid medium obtained from Equation (2.2) is

$$u_{1,zz} = \delta T_{1,z} + \frac{1}{C_{L_1}^2} u_{1,tt} \quad , \quad 0 \leq z \leq b \; , \; t > 0 \tag{2.14}$$

where $u_1(z,t)$ is the displacement field in the solid medium, C_{L_1} is the longitudinal wave speed in the solid medium,

$$C_{L_1} = \left(\frac{\lambda + 2\mu}{\rho_1}\right)^{1/2} \quad , \tag{2.15}$$

$$\delta = \left(\frac{3\lambda + 2\mu}{\lambda + 2\mu}\right)\gamma \quad , \tag{2.16}$$

and γ is the coefficient of linear thermal expansion. The one-dimensional wave equation for the liquid obtained from Equation (2.3) is

$$u_{2,zz} = \beta T_{2,z} + \frac{1}{C_{L_2}^2}\, u_{2,tt} \quad , \quad -\infty \leq z \leq 0 \;,\; t > 0 \tag{2.17}$$

where $u_2(z,t)$ is the displacement in the liquid medium, C_{L_2} is the acoustic wave speed in the liquid medium, and

$$C_{L_2} = \left(\frac{B}{\rho_2}\right)^{1/2} \quad . \tag{2.18}$$

Imposing the displacement and stress continuities respectively at the interface, the conditions

$$u_1(0,t) = u_2(0,t) \tag{2.19}$$

and

$$\sigma^1_{zz}(0,t) = \sigma^2_{zz}(0,t) \tag{2.20}$$

are obtained.

Requiring the back surface of the solid medium to be traction free,

$$\sigma^1_{zz}(b,t) = 0 \quad . \tag{2.21}$$

Requiring that displacement be bounded as z approaches $-\infty$ gives

$$\lim_{z \to -\infty} u_2(z,t) \text{ is bounded.} \tag{2.22}$$

The initial conditions are

$$u_1(z,0) = u_{1,t}(z,0) = 0 \tag{2.23}$$

and

$$u_2(z,0) = u_{2,t}(z,0) = 0 \quad . \tag{2.24}$$

Equations (2.20) and (2.21) can be expressed in terms of displacements using the proper constitutive equations. From the one-dimensional form of the Duhamel-Neumann constitutive equations for isotropic materials (Equation (2.4)):

$$\sigma^1_{zz} = (\lambda + 2\mu)u_{1,z} - \gamma(3\lambda + 2\mu)T_1(z,t) \quad . \tag{2.25}$$

For liquid media, stress in terms of displacement is given by (Equation (2.5)):

$$\sigma^2_{zz} = \mathcal{B}\left(u_{2,z} - \beta T_2(z,t)\right) \tag{2.26}$$

since stress is the negative of pressure.

Equations (2.20) and (2.21) with the help of Equations (2.25) and (2.26) become

$$u_{1,z}(0,t) - (\delta - \varsigma\beta)T_1(0,t) = \Omega u_{2,z}(0,t) \tag{2.27}$$

and

$$u_{1,z}(b,t) = \delta T_1(b,t) \tag{2.28}$$

where

$$\varsigma = \frac{\mathcal{B}}{(\lambda + 2\mu)} \quad . \tag{2.29}$$

Making use of the conditions (2.8) through (2.13), the equations (2.6) and (2.7) are solved for the temperature distribution in the two media, in the next chapter. In Chapter 4, Equations (2.14) and (2.17) are solved for the displacement field using the boundary conditions (2.19), (2.22), (2.27) and (2.28) and the initial conditions (2.23) and (2.24).

CHAPTER 3

TEMPERATURE SOLUTION

Method of Solution

The temperature distribution in the two media, for a laser pulse of arbitrary temporal shape, is obtained in two steps. First, the function $f(t)$ which represents the laser pulse shape in time is replaced by a Dirac delta function $\delta(t)$ and the two equations (2.6) and (2.7) are solved for $T_1(z,t)$ and $T_2(z,t)$. The solutions (Green's functions in time) obtained give temperature response of the two media at position z and and time t due to a thermal source acting instantaneously at time $t = 0$. Replacing t in the solutions by $t - \tau$, then gives the temperature response due to a thermal source acting instantaneously at $t = \tau$. This follows from the translation property of the Green's functions (see [17]).

