The combinatorial theory of single-elimination tournaments
by Christopher Todd Edwards

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Statistics
Montana State University

Abstract:
This thesis is about single-elimination tournaments. These tournaments are popularly used in sporting events, but are also used in paired comparison procedures when the number of treatments is too large to use a round-robin tournament. The nature of this work is experimental design, and thus can help researchers to clarify pre-experiment issues.

Combinatorial methods are used to generate and count the distinct tournament structures that exist. Recursive formulas are derived. Various criteria for "best" are considered. A tournament draw is said to be ordered when the probabilities of teams winning the tournament are ordered by the relative strengths of the teams. Numerous theorems are given concerning ordered tournaments, subtournaments, and "best" tournaments. The results developed apply to any number of teams, not just when the number of teams is a power of two. Assuming a transitive preference structure for the pairwise probabilities of teams beating each other, results on ordered tournaments and "best" tournaments are given, some of which make use of optimization routines on the computer. Results show that there exist categories of ordered tournaments which are best under certain criteria. It is proven that ordered tournament draws can be generated by iteratively combining subtournaments in just two specific ways.

One important conclusion is that the popular "seeded" tournament is not optimal under several reasonable criteria, and hence many tournaments currently conducted, such as high school basketball tournaments, are perhaps inappropriate.

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A thesis submitted in partial fulfillment of the
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Christopher Todd Edwards

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July 18, 1991
Date

Warren W. Esty
Chairperson, Graduate Committee

Approved for the Major Department

7/18/91
Date

K J Tiahrt
Head, Major Department

Approved for the College of Graduate Studies

July 25, 1991
Date

Henry L Parsons
Graduate Dean

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ABSTRACT

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CHAPTER 1

INTRODUCTION

Frequently, a researcher wants to choose the best treatment from a set of treatments. In situations where the treatments can only be compared in pairs, paired comparison techniques are well suited to choose the best treatment. A common example of such a situation is a consumer preference study, where people indicate pairwise preferences for a set of products. A similar situation occurs in sporting events where the best team or player is chosen from a set of teams or players. For convenience, we will use in this thesis the terminology of sports; however, the reader should be aware that all results apply to any situation where paired comparisons are appropriate.

A *tournament* is a rule which specifies how the teams or players are to be compared to choose a winner or winners. In the sports setting, it is customary to use the terms *games*, *competitions,* and *contests* interchangeably to represent experimental units. Similarly, the terms *teams*, *players*, *competitors,* and *contestants* are interchangeable and represent the treatments. We assume in this thesis that games always result in either a win or a loss; ties are not allowed. Further, the score or the size of the victory in a contest is not considered in future contests. The rule for a tournament may also include the method of awarding additional *places*, e.g. second place, third place, etc. Nevertheless, many tournaments are only concerned with choosing the winner, the first-place team.

The tournament most discussed in the statistical literature is the *round-robin tournament*, where every team competes once with every other team. The games specified in a round-robin tournament can be played in any order; the results of prior games do not influence the future games to be played. In *repeated round-robin tournaments*, each pair of teams plays more than one game. *Knockout tournaments* differ from round-robin tournaments in that not all possible pairs of games occur. The games to be played depend on the results of prior games. In a knockout tournament, games are played until all but one player have been "knocked out", or eliminated. The games must be played in a specific order because the results of the earlier games designate the contestants in later games; a player who is eliminated does not compete again. The simplest example of a knockout tournament, and the topic of this thesis, is the single-elimination tournament.

Definition 1.1. A *single-elimination tournament* is a knockout tournament where teams are eliminated after losing one game, and game winners continue to play until all but one team have been eliminated. This single remaining team is the winner of the tournament. We will discuss only single-elimination tournaments in which the pairings of game winners do not depend upon which contestants win, and are determined in advance of the outcomes of the games.

An example of another knockout tournament is the *l-elimination* tournament, in which players are knocked out after l losses, such as in the well-known double-elimination tournament.

Knockout tournaments require fewer games than round-robin tournaments. For a round-robin tournament with t teams, $\binom{t}{2}$ games are played, while in a single-elimination tournament with t teams, only $t-1$ games are played. Thus, if comparisons, or games, are expensive or time-consuming, or the number of teams is large, a round-robin tournament may be infeasible. On the other hand, one upset in a knockout tournament may cause the strongest team to be eliminated. These two conflicting concerns must be balanced wisely. Often, *repeated knockout tournaments* are conducted, where each pair of teams plays more than one game against each other, for example, as in a "best-two-out-of-three" competition. In a repeated knockout tournament, an upset is less likely since more than one victory is required before a team advances in the tournament. Once the decision has been made to use a single-elimination tournament instead of an alternative design, some criteria must be used to choose a desirable scheme.

The goals of this dissertation are to develop notation and terminology for tournaments, to count the number of elements in certain classes of tournaments, and to explore various properties of tournaments.

Terminology

As in any subject, basic terminology must be addressed first. Some, but not all, of the terminology and notation used in the literature is standard. In this thesis, we have taken care to be as consistent as possible with the standard terminology. For convenience, and because the topic of this thesis is such tournaments, hereafter the term *tournament* will refer exclusively to a single-elimination tournament.

Tournaments are specified by the initial games to be played and the rules for pairing the winners in the following games. It is sometimes convenient to refer to the number of rounds in a tournament. The number of *rounds* in a tournament is the maximum number of games that any of the teams must play to win the tournament. In a tournament with k rounds, the last game is round k. The game(s) that produce the team(s) paired in the final round constitute the game(s) in round $k-1$. A *bye* occurs when a team does not play in a round. In many tournaments, some teams will get a bye in earlier rounds. In tournaments with byes, not all teams will play in the first round. A popular type of tournament is the *classic* tournament, which we define below.

Definition 1.2. A *classic* tournament is a tournament where the number of teams is a power of two and there are no byes.

One of the most basic distinctions we make concerning tournaments is that between tournament structures and tournament draws.

Definition 1.3. A *tournament structure* is that part of a tournament rule which specifies *how* the contestants will be paired without specifying *which* contestants will be paired.

The tournament structure shows the method in which the winners of games will be paired with other winners or with teams receiving byes. An example of a tournament structure, displayed in Figure 1 below, is a five-team tournament where two teams play in the first round and the other three teams receive first-round byes. Two of the teams with byes will play in one of the second-round games. The other second-round game will be between the remaining team with a bye and the winner of the first-round game. The two winners of the second-round games will play the third-round game, which will determine the winner of the tournament. The first two horizontal lines on the left in Figure 1 indicate the first-round game. Each pair of horizontal lines connected by a vertical line represents a game. The horizontal line on the right represents the winner of the tournament. Horizontal lines with open left ends represent the initial locations of the teams.

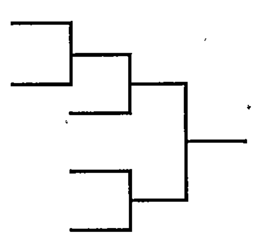


Figure 1. Example of a Tournament Structure for Five Teams.

Definition 1.4. In a tournament structure, a *slot* is one of the horizontal lines with an open left end, and represents the first appearance of a team in a tournament.

Definition 1.5. A *tournament draw* is a tournament structure where the players have been identified and assigned initial slots.

If the initial games of two or more teams or sets of teams play a symmetric role in the structure, such that all pairings would remain the same if the team labels in the slots were interchanged, then the different labelings are not regarded as determining different draws. Thus, distinct draws are distinguishable by the pairings resulting, regardless of their order in a tournament structure diagram.

Figure 2 below shows several different tournament draws. The first picture shows one possible tournament draw for the structure represented in Figure 1. The other two pictures show equivalent tournament draws.

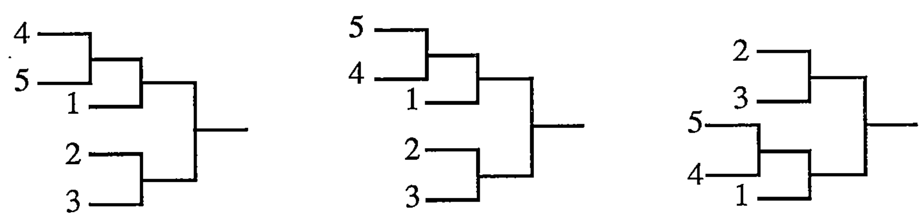


Figure 2. Examples of Tournament Draws.

Sometimes, we will want to refer to the slot a team occupies in a tournament structure; at other times we will identify a team by its *relative strength*. Relative strengths are integers between 1 and t, the number of teams in the tournament. The relative strength of a player represents the ranking of a player. The strongest player is assigned a relative strength of 1 and the weakest player is assigned a relative strength of t. Throughout this thesis, we will identify teams by their relative strengths. For example, using Figure 2 above, the team labeled "1" is the strongest team and the team labeled "5" is the weakest team.

