A continuum mixture theory applied to stress waves in snow

by George Edward Austiguy

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Engineering Mechanics

Montana State University

Abstract:

In avalanche control work the types of explosives and delivery methods used are primarily determined by trial and error. Understanding the propagation of stress waves in snow is a step towards eliminating some of this guesswork.

A continuum theory of mixtures is applied to model snow as a mixture of an elastic solid and an elastic fluid. Three wave types, two dilational and one rotational wave are shown to exist. Theoretical expressions are developed for the wave attenuation and propagation velocity of each of the wave types.

Numerical evaluation shows velocity and attenuation increasing with frequency for all three waves. Wave velocity increases with increasing density while attenuation decreases with increasing density for all three waves. The first dilational wave has a slow wave speed and is highly attenuated. This wave exhibits diffusive behavior at low frequencies and nondispersive behavior at high frequencies. The second dilation wave is the fastest of the three wave types and does not appreciably attenuate. Nondispersive wave behavior characterizes this wave at low and high frequencies. The rotational wave is the least attenuated of all three waves and propagates at velocities greater than that of the first dilational . wave but less than that of the second dilational wave. The rotational wave exhibits nondispersive behavior at low and high frequencies. Wave velocities and attenuation show behavior that is in agreement with existing experimental data.

A CONTINUUM MIXTURE THEORY
APPLIED TO STRESS WAVES
IN SNOW

by

George Edward Austiguy Jr.

A thesis submitted in partial fulfillment
of the requirements for the degree

of

Master of Science

in

Engineering Mechanics

MONTANA STATE UNIVERSITY
Bozeman, Montana

May 1991

Approval

of a thesis submitted by

George Edward Austiguy Jr.

This thesis has been read by each member of the thesis committee and has been found to be satisfactory regarding content, English usage, format, citations, bibliographic style, and consistency and is ready for submission to the College of Graduate Studies.

May 6, 1991 — R L Brown

Date — Chairperson, Graduate Committee

Approved for the Major Department

7 May 1991 — Theodore E. Lang

Date — Head, Major Department

Approved for the College of Graduate Studies

May 17, 1991 — Henry L Parsons

Date — Graduate Dean

STATEMENT OF PERMISSION TO USE

In presenting this thesis in partial fulfillment of the requirements for a master's degree at Montana State University, I agree that the Library shall make it available to borrowers under rules of the Library. Brief quotations from this thesis are allowable without special permission, provided that accurate acknowledgment of source is made.

Permission for extensive quotation from or reproduction of this thesis may be granted by my major professor, or in his absence, by the Dean of Libraries when, in the opinion of either, the proposed use of the material is for scholarly purposes. Any copying or use of the material in this thesis for financial gain shall not be allowed without my written permission.

Signature George E. [illegible]

Date May 7, 1991

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF FIGURES - continuued

LIST OF FIGURES - continuued

ABSTRACT

In avalanche control work the types of explosives and delivery methods used are primarily determined by trial and error. Understanding the propagation of stress waves in snow is a step towards eliminating some of this guesswork.

A continuum theory of mixtures is applied to model snow as a mixture of an elastic solid and an elastic fluid. Three wave types, two dilational and one rotational wave are shown to exist. Theoretical expressions are developed for the wave attenuation and propagation velocity of each of the wave types.

Numerical evaluation shows velocity and attenuation increasing with frequency for all three waves. Wave velocity increases with increasing density while attenuation decreases with increasing density for all three waves. The first dilational wave has a slow wave speed and is highly attenuated. This wave exhibits diffusive behavior at low frequencies and nondispersive behavior at high frequencies. The second dilation wave is the fastest of the three wave types and does not appreciably attenuate. Nondispersive wave behavior characterizes this wave at low and high frequencies. The rotational wave is the least attenuated of all three waves and propagates at velocities greater than that of the first dilational wave but less than that of the second dilational wave. The rotational wave exhibits nondispersive behavior at low and high frequencies. Wave velocities and attenuation show behavior that is in agreement with existing experimental data.