In the second step, the temperature response at any time t due to a laser pulse of arbitrary temporal shape $f(t)$ is obtained by integrating the above Green's functions multiplied by $f(\tau)$ from 0 to t.

Since the Laplace transform method is used in this work for solving differential equations, it is to be noted that only the transformed temperature solutions are needed for solving the wave equations. However, for completeness and for later analysis, the complete solution is carried out.

Temperature Solution

Temperature solutions are obtained by using the Laplace transform technique in the time variable t. The Laplace transform pairs are defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\exp(-st)dt$$

and

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \int_{a-i\infty}^{a+i\infty} F(s)\exp(st)dt$$

where a is chosen so that all the singularities of $F(s)$ are to the left of $Re(s) = a$ in the complex plane.

With this definition, the Laplace transforms of Equations (2.6) and (2.7) with $\delta(t)$ replacing $f(t)$ become

$$\bar{T}_{1,zz}(z,s) + \frac{Q_0\alpha}{c_1}\exp(-\alpha z) = \frac{s}{k_1}\bar{T}_1(z,s) \quad , \quad 0 \leq z \leq \infty \tag{3.1}$$

and

$$\bar{T}_{2,zz}(z,s) = \frac{s}{k_2}\bar{T}_2(z,s) \quad , \quad -\infty \leq z \leq 0 \tag{3.2}$$

where $T_i(z,t)$, $i = 1,2$, represents the Green's function for temperature distribution in the i^{th} medium due to an instantaneous heat source applied at time equals zero i.e., the Green's functions are the solutions of Equations (2.6) and (2.7) when the function in time in these equations is replaced by a Dirac delta function at time equals zero. $\bar{T}_i(z,s)$, $i = 1,2$, represents the corresponding Green's function in the i^{th} medium in the transformed space.

The boundary conditions (2.8) through (2.11) in the transformed space become

$$\bar{T}_1(0,s) = \bar{T}_2(0,s) \quad , \tag{3.3}$$

$$c_1\bar{T}_{1,z}(0,s) = c_2\bar{T}_{2,z}(0,s) \quad , \tag{3.4}$$

$$\lim_{z\to\infty} \bar{T}_1(z,s) \text{ is bounded} \tag{3.5}$$

and

$$\lim_{z\to-\infty} \bar{T}_2(z,s) \text{ is bounded.} \tag{3.6}$$

Equations (3.1) and (3.2) are ordinary differential equations and their solutions are given by

$$\bar{T}_1(z,s) = A_1(s)\exp\left(\sqrt{\frac{s}{k_1}}\,z\right) + B_1(s)\exp\left(-\sqrt{\frac{s}{k_1}}\,z\right) + \bar{T}_{1p}(z,s) \quad , \quad 0 \le z \le \infty \tag{3.7}$$

and

$$\bar{T}_2(z,s) = A_2(s)\exp\left(\sqrt{\frac{s}{k_2}}\,z\right) + B_2(s)\exp\left(-\sqrt{\frac{s}{k_2}}\,z\right) \quad , \quad -\infty \le z \le 0 \tag{3.8}$$

where $A_1(s)$, $A_2(s)$, $B_1(s)$ and $B_2(s)$ are to be determined from the boundary conditions. The last term on the right-hand side of the first equation above is the particular solution of the equation (3.1). Using undetermined coefficients, this is obtained as

$$\bar{T}_{1p}(z,s) = \frac{Q_0\,\alpha k_1}{c_1}\,\frac{\exp(-\alpha z)}{(s-\alpha^2 k_1)} \quad . \tag{3.9}$$

The boundary conditions (3.5) and (3.6) together with Equations (3.7), (3.8) and (3.9) require that $A_1(s)$ and $B_2(s)$ be zero. The conditions (3.3) and (3.4) then give

$$B_1(s) = \frac{PT_0}{\sqrt{s}\,(\alpha^2 k_1 - s)}\left(\alpha + \frac{c_2}{c_1}\sqrt{\frac{s}{k_2}}\right) \tag{3.10}$$

and

$$A_1(s) = B_1(s) + \frac{T_0}{s-\alpha^2 k_1} \tag{3.11}$$

where

$$P = \frac{c_1}{\frac{c_1}{\sqrt{k_1}} + \frac{c_2}{\sqrt{k_2}}} \tag{3.12}$$

and

$$T_0 = \frac{Q_0 \alpha k_1}{c_1} \quad . \tag{3.13}$$

Thus,

$$\bar{T}_1(z,s) = B_1(s)\exp\left(-\sqrt{\frac{s}{k_1}}\, z\right) + \bar{T}_{1p}(z,s) \quad , \quad 0 \le z \le \infty \tag{3.14}$$