Relative strengths, however, are not always easy or possible to assign. A related paradox associated with tournaments and relative strengths can be illustrated by considering the following situation. Suppose team A has beaten team B, team B has beaten team C, and team C has beaten team A. This intransitivity would seem to contradict our intuition about relative strengths, but it is easy to invent an underlying probability structure for which this kind of intransitivity is expected. The following example shows four six-sided dice which have a cycle of probability dominance. Let die 1 have these sides: 4, 4, 4, 4, 4, 4. Let die 2 have these sides: 3, 3, 3, 3, 7, 7. Let die 3 have these sides: 2, 2, 2, 6, 6, 6. Let die 4 have these sides: 1, 1, 5, 5, 5, 5. Consider rolling two dice at a time and declaring the "winner" to be the die with the larger number. Then die 1 beats die 2 with probability $\frac{2}{3}$, die 2 beats die 3 with probability $\frac{2}{3}$, die 3 beats die 4 with probability $\frac{2}{3}$, and die 4 beats die 1 with probability $\frac{2}{3}$. For this example, there is no system of competition which will choose the "best" die, because each die is dominated by another die and thus there can be no "best" die.

To avoid situations such as the intransitive dice, it is customary to assume a particular stochastic structure for the outcomes. The stochastic structure can be summarized in a *preference matrix*, an array of elements that indicates the pairwise probabilities of teams beating each other. The probability that team i beats team j in a game is denoted by p_{ij}, and because ties are not allowed, $p_{ij} = 1 - p_{ij}$. For convenience, we have defined p_{ii} to be .5. The preference matrix for the example of the intransitive dice would be:

$$\mathbf{P} = \begin{bmatrix} .50 & .67 & .50 & .33 \\ .33 & .50 & .67 & .50 \\ .50 & .33 & .50 & .67 \\ .67 & .50 & .33 & .50 \end{bmatrix}.$$

A common assumption that is placed on the preference matrices associated with tournaments is the assumption of *strong stochastic transitivity* (SST). The version of SST we adopt was given by David (1963).

Definition 1.6. (David, 1963) A preference matrix **P** satisfies *strong stochastic transitivity* if and only if $p_{ik} \geq \max(p_{ij}, p_{jk})$ whenever $p_{ij} \geq .5$ and $p_{jk} \geq .5$ for each trio of players i, j, and k.

We assume throughout this thesis that the first row in a preference matirx represents the strongest team, the second row represents the second strongest team, etc. Under this assumption, the entries in a preference matrix are non-decreasing from left to right and non-increasing from top to bottom. Therefore, in the SST preference matrices that we use, the largest element represents the chance that the strongest team beats the weakest team, and is the element in the upper-right corner of the matrix. It is customary to assume independence between games so that mathematical results may be formulated. While this assumption may not be entirely warranted in real life, we feel it is an appropriate mathematical assumption.

Satisfaction of David's definition of transitivity allows us to use the preference matrix to rank the players and assign relative strengths. For the example of the intransitive dice, the preference matrix does not satisfy strong stochastic transitivity, and thus we cannot call any of the dice the "best" die.

An interesting question which might naturally arise at this point is how the initial placement of the teams (the tournament draw) affects the probabilities of the teams winning under a given tournament structure. This is addressed in Chapter 2. Another interesting but unrelated question is how many tournament structures and tournament draws exist for single-elimination tournaments with t teams. The number of structures is counted in Chapter 3. After the number of tournament structures and draws have been counted, it might be reasonable to ask which tournament structures and draws are "best." The definition of "best" will be more fully discussed later in this chapter, but many possibilities exist. One possible criterion for deciding if a tournament is "best" is the concept of *order*, first used by Chung and Hwang (1978) using the equivalent term *monotone* and later more fully detailed by Horen and Riezman (1985) using the equivalent term *fair*. Because the terms monotone and fair have other connotations, we prefer to use the term *ordered*, and we give its definition next.

Definition 1.7. A tournament draw is *ordered* if for all preference matrices satisfying strong stochastic transitivity, the probabilities of the teams winning the tournament are ordered by their relative strengths. That is, if $i < j$ then Pr(team i wins the tournament) $\geq$ Pr(team j wins the tournament). A tournament draw is *unordered* if these probabilities are not completely ordered by the team relative strengths. More precisely, a tournament draw is unordered if there exists a preference matrix and two teams, $i < j$, such that Pr(team i wins the tournament) < Pr(team j wins the tournament).

In an ordered tournament, the strongest team has the largest probability of winning the tournament, the second strongest team has the next largest probability of winning the tournament, etc. In an unordered tournament, all but two of the teams might be correctly ranked. It is important to recognize that this definition of ordered is a condition that holds for all preference matrices that satisfy SST. Showing that a tournament draw is unordered is much easier than showing that a tournament draw is ordered, because only one example of a preference matrix must be found to show that a tournament draw is unordered.

An intuitive discussion of this choice of order as a criterion for tournaments follows. The motivating concept is that of an intuitively "fair" tournament, but unfortunately the term "fair" cannot be fully described in one simple condition. The definition of ordered presented here can be regarded as the negation of a condition which would be contrary to our intuitive idea of "fair". It can be argued that the condition we have chosen for ordered is sufficient to be "fair", but it is by no means necessary. For example, if two teams are of equal strength, we might expect that in a "fair" tournament their probabilities of winning would also be equal. Our condition of ordered does not address this aspect of "fairness".

Suppose teams i and j are of unequal strengths. To avoid being "unfair", should we require that Pr(team i wins the tournament) > Pr(team j wins the tournament), or is ≥ sufficient? The former is too strong a condition to obtain general results for the following reason. Imagine a large tournament with teams i and j in the same half such that their strengths relative to every team in their half are identical. Suppose also that team i is technically stronger than team j due to its strength versus the weakest team, team t, which plays in the other half of the tournament. If team t is beaten with certainty before reaching the finals, then Pr(team i wins the tournament) = Pr(team j wins the tournament), even though team i is stronger. To prevent the existence of a weak team destroying the generality of theorems, we use "≥" in our definition instead of ">".

It is desirable to know which structures can produce ordered tournament draws. Some structures do not have draws which are ordered.

Definition 1.8. A tournament structure is *orderable* if there is a particular tournament draw using the tournament structure that is ordered. A tournament structure is *unorderable* if *all* tournament draws using the tournament structure are unordered.

Often researchers are not interested in the probabilities of the weakest teams winning the tournament. In such cases, it may be reasonable to only consider the criterion of order as it applies to the strongest teams. A tournament draw is *partially ordered of degree f* if the probabilities of the f strongest teams winning the tournament are ordered by their relative strengths and none of the $t-f$ weakest teams have a greater probability of winning the tournament than any of the f strongest teams. Thus, in a partially ordered tournament, the two weakest teams may have unordered probabilities of winning the tournament, but both values may be so small as to be of no interest. Note that an ordered t-team tournament is always a partially ordered tournament of degree t or less.

Other questions which might be asked concerning tournaments are detailed in Table 1 below. This list is not meant to be exhaustive, but it does represent a collection of intriguing questions. Various authors have addressed some of the questions listed in Table 1, such as the issue of ranking contestants and the issue of the number of tournament structures. Many of these questions, however, are unanswered.

Table 1. Some Questions about Tournaments.

1. *	Is there a desirable method or notation for listing tournament structures?
2. *	Which tournament structures are "best"?
3. *	Which tournament draws are "best"?
4. *	What are some criteria for "best"?
5. *	If additional restrictions are imposed on a tournament, such as a limit on the number of rounds allowed, are there still "best" tournaments.
6. *	How many different tournament structures are orderable for a given number of contestants?
7. *	How many different tournament draws are ordered for a given tournament structure?
8. *	Is there always an ordered tournament draw or an orderable tournament structure for any number of contestants?
9.	How can the degree of a partially ordered tournament be determined?
10.	Is there a general notation for partially ordered tournaments?
11.	How would we define an "effective tournament"?
12.	How do incorrect *a priori* rankings influence "tournament effectiveness"?
13.	How does repetition affect "tournament effectiveness"?
14.	Can contestants be ranked from tournament results?

* Answered or partially answered in this dissertation.

To assist in defining the "best" tournament, Table 2 below was constructed. This table lists some possible ideas for how "best" could be defined when comparing two tournament draws.

Table 2. Some Definitions for "Best."

1. *	Highest probability of selecting strongest contestant.
2.	Highest probability of selecting a subset containing the strongest contestant.
3. *	Highest probability of the two strongest contestants meeting in the final round.
4. *	Ordered
5.	Partially ordered
6.	Fewest rounds
7.	Fewest expected number of contests (only appropriate in repeated single-elimination and *l*-elimination tournaments)

* Results provided in this dissertation.