CHAPTER 1

INTRODUCTION

Snow is a material which affects the activities of life in the temperate and polar latitudes causing significant and diverse effects on the environment and society. In its capacity as a water storage source, below normal snowpacks can result in drought while above normal snowpacks can bring on flooding. Blowing and drifting snow causes traveling hazards, transportation problems, and increased loading on buildings and other structures (McKay,1981). These effects of snow can lead to destruction of property, economic hardship and loss of life. In the winter of 1976 - 77 it was estimated that snow and cold weather reduced the United States GNP by $20 billion (McKay,1981). In spite of these problems snow is an indispensable resource as a water supply for homes, livestock, wildlife, agricultural production and hydropower (McKay,1981). It is a recreational resource for activities such as skiing and snowmobiling and is used as a construction material in polar regions for roads and airstrips (McKay,1981). Avalanches are one of snow's more violent manifestations, producing powerful forces which can transport rock, soil and vegetation as well as ice and snow. When man's interests and avalanche terrain coincide, results can be devastating. Many instances are beyond human control such as the massive, earthquake triggered avalanche which fell from the summit of North Huascaran, Peru in 1970 destroying the town of Yungay and killing 20,000 people (Perla and Martinelli, 1976). However many potential disasters can be avoided with proper planning and avalanche control measures. Due to increased recreation and development in mountain areas the recorded incidence of avalanches is becoming greater and the number of people affected by avalanche activity is increasing (SNHM, 1990). Avalanche activity is a significant hazard in the western United States having negative economic effects and resulting in destruction of life and property (SNHM, 1990). Present average annual mortality rates for snow avalanches exceed those due to earthquakes and all other forms of slope failure combined (SNHM, 1990). In North America and Europe, recreational, residential, and commercial use of alpine terrain has required an evolution of avalanche control techniques in order to limit or mitigate the damage caused by avalanches. These techniques consist of a variety of procedures. One of the more active measures involves the initiation of avalanches with explosives, thereby either avoiding large avalanches or eliminating the potential of the slope to avalanche unexpectedly. This technique is widely used in ski areas and over threatened highways. Over 100,000 explosive charges are detonated annually for avalanche control (Perla, 1978). The type of explosive, size of the charge and the style of delivery are primarily determined by trial and error. Information leading to a more efficient determination of how to employ explosives in releasing avalanches is thus desirable. When an explosive charge is detonated in or closely above the snowpack, inelastic deformation in the immediate vicinity of the explosion takes place forming a crater and a region of cracks surrounding the crater. Outside of this region, inelastic deformation of the snow becomes insignificant but attenuation of the stress waves still occur through geometric spreading, porous structure effects and effects of ice structure/pore material interaction(Brown,1980). It is the propagation of stress waves in the region outside of the crater which this investigation seeks to address.

The continuum theory of mixtures is a theory of mixtures based on rational thermodynamics. The fundamental premise is that the constituents of the mixture can be modeled as superimposed continua such that each point in the mixture is occupied by a material point of each constituent. Balance statements are then postulated for each constituent modeled on those of a single continua but contain additional supply terms to account for the interactions between the constituents. In the last twenty years much effort has been devoted to preserving the generality of the theory, in particular with respect to the constitutive equations. This generality has resulted in complex and cumbersome constitutive relations which inhibit the application process. Bedford and Drumheller [1983] cite this as one of the factors why applications of the theory have been so slow in developing and that there is not a single comparison of theoretical results with experimental data in extensive reviews by Atkin and Craine [1976] and Bowen [1976]. While the main impetus is to study wave propagation in snow, it is felt that in light of the above statements the application process has merit in and of itself and is an appropriate and constructive course to pursue. As procedures used in applying the continuum theory of mixtures can be extended to mixtures other than snow, the application process has benefits outside the realm of snow mechanics.

Applications of the continuum theory of mixtures to model snow has seen relatively little attention. Decker and Brown [1983, 1985] used it to model a turbulent snow and air mixture. Adams and Brown [1989, 1990] applied a continuum mixture theory to model physical phenomena associated with snow metamorphism. There is no evidence of modern mixture theory being applied to the propagation of stress waves in snow. Bowen [1972, 1978, 1975] has investigated general wave fronts, shock waves and harmonic plane waves in binary mixtures of elastic materials, but these investigations have remained on the theoretical level. Atkins [1968] examined the propagation of small amplitude waves for a fictitious mixture of an isotropic elastic solid and an inviscid fluid with an early version of a continuum mixture theory. Brown [1980] studied plastic waves in snow but did not employ a mixture theory formulation in his work. Johnson [1978] in his Ph.D. thesis investigated elastic stress waves in snow using an early porous media model developed by Biot. Biot's theory was specifically formulated to model the linear elastic behavior of a fluid saturated porous media rather than being a specialized case of a more general mixture theory. While his work was a significant contribution to porous media modeling there exists certain technical problems with it (Bowen and Wright, 1972). Specifically the constitutive relations for the momentum supplies do not satisfy certain thermodynamic constraints.

This investigation seeks to determine the propagating characteristics of stress waves in snow by using the continuum theory of mixtures to model snow as an isothermal mixture of an elastic ice frame and an inviscid fluid. Because the effects of heat conduction are most pronounced at high frequencies (Atkins, 1968) results for the low frequency regime are the most physically significant. A brief introduction to the continuum theory of mixtures is presented in Chapter 2. Because this is an isothermal model the thermodynamics of mixtures will not be addressed, although results of the second law will be used in implementing the theory. In Chapter 3 the specialized form of the mixture theory equations are presented for a binary mixture which is applicable to snow. The work which follows develops the relationships which govern propagation velocities and attenuation of a harmonic wave for one dimension in the low and high frequency regimes. Finally the results are presented in a graphical form.

