Some planar embeddings of chainable continua can be expressed as inverse limit spaces
by Susan Pamela Schwartz

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics
Montana State University

Abstract:
It is well known that chainable continua can be expressed as inverse limit spaces and that chainable continua are embeddable in the plane. We give necessary and sufficient conditions for the planar embeddings of chainable continua to be realized as inverse limit spaces.

As an example, we consider the Knaster continuum. It has been shown that this continuum can be embedded in the plane in such a manner that any given composant is accessible. We give inverse limit expressions for embeddings of the Knaster continuum in which the accessible composant is specified. We then show that there are uncountably many non-equivalent inverse limit embeddings of this continuum.

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ABSTRACT

It is well known that chainable continua can be expressed as inverse limit spaces and that chainable continua are embeddable in the plane. We give necessary and sufficient conditions for the planar embeddings of chainable continua to be realized as inverse limit spaces.

As an example, we consider the Knaster continuum. It has been shown that this continuum can be embedded in the plane in such a manner that any given composant is accessible. We give inverse limit expressions for embeddings of the Knaster continuum in which the accessible composant is specified. We then show that there are uncountably many non-equivalent inverse limit embeddings of this continuum.

CHAPTER 1

INTRODUCTION

This thesis presents the necessary and sufficient conditions for a planar embedding of a chainable continuum to be expressed as an inverse limit space. The theorem is illustrated by giving inverse limit expressions for embeddings of the Knaster bucket handle continuum. In the process, results are obtained concerning the accessibility of composants and the equivalence of embeddings.

Chainable continua are essentially planar. Bing [1] has demonstrated that every chainable continuum can be embedded in the plane in such a manner that the defining sequence of chains are comprised of interiors of rectangles, i.e. topological disks. We wish to distinguish a similar type of chainability. We will say that a continuum in the plane is disk-chainable if it can be chained in such a manner that for every $\epsilon > 0$ there is an ϵ-chain whose links are topological disks and all intersections of those links are topological disks.

Furthermore, Bing [1] has shown that given a chainable continuum X, there are uncountably many mutually exclusive homeomorphic copies of X in the plane $\mathbb{R}^2$. If Y and Z are elements of this collection, there exists a homeomorphism Φ which takes Y onto Z. If, in addition, Φ extends to a homeomorphism of the plane then Y and Z are equivalently embedded. Bing [1] has given the following example to demonstrate that a chainable continuum may have non-equivalent embeddings. This example also serves as an illustration that there exist non-disk-chainable embeddings of a continuum.

> Let $f_1(x)$ be the function whose graph is the sum of (a) the graph A_1 of $y = 2\sin 3x/2 \quad (0 \leq x \leq 2\pi/3)$, (b) the graph A_2 of $y = 3\sin 6x \quad (2\pi/3 \leq x \leq 5\pi/6)$, (c) the graph A_3 of $y = -\cos 3x \quad (\pi/2 \leq x \leq 5\pi/6)$, and (d) the set $B_1 + B_2 + B_3$, where B_i is symmetric to A_i with respect to the point $(\pi/2, 0)$. Then $f_1(x)$ is a single valued function for some values, triple valued for others, and double valued for two values. The closure M_1 of the graph of $y = f_1(\pi/x \quad mod \quad \pi) \quad (0 < x \leq 1)$ is a snake-like continuum but it does not have the property that for each positive number ϵ there is an ϵ-chain covering it whose links are connected.
>
> Let $f_2(x)$ be the single valued function whose graph is the sum of A_1, A_2, the graph A_4 of $y = \cos 3x \quad (5\pi/6 \leq x \leq 7\pi/6)$ and the set $C_1+C_2+C_4$, where C_i is symmetric to A_i with respect to the point $(7\pi/6, 0)$. The closure M_2 of the graph of $y = f_2(\pi/x \quad mod \quad 7\pi/3) \quad (0 < x \leq 3/7)$ is homeomorphic with M_1 but M_2 can be covered by ϵ-chains with connected links.

It is well known [2] that a chainable continuum, being arc-like, is homeomorphic to an inverse limit of arcs with onto bonding maps. But what of the embedding of a chainable continuum? Must it always be expressable as an inverse limit space? In seeking to explain what we mean by an embedding expressed as an inverse limit we have considered three things: (*i*) the fact that chainable continua are inverse limits of arcs, (*ii*) a theorem of Brown [3] which says that the inverse limit of copies of the same compact metric space with bonding maps which are near-homeomorphisms is itself that compact metric space, and (*iii*) a technique of Martin and Barge [4] for constructing global attractors in the plane. We then make the following definition:

The bounded planar continuum X is embeddable using inverse limits provided

(i) there exists a sequence of near homeomorphisms $\{F_n\}_{n\geq 0}$ from the closed disk D onto itself where $F_n(I) = I$ for I, an arc in D; and

(ii) there exists a homeomorphism $\Phi : (D, F_n) \to \hat{D}$, a closed disk containing X, such that $\Phi\big((I, F_{n|I})\big) = X$.