Many authors have used the first three definitions of "best" in their works. Horen and Riezman (1985) detailed results using the fourth definition, that of *order*. Searls (1963) and Glenn (1960) compared the effectiveness of tournaments using the first definition and the sixth, the expected number of games in a tournament. In this thesis, the first, third, and fourth definitions will be examined.

Many of the above questions and ideas will be discussed in the following chapters. The question of notation will be addressed in Chapter 2 of this thesis, as well as methods of calculating probabilities of winning tournaments. Chapter 3 will deal with counting the number of distinct tournament structures and other combinatorial questions, and will include a proof of a recursive relationship stated without proof by Maurer (1975). The subject of order constitutes much of the material in Chapter 4. Chapter 5 will explore tournament effectiveness among the various criteria for "best," using some of the ideas expressed in Table 2.

Prior Work

Much work has been done in the fields of tournaments and paired comparisons. The work can be divided into two major categories: results concerning round-robin tournaments, and results concerning knockout tournaments. David (1959), Glenn (1960), and Searls (1963) worked on the effectiveness of various types of tournaments, including round-robin tournaments, but, due to computer limitations at that time, they only looked at a few preference matrices among the contestants. Searls also discussed incorrect *a priori* rankings. Harary and Moser (1966) detailed many results for round-robin tournaments, and their work was extended by Moon (1968).

Kendall (1955), David (1963), and many others have worked on the problem of paired comparisons in general. These works are closely related to the work in round-robin tournaments. Davidson and Farquhar (1976) compiled a lengthy bibliography on the subjects of paired comparisons and tournaments. This is an excellent listing of articles and books on the topic of paired comparisons and the subtopics of round-robin tournaments and knockout tournaments.

Narayana (1968), Narayana and Zidek (1969a) and (1969b), Narayana and Hill (1974), Narayana and Agyepong (1979), and Handa and Maitri (1984) have done work on random knockout tournaments. In *random knockout tournaments* the structure of the knockout tournament is not deterministic; rather, contestants are chosen randomly for competitions from among all remaining contestants, regardless of previous pairings. The early work on knockout tournaments concentrated mostly on classic tournaments. Glenn (1960) also briefly discussed first-round byes.

Maurer (1975) worked on the problem of tournament structures. His work is important because it was the first attempt to analyze different tournament structures instead of using random or classic tournaments. Chung and Hwang (1978) and Hwang (1982) contributed to the theory by considering the assignment of relative strengths, or "seedings," in knockout tournaments. Horen and Riezman (1985) looked at ordered tournaments for four and eight contestants. Their work was significant because they considered *all* preference matrices, and not just specific examples.

Ford and Johnson (1959), David (1971), and others have worked on the problem of ranking the participants from the results of a tournament. These results are often concerned with round-robin tournaments, but Ford and Johnson invented an interesting method of ranking without examining all $\binom{t}{2}$ games.

Gilbert (1961), Freund (1956), Carroll (1947) and many others have worked on problems of scheduling round-robin and knockout tournaments. Carroll's work is interesting in that he focused his criticisms of knockout tournaments on their inability to select the second place team accurately. He designed an elaborate scheme to ensure that the second place team was chosen more equitably. Wiorkowski (1972) commented on this problem, and Fox (1973) pointed out that l-elimination tournaments are much better at choosing the other $l-1$ places than are single-elimination tournaments. The issue of tournament scheduling in round-robin tournaments is closely related to the topic of balanced incomplete block designs in experimental design. We feel that while tournament scheduling is interesting, it is not relevant to this discussion, and is better left to future considerations.

CHAPTER 2

NOTATION AND PRELIMINARY RESULTS

This chapter will outline some of the notation used in the rest of this dissertation and will include some preliminary results. The primary result in this chapter is Theorem 2.6, which gives a formula for calculating the probability of any given team winning a tournament. The results of Theorem 2.6 can also be written in matrix notation, which is often easier to write down, but more complicated in interpretation. This chapter will also detail a labeling system for tournaments and will give definitions for such terms as *subtournament*, *tournament half*, and *bracket*.

A labeling notation for the various tournament structures is particularly useful for determining the number of tournament structures. Other authors have devised notations for labeling tournament structures, among them Maurer (1975). Maurer's notation, however, was developed mostly for typographical reasons, and not necessarily for descriptive purposes.

Pictures such as those in Figures 1 and 2 of Chapter 1 are easy to understand and can be used to specify tournament structures. We want the labeling of tournament structures, however, to uniquely specify structures that are distinct. For example, we do not care if the first-round game is displayed at the bottom or the top of a figure. Figure 3 below shows two equivalent structures for three teams. In each structure, there is a first-round game and a team with a first-round bye.

Figure 3. Two Equivalent Structures for Three Teams.

To create uniqueness in our labeling scheme, structures will be of the first type in Figure 3, that is, the teams that play in the first round will be pictured above the teams with byes. Our notation also does not allow a team to have a bye after playing a game. As an example, consider the six-team tournament displayed to the left in Figure 4. From one point of view, there could be three first-round games, one second-round game, and a third-round game. With this viewpoint, a bye would follow one of the first-round games. To avoid this second-round bye, we will write the structure as displayed to the right in Figure 4. This tournament has only two first-round games, two second-round games, and a third-round game.

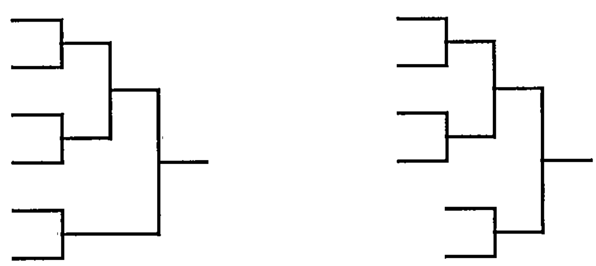


Figure 4. Example of Representations of Byes.

In using the labeling notation, we will have occasion to mention subtournaments and halves.

Definition 2.1. A *subtournament* is a subset of games from a tournament which is a tournament in its own right. It is determined by its final game.

Definition 2.2. A *half* of a tournament is one of the two subtournaments which directly precede the final game of the tournament.

The set of all tournament structures can be generated by combining subtournaments. Also, the method used in counting the distinct tournament structures in Chapter 3 will make use of tournament halves.

Definition 2.3. A *bracket of n rounds* is a subtournament of a classic tournament which has 2^n teams.

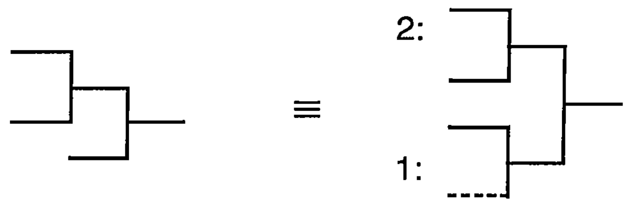
There are 2^{r-n} brackets of n rounds in a tournament with r rounds. Brackets can be divided into halves, called the first half and the second half. As an example of brackets, consider a classic tournament of eight teams and three rounds. In the first round, there are 2^{3-1}, or four brackets of one round, each with two teams. These first-round brackets are simply the first-round games. In the second round, there are two brackets of two rounds, each with four teams, and in the third round there is one bracket of three rounds with eight teams. This third-round bracket is precisely the entire tournament.

A labeling notation for tournament structures with t teams can be defined by the rules shown in Table 3 below. In Chapter 3, we will show a technique for counting tournament structures; in the course of this counting method, we will show the uniqueness of this labeling system. For immediate purposes, this labeling system is merely a shorthand for the pictures of tournament structures.

Table 3. Rules for Labeling Tournament Structures with r Rounds and t Teams

1. A label consists of a string of zeroes, ones, and twos of length 2^{r-1}.
2. The sum of the digits in a label is t.
3. Every label starts with a two.
4. A two cannot be followed by a zero.
5. The sum of the digits in the first half of a label is greater than or equal to the sum of the digits in the second half of a label. This rule also applies to each quarter, eighth, etc., that is, the sum of the digits in the first quarter of a label is greater than or equal to the sum of the digits in the second quarter of a label and the sum of the digits in the third quarter of a label is greater than the sum of the digits in the fourth quarter of a label.
6. If the sum of the digits is the same in both halves of a label, then the sum of the digits in the first quarter is greater than or equal to the sum of the digits in the third quarter. This rule also applies to eighths, sixteenths, etc., that is, if the sum of the digits is the same in all four quarters of a label, then the sum of the digits in the first eighth of a label is greater than or equal to the sum of the digits in the fifth eighth of a label and the sum of the digits in the first sixteenth of a label is greater than the sum of the digits in the ninth sixteenth of a label.