and

$$\bar{T}_2(z,s) = A_2(s)\exp\left(\sqrt{\frac{s}{k_2}}\, z\right) \quad , \quad -\infty \le z \le 0 \tag{3.15}$$

where $B_1(s)$ and $A_2(s)$ are given by (3.11) and (3.12). Using partial fractions,

$$\frac{1}{s - \alpha^2 k_1} = \frac{1}{2\alpha\sqrt{k_1}}\left(\frac{1}{\sqrt{s} - \alpha\sqrt{k_1}} - \frac{1}{\sqrt{s} + \alpha\sqrt{k_1}}\right) \quad . \tag{3.16}$$

Substituting this in Equations (3.14) and (3.15) and using the following inverse Laplace transform formulae [18],

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{\exp(-q\sqrt{s})}{p+\sqrt{s}}\right\} &= \frac{1}{\sqrt{\pi t}}\exp\left(\frac{-q^2}{4t}\right) \\ &\quad - p\exp(pq + p^2 t)erfc\left(p\sqrt{t} + \frac{q}{2\sqrt{t}}\right) \quad , \quad q \ge 0 \quad , \end{aligned} \tag{3.17}$$

$$\mathcal{L}^{-1}\left\{\frac{\exp(-q\sqrt{s})}{\sqrt{s}(p+\sqrt{s})}\right\} = \exp(pq + q^2 t)erfc\left(p\sqrt{t} + \frac{q}{2\sqrt{t}}\right) \quad , \quad q \ge 0 \tag{3.18}$$

and

$$\mathcal{L}^{-1}\left\{\frac{1}{s-p}\right\} = \exp(pt) \quad , \tag{3.19}$$

the Green's functions in time for the temperature are obtained from Equations (3.14) and (3.15) as

$$\begin{aligned} T_1(z,t) = T_0 \exp\left(\frac{-z^2}{4k_1 t}\right) \big[&\exp(x_1^2)erfc\, x_1 \\ &+ (1 - 2P_1 n)\exp(x_2^2)erfc\, x_2\big]\, H(t) \end{aligned} \tag{3.20}$$

and

$$T_2(z,t) = T_0 P_1 \exp\left(\frac{-z^2}{4k_2 t}\right)\exp(x_3^2)erfc(x_3)H(t) \tag{3.21}$$

where $H(t)$ is the Heaviside's function and

$$x_1 = \left(\alpha\sqrt{k_1 t} - \frac{z}{\sqrt{4k_1 t}}\right) \quad , \tag{3.22}$$

$$x_2 = \left(\alpha\sqrt{k_1 t} + \frac{z}{\sqrt{4k_1 t}}\right) \quad , \tag{3.23}$$

$$x_3 = \left(\alpha\sqrt{k_1 t} - \frac{z}{\sqrt{4k_2 t}}\right) \quad , \tag{3.24}$$

$$P_1 = \frac{P}{\sqrt{k_1}} \quad , \tag{3.25}$$

and

$$n = \frac{c_2}{c_1}\sqrt{\frac{k_1}{k_2}} \quad . \tag{3.26}$$

Equations (3.20) and (3.21) give the thermal response of the two media due to an instantaneous application of thermal energy in the solid medium at time equal to zero. The thermal response due to an instantaneous heat source applied at time τ_0, i.e. when $\delta(t)$ is replaced by $\delta(t-\tau_0)$, is then obtained from the translation property of the Green's functions, by shifting t in these equations by τ_0.

Thus, the temperature distributions in the two media due to a thermal source applied instantaneously at τ_0 are given by $T_1(z, t-\tau_0)$ and $T_2(z, t-\tau_0)$.

Now, the temperature distribution $\theta_i(t)$, $i = 1, 2$, in the two media due to a laser pulse of temporal shape $f(t)$ is obtained from the superposition principle of Green's functions as follows:

$$\theta_1(z,t) = \int_0^\infty T_1(z, t-\tau_0) f(\tau_0) d\tau_0 \tag{3.27}$$

and

$$\theta_2(z,t) = \int_0^\infty T_2(z, t-\tau_0) f(\tau_0) d\tau_0 \quad . \tag{3.28}$$

Since the two Green's functions are multiplied by $H(t-\tau_0)$, the upper limit in the above integrals actually reduces to t, which physically means that the temperature distribution at a time t is only due the superposition of all the thermal sources that acted only before and at t. Any future thermal sources do not have any influence on the current temperature.