CHAPTER 2

THE CONTINUUM THEORY OF MIXTURES

In this chapter the basic definitions and the development of the general continuum theory of mixtures is presented. For more detail on the development of mixture theory the reader is referred to Bowen [1976].

Tensors are denoted by boldface uppercase format while vectors are represented by lower case bold type and scalars as plain text.

Kinematics

A mixture is defined as a body $\mathfrak{B}$ consisting of a combination of different materials. Each different material is considered a body in its own right and is denoted as $\mathfrak{B}_a$; a = 1, 2, ..., N, where N is the total number of materials in the mixture. Each body $\mathfrak{B}_a$ is called a phase or constituent. For every phase $\mathfrak{B}_a$ a fixed but otherwise arbitrary reference configuration and a motion can be assigned. In describing the motion of a continuum there are four formulations which are commonly used (Malvern,1969). The material description, the Lagrangian description, the spatial description and the relative description. The Lagrangian and the spatial descriptions are the ones commonly used in the theory of elasticity and thus will be used in this development. The Lagrangian formulation is in terms of the undeformed configuration which is customarily used as the reference configuration. The spatial formulation is in terms of the deformed configuration. In the reference position the particle X_a occupies the position $\mathbf{X}_a$ while in the deformed configuration the particle X_a at time t has the position:

$$\mathbf{x}_a = \chi_a(\mathbf{X}_a,t) \tag{1}$$

Where $\mathbf{X}_a$ is the coordinate in the reference configuration of a particle in the ath body and χ_a is the deformation function for the ath body. The deformation function χ_a is assumed invertible so that:

$$\mathbf{X}_a = \chi_a^{-1}(\mathbf{x}_a,t) \tag{2}$$

This relationship is shown in Figure 1

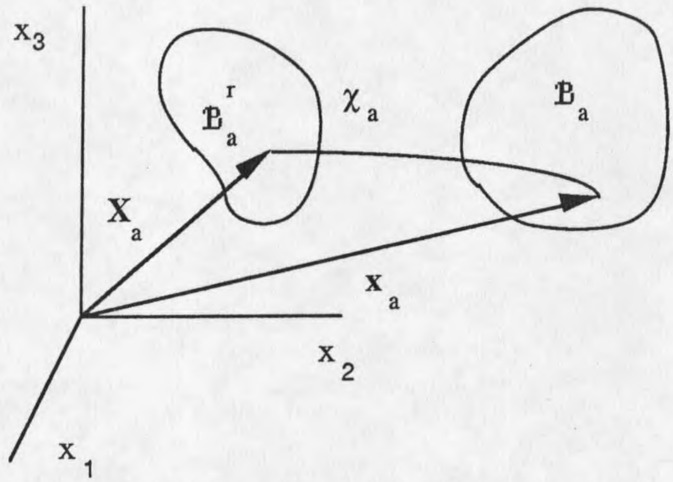

Figure 1: Mapping of the deformation function.

In Figure 1 the superscript r denotes the reference state. The velocity and acceleration of the particle $\mathbf{X}_a$ at time t are defined by:

$$\acute{\mathbf{x}}_a = \frac{\partial \chi_a(\mathbf{X}_a,t)}{\partial t} \qquad (3)$$

$$\ddot{\mathbf{x}}_a = \frac{\partial^2 \chi_a(\mathbf{X}_a,t)}{\partial t^2} \qquad (4)$$

The prime is indicative of the material derivative following the motion of the ath constituent. In a mixture of N constituents the bodies $\mathcal{B}_a$, a = 1, 2,..., N can occupy the same portions of space. It is assumed that each spatial position $\mathbf{x}$ is occupied by a particle from each constituent. This is shown for N=2 in Figure 2.

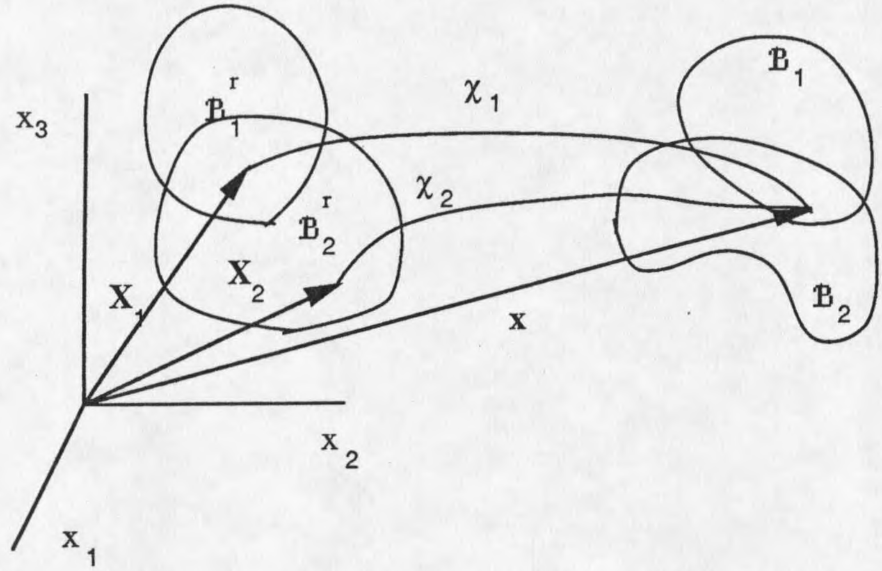

Figure 2: Constituent spatial position.