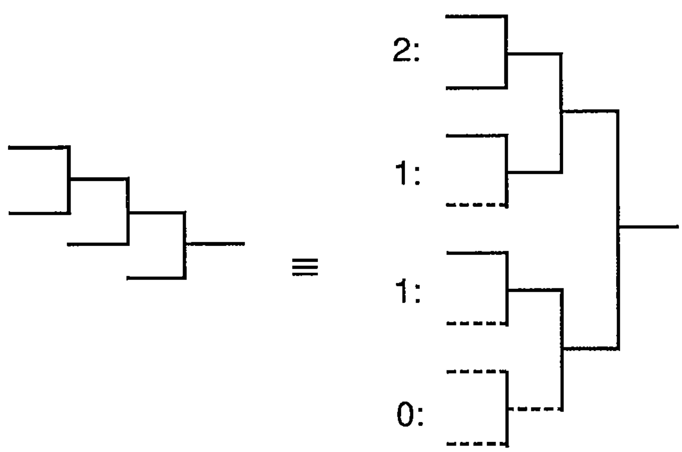
The label of digits is interpreted as follows. If the number of rounds in the tournament structure is r, then the length of the label is 2^{r-1}. Consider a classic tournament divided into 2^{r-1} brackets of two teams each. All tournament structures of at most r rounds can be derived from the classic tournament structure by letting "dummy" teams occupy appropriate initial slots. A "dummy" team is a team that can be beaten with certainty by any original team. In the label of digits, each digit represents one of the two-team brackets. Then, a "2" means there are two original teams (and no "dummy" teams) in the bracket, a "1" means there is one original team (and one "dummy" team) in the bracket, and a "0" means there are no original teams (and two "dummy" teams) in the bracket. To distinguish these labels from other numbers, labels will always be underlined. Examples using this notation are displayed in Figures 5, 6, and 7 below. In Figures 5 and 6 the picture on the left is our representation of the tournament structure, and the picture on the right demonstrates the labeling notation.



where ----- represents a dummy team.

Figure 5. Tournament Structure 21.

The picture to the left in Figure 5 shows the tournament structure for a tournament of two rounds and three teams, labeled 21. The two represents the bracket with two teams, and the one represents the bracket with one team. Note that the team in the bracket with one team received a first-round bye. The picture on the right in Figure 5 demonstrates the use of dummy teams to create the label 21.



where ----- represents a dummy team.

Figure 6. Tournament Structure 2110.

Figure 6 represents the structure labeled 2110. The 21 half of the tournament structure is identical to the tournament structure for three teams. The 10 half of the tournament structure represents the team with the two-round bye.

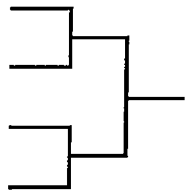


Figure 7. Tournament Structure 22.

Figure 7 shows the tournament structure labeled 22. This is a classic tournament because it has no byes, and therefore, all of the digits in the label are twos.

This labeling system has several desirable features. First, the sum of the digits is the number of teams in the tournament. Second, by looking at the length of the label, we can immediately see how many rounds the tournament requires. Third, the rules presented in Table 3 lend themselves to an iterative technique to generate the labels. Using this technique, all labels that satisfy the rules can be listed. In Chapter 3 we show, by way of our counting technique, that the above labeling system is unique in the sense that each distinct structure has only one label, and each label that satisfies the rules describes only one tournament structure.

We now introduce further notation en route to calculating the probability that any given team wins a tournament.

Classic Tournaments

The equations in this subsection will apply only to classic tournaments. Details for the case when the number of teams is not a power of two are discussed in the next subsection. Let q_{ij} be the probability that the team in slot i wins in round j. Given a tournament with r rounds, note that the probability that the team in slot i wins the tournament is q_{ir}. To calculate q_{ij}, we need to be able to identify the slot numbers of the potential opponents of a team. Proposition 2.5 below will give an equation for $S(i,n)$, the lowest slot number of the potential opponents of the team in slot i in round n. The slot numbers for the other potential opponents will be the next $\left(2^{n-1} - 1\right)$ consecutive integers. In the proof of the proposition, a function g will be created which identifies the initial team in a bracket, and Lemma 2.4 shows that g is non-increasing.

Lemma 2.4. Let i and n be positive integers. Then,

$$g(i,n) = 1 + \text{int}\left(\frac{i-1}{2^n}\right) \times 2^n \text{ is non-increasing in } n.$$

Proof: Let $j < k$. Then,

$$g(i,j) - g(i,k) = \left[1 + 2^j \times \text{int}\left(\frac{i-1}{2^j}\right)\right] - \left[1 + 2^k \times \text{int}\left(\frac{i-1}{2^k}\right)\right]$$

$$= 2^j \times \left[\text{int}\left(\frac{i-1}{2^j}\right) - 2^{(k-j)} \times \text{int}\left(\frac{(i-1) \times 2^{(j-k)}}{2^j}\right)\right]. \qquad (1)$$

There are now two cases: $i - 1 < 2^j$, or $i - 1 \geq 2^j$.

Case I: If $i - 1 < 2^j$, then the first term in the brackets in equation (1) is zero and thus,

$$g(i,j) - g(i,k) = 2^j \times \left[-\, 2^{(k-j)} \times \text{int}\left(\frac{(i-1) \times 2^{(j-k)}}{2^j}\right)\right] \leq 0.$$

Case II: If $i - 1 \geq 2^j$, then let $m = i - 2^j$, so that $i - 1 = m - 1 + 2^j$. Substituting in equation (1) gives:

$$g(i,j) - g(i,k) =$$

$$2^j \times \left[\text{int}\left(\frac{m-1+2^j}{2^j}\right) - 2^{(k-j)} \times \text{int}\left(\frac{(m-1+2^j) \times 2^{(j-k)}}{2^j}\right)\right]$$

$$= 2^j \times \left[\text{int}\left(\frac{m-1}{2^j}\right) + 1 - 2^{(k-j)} \times \text{int}\left(\frac{(m-1) \times 2^{(j-k)}}{2^j}\right) - 1\right]$$

$$= 2^j \times \left[\text{int}\left(\frac{m-1}{2^j}\right) - 2^{(k-j)} \times \text{int}\left(\frac{(m-1) \times 2^{(j-k)}}{2^j}\right)\right].$$

Now by replacing i with m in cases I and II, and by repeating the procedure if necessary, we will eventually have $m - 1 < 2^j$, and Case I will apply. Therefore, $g(i, j) - g(i,k) \leq 0$ in either case, and thus, $g(i,n)$ is non-increasing in n. □

Proposition 2.5. In a classic tournament structure, the lowest slot number for the potential opponents of the team in slot i in round n is given by:

$$S(i,n) = 1 + 2^{n+1}\text{int}\left(\frac{i-1}{2^n}\right) + 2^{n-1} - 2^{n-1}\text{int}\left(\frac{i-1}{2^{n-1}}\right) . \quad (2)$$

Proof: In a classic tournament, there are 2^n teams in each bracket of n rounds. The lowest slot number of the teams in each of these brackets is one more than a multiple of the number of teams in each bracket, or $1 + k \times 2^n$, for some k. Define $g(i,n)$ to be the lowest slot number of the teams in the bracket containing slot i. Then, $k = \text{int}\left(\frac{i-1}{2^n}\right)$, and

$$g(i,n) = 1 + \text{int}\left(\frac{i-1}{2^n}\right) \times 2^n .$$

Lemma 2.4 shows that $g(i,n)$ is non-increasing in n, and therefore, either $g(i,n) = g(i,n-1)$, or $g(i,n) < g(i,n-1)$.

Case I: $g(i,n) = g(i,n-1)$.

In this case, the team in slot i is in the first half of its bracket, because the lowest slot number is the same for the bracket of n rounds and the bracket of $n-1$ rounds. Thus, the lowest slot number of teams in the second half of the bracket is $g(i,n) + 2^{n-1}$.

Case II: $g(i,n) < g(i,n-1)$.

In this situation, the team in slot i is in the second half of its bracket and thus, the lowest slot number of teams in the first half of the bracket is $g(i,n)$.

If team i is in the second half of its bracket, $S(i,n)$ will be the lowest slot number of teams in the first half, or $g(i,n)$. If team i is in the first half, $S(i,n)$ will be the lowest slot number of teams in the second half of the bracket, or $g(i,n) + 2^{n-1}$. In either case, $S(i,n) = g(i,n) + 2^{n-1} + [g(i,n) - g(i,n-1)]$. (The quantity in brackets is 0 in case I and equals $\left(-2^{n-1}\right)$ in case II.) Thus,

$$S(i,n) = 2 \times \left[1 + 2^n \operatorname{int}\left(\frac{i-1}{2^n}\right)\right] + 2^{n-1} - \left[1 + 2^{n-1} \operatorname{int}\left(\frac{i-1}{2^{n-1}}\right)\right]$$

$$= 1 + 2^{n-1} + 2^{n+1} \operatorname{int}\left(\frac{i-1}{2^n}\right) - 2^{n-1} \operatorname{int}\left(\frac{i-1}{2^{n-1}}\right). \qquad \square$$

$S(i,n)$ gives the slot numbers for the teams that could face the team in slot i in round n. As an example, consider the classic tournament structure with eight teams and three rounds, displayed in Figure 8 below.