We are concerned with the question: which embeddings of chainable continua are embeddable using inverse limits? In answer, we present our main theorem, whose proof is given in Chapter 3:

Theorem 3.1: A continuum in the plane is disk-chainable if and only if it is embeddable using inverse limits.

Rephrasing Bing's results in our language and applying our results, we have that every chainable continuum has a disk-chainable embedding. That is, every chainable continuum has an embedding which can be expressed as an inverse limit. In addition, a chainable continuum may have other, non-equivalent embeddings. These may not have an inverse limit expression, as in Bing's example M_1. Or it may be the case that there are non-equivalent disk-chainable embeddings. We demonstrate the latter in Chapter 4, where we embed the Knaster continuum, M, using inverse limits. In the process we obtain a second affirmative answer to a question of Martin and Barge as to whether M can be embedded in the plane with any composant accessible.

CHAPTER 2

TERMS AND NOTATION

A continuum is a nondegenerate, compact, connected metric space. A continuum, X, is chainable or snake-like, if for every $\epsilon > 0$ there exists an ϵ-chain of X; an ϵ-chain being a finite open cover of X whose elements, links, have the property that (i) the diameter of each link is smaller than ϵ and (ii) only adjacent links have non-empty intersection. That is, let $C = \{l_1, \ldots, l_n\}$ be an ϵ-chain of X. Then $diam \quad l_j < \epsilon \quad \forall j \in \{1, \ldots, n\}$ and $l_i \cap l_j \neq \emptyset$ iff $|i - j| \leq 1$. We note that the links need not be connected. Also, every chainable continuum has a defining sequence of chains [5]. That is, there exists $\{C^j\}_{j=0}^{\infty}$, a sequence of ϵ_j-chains of X where $\epsilon_{j+1} < \epsilon_j$ and $\lim_{j \to \infty} \epsilon_j = 0$. In addition, the closure of each link of C^{j+1} is contained in a link of C^j.

In its broadest definition, the inverse limit space X_∞ associated with the inverse limit sequence $\{X_\alpha, f_{\alpha\beta}\}$ is as follows: let $\mathcal{A}$ be a directed set ordered by "$>$"; X_α a topological space; $\alpha, \beta, \gamma \in \mathcal{A}$; $\gamma \geq \beta \geq \alpha$; $\{f_{\alpha\beta} : X_\beta \to X_\alpha\}$ a collection of bonding maps with the properties (i) $f_{\alpha\alpha} = Id_{|X_\alpha}$ and (ii) $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$. Then X_∞ is a subset of the product space $\prod_{\alpha \in \mathcal{A}} X_\alpha$. With the product topology and projection maps $\pi_\beta : \prod_{\alpha \in \mathcal{A}} X_\alpha \to X_\beta$ we describe the inverse limit space X_∞ as

$$X_\infty = \{\underline{x} \in \prod_{\alpha \in \mathcal{A}} X_\alpha | f_{\alpha\beta} \circ \pi_\beta(\underline{x}) = \pi_\alpha(\underline{x}) \text{ whenever } \beta \geq \alpha\}.$$

For our purpose we only consider the countable indexing set $\mathbb{N}$; our factor spaces $\{X_n\}_{n=0}^{\infty}$ are metric spaces; and the bonding maps are continuous. For simplicity of notation we denote the bonding maps by $\{f_n : X_{n+1} \to X_n\}_{n=0}^{\infty}$ so that for $m \geq n$, f_{nm} is written as $f_n \circ f_{n+1} \ldots \circ f_{m-1}$. We choose the standard notation (X_n, f_n) for the inverse limit space X_∞ and casually refer to it as the "inverse limit". By x_n we mean $\pi_n(\underline{x})$. Then

$$(X_n, f_n) = \{\underline{x} \in \prod_{n=0}^{\infty} X_n | f_n(x_{n+1}) = x_n\}$$

with the induced metric topology is a metric space which is compact (connected) whenever the X_n are compact (connected). In fact, if $\{X_n\}$ are $\mathcal{P}$-like and $\{f_n\}$ are surjections $\forall n \in \mathbb{Z}^+$ then (X_n, f_n) is $\mathcal{P}$-like [2]. This implies that the inverse limit of arc-like continua is arc-like. Since arc-like is synonymous with chainable [2], we see that the inverse limit of chainable continua with onto bonding maps is a chainable continuum.

In the case where we have an inverse limit (X_n, f_n) with a single bonding map, there is an induced homeomorphism $\hat{f}$ on (X_n, f_n) called the shift map, defined by $\hat{f}((x_0, x_1, \ldots,)) = (f(x_0), x_0, x_1, \ldots)$.

A continuous function is a near-homeomorphism if it can be uniformly approximated by a sequence of homeomorphisms.

In this paper we appeal to the following theorem of Morton Brown [3]:

> Let $S = (X_n, f_n)$ where the X_n are all homeomorphic to a compact metric space X, and for all n, f_n is a near-homeomorphism. Then S is homeomorphic to X.

CHAPTER 3

MAIN THEOREM

In this chapter we prove the necessary and sufficient conditions for our theorem:

> **Theorem 3.1** A continuum in the plane is disk-chainable if and only if it is embeddable using inverse limits.