CHAPTER 4

DISPLACEMENT SOLUTION

Similar to the temperature solution, displacement solution is also obtained in two steps. In the first step, displacement due to a sudden application of laser energy at time equals zero (with the spatial distribution being an exponential decay along the thickness as given in Assumption 9, Chapter 2) is obtained. This gives a Green's function in time for the displacement. The deformation response due to an instantaneous laser source acting only at $t = \tau$ is then obtained by using the translation property of Green's functions.

In the second step using the superposition principle of the Green's functions, displacement fields in the two media due to a laser pulse of arbitrary temporal shape are obtained.

Displacement Solution

In the first step, Green's functions in time for the displacement fields in the two media are obtained by substituting for temperature terms in the wave equations (2.14) and (2.17) and in the boundary conditions (2.27) and (2.28), the Green's functions for temperature obtained in the last chapter.

Laplace transform technique is again used to obtain the required solutions. The Laplace transforms of Equations (2.14) and (2.17) (after making the above mentioned replacements) with the help of the initial conditions (2.23) and (2.24), the following differential equations are obtained:

$$\frac{d^2 \bar{w}_1}{dz^2}(z,s) = \delta \frac{d\bar{T}_1}{dz}(z,s) + \frac{s^2}{C_{L_1}^2} \bar{w}_1(z,s) \tag{4.1}$$

and

$$\frac{d^2\bar{w}_2}{dz^2}(z,s) = \beta\frac{d\bar{T}_2}{dz}(z,s) + \frac{s^2}{C_{L_2}^2}\bar{w}_2(z,s) \tag{4.2}$$

where $w_i(z,t)$, $i = 1,2$ represent the Green's functions in time for the displacements and $\bar{w}_i(z,s)$, $i = 1,2$ represent their Laplace transforms in the same order.

The boundary conditions (2.19), (2.22), (2.27) and (2.28), after replacing the temperatures in (2.27) and (2.28) by their Green's functions, upon taking Laplace transforms become:

$$\bar{w}_1(0,s) = \bar{w}_2(0,s) \quad , \tag{4.3}$$

$$\lim_{z\to-\infty} \bar{w}_2(z,s) \text{ is bounded,} \tag{4.4}$$

$$\bar{w}_{1,z}(0,s) - (\delta - \varsigma\beta)\bar{T}_1(0,2) = \varsigma\bar{w}_{2,z}(0,s) \quad , \tag{4.5}$$

and

$$\bar{w}_{1,z}(b,s) - \delta\bar{T}_1(b,s) = 0 \quad . \tag{4.6}$$

The complete solutions of (4.1) and (4.2) are given by:

$$\bar{w}_1(z,s) = d_1(s)\cosh(sz/C_{L_1}) + d_2(s)\sinh(sz/C_{L_1}) + \bar{w}_{1p}(z,s) \tag{4.7}$$

and

$$\bar{w}_2(z,s) = d_3(s)\cosh(sz/C_{L_2}) + d_4(s)\sinh(sz/C_{L_2}) + \bar{w}_{2p}(z,s) \tag{4.8}$$

where $d_i(s)$, $i = 1,2,3,4$ are to be determined from the boundary conditions (4.3) through (4.6) and $w_{1p}(z,s)$ and $w_{2p}(z,s)$ are the particular solutions of (4.1) and (4.2) respectively, found by the method of undetermined coefficients. These are

$$\bar{w}_{1p}(z,s) = \frac{-B_1(s)\delta\sqrt{(s/k_1)}}{s/k_1 - s^2/C_{L_1}^2}\exp\left(-\sqrt{\frac{s}{k_1}}\,z\right) + \frac{C_{L_1}^2\delta T_0\alpha\exp(-\alpha z)}{(s-\alpha^2 k_1)(s^2-\alpha^2 C_{L_1}^2)} \tag{4.9}$$

and

$$\bar{w}_{2p}(z,s) = \frac{\beta A_2(s)\sqrt{(s/k_2)}}{s/k_2 - s^2/C_{L_2}^2}\exp\left(\sqrt{\frac{s}{k_2}}\,z\right) \tag{4.10}$$

where $A_1(s)$ and $B_2(s)$ are as given by Equations (3.10) and (3.11).