The density for the mixture is defined by:

$$\rho = \sum_{a=1}^{N} \rho_a \tag{5}$$

where ρ_a is the partial density with the definition:

$$\rho_a = \phi_a \gamma_a \tag{6}$$

ϕ_a is the volume fraction of the ath constituent and γ_a is the material density of the ath constituent. A more general development considers the volume fraction of each constituent as an independent kinematic variable and generates additional balance equations to govern changes in the volume fraction. In the development presented here the volume fraction is considered constant. For the more general development see papers by Bowen [1982] or Passman, Nunziato, and Walsh [1984]. The velocity of the mixture is defined as the mass weighted average of the constituent velocities.

$$\rho \dot{x} = \sum_{a=1}^{N} \rho_a \acute{x}_a \quad (7)$$

The diffusion velocity of the ath constituent is defined as:

$$u_a = \acute{x}_a - \dot{x} \quad (8)$$

In the Lagrangian coordinates the deformation is described in terms of the deformation gradient of X_a at time t and is given by:

$$F_a = x_a \overleftarrow{\nabla} = \chi_a(X_a,t)\overleftarrow{\nabla} \quad (9)$$

∇ is the gradient operator with respect to the Lagrangian coordinates X_a . Eq.(9) has the inverse:

$$\mathbf{F}_a^{-1} = \chi_a^{-1}(\mathbf{x}_a,t)\overleftarrow{\nabla}_x = \mathbf{X}_a\overleftarrow{\nabla}_x \tag{10}$$

Where ∇_x is the gradient operator with respect to the deformed coordinates $\mathbf{x}_a$. The velocity gradient for the ath constituent is defined in terms of the spatial coordinates as:

$$\mathbf{L}_a = \dot{\mathbf{x}}_a(\mathbf{x},t)\overleftarrow{\nabla}_x \tag{11}$$

This can be written in terms of the deformation gradient as:

$$\mathbf{L}_a = \dot{\mathbf{F}}_a\mathbf{F}_a^{-1} \tag{12}$$

<u>Mass Balance Equation</u>

In mixture theory balance principles follow what is known as the Mixture Balance Principle. This principle states that when the statement of balance for each constituent is summed over all the constituents, the sum must have the same form as the balance equation for a single constituent continuum. For a fixed spatial volume V the global balance of mass for the ath constituent is postulated as:

$$\frac{\partial}{\partial t}\left(\int_V \rho_a dV\right) = -\int_{\partial V} \rho_a \dot{\mathbf{x}}_a \cdot \mathbf{n}\, ds + \int_V \hat{c}_a dV \tag{13}$$

This states that the rate of change in mass for the ath constituent contained in the region is due to the flux of mass across the boundary, plus the mass supply from the other constituents. In this balance statement the term $\hat{c}_a$ is called the mass supply term for the ath constituent and represents the rate at which the ath constituent is gaining mass from the other constituents. Thus if there are no chemical reactions or phase changes $\hat{c}_a = 0$.

The balance statement for the mixture is:

$$\frac{\partial}{\partial t}\left(\int_V \rho dV\right) = -\int_{\partial V} \rho \dot{x} \cdot n \, ds \qquad (14)$$

Note that Eq.(14) has the same form as that for a single constituent material. Physically it states that the rate of change of mass for the mixture is due to the mass flux across the boundary. Thus there is no net mass production in the mixture. By applying the divergence theorem the local form of the balance statements can be obtained for an arbitrary fixed volume. For the ath constituent this yields:

$$\frac{\partial \rho_a}{\partial t} + \vec{\nabla} \cdot (\rho_a \dot{x}_a) = \hat{c}_a \qquad (15)$$

and for the mixture:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \dot{x}) = 0 \tag{16}$$

Finally, when Eq.(14) is summed over N constituents and Eq.(5), Eq.(7) and Eq.(15) are considered, the mixture balance principle requires that:

$$\sum_{a=1}^{N} \hat{c}_a = 0 \tag{17}$$

Balance of Momentum

The balance of momentum equations take the form of balance statements for linear momentum and angular momentum for both the mixture and the constituents. For a mixture of two or more materials it is advantageous to derive the momentum balance statements on a given spatial domain (Malvern,1969). Thus consider the fixed volume V in space represented in Figure 3.