Before proceeding with this proof, we are in need of some vocabulary, a technical proposition and two supporting lemmas.

Suppose (X, d) is a compact metric space and

$$\mathcal{C}(X) = \{C | C \text{ is a compact subset of } X\}.$$

Then $d_H : \mathcal{C}(X) \times \mathcal{C}(X) \to \mathbb{R}^+$, the Hausdorf metric on $\mathcal{C}(X)$ is given by

$$d_H(C_1, C_2) = \inf\{r | C_i \subseteq N_r(C_j) \qquad i \neq j \in \{0,1\}\}$$

where

$$N_r(C_j) = \cup_{x \in C_j} B_r(x)$$

and

$$B_r(x) = \{y \in X | d(x,y) < r\}.$$

That is, $N_r(C_j)$ is an open neighborhood about the set C_j in the d-metric topology on X.

An equivalent definition is useful:

$$d_H(C_1, C_2) = \max\{d(x_i, C_j) | x_i \in C_i, i \neq j \in \{0, 1\}\}$$

where $d(x_i, C_j)$ is the usual distance between a point and a set.

Lemma 3.2: If (X, d) is a compact metric space, then $(\mathcal{C}(X), d_H)$ is also a compact metric space.

Proof of lemma. It is known that $(\mathcal{C}(X), d_H)$ is a complete metric space [6]. Therefore it suffices to show that $(\mathcal{C}(X), d_H)$ is totally bounded.

In the metric topology on $(\mathcal{C}(X), d_H)$, let $B_\epsilon(C)$ be an open ϵ-ball about the point C, that is

$$B_\epsilon(C) = \{D \in \mathcal{C}(X) | d_H(D, C) < \epsilon\}.$$

Given $\epsilon > 0$, we will produce a finite number of elements of $\mathcal{C}(X)$, $\{C_i\}_{i=1}^{s}$, such that $\cup_{i=1}^{s} B_\epsilon(C_i) = \mathcal{C}(X)$.

With a given $\epsilon > 0$, cover X with $\epsilon/4$-balls, $\{X_i\}$, and consider a finite subcover $\{X_1, X_2, \ldots, X_n\}$ where $n = n(\epsilon)$. Let $x_i \in X_i$, $i \in \{1, 2, \ldots, n\}$. Consider the power set $\mathcal{P}(\{x_1, \ldots, x_n\})$. $\# \mathcal{P}(\{x_1, \ldots, x_n\}) = \sum_{k=0}^{n} \binom{n}{k} = 2^n$. Let $\{C_i\}_{i=1}^{2^n}$ be the 2^n elements of the power set. Claim:

$$\mathcal{C}(X) = \cup_{i=1}^{2^n} B_\epsilon(C_i).$$

The proof of the claim is as follows.

(i) Clearly $\cup_{i=1}^{2^n} B_\epsilon(C_i) \subseteq \mathcal{C}(X)$.

(ii) To show containment in the other direction, let $D \in \mathcal{C}(X)$ and then consider

D as a subset of X. $\exists\{i_0, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ such that $D \cap X_i \neq \emptyset$, $i \in \{i_0, \ldots, i_k\}$ and $D \cap X_i = \emptyset$ for $i \notin \{i_0, \ldots, i_k\}$.

Let $C_{\hat{i}} = \{x_{i_0}, \ldots, x_{i_k}\}$. We show that $D \in B_\epsilon(C_{\hat{i}})$ as follows:

(a) $D \subseteq \cup_{j=0}^{k} X_{i_j} \Rightarrow$

$$\forall x \in D \qquad \exists j \text{ such that } d(x, x_{i_j}) < \epsilon/2$$

which implies

$$D \subseteq N_{\epsilon/2}(C_{\hat{i}})$$

since $C_{\hat{i}} = \{x_{i_0}, \ldots, x_{i_k}\}$ and $N_{\epsilon/2}(C_{\hat{i}}) = \cup_{j=0}^{k} B_{\epsilon/2}(x_{i_j})$.

(b) On the other hand,

$$N_{\epsilon/2}(D) = \cup_{x \in D} B_{\epsilon/2}(x)$$

and we have that

$$D \cap X_{i_j} \neq \emptyset \qquad \forall j \in \{0, \ldots, k\}.$$

We recall that X_{i_j} is an open $\epsilon/4$-ball in (X, d). Thus

$$\exists x \in D \text{ such that } d(x, x_{i_j}) < \epsilon/2$$

implies that for each $j \in \{0, \ldots, k\}$

$$x_{i_j} \in B_{\epsilon/2}(x) \text{ for some } x \in D,$$

which implies that

$$C_{\hat{i}} \subseteq N_{\epsilon/2}(D).$$

Then we see that

$$d_H(D, C_{\hat{i}}) = \inf\{r | C_{\hat{i}} \subseteq N_r(D) \text{ and } D \subseteq N_r(C_{\hat{i}})\} \leq \epsilon/2 < \epsilon,$$

$\Rightarrow$

$$D \in B_\epsilon(C_{\hat{i}})$$

$\Rightarrow$

$$\mathcal{C}(X) \subseteq \cup_{i=1}^{2^n} B_\epsilon(C_i).$$

We have shown that $(\mathcal{C}(X), d_H)$ is totally bounded in addition to being complete, therefore $(\mathcal{C}(X), d_H)$ is compact. ■

Our object here is this: $(\mathcal{C}(X), d_H)$ being a compact metric space is sequentially compact.