Upon substituting (4.9) and (4.10) into the boundary conditions (4.3), (4.5) and (4.6), the following equations are obtained after some algebraic manipulations:

$$d_1(s) - d_3(s) = \bar{w}_{2p}(0,s) - \bar{w}_{1p}(0,s) = M_1(s) \quad , \tag{4.11}$$

$$\begin{aligned}\left(\frac{s}{C_{L_1}}\right) d_2(s) - \varsigma\left(\frac{s}{C_{L_2}}\right) d_4 &= \varsigma \bar{w}_{2p,z}(0,s) + (\delta - \varsigma\beta)\bar{T}_1(0,s) \\ &- \bar{w}_{1p}(0,s) = M_2(s) \quad ,\end{aligned} \tag{4.12}$$

$$\begin{aligned}\left(\frac{s}{C_{L_1}}\right) d_1(s)\sinh\left(\frac{sb}{C_{L_1}}\right) + \left(\frac{s}{C_{L_1}}\right) d_2(s)\cosh\left(\frac{sb}{C_{L_1}}\right) &= \delta\bar{T}_1(b,s) \\ &- \bar{w}_{1p,z}(b,s) = M_3(s)\end{aligned} \tag{4.13}$$

and

$$d_3(s) = d_4(s) \quad . \tag{4.14}$$

Equations (4.11), (4.12) and (4.13) are three simultaneous equations in three unknowns d_1, d_2 and d_3. Solving these,

$$d_1(s) = C_{L_1}\Omega\left[\left(\frac{\varsigma}{C_{L_2}} M_1(s) - \frac{M_2(s)}{s}\right)\cosh\frac{sb}{C_{L_1}} + \frac{M_3(s)}{s}\right] \quad , \tag{4.15}$$

$$d_2(s) = C_{L_1}\Omega\left[\left(-\frac{\varsigma}{C_{L_2}} M_1(s) + \frac{M_2(s)}{s}\right)\sinh\frac{sb}{C_{L_1}} + \frac{\varsigma}{s}\frac{C_{L_1}}{C_{L_2}} M_3(s)\right] \quad , \tag{4.16}$$

$$d_3(s) = C_{L_1}\Omega\left[-\frac{1}{C_{L_1}}\sinh\frac{sb}{C_{L_1}} + \cosh\frac{sb}{C_{L_1}}\left(\frac{M_2(s)}{s}\right) + \frac{M_3(s)}{s}\right] \tag{4.17}$$

and

$$d_4(s) = d_3(s) \tag{4.18}$$

where

$$\Omega = \frac{1}{\left(\varsigma\frac{C_{L_1}}{C_{L_2}}\cosh\frac{sb}{C_{L_1}} + \sinh\frac{sb}{C_{L_1}}\right)} \quad . \tag{4.19}$$

Substituting (4.15) and (4.16) into (4.7), and (4.17) and (4.18) into (4.8), the Green's functions in the transformed space are obtained as

$$\bar{w}_1(z,s) = \Omega \left[R_1(s) \cosh \frac{s(b-z)}{C_{L_1}} + R_2(s) \cosh \frac{sz}{C_{L_1}} + \varsigma \frac{C_{L_1}}{C_{L_2}} \sinh\left(\frac{sz}{C_{L_1}}\right) \right] + \bar{w}_{1p}(z,s) \tag{4.20}$$

and

$$\bar{w}_2(z,s) = \frac{\Omega}{2} \left[\{R_4(s) + R_3(s)\} \exp\left(\frac{sb}{C_{L_1}}\right) + \{R_4(s) - R_3(s)\} \exp\left(-\frac{sb}{C_{L_1}}\right) + 2R_2(s) \right] \exp\left(\frac{sz}{C_{L_2}}\right) + w_{2p}(z,s) \tag{4.21}$$

where

$$R_1(s) = \left[\frac{\varsigma}{C_{L_2}} M_1(s) - M_2(s) \right] C_{L_1} \quad , \tag{4.22}$$