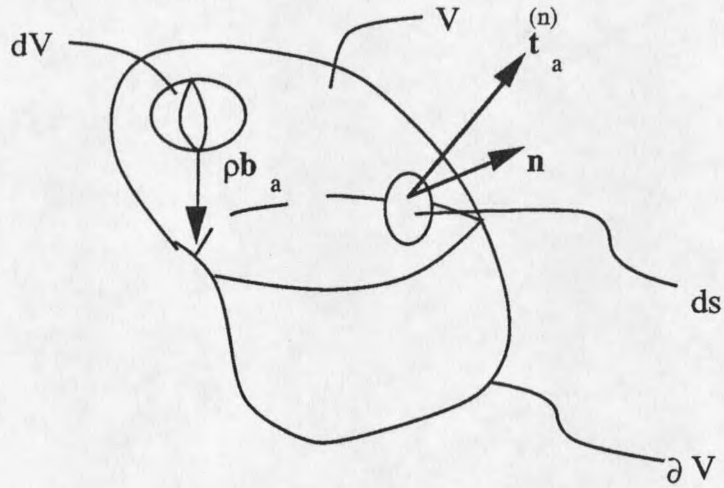

Figure 3: Fixed volume in space.

The balance of linear momentum for the ath constituent is postulated as:

$$\frac{\partial}{\partial t}\left(\int_V \rho_a \dot{\mathbf{x}}_a dV\right) = -\int_{\partial V} \{\rho_a \dot{\mathbf{x}}_a(\dot{\mathbf{x}}_a \cdot \mathbf{n}) - \mathbf{t}_a^{(n)}\} ds$$

$$+ \int_V (\rho_a \mathbf{b}_a + \hat{\mathbf{p}}_a + \hat{c}_a \dot{\mathbf{x}}_a) dV \tag{18}$$

The first term on the right hand side accounts for the momentum flux across the boundary ∂V plus the contact force (i.e. surface loading) on the ath constituent at the boundary ∂V, where $\mathbf{t}_a^{(n)}$ is the stress vector acting on the ath constituent. The second term on the right hand side accounts for the body forces acting on the ath constituent plus the complete local interaction force on the ath constituent from the other constituents in V. $\mathbf{b}_a$ is the body force acting on the ath constituent, $\hat{\mathbf{p}}_a$ is called the momentum supply, which represents interactions with other phases and $\hat{c}_a\dot{\mathbf{x}}_a$ represents the change in momentum due to a change in mass for the ath constituent. Substituting $\mathbf{t}_a^{(n)} = \mathbf{n} \circ \mathbf{T}_a$ for the stress vector allows application of the divergence theorem and consideration of an arbitrary volume yields the local form of Eq.(18). Finally making use of Eq.(15) yields :

$$\rho_a\ddot{\mathbf{x}}_a = \vec{\nabla}\cdot\mathbf{T}_a + \hat{\mathbf{p}}_a + \rho_a\mathbf{b}_a \tag{19}$$

The linear momentum balance statement for the mixture is postulated as:

$$\rho\ddot{\mathbf{x}} = \vec{\nabla}\cdot\mathbf{T} + \rho\mathbf{b} \tag{20}$$

In consideration of Eq.(19) and Eq.(20) the summation rule of the mixture balance principle requires:

$$\mathbf{T} = \sum_{a=1}^{N} \{ \mathbf{T}_a - \rho_a(\mathbf{u}_a\mathbf{u}_a) \} \tag{21}$$

$$\sum_{a=1}^{N} \hat{\mathbf{p}}_a + \hat{\mathbf{c}}_a\mathbf{u}_a = 0 \tag{22}$$

The balance of moment of momentum for the ath constituent is postulated as:

$$\frac{\partial}{\partial t}\int_V \mathbf{x}\times\rho_a\dot{\mathbf{x}}_a\,dV = -\int_{\partial V}(\mathbf{x}\times\rho_a\dot{\mathbf{x}}_a)(\dot{\mathbf{x}}_a\cdot\mathbf{n})\,ds$$

$$+\int_{\partial V}\mathbf{x}\times\mathbf{t}_a^{(n)}\,ds + \int_V\{\mathbf{x}\times(\rho_a\mathbf{b}+\hat{\mathbf{p}}_a+\hat{c}_a\dot{\mathbf{x}}_a)+\hat{\mathbf{m}}_a\}\,dV \qquad (23)$$

The term $\hat{\mathbf{m}}_a$ is called the moment of momentum supply vector. The term $\int_V\{\mathbf{x}\times(\hat{\mathbf{p}}_a+\hat{c}_a{}'\mathbf{x}_a)+\hat{\mathbf{m}}_a\}\,dV$ accounts for the moment due to the local interaction of the ath constituent with the other constituents in V. Using the divergence theorem the flux across the boundary can be written as:

$$-\int_{\partial V}(\mathbf{x}\times\rho_a\dot{\mathbf{x}}_a)(\dot{\mathbf{x}}_a\cdot\mathbf{n})\,ds = -\int_V\{[(\mathbf{x}\times\rho_a\dot{\mathbf{x}}_a)\dot{\mathbf{x}}_a]\cdot\overleftarrow{\nabla}\}dV \qquad (24)$$