A function is called monotone if point inverses are connected. We next show that in a certain setting, near-homeomorphisms are monotone.

Proposition 3.3: If (X, d) is a compact, locally connected metric space and $H : X \to X$ is a near- homeomorphism, then H is monotone.

Proof. We wish to show that any point inverse is connected. We will do so by demonstrating that in $(\mathcal{C}(X), d_H)$ a point inverse of H is the limit of a sequence of (compact) connected sets in X, and is therefore itself (compact) connected [6].

Let $c \in X$ and $H = \lim_{n \to \infty} H_n$ where $H_n : X \to X$ are homeomorphisms. Given $\epsilon' > 0$, by local connectedness $\exists 0 < \epsilon \leq \epsilon'$ and a U_ϵ, open and connected with *diam* $U_\epsilon < \epsilon$ and $c \in U_\epsilon \subseteq B_{\epsilon'}(c)$. We note that $\bar{U}_\epsilon$ is compact and connected, as is $H_n^{-1}(\bar{U}_\epsilon)$. Also, $x \in H^{-1}(c)$ if and only if given any $\eta > 0$, there is an integer $N(\eta)$, dependent on η, such that $n \geq N \Rightarrow d(H_n(x), c) < \eta$.

Define

$$A_k = H_{n(k)}^{-1}(\bar{U}_{1/k}) \text{ for } k > 0 \text{ and } n(k) \geq N(1/k).$$

We see that $\{A_k\}_{k=1}^{\infty}$ has a convergent subsequence because in the previous lemma $(\mathcal{C}(X), d_H)$ was shown to be sequentially compact. For simplicity of notation, call this subsequence $\{A_k\}_{k=1}^{\infty}$. It is now possible to find a subsequence of this $\{A_k\}_{k=1}^{\infty}$ such that if $k > j$ then $n(k) > n(j)$. Again call this subsequence $\{A_k\}_{k=1}^{\infty}$ and let $A = \lim_{k\to\infty} A_k$.

Claim:

$$A = H^{-1}(c).$$

The claim is proven as follows. (i) Suppose $x \in H^{-1}(c)$. Then by definition, $x \in A_k \quad \forall k$. If $x \notin A$, then because A is compact we can say that $d(x, A) = r > 0$. $\exists k(r)$ such that $k \geq k(r) \Rightarrow d_H(A, A_k) < r/2$. This yields $A_k \subseteq N_{r/2}(A)$. But x cannot be in an $r/2$-neighborhood about A since the distance between x and A is r which is larger than $r/2$. Then it cannot be that x is in A_k, which by definition it is. Hence a contradiction. Thus $x \in H^{-1}(c)$ implies $x \in A$ which implies $H^{-1}(c) \subseteq A$.

(ii) Suppose $x \in A$. We wish to show that x must be in $H^{-1}(c)$. To show $x \in H^{-1}(c)$ it suffices to do the following: given $\epsilon > 0$, produce a p such that $d(H_n(x), c) < \epsilon \quad \forall n \geq n(p)$.

Let $\epsilon > 0$.

(a) Because $\{H_n\}$ is a uniformly Cauchy sequence,

$$\exists I \text{ such that } i_1, i_2 \geq I$$

$\Rightarrow$

$$d(H_{i_1}(y), H_{i_2}(y)) < \epsilon/6 \qquad \forall y \in X.$$

(b) Choose k such that

$$1/k < \epsilon/4$$

and

$$n(k) \geq I.$$

(c) Since $H_{n(k)}$ is continuous, we have the existence of a $\delta(k, \epsilon)$ such that

$$d(x, y) < \delta \Rightarrow d(H_{n(k)}(x), H_{n(k)}(y)) < \epsilon/4.$$

(d) Because $\{A_k\}$ converges we have a J such that

$$j \geq J \Rightarrow d_H(A, A_j) < \delta.$$

Then

$$\max\{d(y, A_j), d(y_j, A) | y \in A, y_j \in A_j\} < \delta$$

$\Rightarrow$

$$d(x, A_j) < \delta$$

$\Rightarrow$

$$\exists y_j \in A_j \text{ such that } d(x, y_j) < \delta.$$

Now let $p > \max\{J, k\}$. Then

$$p > k \Rightarrow 1/p < 1/k < \epsilon/4$$

and

$$n(p) > n(k) \geq I.$$

Further,

$$\exists y \in A_p \text{ such that } d(x, y) < \delta.$$

This implies

$$d(H_{n(k)}(x), H_{n(k)}(y)) < \epsilon/4$$

by part *(c)*. By the triangle inequality we obtain

$$d(H_{n(p)}(x), c) < 5\epsilon/6$$

as follows:

$$d(H_{n(p)}(x), c) \leq d(H_{n(p)}(x), H_{n(k)}(x))$$

$$+d(H_{n(k)}(x), H_{n(k)}(y)) + d(H_{n(k)}(y), H_{n(p)}(y)) + d(H_{n(p)}(y), c)$$

$$< \epsilon/6 + \epsilon/4 + \epsilon/6 + \epsilon/4 = 5\epsilon/6.$$

And by part *(a)*

$$d(H_n(x), H_{n(p)}(x)) < \epsilon/6 \qquad \forall n \geq n(p) > n(k) \geq I$$

so

$$d(H_n(x), c) \leq d(H_n(x), H_{n(p)}(x)) + d(H_{n(p)}(x), c) < 5\epsilon/6 + \epsilon/6 = \epsilon.$$