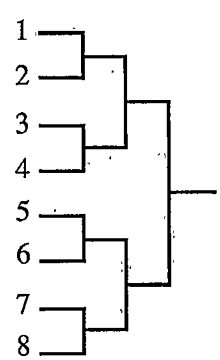


Figure 8. Eight-team Classic Tournament Structure with Slot Numbers Identified.

In Figure 8, the slot numbers, and not the relative strengths of the teams, have been labeled in the tournament structure. In this tournament, the team in slot one plays the team in slot two in the first round. In the second round, the team in slot one would play either the team in slot three or the team in slot four. From equation (2),

$S(1,1) = 1 + 0 + 1 - 0 = 2$, which agrees with the fact that the team in slot one plays the team in slot two in round one. Similarly, $S(1,2) = 1 + 0 + 2 - 0 = 3$, the lowest slot number of the two potential opponents of the team in slot one in the second round. For team six in round three, the potential opponents are the teams in slots one through four. Correspondingly, $S(6,3) = 1$. Table 4 below shows the values for $S(i,n)$ for tournament structure 2222, that is, for $i = 1, 2, ..., 8$, and $n = 1, 2$, and 3.

Table 4. Values of $S(i,n)$, Lowest Slot Number of Potential Opponents of the Team in Slot i in Round n for Tournament Structure 2222.

	Values of n		
Values of i	1	2	3
1	2	3	5
2	1	3	5
3	4	1	5
4	3	1	5
5	6	7	1
6	5	7	1
7	8	5	1
8	7	5	1

The general notation for q_{ij}, the probability that the team in slot i wins in round j, can now be formulated. Let $P(i, k)$ be the probability that the team in slot i beats the team in slot k. For the first round then, $q_{i1} = P(i, S(i,1))$. For the other rounds, the probability of the team in slot i winning in round j is given by Theorem 2.6.

Theorem 2.6. For $j \geq 2$, $q_{ij} = q_{i,j-1}\left[\sum_{k=l}^{u} P(i,k)q_{k,j-1}\right]$,

where $l = S(i, j-1)$, $u = l + 2^{j-1} - 1$, and $q_{i1} = P(i, S(i,1))$.

Proof: The chance of a team winning in any round is the chance that the team wins in the previous round times the chance that the team beats its opponent in the current round.

The team's opponent in the current round is conditional on which team wins the other subtournament. Thus we must sum over all possible opponents. The index for the summation is found by using Proposition 2.5 and using the next $\left(2^{n-1}-1\right)$ consecutive integers. □

The notation for calculating the probability of a team winning the tournament can also be expressed using matrices. The matrix notation is simpler because we will not have to worry about the slot number a team occupies. Let $\mathbf{O}_{ij}$ be a square matrix identifying the potential opponents to team i in round j. $\mathbf{O}_{ij}$ has zeroes on the off-diagonals and ones and zeros on the diagonals. The ones correspond to teams that could possibly meet team i in round j. Note that in $P(i\,,j)$ the argument i refers to the slot number whereas in $\mathbf{O}_{ij}$ the argument i refers to the relative strength. This distinction makes $\mathbf{O}_{ij}$ easier to work with, because we often know the team numbers but do not care to know the specific slot numbers. As an example, consider Figure 9 below which shows the "seeded" eight-team classic tournament. This tournament is very popular in sporting events. The numbers in Figure 9 are the relative strengths, not the slot numbers.

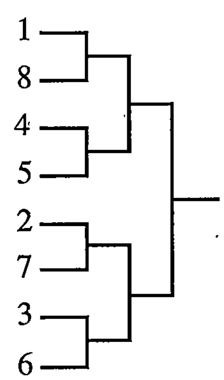


Figure 9. "Seeded" Eight-team Tournament.

In this tournament, the potential opponents for team five, for example, in the three rounds are team four in round one, either team one or team eight in round two, and either team two, team three, team six, or team seven in round three. Correspondingly,

$$\mathbf{O}_{51} = \begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}, \mathbf{O}_{52} = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}, \text{ and}$$

$$\mathbf{O}_{53} = \begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}.$$

Note that the trace of $\mathbf{O}_{ij}$ is 2^{j-1}.

Let $\mathbf{Q}_j$ be the $t \times 1$ vector of probabilities of the teams winning in round j. That is, $\mathbf{Q}_j$ is composed of the q_{ij}'s, but not necessarily sequentially, because q_{ij} refers to the team in *slot i*, not the team with *relative strength i*. To deal with this notational problem, we define $L(i)$ to be a link function between slot number and relative strength. For example, $L(2)$ is the slot number of the second strongest team. Then $\mathbf{Q}_j = [\, q_{L(1),j}\; q_{L(2),j}\; q_{L(3),j} \cdots q_{L(t),j}\,]'$. So $q_{L(i),r} = \mathbf{Q}_r' e_i$, where e_i is a column indicator vector having all zeroes, except for a one in row i. Row i of the matrix $\mathbf{P}$ can then be written as $e_i'\mathbf{P}$. The elements from this vector that correspond to the potential opponents of team i are the non-zero elements in the vector $e_i'\mathbf{PO}_{ij}$. The scalar $e_i'\mathbf{PO}_{ij}\mathbf{Q}_{j-1}$ is the sum of the probabilities of team i beating its potential opponents times the probability that the potential opponents win in round $j-1$. Then, $q_{L(i),j} = e_i'\mathbf{PO}_{ij}\mathbf{Q}_{j-1}\mathbf{Q}_{j-1}'e_i$. Note that because $q_{L(i),j}$ depends on $\mathbf{O}_{ij}$, $\mathbf{Q}_j$ cannot be written conveniently in matrix notation. It is possible, however, to write a matrix expression for $\mathbf{Q}_j$ using Hadamard products, but the notation is cumbersome, so we have not pursued it here.

Non-classic Tournaments

If the number of teams is not a power of two, it is still possible to use the notation developed by creating "dummy" teams until the number of teams is a power of two. Both Searls (1963) and Chung and Hwang (1978) have used this technique. When dummy teams are created, the preference matrix must be supplemented with ones and zeroes. Each added dummy team has probability zero of beating any original team, and therefore, each original team has probability one of beating a dummy team. Note that the performance of dummy teams against other dummy teams is irrelevant to the outcome of the tournament, so any probabilities consistent with strong stochastic transitivity can be assigned to the meetings between dummy teams.

To see how these dummy teams would be created, consider the tournament draw for the tournament 2111 given in Figure 2 of Chapter 1. This tournament has five teams and thus requires three dummy teams to form a classic tournament. If these three dummy teams are given relative strengths of six, seven, and eight, the preference matrix is:

$$\begin{bmatrix} \frac{1}{2} & p_{12} & p_{13} & p_{14} & p_{15} & 1 & 1 & 1 \\ p_{21} & \frac{1}{2} & p_{23} & p_{24} & p_{25} & 1 & 1 & 1 \\ p_{31} & p_{32} & \frac{1}{2} & p_{34} & p_{35} & 1 & 1 & 1 \\ p_{41} & p_{42} & p_{43} & \frac{1}{2} & p_{45} & 1 & 1 & 1 \\ p_{51} & p_{52} & p_{53} & p_{54} & \frac{1}{2} & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

This chapter has detailed some of the basic notation necessary to discuss tournaments, such as a labeling system and a formula to find the chance that a given team wins the tournament. Other notation used in this thesis will be introduced as necessary.

CHAPTER 3

THE NUMBER OF TOURNAMENT STRUCTURES

When a researcher uses a tournament to choose the best treatment, or when a tournament director schedules a sporting event, a question that must be addressed is which tournament structure to use. Historically, in sporting events, classic tournaments have been used extensively. Leagues and divisions are often specifically formed so that the number of teams is a power of two. There are situations, however, where non-classic tournaments have been used, the NCAA basketball tournament in the early eighties being an example. In that tournament, a number of higher ranked teams received first-round byes. The Pro Bowlers Tour uses a ladder-like tournament (as in Figure 6 of Chapter 2) to select their weekly champion. This chapter makes use of the labeling notation introduced in Chapter 2 to list and count the number of tournament structures. An equation for the number of tournament structures, including the number of tournaments in a restricted class of tournament structures, is given in Theorem 3.9. The proof of Theorem 3.9 proves one of the recursive relationships stated without proof by Maurer (1975).

Recall that the labeling notation introduced in Chapter 2 consists of strings of the digits zero, one, and two. The length of a label is a power of two, and labels can therefore be broken into halves. Throughout this chapter, capital Roman letters will denote sets, while capital Roman letters followed by a $ will indicate a label, a string of zeroes, ones, and twos.