$$R_2(s) = (M_3(s) C_{L_1})/s \quad , \tag{4.23}$$

$$R_3(s) = -M_1(s) \tag{4.24}$$

and

$$R_4(s) = \frac{M_2(s) C_{L_1}}{s} \quad . \tag{4.25}$$

The functions (including Ω) multiplying $R_i(s)$, $i = 1,2,3,4$ are either exponentials or functions of exponentials. By the second shifting theorem of Laplace transforms,

$$\mathcal{L}^{-1}\{\exp(-as)F(t)\} = F(t-a)H(t-a) \quad .$$

Therefore the only contribution of these functions is towards Heaviside's functions and a shift in the time variable. However these functions bring out clearly the wave nature of the problem, so the exact contributions of these to the displacement solution will be considered first. From Equation (4.19),

$$\Omega = \frac{1}{\varsigma \frac{C_{L_1}}{C_{L_2}} \cosh \frac{sb}{C_{L_1}} + \sinh \frac{sb}{C_{L_1}}} \quad . \tag{4.26}$$

the wave nature of the problem, so the exact contributions of these to the displacement solution will be considered first. From Equation (4.19),

$$\Omega = \frac{1}{\varsigma\frac{C_{L_1}}{C_{L_2}}\cosh\frac{sb}{C_{L_1}} + \sinh\frac{sb}{C_{L_1}}} \quad . \tag{4.26}$$

This can be rewritten as

$$\Omega = \frac{\exp(-sb/C_{L_1})}{B\left\{1+\frac{A}{B}\exp(-2sb/C_{L_1})\right\}} \tag{4.27}$$

where

$$A = \varsigma\frac{C_{L_1}}{C_{L_2}} - 1 \tag{4.28}$$

and

$$B = \varsigma\frac{C_{L_1}}{C_{L_2}} + 1 \quad . \tag{4.29}$$

Noticing that A/B is less than one, the terms inside the brackets in the denominator of the above expression for Ω can be expanded using binomial theorem as follows:

$$\begin{aligned}\Omega = \frac{2\exp(-sb/C_{L_1})}{B}&\left[1-\left(\frac{A}{B}\right)\exp\left(\frac{-2sb}{C_{L_1}}\right)\right.\\ &\left.+\left(\frac{A}{B}\right)^2\exp\left(\frac{-4sb}{C_{L_1}}\right)-\cdots\right] \quad .\end{aligned} \tag{4.30}$$

With this, substituting Ω in the expressions for the displacement of fields,

$$\begin{aligned}\bar{w}_1(s,z) = \frac{R_1(s)}{B}&\{H_{11}(z,s)+H_{12}(z,s)\}\\ &+R_2(s)\{H_{21}(z,s)+H_{22}(z,s)\}+\bar{w}_{1p}(z,s)\end{aligned} \tag{4.31}$$

and

$$\begin{aligned}\bar{w}_2(s,z) = R_3(s)H_{31}(z,s)&+\{R_4(s)+R_3(s)\}H_{32}(z,s)\\ &+\{R_4(s)-R_3(s)\}H_{33}(z,s)+\bar{w}_{2p}(z,s)\end{aligned} \tag{4.32}$$

where

$$H_{11}(z,s) = Q\exp(-sz/C_{L_1}) \quad , \tag{4.33}$$

$$H_{12}(z,s) = Q \exp\left(-s\left(\frac{b}{C_{L_1}} + \frac{b-z}{C_{L_1}}\right)\right) \quad , \tag{4.34}$$

$$H_{21}(z,s) = Q \exp\left(-s\left(\frac{b-z}{C_{L_1}}\right)\right) \quad , \tag{4.35}$$

$$H_{22}(z,s) = \frac{Q}{B} \exp\left(-s\left(\frac{b+z}{C_{L_1}}\right)\right) \quad , \tag{4.36}$$

$$H_{31}(z,s) = \frac{Q}{B} \exp\left(\frac{sz}{C_{L_2}}\right) \quad , \tag{4.37}$$

$$H_{32}(z,s) = \frac{Q}{B} \exp\left(-s\left(\frac{2b}{C_{L_1}} - \frac{z}{C_{L_2}}\right)\right) \tag{4.38}$$

and

$$H_{33}(z,s) = \frac{2Q}{B} \exp\left(-s\left(\frac{b}{C_{L_1}} - \frac{z}{C_{L_2}}\right)\right) \tag{4.39}$$

with

$$Q = \left[1 - \frac{A}{B} \exp\left(-\frac{2sb}{C_{L_1}}\right) + \frac{A^2}{B^2} \exp\left(-\frac{4sb}{C_{L_1}}\right) - \cdots\right] \quad . \tag{4.40}$$

The physical interpretation that can be obtained from the Heaviside functions will now be discussed.