Hence we have show that given $\epsilon > 0$ and $x \in A$, $\exists p$ such that $n \geq n(p)$ implies

$$d(H_n(x), c) < \epsilon$$

$\Rightarrow$

$$x \in H^{-1}(c)$$

$\Rightarrow$

$$A \subseteq H^{-1}(c).$$

Parts (i) and (ii) prove the claim that $A = H^{-1}(c)$. But A being the limit of (compact) connected sets is (compact) connected. Thus $H^{-1}(c)$ is connected, and H is monotone. ∎

Lemma 3.4: If H is a near-homeomorphism and U is an open disk, then $H^{-1}(U)$ is simply connected.

Proof. Let $H = \lim_{t\to 0} H_t$ where H_t are homeomorphisms. Let S be a simple closed curve in the open set $H^{-1}(U)$. S is compact, thus $H(S)$ is a compact subset of $U = H(H^{-1}(U))$. There exist a $\delta > 0$ and a $T > 0$ such that for all $t \leq T$, $H_t(S) \subseteq U$; in fact, $d(H_t(S), \partial U) > \delta > 0$.

Now the region B bounded by S is an open disk. So $H_t(B)$ is an open disk contained in U.

$$H(B) = \lim_{t\to 0} H_t(B)$$

and

$$d(H_t(B), \partial U) = d(H_t(S), \partial U) > \delta > 0 \quad \text{for } t \leq T.$$

This implies that $H(B) \subseteq U$ which implies that

$$B \subseteq H^{-1}(H(B)) \subseteq H^{-1}(U).$$

Thus every simple closed curve in $H^{-1}(U)$ bounds a region which is entirely contained in $H^{-1}(U)$. This implies that $H^{-1}(U)$ is simply connected. ■

We now prove the necessary conditions of Theorem 3.1:

> If a continuum is embeddable using inverse limits then it is disk-chainable.

Proof. Let D be a compact disk and I be a closed arc in D; $D_n = D$, $I_n = I$ and $I_n \subseteq D_n \quad \forall n \in \mathbb{Z}^+$. Suppose there are near-homeomorphisms $F_n : D_{n+1} \to D_n$, where $F_{n|I_{n+1}} = f_n : I_{n+1} \to I_n$ is a surjection. We know that $(D_n, F_n) \cong D$. Let $\Phi\Big((D_n, F_n)\Big) = D$ where Φ is a homeomorphism. Then (I_n, f_n) and hence $\Phi\Big((I_n, f_n)\Big)$ is a chainable continuum [2]. $\Phi\Big((I_n, f_n)\Big)$ is embeddable using inverse limits. We will show that $\Phi\Big((I_n, f_n)\Big)$ is a disk-chainable embedding by chaining (I_n, f_n) in (D_n, F_n) with chains whose links are topological disks, and the intersections of the links are also disks.

We notice the following. Given any n, let $C = \{l_1, \dots, l_m\}$ be a δ-chaining of I_n by disks where $l_i \cap l_j$ is a disk $\forall i, j \in \{1, \dots, m\}$.

(*i*) $F_n^{-1}(l_j)$ is non-empty and open because F_n is continuous; $F_n^{-1}(l_j)$ is connected by proposition 3.3; and $F_n^{-1}(l_j)$ is simply connected by lemma 3.4. That is, $F_n^{-1}(l_j)$ is a topological disk.

(*ii*) $\cup_{j=1}^{m} F_n^{-1}(l_j)$ covers I_{n+1}.

(iii) $\left(l_i \cap l_j \neq \emptyset \text{ iff } F_n^{-1}(l_i) \cap F_n^{-1}(l_j) \neq \emptyset\right) \Rightarrow \{F_n^{-1}(l_1), \ldots, F_n^{-1}(l_m)\}$ is a chaining of I_{n+1}.

(iv) $F_n^{-1}(l_i) \cap F_n^{-1}(l_j) = F_n^{-1}(l_i \cap l_j)$ is a disk for the same reasons as given in (i).