Before deriving the formula to count tournament structures, a few definitions must be addressed. The *primary sum* of the character string *A*$ is the sum of the digits in *A*$. For example, the label <u>21101000</u> has a primary sum of 5. The primary sum of a string is simply the number of teams in the tournament structure. To define the secondary sum of $A\$$, write $A\$$ as $A1\$A2\$$, where $A1\$$ is the first half and $A2\$$ is the second half. The *secondary sum* of $A\$$ equals the primary sum of $A1\$$. The secondary sum of $A\$$ gives the number of teams in the first half of the tournament. For example, the label $\underline{2110}$ would have $A1\$ = \underline{21}$ and $A2\$ = \underline{10}$. The secondary sum of $A\$$ is then the primary sum of $\underline{21}$, or 3.

The technique for listing the tournament structures will combine subtournament labels. At times in listing the tournament labels, illegal labels will be created; that is, some labels will not satisfy all of the rules given in Table 3 of Chapter 2. These illegal labels will be counted separately in a duplicate set and subtracted from the total count.

Definition 3.1. The *duplicate set* of A, D_A, is the set $\{A\$B\$$ such that $A\$ \in A$ and $B\$ \in A$ and the secondary sum of $A\$ <$ the secondary sum of $B\$\}$.

D_A is a set of duplicate tournament labels, and in fact contains only illegal labels. These illegal labels do not have the proper ordering of secondary sums, violating rule 6 in Table 3. In cases where the length of $A\$$ is greater than the length of $B\$$, the label $B\$$ is "expanded" to have the same length as $A\$$. Because the length of labels is a power of two, the expansion of a label will double its length. The expansion rules are as follows. A $\underline{2}$ becomes $\underline{11}$, a $\underline{1}$ becomes $\underline{10}$, and a $\underline{0}$ becomes $\underline{00}$. If necessary, expanding continues until the length of $B\$$ is the same as the length of $A\$$. Similarly, we expand $A\$$ if the length of $A\$$ is less than the length of $B\$$.

For $A \cap B = \phi$, define $A*B$ to be $\{A\$B\$$, where $A\$ \in A$ and $B\$ \in B\}$. Define $A*A$ to be $\{A\$B\$$ where $A\$ \in A$ and $B\$ \in A\} \sim D_A$, where $\sim$ represents the usual set difference operation. Thus, in $A*A$ we are excluding the duplicate tournament labels. As an example of illegal labels in D_A, consider $A = \{\underline{22}, \underline{2110}\}$. Then, using the expanded label $\underline{1111}$ for $\underline{22}$, $D_A = \{\underline{11112110}\}$. Note that $\underline{11112110}$ is equivalent to $\underline{21101111}$, and because the leading digit is not a two, nor are the secondary sums properly ordered, 11112110 is not a legitimate label. Thus, $A*A = \{\underline{2222}, \underline{21101111}, \underline{21102110}\}$.

Now we are prepared to define sets of tournament structures.

Definition 3.2. $T(t, r)$ is the set of tournament structures of t teams in at most r rounds.

The set $T(t, r)$ will be enumerated by breaking each tournament structure into halves, and iteratively counting the number of tournament structures in each half. The final result of this counting is given in Theorem 3.7 below, but we first need some preliminary results. The following theorem details how the sets of tournament structures are constructed.

Theorem 3.3. For $r \geq 2$, $T(t, r) = \bigcup_{j=l}^{u} \left[T(t-j, r-1) * T(j, r-1) \right]$,

where $u = \text{int}\left(\frac{t}{2}\right)$ and $l = \max\left(1, t - 2^{r-1}\right)$. The initial conditions are $T(1,1) = \{\underline{1}\}$ and $T(2,1) = \{\underline{2}\}$.

Proof: We will break the t teams into two subtournaments and then take the union of all sets thus created. The union of the sets created will be

$$\bigcup_{j=l}^{u} \left[T(t-j, r-1) * T(j, r-1) \right].$$

To avoid duplication, we will place the larger number of teams in the first half of the set combining process. Thus, the upper limit in the union is $u = \text{int}\left(\frac{t}{2}\right)$. We also need to restrict the lower limit of the union if $t - j$ teams cannot be scheduled in $r - 1$ rounds. There are at most 2^{r-1} teams in round $r - 1$, so $t - 2^{r-1}$ is the smallest possible index. Of course, if this number is negative, the counting will start at one. Thus, the lower limit in the union is $l = \max\left(1, t - 2^{r-1}\right)$. Therefore,

$$T(t\,,r) = \bigcup_{j=l}^{u} \left[T(t-j,\, r-1) * T(j,\, r-1) \right], \tag{1}$$

where $u = \text{int}\left(\frac{t}{2}\right)$ and $l = \max\left(1, t - 2^{r-1}\right)$. □

Now, we will count the number of elements in $T(t\,,r)$. Define $N(A)$ to be the number of unique elements in A. In the following propositions, we assume $A = T(i, k)$ and $B = T(j, k)$, where $i \neq j$. Thus the primary sum is i for all elements in A.

Proposition 3.4. If $A \cap B = \phi$, then $N(A*B) = N(A)\,N(B)$.

Proof: Because the sets are disjoint, and because of the construction of $A*B$, the number of elements in $A*B$ is the product of the number of elements in A and the number of elements in B. □

Proposition 3.5. $N(A*A) = N(A)^2 - N(D_A)$.

Proof: If we paired each of the elements of A with every other element of A, we would have $N(A)^2$ elements. Because the duplicate set D_A is contained in the set of pairs of elements in A, the number of elements in $A*A$ is $N(A)^2 - N(D_A)$. □

Proposition 3.6. $N(D_A) = \binom{N(A)}{2}$.

Proof: First we note that we have assumed that all elements of A have the same primary sum. There are $N(A)^2$ ways to pair the elements of A together. However, some of these are violations of the labeling notation because the secondary sum of the second element is larger than the secondary sum of the first element. For each two elements in A, there is only one legitimate way to pair them. Consider placing the $N(A)^2$ elements in a square array. In the first row and column, there are $N(A) - 1$ pairs of elements that are violations of the labeling system. In the second row and the second column, there are

$N(A) - 2$ more pairs of elements that are violations of the labeling system. Continuing this argument, there are

$$\sum_{i=1}^{N(A)-1} i$$

pairs of elements where the secondary sums are not properly ordered. Thus,

$$N(D_A) = \sum_{i=1}^{N(A)-1} i = \frac{(N(A)-1) * N(A)}{2} = \binom{N(A)}{2}. \qquad \square$$

Using Proposition 3.5, we see that $N(A*A) = N(A)^2 - \binom{N(A)}{2}$

$$= N(A)^2 - \left(\frac{N(A)^2 - N(A)}{2}\right) = \frac{N(A)(N(A)+1)}{2}.$$

Because $T(t, r)$ in (1) is the union of disjoint sets, the number of elements in the set is the sum of the number of elements in each set. Then,

$$N(T(t, r)) = \sum_{j=l}^{u} N\left[T(t-j, r-1) * T(j, r-1)\right],$$

where $l = \max\left(1, t - 2^{r-1}\right)$ and $u = \operatorname{int}\left(\frac{t}{2}\right)$. By using Proposition 3.4, Proposition 3.6, the definition of $A*B$, and some algebraic manipulation, we have Theorem 3.7, given below.

Theorem 3.7. In a tournament of t teams and at most r rounds, the number of tournament structures is given by

$$N(T(t, r)) =$$

$$\sum_{j=l}^{u} \left[N(T(t-j, r-1))\right] \left[N(T(j, r-1)) + \delta_{t-j,j}\right] \left[1 - \frac{\delta_{t-j,j}}{2}\right], \qquad (2)$$

where $\delta_{i,j}$ is Kronecker's delta, $l = \max\left(1, t - 2^{r-1}\right)$, and $u = \text{int}\left(\frac{t}{2}\right)$.

Equation (2) can be used to count the total number of tournament structures for a given number of teams, or it can be used to count tournament structures which are restricted by a maximum number of rounds. For instance, a basketball tournament may be required to be finished in four days, with only one game played per day by each team. Tournaments which have five or more rounds would then be unacceptable.

To count the total number of tournament structures for a given number of teams, note that there are at most $t-1$ rounds in a tournament with t teams, and therefore $T(t, t-1)$ is the set of all tournament structures for t teams. Let $a_t = N(T(t, t-1))$. Note that the initial condition for this sequence is $a_1 = 1$. Then,

$$a_t = \sum_{j=l}^{u} \left[N(T(t-j, t-2))\right] \left[N(T(j, t-2)) + \delta_{t-j,j}\right] \left[1 - \frac{\delta_{t-j,j}}{2}\right],$$

from (2) above. A simpler notation can be derived for a_t and is given in Theorem 3.9 below. First, however, Lemma 3.8 shows that $N(T(t, j))$ can be simplified.