Where the definition of the tensor product has been used. Applying the relationship $\mathbf{t}_a^{(n)}=\mathbf{n}\cdot\mathrm{T}_a$ and the tensor identities:

$$\mathbf{v}\cdot\mathbf{A}=\mathbf{A}^{T}\cdot\mathbf{v} \qquad (25)$$

$$(\mathbf{x}\times\mathbf{A})\cdot\mathbf{v}=(\mathbf{x}\times\mathbf{A}\cdot\mathbf{v}) \qquad (26)$$

Where A is any tensor and v is any vector, the second surface integral in Eq.(23) can be written:

$$\int_{\partial V} (\mathbf{x} \times \mathbf{t}_a^{(n)})\, ds = \int_{\partial V} \{(\mathbf{x} \times \mathbf{T}_a^T)\cdot\mathbf{n}\}\, ds \tag{27}$$

Then applying the divergence theorem transforms it to a volume integral yielding:

$$\int_{\partial V} \{(\mathbf{x} \times \mathbf{T}_a^T)\cdot\mathbf{n}\}\, ds = \int_{V} \{(\mathbf{x} \times \mathbf{T}_a^T)\cdot\overleftarrow{\nabla}\}\, dV \tag{28}$$

All terms in Eq.(23) can now be written within a volume integral Since Eq.(23) is valid for an arbitrary volume the integral must vanish. Therefore:

$$\frac{\partial}{\partial t}(\mathbf{x} \times \rho_a \dot{\mathbf{x}}_a) = -\{(\mathbf{x} \times \rho_a \dot{\mathbf{x}}_a)\dot{\mathbf{x}}_a\}\cdot\overleftarrow{\nabla} + (\mathbf{x} \times \mathbf{T}_a^T)\cdot\overleftarrow{\nabla} + \mathbf{x} \times (\rho_a \mathbf{b} + \widehat{\mathbf{p}}_a + \widehat{c}_a \dot{\mathbf{x}}_a) + \widehat{\mathbf{m}}_a \tag{29}$$

The second term on the right hand side can be rewritten using the identity:

$$(\mathbf{x} \times \mathbf{T}_a^T)\cdot\overleftarrow{\nabla} = \mathbf{x} \times (\mathbf{T}_a^T\cdot\overleftarrow{\nabla}) + \mathbf{t}_{a_A} \tag{30}$$

The vector $\mathbf{t}_{a_A}$ has the Cartesian form:

$$\mathbf{t}_{a_A} = (T_{a_{23}} - T_{a_{32}})\mathbf{e}_1 + (T_{a_{31}} - T_{a_{13}})\mathbf{e}_2 + (T_{a_{12}} - T_{a_{21}})\mathbf{e}_3 \tag{31}$$

Where $\mathbf{e}_i$, i =1,2,3 are the unit vectors along the coordinate axes. Then differentiating the first term in Eq.(29) and reorganizing yields:

$$\begin{aligned} &\mathbf{x} \times \{\rho_a\ddot{\mathbf{x}}_a - \vec{\nabla}\cdot\mathbf{T}_a - \hat{\mathbf{p}}_a - \rho_a\mathbf{b}\} \\ &+\mathbf{x} \times \{\frac{\partial\rho_a}{\partial t}\dot{\mathbf{x}}_a + (\vec{\nabla}\cdot\rho_a\dot{\mathbf{x}}_a)\dot{\mathbf{x}}_a - \hat{c}_a\dot{\mathbf{x}}_a\} \\ &\qquad = \mathbf{t}_{a_A} + \widehat{\mathbf{m}}_a \end{aligned} \tag{32}$$

Considering Eq.(15) and Eq.(19) it is evident that the left hand side is zero leaving :

$$\mathbf{t}_{a_A} + \widehat{\mathbf{m}}_a = 0 \tag{33}$$

A more convenient form of (33) is:

$$\widehat{\mathbf{M}}_a = \mathbf{T}_a - \mathbf{T}_a^T \tag{34}$$

In this expression $\widehat{\mathbf{M}}_a$ is a skew symmetric linear transformation with the components:

$$\widehat{M}_{a11} = \widehat{M}_{a22} = \widehat{M}_{a33} = 0$$
$$\widehat{M}_{a32} = -\widehat{M}_{a23} = \widehat{m}_{a1}$$
$$\widehat{M}_{a13} = -\widehat{M}_{a31} = \widehat{m}_{a2}$$
$$\widehat{M}_{a21} = -\widehat{M}_{a12} = \widehat{m}_{a3} \tag{35}$$