Let $\epsilon > 0$ be given. Let n be large enough such that

$$\frac{1}{2^{n-3}} < \frac{\epsilon}{diam \quad D}.$$

Choose

$$\delta < \min\{(1 - \frac{1}{2^n}) \cdot \epsilon, \alpha\}$$

where $\alpha > 0$ is chosen in a manner such that if

$$C = \{l_1, \ldots, l_m\} \quad \text{is an } \alpha - \text{chain}$$

then

$$diam \quad F_{n-1-k} \circ \cdots \circ F_{n-2} \circ F_{n-1}(l_j) < \frac{2^{n-2}}{2^n - 1} \cdot \epsilon \qquad \text{for} \qquad k = 0, 1, \ldots, n-1.$$

For this n, $\exists\delta$-chain of I_n whose links are connected. Call this δ-chain of I_n

$$C_n = \{l_1, l_2, \ldots, l_{m_n}\}.$$

Let

$$\hat{l}_j = \{\underline{x} \in (D_k, F_k) | x_n \in l_j\} \qquad \text{for} \quad j \in \{1, 2, \ldots, m_n\}$$

and let

$$\hat{C}_n = \{\hat{l}_1, \hat{l}_2, \ldots, \hat{l}_{m_n}\}.$$

Claim: $\hat{C}_n$ is an ϵ-chain of (I_k, f_k) whose links, and the intersections of those links, are connected. This claim is established in the following manner.

(i) We first establish that

$$diam \ \hat{l}_j < \epsilon \qquad \forall j \in \{1, 2, \ldots, m_n\}.$$

Let $\underline{x}, \underline{y} \in \hat{l}_j$. Then

$$\begin{aligned} d(\underline{x}, \underline{y}) &= \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i} \\ &\leq \sum_{i=0}^{n-1} \frac{|x_i - y_i|}{2^i} + \sum_{i=n}^{\infty} \frac{diam \quad D}{2^i} \\ &\leq \sum_{i=0}^{n-1} \frac{diam \quad F_{n-1-i} \circ \cdots \circ F_{n-2} \circ F_{n-1}(l_j)}{2^{n-1-i}} + (diam \quad D)(\frac{1}{2^{n-1}}) \\ &< \frac{2^{n-2}}{2^n - 1} \cdot \epsilon \sum_{i=0}^{n-1} \frac{1}{2^i} + \frac{\epsilon}{4} \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{4} = 3\epsilon/4. \end{aligned}$$

Thus

$$diam \quad \hat{l}_j = \sup_{\underline{x}, \underline{y} \in \hat{l}_j} d(\underline{x}, \underline{y}) \leq 3\epsilon/4 < \epsilon.$$

(ii) We next observe the $\hat{C}_n$ covers (I_k, f_k). Let

$$\underline{x} = (x_0, x_1, \ldots) \in (I_k, f_k) \subseteq (D_k, F_k).$$

Then

$$\pi_n(\underline{x}) = x_n \in I_n \Rightarrow \exists j \in \{1, 2, \ldots, m_n\} \text{ such that } x_n \in l_j.$$

By definition

$$\hat{l}_j = \{\underline{y} \in (D_k, F_k) | y_n \in l_j\} \Rightarrow \underline{x} \in \hat{l}_j.$$

(iii) We also see that $\hat{l}_j \cap \hat{l}_i \neq \emptyset$ iff $|j - i| \leq 1$, as follows: $\hat{l}_j \cap \hat{l}_i \neq \emptyset$ iff $\exists \underline{x} \in (\hat{l}_j \cap \hat{l}_i)$ iff $\exists n$ such that $x_n \in (l_j \cap l_i)$ iff $(l_j \cap l_i) \neq \emptyset$ iff $|j - i| \leq 1$ since $C_n = \{l_1, \ldots, l_{m_n}\}$ is a δ-chain of I_n.

(iv) Finally we see that the elements of $\hat{C}_n$ are open and connected sets with connected intersections.

Let $\underline{x} \in \hat{l}_j$ for any $\hat{l}_j \in \hat{C}_n$. Consider the map $\underline{x} = (x_0, x_1, \ldots) \rightarrow (x_n, x_{n+1}, \ldots)$ i.e. $h : (D_k, F_k) \rightarrow (D_k, F_k)$ by $h(\underline{x}) = \underline{y}$ where $y_k = x_{n+k}$. Clearly h is a homeomorphism. For simplicity of notation, let $\hat{l}_j$ now denote $h(\hat{l}_j)$ and $\hat{C}_n$ denote $h(\hat{C}_n)$ as follows:

$$\hat{C}_n = \{\hat{l}_j\}_{j=1}^{m_n} \text{ where } \quad \hat{l}_j = \{\underline{y} \in h((D_k, F_k)) | y_0 \in l_j\}.$$

Then

$$\pi_k(\hat{l}_j) = \begin{cases} l_j & \text{for } k = 0 \\ F_{n+k-1}^{-1} \circ \cdots \circ F_{n+1}^{-1} \circ F_n^{-1}(l_j) & \text{for } k > 0 \end{cases}.$$

Thus $\pi_k(\hat{l}_j)$ is a disk. And we see that

$$\hat{l}_j = (\pi_k(\hat{l}_j), F_{n+k})$$

and

$$\overline{\hat{l}_j} = \overline{(\pi_k(\hat{l}_j), F_{n+k})} = (\overline{\pi_k(\hat{l}_j)}, F_{n+k}).$$

Invoking Brown's theorem, we have $\hat{l}_j = \mathrm{int}\overline{\hat{l}_j}$ is an open disk. This makes $\hat{C}_n$ a chaining by disks. Furthermore, since $l_i \cap l_j$ is a disk, the same argument gives us $\hat{l}_i \cap \hat{l}_j = \widehat{l_i \cap l_j}$ is a disk.