Lemma 3.8. $N(T(t, j)) = N(T(t, t-1))$ for all $j \geq t-1$, that is, when $j - t + 1 \geq 0$.

Proof: If $j \geq t-1$, then there are more rounds than the tournament structure with the maximum number of rounds. Thus, these extra rounds are artificial and do not create additional tournament structures. Therefore, $N(T(t, j)) = N(T(t, t-1))$ for all $j \geq t-1$. □

Using Lemma 3.8, with $j \geq 1$, $N(T(t-j, t-2)) = N(T(t-j, t-j-1)) = a_{t-j}$, because $(t-2) - (t-j) + 1 = j - 1 \geq 0$. Similarly, with $j \leq \text{int}\left(\frac{t}{2}\right)$,

$N(T(j, t-2)) = N(T(j, j-1)) = a_j$, because $(t-2) - j + 1 = t - j + 1 \geq t - \text{int}\left(\frac{t}{2}\right) - 1 \geq 0$.

$$\text{Thus, } a_t = \sum_{j=1}^{\text{int}\left(\frac{t}{2}\right)} a_{t-j}\left[a_j + \delta_{t-j,j}\right]\left[1 - \frac{\delta_{t-j,j}}{2}\right].$$

To simplify a_t further, we prove Theorem 3.9.

Theorem 3.9. (Maurer, 1975) The number of tournament structures for t teams is given by

$$a_t = \frac{1}{2}\sum_{j=1}^{t-1} a_{t-j}\left(a_j + \delta_{t-j,j}\right), \tag{3}$$

where $a_1 = 1$.

Proof: t is either odd or even. If t is even, then $t = 2m$ for some m, and $\text{int}\left(\frac{t}{2}\right) = m$. Then,

$$a_t = \sum_{j=1}^{m} a_{t-j}\left[a_j + \delta_{t-j,j}\right]\left[1 - \frac{\delta_{t-j,j}}{2}\right]$$

$$= \sum_{j=1}^{m-1} a_{t-j}a_j + a_{t-m}\left(a_m + 1\right)\left(\frac{1}{2}\right)$$

$$= \sum_{j=1}^{m-1} a_{t-j}a_j + a_m\left(a_m + 1\right)\left(\frac{1}{2}\right)$$

$$= \sum_{j=1}^{m-1} \frac{a_{t-j}a_j}{2} + \sum_{j=1}^{m-1} \frac{a_{t-j}a_j}{2} + a_m\left(a_m + 1\right)\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\left[\sum_{j=1}^{m-1} a_{t-j}a_j + \sum_{k=m+1}^{t-1} a_k a_{t-k} + a_m\left(a_m + 1\right)\right]$$

$$= \frac{1}{2}\sum_{j=1}^{t-1} a_{t-j}\left(a_j + \delta_{t-j,j}\right).$$

If t is odd, then $t = 2m + 1$ for some m, and hence $\mathrm{int}\left(\frac{t}{2}\right) = m$.

$$a_t = \sum_{j=1}^{\mathrm{int}\left(\frac{t}{2}\right)} a_{t-j}\left[a_j + \delta_{t-j,j}\right]\left[1 - \frac{\delta_{t-j,j}}{2}\right]$$

$$= \sum_{j=1}^{m} a_{t-j}a_j$$

$$= \frac{1}{2}\left[\sum_{j=1}^{m} a_{t-j}a_j + \sum_{j=1}^{m} a_{t-j}a_j\right]$$

$$= \frac{1}{2}\left[\sum_{j=1}^{m} a_{t-j}a_j + \sum_{k=m+1}^{2m} a_{t-k}a_k\right]$$

$$= \frac{1}{2}\left[\sum_{j=1}^{m-1} a_{t-j}a_j + \sum_{k=m+1}^{t-1} a_k a_{t-k} + a_m\left(a_m + 1\right)\right]$$

$= \frac{1}{2}\sum_{j=1}^{2m} a_{t-j}\left(a_j + \delta_{t-j,j}\right)$, because $t - j$ never equals j. Then,

$$a_t = \frac{1}{2}\sum_{j=1}^{t-1} a_{t-j}\left(a_j + \delta_{t-j,j}\right).$$ □

Theorem 3.7 is more useful than Theorem 3.9 for our purposes due to its capability of counting tournament structures which are restricted by the number of rounds. The following examples show the tournament structures for up to seven teams.

2

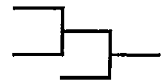


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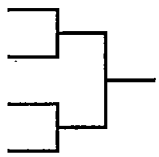
Figure 10. Tournament Structures for $t = 2$ and $t = 3$ Teams.



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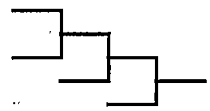

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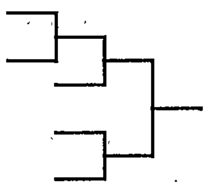
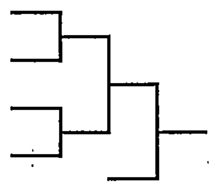


2111



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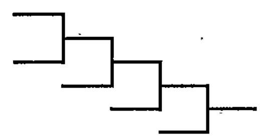

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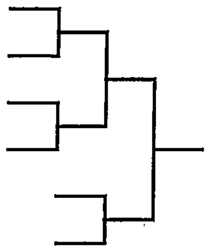
Figure 11. Tournament Structures for $t = 4$ and $t = 5$ Teams.



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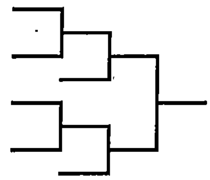

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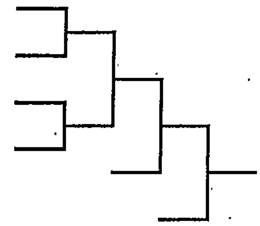

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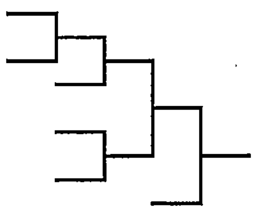

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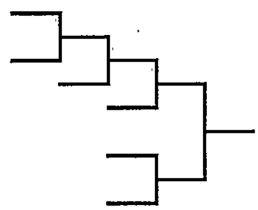

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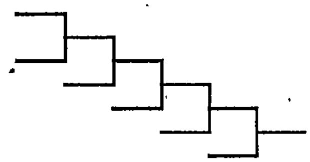

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Figure 12. Tournament Structures for $t = 6$ Teams.