Eq.(34) is taken to be the axiom of the balance of moment of momentum for the ath constituent and shows that the stress tensor is not symmetric, unless of course, $\widehat{\mathbf{M}}_a$ is zero. Local representation of the moment of momentum balance statement for the mixture is given by:

$$\rho\overline{\mathbf{x} \times \dot{\mathbf{x}}} = \vec{\nabla}\cdot(\mathbf{x} \times \mathbb{T}) + \rho\mathbf{x} \times \mathbb{b} \tag{36}$$

Use of the same argument yields the axiom of the balance of the moment of momentum for the mixture:

$$\mathbb{T} = \mathbb{T}^T. \tag{37}$$

Thus the stress tensor for the mixture is symmetric even though the constituent stress tensor is not. The mixture balance principle requires the following summation rule.

$$\sum_{a=1}^{N} \widehat{M}_a = 0 \tag{38}$$

Summary

As a summary and for ease of reference the field equations and summation constraints which result from the continuum theory of mixtures are collected and presented below.

Balance of Mass

$$\frac{\partial \rho_a}{\partial t} + \vec{\nabla} \cdot (\rho_a \dot{x}_a) = \hat{c}_a \qquad \text{Constituent} \tag{39}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \dot{x}) = 0 \qquad \text{Mixture} \tag{40}$$

Balance of Linear Momentum

$$\rho_a \ddot{x}_a = \vec{\nabla} \cdot T_a + \hat{p}_a + \rho_a b_a \qquad \text{Constituent} \tag{41}$$

$$\rho \ddot{\mathbf{x}} = \vec{\nabla} \cdot \mathbf{T} + \rho \mathbf{b} \qquad \text{Mixture} \qquad (42)$$

Balance of Moment of Momentum

$$\widehat{\mathbf{M}}_a = \mathbf{T}_a - \mathbf{T}_a^T \qquad \text{Constituent} \qquad (43)$$

$$\mathbf{T} = \mathbf{T}^T \qquad \text{Mixture} \qquad (44)$$

Summation Constraints

$$\sum_{a=1}^{N} \hat{c}_a = 0 \qquad (45)$$

$$\mathbf{T} = \sum_{a=1}^{N} \{ \mathbf{T}_a - \rho_a(\mathbf{u}_a \mathbf{u}_a) \} \qquad (46)$$

$$\sum_{a=1}^{N} \widehat{\mathbf{p}}_a + \widehat{\mathbf{c}}_a \mathbf{u}_a = 0 \qquad (47)$$

$$\sum_{a=1}^{N} \widehat{\mathbf{M}}_a = 0 \qquad (48)$$

CHAPTER 3

APPLICATION TO SNOW

Snow may be modeled as a binary mixture consisting of a solid phase and a fluid phase. The mixture is comprised of an ice matrix or skeleton in which the void spaces are filled with air. Thus the solid phase is represented by the ice matrix while the fluid phase is represented by the air. In general snow is considered to be an elastic-visco-plastic material. However when deformation occurs rapidly (where the yield stress is not exceeded) and then stops inelastic strains do not have time to form appreciably and thus are small enough to be negligible (private communication, R.L. Brown, 1991). Thus for high strain rates of short duration the snow responds in a linear elastic type manner. Langham [1981] assures us that snow may be deformed elastically when subjected to a small load applied over a short duration of time. Mellor [1977] further states that: "A rapidly generated pulse of stress or strain propagates as an elastic wave when the amplitude is less than the failure stress or the failure strain of the material." For snow this is not entirely true. While the material may respond in an elastic type manner when inelastic strains are negligible, the wave does attenuate due to porous structure effects and interaction of the ice and fluid phases (Brown, 1980). Thus the wave is not a true elastic wave in the strictest sense. Mellor goes on to say that for the case of snow :"Maximum sustainable amplitude for an elastic wave probably corresponds quite closely to the uniaxial strength of snow for one dimensional propagation in a long bar and to the collapse strength of a wide layer for plane wave propagation". Thus it is apparent that snow can behave elastically under certain types of loading. It is desirable to have as simple a formulation as is reasonable. To this end the snow is assumed to be of a simple form, i.e. well sintered grains of equal size with homogeneous and isotropic properties. As a further simplification chemical reactions are omitted and the effects of body forces are neglected. It will be convenient to express the field equations in terms of the displacements. The displacement of the ath constituent is defined as:

$$\mathbf{w}_a = \mathbf{x}_a - \mathbf{X}_a \tag{49}$$

Using the standard linearization process from the theory of elasticity, the equations of motion for the ath constituent take the form:

$$\rho_{aR}\frac{\partial^2 \mathbf{w}_a}{\partial t^2} = \vec{\nabla}\cdot\mathbf{T}_a + \hat{\mathbf{p}}_a \tag{50}$$