$H_{11}(z,s)$ represents waves travelling to the right from $z = 0$ to $z = b$ starting at times $t = 0$, $2b/C_{L_2}$, $4b/C_{L_2}$, etc. $H_{22}(z,s)$ represents waves travelling to the right from $z = 0$ to $z = b$ starting at times $t = b/C_{L_1}$, $3b/C_{L_1}$, etc. $H_{12}(z,s)$ represents waves travelling to the left from $z = b$ to $z = 0$ starting at times $t = b/C_{L_1}$, $3b/C_{L_1}$, etc. $H_{21}(z,s)$ represents waves travelling to the left from $z = b$ to $z = 0$ at times $t = 0$, $2b/C_{L_1}$, $4b/C_{L_1}$, etc.

$H_{31}(z,s)$ represents waves travelling from $z = 0$ to the left in the liquid medium starting from $t = 0$, $2b/C_{L_1}$, $4b/C_{L_1}$, etc. $H_{32}(z,s)$ and $H_{33}(z,s)$ also represent the waves travelling from $z = 0$ to the left in the liquid medium but starting respectively at times $t = 2b/C_{L_1}$, $4b/C_{L_1}$, etc. and at $t = b/C_{L_1}$, $3b/C_{L_1}$, etc.

With this interpretation, the $R_i(s)$ terms in Equations (4.31) and (4.32) will now be inverted.

Substituting Equations (4.11) to (4.13) in Equations (4.22) to (4.25) and then making use of Equations (4.9), (4.10), (3.10) and (3.11), the following are obtained:

$$R_1(s) = \frac{-\varsigma\beta C_{L_1} P T_0}{\sqrt{k_1}} \frac{1}{s\left(\sqrt{s}+\frac{C_{L_2}}{\sqrt{k_2}}\right)} \frac{1}{\left(\sqrt{s}+\frac{C_{L_2}}{\sqrt{k_2}}\right)}$$
$$- \frac{T_0 P \delta C_{L_1}}{\sqrt{k_1}} \frac{1}{\sqrt{s}(\sqrt{s}+\alpha\sqrt{k_1})} \frac{1}{\left(s-\frac{C_{L_1}^2}{k_1}\right)}$$
$$+ \frac{T_0 \delta C_{L_1}^3}{k_1} \frac{1}{(s^2-\alpha^2 C_{L_1}^2)} \frac{1}{\left(s-\frac{C_{L_1}^2}{k_1}\right)} \tag{4.40}$$
$$- \frac{\delta T_0 \alpha\varsigma C_{L_1}^3}{C_{L_2}} \left\{ \frac{P}{\sqrt{k_1}} \frac{1}{\left(\frac{C_{L_1}^2}{k_1}-s\right)s} \left(1+\frac{C_2}{\alpha C_1}\sqrt{\frac{s}{k_2}}\right) + \frac{1}{s^2-\alpha^2 C_{L_1}^2} \right\} ,$$

$$R_2(s) = \frac{C_{L_1}\delta T_0 P}{s-\frac{C_{L_1}^2}{k_1}} \left(\alpha+\frac{C_2}{C_1}\sqrt{\frac{s}{k_2}}\right) \frac{\exp\left(-\sqrt{\frac{s}{k_1}}\, b\right)}{\sqrt{s}(\alpha^2 k_1 - s)}$$
$$+ \frac{C_{L_1}\delta T_0 \exp(-\alpha b)}{(k_1\alpha^2 - s)} \frac{s}{(C_{L_1}^2\alpha^2 - s^2)} , \tag{4.41}$$

$$R_3(s) = \frac{-B_1\delta\sqrt{\frac{s}{k_1}}}{\frac{s}{k_1}-\frac{s^2}{C_{L_1}^2}} + \frac{T_0\alpha\delta C_{L_1}^2}{(s-k_1\alpha^2)(s^2-\alpha^2 C_{L_1}^2)}$$
$$- \frac{\beta A_2\sqrt{(s/k_2)}}{\frac{s}{k_2}-\frac{s^2}{C_{L_2}^2}} \tag{4.42}$$

and