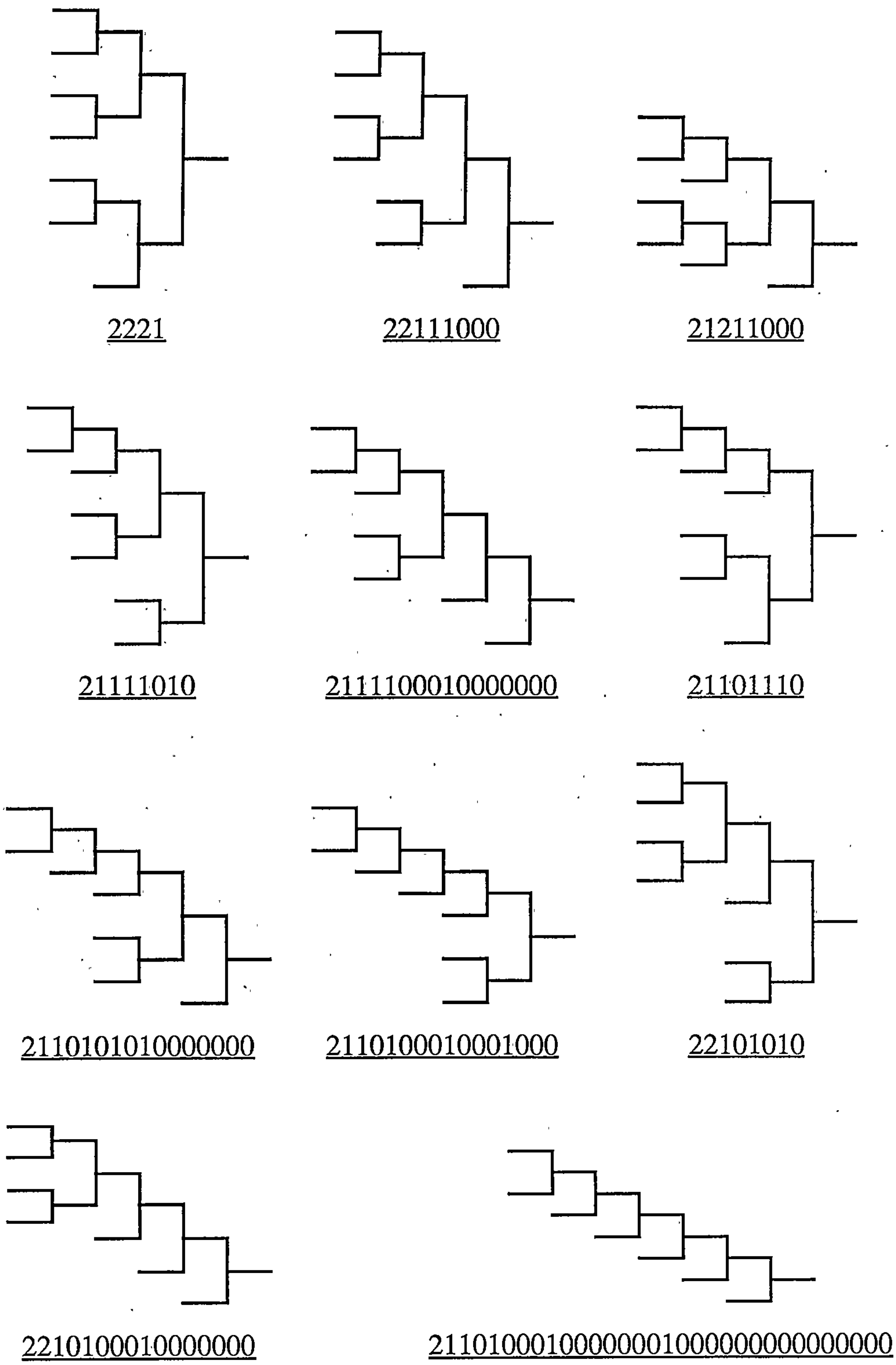

Figure 13. Tournament Structures for $t = 7$ Teams.

Table 5 below shows the values for $N(T(t, r))$. The values in Table 5 are useful in situations where the tournament is constrained by time, i.e., the tournament is limited to less than $t - 1$ rounds. As an example from Table 5, for eight teams and at most four rounds, there are eight possible tournament structures. One of these tournament structures has only three rounds, so seven have exactly four rounds. The right most values in Table 5 for t less than or equal to eleven are the values reported by Maurer (1975), because such tournaments take at most ten rounds. For twelve or more teams, more columns are required before the values match Maurer's.

Table 5. The Number of Tournament Structures with t Teams and at Most r Rounds.

	Values of r								
Values of t	2	3	4	5	6	7	8	9	10
3	1	1	1	1	1	1	1	1	1
4	1	2	2	2	2	2	2	2	2
5		2	3	3	3	3	3	3	3
6		2	5	6	6	6	6	6	6
7		1	6	10	11	11	11	11	11
8		1	8	17	22	23	23	23	23
9			8	25	39	45	46	46	46
10			9	38	70	90	97	98	98
11			7	52	118	171	198	206	207
12			7	73	200	325	406	441	450
13			4	93	324	598	811	928	972
14			3	121	526	1,097	1,613	1,951	2,113
15			1	143	825	1,972	3,155	4,046	4,555
16			1	172	1,290	3,531	6,141	8,349	9,795

It is often convenient to know how many tournament structures of *exactly* r rounds exist. This can of course be calculated easily from Table 5 by subtracting adjacent columns. Table 6 below shows the number of tournament structures with t teams and exactly r rounds.

Table 6. The Number of Tournament Structures with t Teams and Exactly r Rounds.

	Values of r								
Values of t	2	3	4	5	6	7	8	9	10
3	1								
4	1	1							
5		2	1						
6		2	3	1					
7		1	5	4	1				
8		1	7	9	5	1			
9			8	17	14	6	1		
10			9	29	32	20	7	1	
11			7	45	66	53	27	8	1
12			7	66	127	125	81	35	9
13			4	89	231	274	213	117	44
14			3	118	405	571	516	338	162
15			1	142	682	1,147	1,183	891	509
16			1	171	1,118	2,241	2,610	2,208	1,446

We conclude this chapter by observing that $2^{t-4} < a_t < 2^{t-3}$. Thus, $\log_2(a_t)$ increases linearly.

At this point, we could count tournament draws. We feel, however, that because many of the possible tournament draws have unrealistic first-round games, such as the two best teams meeting, it is not worthwhile to count tournament draws. We will, however, count ordered tournament draws. In Theorem 4.24 in the next chapter, we show that the tournament draws associated with orderable tournament structures are unique, and thus if we count orderable tournament structures, we have then counted ordered tournament draws.

CHAPTER 4

ORDERED TOURNAMENTS

In many cases, a researcher wants to rank the treatments from the results of an experiment, or would like his method to choose the best treatment with high probability. If the *a priori* relative strengths are correctly assigned to the teams, then no tournament is necessary; we can just choose the strongest team and declare it to be the winner. This situation is of course mathematically uninteresting, so we restrict our attention to the case where a tournament must be held. We then assume that our *a priori* assignment of relative strengths is indeed correct when we calculate the properties of alternative tournament structures.

A reasonable condition that can be imposed on the tournament draws is that of order. Recall that an *ordered tournament draw* requires the probabilities of teams winning the tournament to be ordered by their relative strengths. Thus, the strongest team has the best chance of winning, the second strongest team has the second best chance, etc. This chapter will detail some results concerning ordered tournament draws. Horen and Riezman (1985) showed that the four-team classic tournament structure is orderable, while the eight-team classic tournament structure is unorderable. They also conjectured that for 16 and more teams, the classic tournament structures are unorderable. Later in this chapter we prove this conjecture and completely characterize the orderable tournament structures.

The first result we give concerning ordered tournaments is a simple fact about the strongest teams and their placement in an ordered tournament draw.

Theorem 4.1. If a weaker team can win by playing fewer games than a stronger team, then the tournament draw is unordered.

Proof: Let n equal the number of games the stronger team, team i, must play to win the tournament. Let m equal the number of games the weaker team, team $j > i$, must play to win the tournament. Consider a tournament draw where $m < n$. Because $\frac{m}{n} < 1$, and

$$\lim_{\varepsilon \to 0} \left[\frac{\ln(.5 + \varepsilon)}{\ln(.5 - \varepsilon)} \right] = 1, \text{ there exists an } \varepsilon_0, \text{ such that}$$

$$\frac{m}{n} < \left[\frac{\ln(.5 + \varepsilon_0)}{\ln(.5 - \varepsilon_0)} \right], \text{ or } (.5 + \varepsilon_0)^n < (.5 - \varepsilon_0)^m.$$

Let $p_{ij} = .5 + \frac{j - i}{t - 1}\varepsilon_0$. Then, $q_{L(i),r}$, the probability that team i wins the tournament, is less than $(.5 + \varepsilon_0)^n$ and $q_{L(j),r}$, the probability that team j wins the tournament, is greater than $(.5 - \varepsilon_0)^m$.

Now, $q_{L(i),r} > (.5 - \varepsilon_0)^m > (.5 + \varepsilon_0)^n > q_{L(i),r}$, and therefore the tournament draw is unordered. □

Theorem 4.1 says that in an ordered tournament draw, the strongest team will get the most byes, the second strongest team will get the second most byes, etc. As an example of the application of Theorem 4.1, consider the tournament structure 2111, shown in Figure 14 below.

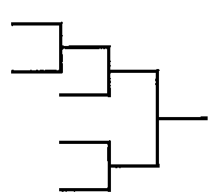

Figure 14. Tournament Structure 2111.

Theorem 4.1 says that the three strongest teams must be in the three slots receiving first-round byes. Teams four and five must play in the first round. The question of how to place the top three teams in this tournament structure will be answered by Theorem 4.3 below. Theorem 4.3 also will help us count the number of ordered tournament structures. The theorem involves the "*top-n*" tournament draw.

Definition 4.2. Let $n < t$ and consider a t-team tournament draw. Its associated *top-n* tournament draw is the n-team tournament draw which results from treating teams in the top n as automatic winners over teams not in the top n, and therefore does not include any games with a team not in the top n.

Theorem 4.3. If a t-team tournament draw is ordered, then its "top-n" tournament draw is also ordered.

Contrapositive: If the "top-n" tournament draw is unordered, then the t-team tournament draw is also unordered.

Proof of the Theorem: The "top-n" tournament draw can be represented with a preference matrix where $p_{ij} = 1$ for all $j > n$, where $j > i$. If $p_{ij} = 1$ for all $j > n$, then the tournament is essentially an n-team tournament. Because the tournament draw is ordered for any preference matrix, it is ordered for any preference matrix which uses probabilities where $p_{ij} = 1$ for all $j > n$. The probability that a team in the "top-n" wins the n-team tournament draw is therefore equal to the probability that the same team wins the t-team tournament draw. Thus, the "top-n" tournament draw must also be ordered. □

Theorem 4.1 and Theorem 4.3 taken together prove the following theorem.

Theorem 4.4. If a tournament draw is ordered, then the two strongest teams play in opposite halves of the tournament.