Or in terms of the solid and gas phase:

$$\rho_{sR}\frac{\partial^2 \mathbf{w}_s}{\partial t^2} = \vec{\nabla}\cdot\mathbf{T}_s + \hat{\mathbf{p}}_s \tag{51}$$

$$\rho_{gR}\frac{\partial^2 \mathbf{w}_g}{\partial t^2} = \vec{\nabla}\cdot\mathbf{T}_g + \hat{\mathbf{p}}_g \tag{52}$$

Where the subscripts g and s are labels for the gas and the solid phase, respectively, and the subscript R denotes the reference state. Constitutive equations for the stress tensors and the momentum supply vector are required to put the equations of motion in a closed form. The derivations of the constitutive equations are both tedious and complex, adding little insight to the application process. Thus they will simply be presented. For a rigorous development of the constitutive equations presented here, the reader is referred to Bowen [1976]. The linearized version of the constitutive equations are:

$$\mathbf{T}_s = -\Pi_{sR}\mathbf{I}+(\sigma_{sg}+\lambda_{gs})(\mathrm{tr}\mathbf{E}_g)\mathbf{I}+(\lambda_s-\sigma_{gs})(\mathrm{tr}\mathbf{E}_s)\mathbf{I}+2\mu_s\mathbf{E}_s \tag{53}$$

$$\mathbf{T}_g = -\Pi_{gR}\mathbf{I}-(\sigma_{sg}-\lambda_g)(\mathrm{tr}\mathbf{E}_g)\mathbf{I}+(\sigma_{gs}+\lambda_{gs})(\mathrm{tr}\mathbf{E}_s)\mathbf{I} \tag{54}$$

$$\hat{\mathbf{p}}_g = -\hat{\mathbf{p}}_s = \sigma_{sg}\vec{\nabla}(\mathrm{tr}\mathbf{E}_g) - \sigma_{gs}\vec{\nabla}(\mathrm{tr}\mathbf{E}_s)-\xi\left(\frac{\partial \mathbf{w}_g}{\partial t} - \frac{\partial \mathbf{w}_s}{\partial t}\right) \tag{55}$$

Where $\mathbf{E}_s$ is the infinitesimal strain tensor for the solid and $\mathbf{E}_g$ is the infinitesimal strain tensor for the gas.

$$\mathbf{E}_s = 1/2(\mathbf{w}_s\overleftarrow{\nabla}+\overrightarrow{\nabla}\mathbf{w}_s) \qquad (56)$$

$$\mathbf{E}_g = 1/2(\mathbf{w}_g\overleftarrow{\nabla}+\overrightarrow{\nabla}\mathbf{w}_g) \qquad (57)$$

$\mathbf{I}$ is the identity tensor, Π_{g_R} and Π_{s_R} are constants which represent the pressure in each constituent when there is no strain measured relative to the reference state. σ_{sg}, σ_{gs}, λ_{gs}, μ_s, λ_g, λ_s, are material constants. σ_{sg} and σ_{gs} are coupling coefficients which arise because of a dependence of the momentum supplies on the strain gradients. They account for the local interaction of the solid and the fluid that occurs even in static situations. The term $(\lambda_s - \sigma_{gs})$ is analogous to the lame parameter λ in elasticity. μ_s is the shear modulus for the solid phase. $(\sigma_{sg}+\lambda_{gs})$ accounts for the dependence of the stress on the solid due to the strain of the fluid. $(\sigma_{gs}+\lambda_{gs})$ accounts for the stress on the fluid due to the strain of the solid. The term $(\lambda_g-\sigma_{sg})$ is related to the modulus of elasticity for the fluid. ξ is the Stokes drag coefficient and arises due to the momentum supplies dependence on the relative velocities. ξ accounts for the drag force between the constituents due to the relative motion. It is a thermodynamic result that :

$$\xi > 0 \tag{58}$$

To insure that the isothermal strain energy of the mixture is positive definite (Bowen, 1976) λ_g, λ_s, λ_{gs} and μ_s must satisfy the following inequalities:

$$\lambda_g > 0 \tag{59}$$

$$\mu_s > 0 \tag{60}$$

$$\lambda_g(\lambda_s + \frac{2}{3}\mu_s) > \lambda_{gs}^{\;2} \tag{61}$$

Substituting the constituent equations and using the identity:

$$\mathrm{tr}\mathbb{E}_a = \vec{\nabla}\cdot \mathbf{w}_a \tag{62}$$

yields the isothermal linear equations of motion for the solid and gas phases.

$$\rho_{SR}\frac{\partial^2 \mathbf{W}_s}{\partial t^2} = (\lambda_s+\mu_s)\vec{\nabla}(\vec{\nabla}\cdot\mathbf{w}_s) + \mu_s\vec{\nabla}\cdot(\mathbf{w}_s\overleftarrow{\nabla})$$

$$+ \lambda_{gs}\vec{\nabla}(\vec{\nabla}\cdot\mathbf{w}_g) + \xi(\frac{\partial \mathbf{w}_g}{\partial t} - \frac{\partial \mathbf{w}_s}{\partial t}) \tag{63}$$