In conclusion, $\forall \epsilon > 0 \quad \exists \hat{C}_n$, an ϵ-chain of (I_n, f_n) whose links and intersections of those links are disks. We have shown that the continuum $\Phi\big((I_n, f_n)\big)$ which is embeddable using inverse limits is disk-chainable. □

We proceed with the proof of the sufficient conditions for theorem 3.1:

> If X is a disk chainable embedding of a chainable continuum, then X is embeddable using inverse limits.

Proof. Let C be a chainable continuum and X a disk chainable embedding of C in the closed disk D. We can make the following construction:

(i) $\{B^j\}_{j=0}^{j=\infty}$ is a defining sequence of chains for X where

(a) the links $\{B^j_m\}_{m=1}^{m(j)}$ of B^j are connected,

(b) the closures of the intersections of adjacent links in a chain are connected,

(c) $diam \quad B^j_m = \bar{\epsilon}_j \quad \forall m \in \{1, 2, \ldots, m(j)\}$ where $\lim_{j\to\infty} \bar{\epsilon}_j = 0$;

(ii) $K_j \subseteq \cup_{m=1}^{m(j)} B^j_m$ is a closed arc such that

(a) $K_j \cap (B^j_m \cap B^j_{m-1})$ is an arc $\forall m \in \{2, 3, \ldots, m(j)\}$ and

(b) the endpoints of K_j are in $X \cap B^j_1$ and $X \cap B^j_{m(j)}$ respectively.

(c) $B^j_1 \cap X$ is not a subset of B^j_2 and $B^j_{m(j)} \cap X$ is not a subset of $B^j_{m(j)-1}$.

We now have "nice" disks with arcs running through them and can construct maps which "collapse" the disks onto the arcs in the following manner:

(iii) $G_j : D \to D$ are Lipschitz near-homeomorphisms with Lipschitz constants $\bar{\delta}_j$; and there are open neighborhoods V_j of $\cup_{m=1}^{m(j)} B^j_m$ where

(a)

$$V_0 \subseteq D, \qquad \cup_{m=1}^{m(j)} B^j_m \subseteq V_j \subseteq \cup_{m=1}^{m(j-1)} B^{j-1}_m \quad \text{for} \quad j = 1, 2, \ldots$$

(b)

$$G_j(V_j) = K_j \quad \forall j = 0, 1, 2, \ldots$$

(c)

$$G_j(B^j_m) \subseteq B^j_m \qquad \forall m \in \{1, 2, \ldots, m(j)\}; \qquad j \in \{0, 1, 2, \ldots, \}$$

and

$$G_j(B^{j-1}_m) \subseteq B^{j-1}_m$$

(d) G_j restricted to the complement of $\cup_{m=1}^{m(j-1)} B^{j-1}_m$ is the identity, $j = 1, 2, \ldots$ and G_0 is the identity outside some neighborhood of V_0.

(a) through *(d)* gives us

$$\|G_j - id\| \leq \bar{\epsilon}_{j-1}$$

and

$$d(g_j(x), x) \leq \bar{\epsilon}_j \quad \text{where} \quad g_j = G_{j|K_{j+1}}.$$

Now we will choose a subsequence $\{C^j_m\}$ of $\{B^j_m\}$. Find a j_0 such that $\bar{\epsilon}_{j_0} < 1/2$. Let

$$\epsilon_0 = \bar{\epsilon}_{j_0} < 1/2; \qquad C^0_m = B^{j_0}_m; \qquad I_0 = K_{j_0}; \qquad U_0 = V_{j_0}.$$

Rename G_{j_0} as F_0 and define

$$\delta_0 = \max\{\bar{\delta}_{j_0}, 1\}.$$

Next choose $j_1 > j_0$ with $\bar{\epsilon}_{j_1} < \frac{1}{2\delta_0}$. Let $\epsilon_1 = \bar{\epsilon}_{j_1}$ and rename $B^{j_1}_m, K_{j_1}, V_{j_1}$ and G_{j_1} as C^1_m, I_1, U_1, F_1, respectively. Define $\delta_1 = \max\{\bar{\delta}_{j_1}, 1\}$.

Choose $j_2 > j_1$ with $\bar{\epsilon}_{j_2} < \frac{1}{2\delta_0\delta_1 2^2}$ and proceed to reindex the chain, arc, open set and function, and define a new δ as above.

Continue thus, so that

$$\epsilon_k < \frac{1}{2\delta_0\delta_1\cdots\delta_{k-1}\cdot k^k} \quad \text{and} \quad \delta_k = \max\{\bar{\delta}_{j_k}, 1\}.$$

Let $f_j = F_{j|I_{j+1}}$ and define

$$\Lambda = (J_j, f_{j|J_{j+1}}) \quad \text{where} \quad J_j = \cap_{k\geq j} f_j \circ \cdots \circ f_k(I_{k+1}).$$

We may also think of J_j as

$$J_j = \cap_{k\geq 0} J_j^k \text{ where } J_j^k = f_j \circ \cdots \circ f_{j+k}(I_{j+k+1}).$$

Given a j, we will show that

$$J_j^k \cap C_1^j \neq \emptyset \quad \forall k \geq 0.$$

From our construction, we have that

$$I_{j+1} \cap C_m^{j+1} \neq \emptyset \quad \forall m \in \{1, 2, \ldots, m(j+1)\},$$

and that

$$\exists m_1 \in \{1, 2, \ldots, m(j+1)\} \text{ such that } cl\ C_{m_1}^{j+1} \subseteq C_1^j.$$

Similarly,

$$I_{j+2} \cap C_m^{j+2} \neq \emptyset \quad \forall m \in \{1, 2, \ldots, m(j+2)\}$$

and $\exists m_2$ in this set with $cl\ C_{m_2}^{j+2} \subseteq C_{m_1}^{j+1}$. Thus given an m_k, there is an $m_{k+1} \in \{1, 2, \ldots, m(j+k+1)\}$ with

$$cl\ C_{m_{k+1}}^{j+k+1} \subseteq C_{m_k}^{j+k} \text{ and } I_{j+k+1} \cap C_{m_{k+1}}^{j+k+1} \neq \emptyset.$$

Consider $J_j^0 = f_j(I_{j+1})$. There is an $a_0 \in I_{j+1} \cap C_{m_1}^{j+1}$ and $b_0 = f_j(a_0) \in f_j(I_{j+1} \cap C_{m_1}^{j+1}) \subseteq f_j(I_{j+1}) \cap f_j(C_{m_1}^{j+1}) \subseteq J_j^0 \cap f_j(C_1^j) \subseteq J_j^0 \cap C_1^j$. Thus $J_j^0 \cap C_1^j \neq \emptyset$, since $b_0 \in J_j^0 \cap C_1^j$.

Suppose we have $m_1, m_2, \ldots, m_{k+1}$ where $cl\ C_{m_{k+1}}^{j+k+1} \subseteq \cdots \subseteq cl\ C_{m_1}^{j+1} \subseteq C_1^j$. Then there is an $a_k \in I_{j+k+1} \cap C_{m_k}^{j+k+1}$ with

$$f_j \circ \cdots \circ f_{j+k}(a_k) \in f_j \circ \cdots \circ f_{j+k}(I_{j+k+1} \cap C_{m_k}^{j+k+1})$$

$$\subseteq f_j \circ \cdots \circ f_{j+k}(I_{j+k+1}) \cap f_j \circ \cdots \circ f_{j+k}(C_{m_{k+1}}^{j+k+1})$$

$$\subseteq J_j^k \cap f_j \circ \cdots \circ f_{j+k-1}(C_{m_k}^{j+k}).$$

The key step here is that

$$f_{j+k}(C_{m_{k+1}}^{j+k+1}) \subseteq f_{j+k}(C_{m_k}^{j+k}) \subseteq C_{m_k}^{j+k}.$$

So we see that

$$b_k = f_j \circ \cdots \circ f_{j+k}(a_k) \in J_j^k \cap C_1^j.$$

Thus

$$J_j^k \cap C_1^j \neq \emptyset \quad \forall k \geq 0;$$

and

$$\lim_{k\to\infty} b_k \in cl\ C_1^j,$$

$$\lim_{k\to\infty} b_k \in \cap_{k\geq 0} J_j^k$$

implies

$$J_j \cap cl\ C_1^j \neq \emptyset.$$

We now see that our choice of the first link C_1^j of C^j was not crucial to the argument. We could in fact have shown

$$J_j \cap cl\ C_m^j \neq \emptyset \quad \forall m \in \{1, 2, \ldots, m(j)\}.$$

All told, J_j is not only a compact, connected, non-empty (nested intersection of non-empty compact sets) subset of an arc, but has interior as well (i.e. is not a point). Thus Λ is a non-degenerate planar continuum.

To complete the proof, it suffices to show that Λ and X are equivalently embedded, and that this makes X embeddable using inverse limits. We define

$$H : (D, F_j) \to D \text{ by } H(\underline{z}) = \lim_{n \to \infty} z_n$$

where $z_n = \pi_n(\underline{z})$. We now establish that H is the homeomorphism we seek by verifying that

(*i*) H is well-defined,

(*ii*) H is continuous,

(*iii*) $H(\Lambda) = X$,

(*iv*) H is one to one,

(*v*) H is onto.

(*i*) *Proof that H is well defined.* We wish to show that $\{z_n\}$ has a unique limit point. Suppose not. Suppose w and y are both cluster points for $\{z_n\}$; and $d(w, y) = \delta > 0$. Then $\exists$ subsequences $\{z_{n_j}\}$ and $\{z_{n_k}\}$ such that $z_{n_j} \to w$, $z_{n_k} \to y$. Thus, given $\delta/3$, $\exists J, K$ such that

$$\{z_{n_j}\} \subseteq B_{\delta/3}(w) \quad \text{for} \quad j \geq J,$$